Numerical Analysis of the evolving surface finite element method for some parabolic problems

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This thesis is based on the following papers (status as of November 1, 2017):

a) B. Kovács and Ch. A. Power Guerra. Higher-order time discretizations with ALE finite elements for parabolic problems on evolving surfaces. *IMA J. Numer. Anal.* (2017). DOI:10.1093/imanum/drw074, arXiv:1410.0486.

b) B. Kovács and Ch. A. Power Guerra. Error analysis for full discretizations of quasilinear parabolic problems on evolving surfaces. *Numer. Methods Partial Differential Equations* **32** (2016), no. 4, p. 1200–1231. arXiv:1503.08990.

c) B. Kovács and Ch. A. Power Guerra. Maximum norm stability and error estimates for the evolving surface finite element method. *Numer. Methods Partial Differential Equations* (2017). DOI:10.1002/num.22212, arXiv:1510.00605.

d) B. Kovács, B. Li, Ch. Lubich, Ch. A. Power Guerra. Convergence of finite elements on an evolving surface driven by diffusion on the surface. *Numer. Math.* **137** (2017), no. 3, p. 643–689. arXiv:1607.07170.

The author's contribution to the articles mentioned above was

a) 50%,

b) 40%,

c) 60%,

d) 20%.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der *evolving surface finite element method* (ESFEM), zu Deutsch, Finite-Elemente-Methode auf bewegten Oberflächen. Eines der vielen Einsatzgebiete der ESFEM ist das Nähern von Lösungen parabolischer partieller Differentialgleichungen auf bewegten Oberflächen. Solche Gleichungen spielen bei der Modellierung von Problemen der mathematischen Biologie eine Rolle. Beispielsweise kann das Wachstum eines Tumors auf diese Weise modelliert werden.

Das Hauptaugenmerk dieser Arbeit liegt auf der numerischen Analyse der ESFEM, angewandt auf parabolische Probleme. Es werden fünf verschiedene Problemstellungen betrachtet.

Die Arbeit ist wie folgt aufgebaut: Zu Beginn werden im Wesentlichen bekannte Ergebnisse zusammengefasst, auf die im Laufe der Arbeit verwiesen wird. Im nachfolgenden Kapitel werden die Problemstellungen, die in dieser Arbeit behandelt werden, vorgestellt. Für jedes Problem wird die partielle Differentialgleichung, die gewählte numerische Methode und das Endresultat angegeben. Nach diesen einleitenden Kapiteln folgt die ausführliche Betrachtung der Beweise der Theoreme. Alle wichtigen Techniken und

Ideen werden vorgestellt und erläutert. Im Anschluss daran wird verdeutlicht, welche dieser Techniken und Ideen neu sind und bisher noch nicht bekannt waren. Zu jeder Problemstellung werden numerische Experimente vorgestellt. Im letzten Kapitel wird noch ein neues Resultat bewiesen, dass während der Entstehung dieser Dissertation entdeckt worden ist.

Die vier bereits publizierte Forschungsresultate, sind im Anhang beigefügt.

Abstract

The present work investigates the *evolving surface finite element method* (ESFEM). One of its many applications is to approximate the solution of a parabolic partial differential equation on an evolving surface. Such equations are of interest for mathematical biology. For example, the growth of a tumor can be modeled with such an equation.

The main interest of this work is the numerical analysis of the ESFEM, applied on some parabolic problems. Five different problems are considered here.

The structure of the work is as follows: The first chapter summarizes results, which are essentially already known in the literature. The subsequent chapter presents the investigated problems. The author explains for each problem the partial differential equation, the chosen numerical method and the final result. These introductory chapters are followed by a detailed summary and discussion of the proofs of the corresponding theorems. All important techniques and ideas are presented and explained. Afterwards, the author emphasizes the originality of the work. Every problem is concluded by some numerical experiments. In the last chapter the author proves

a novel result, which has been discovered during the preparations of this thesis.

The four already published results are provided in the appendix.

Contents

Chapter 1.

Introduction

Let no one ignorant of geometry enter.

— Inscription over the entrance of the worlds first university at a place called Academy, named after an ancient hero Academus.

The present work studies the stability and convergence of the evolving surface finite element method, i.e. a finite element method for surface problems, where the surface is changing in time. In all our problems we are considering advection diffusion equations, which are used to model important problems. We briefly describe our problems. Our first problem consists of analyzing an arbitrary Lagrangian-Eulerian evolving surface finite element method. It is well known that for the standard evolving surface finite element method mesh distortion may quickly happen. Thus, it make sense to analyze numerical schemes, which try to prevent mesh distortion. The second problem consists of analyzing a quasilinear advection diffusion equation. Since many important problem are nonlinear, the author considers analyzing such a problem as interesting. The third problem is the derivation of maximum norm bounds for the evolving surface finite element method. The author thinks that the topic fits well within the current literature, since they are a nontrivial extension of both L^2-based bounds of evolving surface finite elements and maximum-norm bounds of finite elements on Euclidean domains. Our forth problem consists of analyzing finite elements on an evolving surface driven by diffusion. Numerical experiments, done by the author, show that regularizing with a velocity law, like it is done in the problem, is competitive to the standard mean curvature regularization. Hence, analyzing such a scheme is of great interest. The fifth problem consists of analyzing the last mentioned problem for linear finite elements. Since the majority of evolving surface finite element simulations are performed with linear finite elements, the author considers this problem as important. The analysis of this fifth problem has been done during the preparation of this thesis. It has not been submitted yet.

The thesis is organized as follows: First, we introduce notation and recall basic results from the literature. Then, we present the objectives of our four submitted articles. This is followed by a carefully selected summary of the submitted and published results. The summary is enriched by additional content, which for different reasons was not included in the published articles. The last chapter analyzes the fifth problem.

Acknowledgments

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Chapter 2.

Notation and preliminaries

In this chapter we introduce the evolving surface finite element method (ESFEM) and collect results known from the literature. If the reader is familiar with the ESFEM, then the author recommends the following quick guide for the notation: The *generic notation* below, the notation "X is an evolving surface" (2.11), the notation for material derivative (2.13) and velocity (2.14), the notation for bilinear forms (2.19) with their product rule (2.20) and the finite element nodal value notation (2.29).

This chapter is organized as follows: The first section introduces evolving surfaces and cover necessary analytic tools for them. The next two sections are devoted to ESFEM. First, evolving meshes with finite element functions are covered. Then, the lifting process with some geometric perturbation errors are presented. The preliminaries are closed with some finite element error bounds.

Generic notation

The author denotes the generic constant with const. The notation const (1.1) is used, if the author wants to refer to the generic constant from equation (1.1). ${}^{t}v$ means the transpose of v. The symbol $\mathbb{1}_X \colon X \to X$ denotes the identity map. $I_d \in \mathbb{R}^{d \times d}$ is used for the unit matrix of dimension $d \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

2.1. Evolving surfaces

The aim of this section is to give a rigorous definition of an evolving surface together with the necessary analytic tools and identities to efficiently work with them. Since the finite element method, analyzed in this work, does not use level set functions, we restrain ourselves to parametric evolving surfaces. This section is organized as follows: The author begins with a rigorous definition of surfaces. Then, tangent space and normal vector fields are introduced. Then, the oriented distance function is explained. Next, the tangential gradient, the extended Weingarten, the mean curvature, the tangential divergence, the tangential Jacobian, the Laplace-Beltrami operator and the tangential Hessian are defined. Intrinsic definitions are used, which imply the corresponding extrinsic characterizations. Afterwards, Lebesgue and Sobolev spaces on surface are given. Finally, evolving surfaces are defined. The author extends the oriented distance function to the time depending case and concludes with the definition of the material derivative together with some useful product rules.

Surfaces

We write $\Omega \Subset \mathbb{R}^d$, if and only if Ω is an open set with compact support.

For $k \in \mathbb{N}$ we call $\Gamma \subset \mathbb{R}^3$ a **surface** of class C^k, if and only for every point $x \in \Gamma$ there exists an open neighborhood $U \Subset \mathbb{R}^3$ of x, and a parametrization $X \in C^k(\Omega; \mathbb{R}^3)$, with $\Omega \Subset \mathbb{R}^2$, such that X is injective, $X(\Omega) = U \cap \Gamma$ and the Jacobian of X has full rank. X is called **local parametrization** and we set $\theta \coloneqq X^{-1}$.

We say $f \in C^k(\Gamma)$, if for every local parametrization X of Γ we have $f \circ X \in C^k(\Omega; \mathbb{R})$.

The **tangent space** at $x \in \Gamma$ is the linear span of $\partial_1 X$ and $\partial_2 X$, where X is a local parametrization around x. The subspace is denoted with $\mathrm{T}_x\Gamma$ and $\xi \in \mathrm{T}_x\Gamma$ is called **tangent vector**. $\mathrm{T}_x\Gamma$ is well-defined, i.e. it is independent of X. A vector orthogonal to $\mathrm{T}_x\Gamma$ is called **normal vector**, if it has unit length. We call $\mathrm{n} \in C^{k-1}(\Gamma; \mathbb{R}^3)$ a **normal field**, if for every $x \in \Gamma$ we have that $\mathrm{n}(x)$ is a normal vector.

From now on we will always assume that $\Gamma \in C^2$ is a compact surface equipped with the outwards pointing normal field n.

Oriented distance function

For $\mathcal{N} \Subset \mathbb{R}^3$ we say that $d_\Gamma \in C^2(\mathcal{N})$ is an **oriented distance function** for Γ, if $\Gamma \subset \mathcal{N}$ and for each $p \in \mathcal{N}$ there exists only one $x \in \Gamma$ with minimal distance to p and in addition we have the formula

$$p = x + d_\Gamma(p)\mathrm{n}(x). \tag{2.1}$$

For $X \in C^2$ such a function always exists, cf. [46, lemma 14.16]. As a consequence a function $f\colon \Gamma \to \mathbb{R}$ can always be extended to $\bar{f}\colon \mathcal{N} \to \mathbb{R}$ be setting $\bar{f}(p) \coloneqq f(x)$. In particular this applies to n itself and a short calculations reveals

$$\nabla_{\mathbb{R}^3} d_\Gamma = \bar{\mathrm{n}}. \tag{2.2}$$

From now on we will always assume that n is defined on $\mathcal{N}$ like above. Further, we denote with $\bar{f} \in C^k(\mathcal{N})$ an arbitrary extension of $f \in C^k(\Gamma)$.

Derivatives

For $f \in C^1(\Gamma)$ we define the **tangential gradient** at $x \in \Gamma$ via

$$\nabla f(x) = (\nabla^1 f(x), \dots, \nabla^3 f(x)) \coloneqq \sum_{i,j=1}^{2} \partial_i f(\theta) g^{ij}(\theta) \partial_j X(\theta), \tag{2.3}$$

where $(g^{ij}(\theta))_{i,j=1}^2$ is the inverse of $(g_{ij}(\theta))_{i,j=1}^2$ with $g_{ij}(\theta) \coloneqq \partial_i X(\theta) \cdot \partial_j X(\theta)$. Again, this is well defined. We also have the alternative formula

$$\nabla f(x) = (I_3 - \mathrm{n}(x)\, {}^{\mathrm{t}}\mathrm{n}(x)) \nabla_{\mathbb{R}^3} \bar{f}(x) \eqqcolon \mathrm{pr}_{\mathrm{T}_x\Gamma} \nabla_{\mathbb{R}^3} \bar{f}(x), \tag{2.4}$$

where $\bar{f}$ is an extension of f on $\mathscr{N}$, ${}^{\mathrm{t}}\mathrm{n}$ denotes the transpose of n and $\nabla_{\mathbb{R}^3} \bar{f}(x)$ denotes the $\mathbb{R}^3$-gradient of $\bar{f}$ at x. For $f = (f^1, \dots, f^d) \in C^k(\Gamma; \mathbb{R}^d)$ we take

$$\nabla f(x) = \begin{pmatrix} \nabla f^1(x) & \cdots & \nabla f^d(x) \end{pmatrix} \in \mathbb{R}^{3 \times d}.$$

The extended **Weingarten map** is defined as

$$\mathcal{H}(x) \coloneqq \nabla \mathrm{n}(x) = \nabla_{\mathbb{R}^3} \mathrm{n}(x) = \begin{pmatrix} \nabla_{\mathbb{R}^3} \mathrm{n}^1(x) & \cdots & \nabla_{\mathbb{R}^3} \mathrm{n}^3(x) \end{pmatrix}.$$

Obviously, $\mathcal{H}(x)$ is symmetric with $\mathrm{pr}_{\mathrm{T}_x\Gamma} \mathcal{H}(x) = \mathcal{H}(x) \, \mathrm{pr}_{\mathrm{T}_x\Gamma} = \mathcal{H}(x)$. The **mean curvature** is defined as

$$H(x) = \mathrm{trace}(\mathcal{H}(x)).$$

Note that we are following the convention of not taking the arithmetic mean.

For an arbitrary vector field $F \in C^1(\Gamma; \mathbb{R}^3)$, not necessarily tangent to Γ, we define the **tangential divergence** via

$$\operatorname{div} F(x) \coloneqq \sum_{i,j=1}^{2} g^{ij}(\theta) \partial_i X(\theta) \cdot \partial_j F(\theta). \tag{2.5}$$

We also have the alternative formula

$$\operatorname{div} F(x) = \operatorname{trace}\left(\mathrm{pr}_{\mathrm{T}_x\Gamma}\,{}^{\mathrm{t}}(\mathrm{jac}_{\mathbb{R}^3}\bar{F}(x))\right),$$

where trace denotes the trace of a matrix and $\mathrm{jac}_{\mathbb{R}^3}\bar{F}(x)$ denotes the $\mathbb{R}^3$-Jacobian matrix of $\bar{F}$ at x.

For $F\colon \Gamma_1 \to \Gamma_2$ we define the **tangential Jacobian** as

$$\mathrm{jac}\, F(x) \coloneqq \sum_{i,j,k=1}^{2} \partial_i Y(\rho)\partial_k(\rho^i \circ F \circ X)(\theta) g^{jk}\,{}^{\mathrm{t}}(\partial_j X(\theta)), \tag{2.6}$$

where Y is parametrization around $F(x)$ with $\rho = (\rho^1, \rho^2) \coloneqq Y^{-1}$. Using the chain rule for $F = Y \circ (\rho \circ F)$ one easily sees that we have the following alternative formula

$$\mathrm{jac}\, F(x) = \mathrm{pr}_{\mathrm{T}_{F(x)}\Gamma_2}\,\mathrm{jac}_{\mathbb{R}^3}\bar{F}(x)\,\mathrm{pr}_{\mathrm{T}_x\Gamma_1}.$$

We observe

$$\|{}^{\mathrm{t}}(\mathrm{jac}\, F)\| \leq \|\nabla F\|, \tag{2.7}$$

where $\|\cdot\|$ denotes the operator norm induced by the Euclidean norm. Further, we have the following chain rule

$$\nabla_{\Gamma_1}(f \circ F)(x) = {}^{\mathrm{t}}(\mathrm{jac}\, F(x))\nabla_{\Gamma_2} f(F(x)). \tag{2.8}$$

The **Laplace-Beltrami** operator is given via

$$\Delta f \coloneqq \operatorname{div} \nabla f,$$

where $f \in C^2(\Gamma; \mathbb{R})$. For $f = (f^1, \dots, f^d) \in C^2(\Gamma; \mathbb{R}^d)$ we set $\Delta f = (\Delta f^1, \dots, \Delta f^d)$.

Finally, for $f \in C^2(\Gamma)$ we define the **tangential Hessian** as

$$\nabla^2 f(x) \coloneqq \nabla(\nabla f(x)) = \begin{pmatrix}\nabla(\nabla f)^1(x) & \cdots & \nabla(\nabla f)^3(x)\end{pmatrix}.$$

We have the following alternative formula

$$\begin{aligned}\nabla^2 f(x) &= \mathrm{pr}_{\mathrm{T}_x\Gamma}\,\nabla^2_{\mathbb{R}^3}\bar{f}(x)\,\mathrm{pr}_{\mathrm{T}_x\Gamma} - \mathrm{n}(x)\cdot\nabla_{\mathbb{R}^3}\bar{f}(x)\mathcal{H}(x) - \mathcal{H}\nabla_{\mathbb{R}^3}\bar{f}(x)\,{}^{\mathrm{t}}\mathrm{n}(x)\\ &= \mathrm{pr}_{\mathrm{T}_x\Gamma}\,\nabla^2_{\mathbb{R}^3}\bar{f}(x)\,\mathrm{pr}_{\mathrm{T}_x\Gamma} - \mathrm{n}(x)\cdot\nabla_{\mathbb{R}^3}\bar{f}(x)\mathcal{H}(x) - \mathcal{H}\nabla f(x)\,{}^{\mathrm{t}}\mathrm{n}(x).\end{aligned}$$

where $\nabla^2_{\mathbb{R}^3}\bar{f}(x)$ denotes the $\mathbb{R}^3$-Hessian of $\bar{f}$ at x. From this we deduce the commutator relation

$$\nabla^i\nabla^j f(x) - \nabla^j\nabla^i f(x) = (\mathcal{H}\nabla f(x))^j \mathrm{n}^i - (\mathcal{H}\nabla f(x))^i \mathrm{n}^j$$

and further the following useful formula for numerical experiments

$$\Delta f(x) = \Delta_{\mathbb{R}^3} f - {}^{\mathrm{t}}\mathrm{n}(x)\nabla^2_{\mathbb{R}^3} f(x)\mathrm{n}(x) - \mathrm{n}\cdot\nabla_{\mathbb{R}^3} f(x) H,$$

were $\Delta_{\mathbb{R}^3} f(x)$ is the $\mathbb{R}^3$-Laplace of f at x.

Lebesgue and Sobolev spaces on surfaces

We consider on $\Gamma \subset \mathbb{R}^3$ the Hausdorff measure of dimension 2. For $p \in [1, \infty]$ and $f \in C^0(\Gamma)$ we define the **p-Lebesgue norm** for $p < \infty$ as

$$\|f\|^p_{L^p(\Gamma)} := \int_\Gamma |f|^p,$$

and for $p = \infty$ as

$$\|f\|_{L^\infty(\Gamma)} := \inf_N \sup_{x\in\Gamma\setminus N} |f(x)|$$

where $N \subset \Gamma$ ranges over all null sets. $L^p(\Gamma)$ is the completion of $C^0(\Gamma)$ w.r.t. the norm $\|\cdot\|_{L^p(\Gamma)}$. For $k \in \{0, 1, 2\}$ and $f \in C^k(\Gamma)$ we define the **(k, p)-Sobolev norm** as

$$\|f\|^p_{W^{k,p}(\Gamma)} := \sum_{i=0}^{k} \|\nabla^i f\|^p_{L^p(\Gamma)}.$$

$\|\cdot\|_{W^{k,p}(\Gamma)}$ is the completion of $C^k(\Gamma)$ w.r.t. $W^{k,p}(\Gamma)$. We set $H^k(\Gamma) := W^{k,2}(\Gamma)$. The (k, p)-Sobolev seminorm is given by

$$|f|_{W^{k,p}(\Gamma)} = \|\nabla^k f\|_{L^p(\Gamma)}.$$

For explicit computations we have the useful formula

$$\int_\Gamma f = \sum_{i=1}^{N} \int_{U_i} \varphi_i(\theta) f(\theta) \sqrt{g(\theta)}\, \mathrm{d}\theta, \tag{2.9}$$

where $(U_i)_{i=1}^N$ is an open covering of Γ with subordinated partition of unity $(\varphi_i)_{i=1}^N$ and with $\sqrt{g} := \sqrt{\det(g_{ij})_{i,j=1}^2}$.

Integration by parts reads for $f \in C^1(\Gamma)$ or equivalently for $F \in C^1(\Gamma; \mathbb{R}^3)$ as

$$\int_\Gamma \nabla f = \int_\Gamma f H \mathrm{n}, \qquad \int_\Gamma \operatorname{div} F = \int_\Gamma H \mathrm{n} \cdot F. \tag{2.10}$$

Parametric evolving surfaces

For $a < b$ and $l, k \in \mathbb{N}$. We say $f \in C^l(a, b; C^k(\Gamma; \mathbb{R}^3))$, if $f\colon \Gamma \times [a, b] \to \mathbb{R}^3$ and if $\partial_t^{l'} \partial_1^{k_1} \partial_2^{k_2} f(x, t)$ exists for $l' \leq l$ and $k_1 + k_2 \leq k$.

A **dynamic parametrization** is a map $X \in C^1(a, b; C^2(\Gamma; \mathbb{R}^3))$ such that for all $t \in [a, b]$ we have that

$$X_t\colon \Gamma \to \mathbb{R}^3, \quad X_t(x) = X(x, t), \tag{2.11}$$

is injective and the Jacobian of $X_t(\theta)$ has full rank. As a consequence we have that $\Gamma(t) := \Gamma_t := X(\Gamma, t)$ is a surface.

A family of surface $(\Gamma(t))_{t \in [a,b]}$, or simply $\Gamma(t)$, is called a parametrizable **evolving surface**, if and only if there exists a dynamic parametrization for it. We say "$X \in C^1(a, b; C^2(\Gamma; \mathbb{R}^3))$ is an evolving surface", if we want to emphasize that we are considering an evolving surface $\Gamma(t)$ with a fix dynamic parametrization.

Time dependent oriented distance function

We say that $d_{\Gamma(t)}\colon \mathcal{N} \to \mathbb{R}$ is a **time dependent oriented distance function** for $\Gamma(t)$, if and only if $\bigcup_{t \in [a,b]} \mathcal{N}_t \times \{t\} = \mathcal{N} \Subset \mathbb{R}^3 \times \mathbb{R}$

with $\Gamma(t) \subset \mathcal{N}_t \Subset \mathbb{R}^3$ and for each $p \in \mathcal{N}(t)$ there exists only one $x \in \Gamma(t)$ having minimal distance to p, which satisfy

$$p = x + d_{\Gamma(t)}(p,t)\mathrm{n}(p,t), \tag{2.12}$$

where we have used that n is extended on $\mathcal{N}$, cf. (2.1). With the same reference as in the stationary case [46, lemma 14.16] we can easily see existence and observe that $d_{\Gamma(t)}$ inherits its regularity from X.

Time derivative for evolving surfaces

For an evolving surface $\Gamma(t)$ we say $f \in C^l(a,b;C^k(\Gamma_t))$, if and only if $(x,t) \mapsto f(X(x,t),t) \in C^l(a,b;C^k(\Gamma))$. This is well-defined, i.e. it does not depend on X. Similar we define the spaces $L^q(a,b;W^{k,p}(\Gamma_t))$ and $C^l(a,b;W^{k,p}(\Gamma_t))$. The latter two spaces are equipped with the obvious norms.

For an evolving surface X we define the **material derivative** of $f \in C^1(a,b;C^0(\Gamma_t))$ as

$$\partial_t^X f(x,t) := \frac{\mathrm{d}f(X(x_0,t),t)}{\mathrm{d}t} \tag{2.13}$$

and the **velocity** of X via

$$v_X(x,t) := \partial_t^X \mathbb{1}(x,t) = \frac{\partial X(x_0,t)}{\partial t}, \tag{2.14}$$

where we assumed that $x_0 \in \Gamma$ and $x = X(x_0,t)$. For the material derivative we have the alternative formula

$$\partial_t^X f(x,t) = \frac{\partial \bar{f}(x,t)}{\partial t} + \nabla_{\mathbb{R}^3} \bar{f}(x,t). \tag{2.15}$$

Having the definition of material derivative and velocity we want to comment on the definition of evolving surface. For an evolving surface the notion of spatial derivative does not depend on the dynamic parametrization. The notion of time derivative depends on the chosen dynamical system, i.e. for two evolving surfaces X and Y with $X(\Gamma, t) = Y(\Gamma, t)$ we have in general $\partial_t^X \neq \partial_t^Y$. However, in that case we still have

$$v_X(x,t) - v_Y(x,t) \in \mathrm{T}_x\Gamma(t). \tag{2.16}$$

This readily follows from $\frac{\mathrm{d}}{\mathrm{d}t} Y_t^{-1} \circ X_t(x_0) \in \mathrm{T}_{x_0}\Gamma$ and the equation

$$\frac{\partial Y_t^{-1}(x)}{\partial t} = -\operatorname{jac}(Y_t^{-1})(x) v_Y(x,t).$$

As a consequence, (2.16) with (2.15) imply

$$\partial_t^X f(x,t) - \partial_t^Y f(x,t) = \big(v_X(x,t) - v_Y(x,t)\big) \cdot \nabla f(x,t). \tag{2.17}$$

The following commutator relation is useful

$$\partial_t^X \nabla f - \nabla \partial_t^X f = -\nabla v_X \nabla f + \mathrm{n}^{\mathrm{t}} \mathrm{n}^{\mathrm{t}} (\nabla v_X) \nabla f. \tag{2.18}$$

This follows from a tedious calculation with (2.3).

Time derivatives of some bilinear forms

The following bilinear forms are frequently used

$$m(f,g) \coloneqq m(t; f,g) \coloneqq \int_{\Gamma(t)} fg, \tag{2.19a}$$

$$a(f,g) \coloneqq a(t; f,g) \coloneqq \int_{\Gamma(t)} \nabla f \cdot \nabla g, \tag{2.19b}$$

$$a^*(f,g) \coloneqq a^*(t; f,g) \coloneqq m(f,g) + a(f,g), \tag{2.19c}$$

$$(\partial_t^X m)(f,g) \coloneqq (\partial_t^X m)(t; f,g) \coloneqq \int_{\Gamma(t)} \operatorname{div} v_X fg, \tag{2.19d}$$

$$(\partial_t^X a)(f,g) \coloneqq (\partial_t^X a)(t; f,g) \coloneqq \int_{\Gamma(t)} \mathscr{D}(X) \nabla f \cdot \nabla g, \tag{2.19e}$$

$$(\partial_t^X a^*)(f,g) \coloneqq (\partial_t^X a^*)(t; f,g) \coloneqq (\partial_t^X m)(f,g) + (\partial_t^X a)(f,g), \tag{2.19f}$$

where $\mathscr{D}(X) \coloneqq \operatorname{div} v_X - \nabla v_X - {}^{\mathrm{t}}(\nabla v_X)$. We claim the following product rule:

$$\frac{\mathrm{d}}{\mathrm{d}t} m(f,g) = (\partial_t^X m)(f,g) + m(\partial_t^X f, g) + m(f, \partial_t^X g), \tag{2.20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a(f,g) = (\partial_t^X a)(f,g) + a(\partial_t^X f, g) + a(f, \partial_t^X g), \tag{2.20b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a^*(f,g) = (\partial_t^X a^*)(f,g) + a^*(\partial_t^X f, g) + a^*(f, \partial_t^X g). \tag{2.20c}$$

The last equation is obviously a consequence of the first and second one. The first equation follows from (2.9) and (2.5). For the second equation we use additionally (2.18). Of course, if we are in the situation of (2.16), then we interchange the letter X with Y for the product rule.

2.2. Evolving discrete surface

In this section the spatial discrete counterpart of evolving surfaces together with some useful tools and identities are introduced. Geometric perturbation errors are not covered here. This topic is postponed to the subsequent section. This section is organized as follows: First, the definition of an admissible linear mesh is given. Then, the discrete tangent space, discrete normal fields and discrete tangential derivatives are given. Next, Lebesgue spaces and Sobolev spaces together with an useful local formula are given. Afterwards, finite element functions are given. The author continues with the discrete dynamic parametrization and introduces discrete bilinear forms together with some useful product rules. This section is completed with the definition of finite element matrices together with some useful Lipschitz estimates.

Evolving meshes

Denote by $\Delta^2 \subset \mathbb{R}^2$ the standard simplex of dimension 2 and let $[a,b]$ be a time interval. A linear **evolving mesh** or **evolving discrete surface** on $[a,b]$ consist of continuously time depending nodes $(x_i(t))_{i=1}^N \subset \mathbb{R}^3$ and 2-dimensional simplex relations $(E_\imath(t))_{\imath=1}^M$, where we identify $E_\imath(t) \subset \mathbb{R}^3$ with the simplex itself, which we require to satisfy:

- For $\imath = 1, \ldots, M$ we have that $E_\imath(t) = (x_{\imath(1)}, \ldots, x_{\imath(3)})$ is non-degenerated, i.e. the map $T_\imath \colon \Delta^2 \times [a,b] \to E_\imath(t)$,

$$\begin{aligned} T_\imath(\theta, t) &= x_{\imath(1)} + \begin{pmatrix} x_{\imath(3)} - x_{\imath(1)} & x_{\imath(2)} - x_{\imath(1)} \end{pmatrix} \theta \\ &=: x_{\imath(1)} + D^\imath(t)\theta, \end{aligned} \tag{2.21}$$

  is a bijection.

- The intersection of two simplices is a common edge, a common node or empty.

- There are no boundary simplices, i.e. every edge is the intersection of two different simplices.

We set the **mesh width** h as

$$h(t) \coloneqq \max_{\imath=1}^{M} \operatorname{diam}(E_\imath(t)), \quad h \coloneqq \sup_{t\in[a,b]} h(t),$$

where diam denotes the 2-dimensional diameter, the **in-ball radius** at time t as

$$h_{\min}(t) \coloneqq \min_{\imath=1}^{M} \rho(E_\imath(t)),$$

where ρ denotes the radius of the maximum inner circle. We set $x_h(t) \coloneqq (x_i(t))_{i=1}^N \in \mathbb{R}^N \otimes \mathbb{R}^3$ and

$$\Gamma(x_h) \coloneqq \Gamma(x_h(t)) \coloneqq \Gamma_h(t) \coloneqq \Gamma_{h,t} \coloneqq \bigcup_{\imath=1}^{M} E_\imath(t).$$

The degeneracy of $\Gamma_h(t)$ is measured via

$$\sigma(h) := \sup_{t\in[a,b]} \frac{h(t)}{h_{\min}(t)}.$$

A family $(\Gamma_h(t))_{h>0}$ is called **admissible**, if and only if

$$\sup_{h>0} \sigma(h) \leq \text{const}\,. \tag{2.22}$$

From now on we denote with $\Gamma_h(t)$ an admissible family of evolving meshes.

Discrete spatial derivatives

We consider on $\Gamma_h(t) \subset \mathbb{R}^3$ the Hausdorff measure of dimension 2. Since any element has a smooth parametrization, we can define for $\imath = 1, \dots, M$ on $E_\imath(t) \setminus \partial E_\imath(t)$ derivatives as in the smooth surface case. This gives us for almost every point $x \in \Gamma_h(t)$ the notion of a **discrete tangent space**, a **discrete normal field** $\mathrm{n}_h(x,t)$ and an element-wise extended function $\bar{f}(x,t)$. Indeed, the tangent vectors are the first column $D_1^\imath(t)$ and second column $D_2^\imath(t)$ of $D^\imath(t)$, cf. (2.21). Hence, we define

$$\left(g_{ij}^\imath(t)\right)_{i,j=1}^2 = {}^{\mathrm{t}}D^\imath(t)D^\imath(t). \tag{2.23}$$

The inverse of $\left(g_{ij}^\imath(t)\right)_{i,j=1}^2$ is denoted by $(g_\imath^{ij}(t))_{i,j=1}^2$.

The **discrete tangential gradient** of a function $f\colon \Gamma_h(t) \to \mathbb{R}$ on $E_\imath(t) \setminus \partial E_\imath(t)$ is given by

$$\nabla_h f(x,t) := \sum_{i,j=1}^2 \partial_i f(\theta,t) g_\imath^{ij}(t) D_i^\imath(t), \tag{2.24}$$

where $f(\theta,t) = f(T_\imath(\theta,t),t)$. The **discrete tangential divergence** of a function $F\colon \Gamma_h(t) \to \mathbb{R}^3$ on $E_\imath(t) \setminus \partial E_\imath(t)$ is given by

$$\operatorname{div}_h F(x,t) := \sum_{i,j=1}^2 g_\imath^{ij}(t) D_i^\imath(t) \cdot \partial_j F(\theta,t), \tag{2.25}$$

where $F(\theta,t) = F(T_\iota(\theta,t),t)$. Assume that for two evolving discrete surfaces $\Gamma(x_h)$, $\Gamma(y_h)$ the function $F\colon \Gamma(x_h) \to \Gamma(y_h)$ maps the element $E_\iota(t) \subset \Gamma(x_h)$ onto the element $\widetilde{E}_\iota(t) \subset \Gamma(y_h)$. Then, the **discrete tangential Jacobian** of F on $E_\iota(t) \setminus \partial E_\iota(t)$ is given by

$$\operatorname{jac}_h F(x,t) := \sum_{i,j,k=1}^{2} \widetilde{D}_i^\iota(t)\partial_k(\rho^i \circ F \circ X)(\theta,t) g_\iota^{jk}\, {}^{\mathrm{t}}D_j^\iota(t), \tag{2.26}$$

where $\rho = (\rho^1, \rho^2)$ is the inverse of $\widetilde{T}_i\colon \Delta^2 \times [a,b] \to \widetilde{E}_i(t)$, cf. (2.21). As in the smooth case we can give alternative (extrinsic) formulas for the discrete tangential gradient, discrete tangential divergence and discrete tangential Jacobian:

$$\begin{aligned}
\nabla_h f(x,t) &= (I_3 - \mathrm{n}_h(x,t)\,{}^{\mathrm{t}}\mathrm{n}_h(x,t))\nabla_{\mathbb{R}^3}\bar{f}(x,t) \\
&=: \operatorname{pr}_{\mathrm{T}_x\Gamma_{h,t}} \nabla_{\mathbb{R}^3}\bar{f}(x,t), \\
\operatorname{div}_h F(x,t) &= \operatorname{trace}\left(\operatorname{pr}_{\mathrm{T}_x\Gamma}{}^{\mathrm{t}}(\operatorname{jac}_{\mathbb{R}^3}\bar{F}(x,t))\right), \\
\operatorname{jac}_h F(x,t) &= \operatorname{pr}_{\mathrm{T}_{F(x)}\Gamma(y_h)} \operatorname{jac}_{\mathbb{R}^3}\bar{F}(x,t) \operatorname{pr}_{\mathrm{T}_x\Gamma(x_h)},
\end{aligned}$$

where $\nabla_{\mathbb{R}^3}\bar{f}(x,t)$ and $\operatorname{jac}_{\mathbb{R}^3}\bar{F}(x,t)$ are the gradient and Jacobian, respectively, w.r.t. the spatial variable $x \in \Gamma_h(t) \subset \mathbb{R}^3$. Note that these formulas are only almost everywhere defined. Also observe that for the discrete Jacobian the estimate (2.7) and the chain rule (2.8) almost everywhere holds.

Integration

The **Lebesgue space** $L^p(\Gamma_{h,t})$ and even the **Sobolev space** $W^{1,p}(\Gamma_{h,t})$ are readily defined having in mind that the derivative is elementwise given. Formula (2.9) now becomes

$$\int_{\Gamma_h(t)} f = \sum_{i=1}^{M} \int_{\Delta^2} f(\theta,t)\sqrt{g_{h,i}(t)}\,\mathrm{d}\theta, \tag{2.27}$$

where $f(\theta,t) := f(T_i(\theta,t),t)$ and $\sqrt{g_{h,i}(t)} := \sqrt{\det((g^i_{ij}(t))^2_{i,j=1})}$, cf. (2.21) for T_i and (2.23) for $(g^i_{ij}(t))^2_{i,j=1}$. Unfortunately, there is no meaningful integration by parts formula on $\Gamma_h(t)$.

Finite element functions

On an evolving discrete surface $\Gamma_h(t)$ we define for $i = 1,\dots,N$ the i-th **Lagrange basis function** via the requirement

$$\chi_i(t)\colon \Gamma_h(t) \to \mathbb{R}, \quad \chi_i(x_j(t),t) := \chi_i(t)(x_j(t)) := \delta_{ij}, \tag{2.28}$$

where δ_{ij} denotes the Kronecker delta, e.g. we have $\delta_{ij} = 1$ iff $i = j$ and $\delta_{ij} = 0$ else, and for each element $E_j(t)$ we require that $\chi_i(T_j(\theta),t)\colon \Delta^2 \times [a,b] \to \mathbb{R}$ is, for a fix $t \in [a,b]$, a polynomial of degree 1. We define the **finite element space** as

$$S_h(x_h(t)) := S_h(x_h) := S_h(t) := \operatorname{span}\{\chi_1(t),\dots,\chi_N(t)\},$$

where span denotes the linear span. We write $S_h(x_h;\mathbb{R}^d)$, if the finite element function takes values in $\mathbb{R}^d$ for $d \in \mathbb{N}$. If we want to emphasize that $\phi_h \in S_h(x_h)$ is defined on $x_h(t)$, then we write $\phi_h[x_h]$.

For the analysis of some consistency errors we define the following finite element Sobolev dual norm

$$\|\phi_h\|_{H_h^{-1}(\Gamma_h(t))} = \sup_{0\neq\psi_h\in S_h(t)} \frac{m_h(\phi_h,\psi_h)}{\|\psi_h\|_{H^1(\Gamma_h(t))}}.$$

We introduce a special nodal value notation. If

$$\phi_h(t) = \sum_{i=1}^N \phi_i(t)\chi(t) \in S_h(t;\mathbb{R}^d),$$

then we overload the symbol and also denote the nodal value vector with

$$\phi_h(t) := \left(\phi_i(t)\right)_{i=1}^N = \left(\phi_i^1(t),\dots,\phi_i^d(t)\right)_{i=1}^N \in \mathbb{R}^N \otimes \mathbb{R}^d. \tag{2.29}$$

It will be always clear from the context, if we are considering a function or a degree of freedom vector. The reason for introducing this notation is that the authors gets constantly confused, if the degree of freedom vector is denoted by an other unrelated letter, as it is usual in the literature.

Discrete dynamical parametrization

Unlike for $\Gamma(t)$, there is for $\Gamma_h(t)$ only one meaningful dynamical parametrization. The **discrete dynamical parametrization** is given via

$$X_h\colon \Gamma_h(a) \times [a,b] \to \mathbb{R}^3, \quad (x_0,t) \mapsto \sum_{i=1}^{N} x_i(t)\chi_i[x_h(a)](x_0).$$

Just like in (2.13) and (2.14), we define the **discrete material derivative** as

$$\partial_t^{X_h} f(x,t) \coloneqq \frac{\mathrm{d} f(X_h(x_0,t),t)}{\mathrm{d}t}, \tag{2.30}$$

and the **discrete velocity** as

$$v_{X_h}(x,t) \coloneqq v_{x_h}(x,t) \coloneqq \partial_t^{X_h} \mathbb{1}(x,t) = \frac{\partial X_h(x_0,t)}{\partial t},$$

where $x = X_h(x_0,t)$.

We remark, that we have purposely avoided the notation $\partial_t^h f$, because for consistency reasons this would have forced us to define v_h, which is already heavily overused in the literature.

From (2.28) we immediately deduce $\chi_i(X_h(x_j(0),t),t) = \delta_{ij}$, which implies

$$\partial_t^{X_h} \chi_i = 0. \tag{2.31}$$

Having this result and the nodal value vector notation (2.29) in mind we write

$$\frac{\mathrm{d}x_h(t)}{\mathrm{d}t} = v_{x_h}(t).$$

With basically the same proof we can show the discrete counter part of the commutator relation (2.18) is given by

$$\partial_t^{X_h} \nabla_h f - \nabla_h \partial_t^{X_h} f = -\nabla_h v_{X_h} \nabla_h f + \mathrm{n}_h {}^{\mathrm{t}}\mathrm{n}_h {}^{\mathrm{t}}(\nabla_h v_{X_h}) \nabla_h f. \quad (2.32)$$

In this case we use (2.24).

Time derivatives of some discrete bilinear forms

The following discrete bilinear forms are frequently used

$$m_h(f,g) \coloneqq m_h(t;f,g) \coloneqq \int_{\Gamma_h(t)} fg, \quad (2.33a)$$

$$a_h(f,g) \coloneqq a_h(t;f,g) \coloneqq \int_{\Gamma_h(t)} \nabla_h f \cdot \nabla_h g, \quad (2.33b)$$

$$a_h^*(f,g) \coloneqq a_h^*(t;f,g) \coloneqq m_h(f,g) + a_h(f,g), \quad (2.33c)$$

$$(\partial_t^{X_h} m_h)(f,g) \coloneqq (\partial_t^{X_h} m)(t;f,g) \coloneqq \int_{\Gamma_h(t)} \operatorname{div}_h v_{X_h} fg, \quad (2.33d)$$

$$(\partial_t^{X_h} a_h)(f,g) \coloneqq (\partial_t^{X_h} a_h)(t;f,g) \coloneqq \int_{\Gamma_h(t)} \mathscr{D}(X_h) \nabla_h f \cdot \nabla_h g, \quad (2.33e)$$

$$(\partial_t^{X_h} a_h^*)(f,g) \coloneqq (\partial_t^{X_h} a_h^*)(t;f,g) \coloneqq (\partial_t^{X_h} m_h)(f,g) + (\partial_t^{X_h} a_h)(f,g), \quad (2.33f)$$

where $\mathscr{D}(X_h) \coloneqq \operatorname{div}_h v_{X_h} - \nabla_h v_{X_h} - {}^{\mathrm{t}}(\nabla_h v_{X_h})$. The following discrete product rules are proven just like for (2.20):

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(f,g) = (\partial_t^{X_h} m_h)(f,g) + m_h(\partial_t^{X_h} f, g) + m_h(f, \partial_t^{X_h} g), \quad (2.34a)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a_h(f,g) = (\partial_t^{X_h} a_h)(f,g) + a_h(\partial_t^{X_h} f, g) + a_h(f, \partial_t^{X_h} g), \quad (2.34b)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a_h^*(f,g) = (\partial_t^{X_h} a_h^*)(f,g) + a_h^*(\partial_t^{X_h} f, g) + a_h^*(f, \partial_t^{X_h} g). \quad (2.34c)$$

This requires the integral representation (2.27), the definition of tangential divergence (2.25) and discrete commutator relation (2.32).

Finite element matrices

The mass matrix and stiffness matrix are defined as

$$M(t) \coloneqq M(x_h) \coloneqq \left(m_h(\chi_i, \chi_j)\right)_{i,j=1}^N,$$
$$A(t) \coloneqq A(x_h) \coloneqq \left(a_h(\chi_i, \chi_j)\right)_{i,j=1}^N.$$

From (2.31) and (2.34) we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t} M(t) = \frac{\mathrm{d}}{\mathrm{d}t} M(x_h) = \left((\partial_t^{X_h} m_h)(\chi_i, \chi_j)\right)_{i,j=1}^N,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \frac{\mathrm{d}}{\mathrm{d}t} M(x_h) = \left((\partial_t^{X_h} a_h)(\chi_i, \chi_j)\right)_{i,j=1}^N.$$

Further, for a nodal value vector $\phi_h \in S_h(t)$ we write

$$M(t)\phi_h \coloneqq M(t)\phi_h(t).$$

In this work, finite element matrices appear in every time stability lemma. Hence, Lipschitz bounds for them are of interest. Assuming that

$$\|v_{X_h}\|_{L^\infty(a,b;W^{1,\infty}(\Gamma_{h,t}))} \leq \text{const} \tag{2.35}$$

we claim for $\phi_h, \psi_h \in S_h(t)$ and $s, t \in [a, b]$ with $|s - t|$ sufficiently small that

$$|\phi_h \cdot (M(t) - M(s))\psi_h| \leq |t - s| \, |\phi_h|_{M(t)} \, |\psi_h|_{M(t)} \, \text{const}, \tag{2.36a}$$
$$|\phi_h \cdot (A(t) - A(s))\psi_h| \leq |t - s| \, |\phi_h|_{A(t)} \, |\psi_h|_{A(t)} \, \text{const}, \tag{2.36b}$$
$$|\phi_h \cdot (M^{-1}(t) - M^{-1}(s))\psi_h| \leq |t - s| \, |\phi_h|_{M^{-1}(t)} \, |\psi_h|_{M^{-1}(t)} \, \text{const}, \tag{2.36c}$$

where

$$|\phi_h|^2_{M(t)} := \phi_h \cdot M(t)\phi_h = \|\phi_h\|^2_{L^2(\Gamma_{h,t})}, \tag{2.37a}$$

$$|\phi_h|^2_{A(t)} := \phi_h \cdot A(t)\phi_h = |\phi_h|^2_{H^1(\Gamma_{h,t})}, \tag{2.37b}$$

$$|\phi_h|^2_{M^{-1}(t)} := \phi_h \cdot M^{-1}(t)\phi_h, \tag{2.37c}$$

$$\|\phi_h\|^2_{*,t} := \phi_h \cdot \left(M(t) + A(t)\right)^{-1}\phi_h. \tag{2.37d}$$

(2.37e)

We sketch how to prove (2.36). Consider

$$M(t) - M(s) = \int_s^t \frac{\mathrm{d}}{\mathrm{d}\tau} M(\tau)\,\mathrm{d}\tau.$$

For $\psi_h = \phi_h$ we get

$$|\phi_h|^2_{M(t)} \leq |\psi_h|^2_{M(s)} + \int_s^t |\psi_h|^2_{M(\tau)}\,\mathrm{d}\tau \,\mathrm{const}\,(2.35).$$

Using a Gronwall estimate together with a size restriction on $|t-s|$ gives us the desired bound. The other bounds are proven with similar arguments.

2.3. Lift

In this section we provide tools to compare functions and bilinear forms on the continuous surface with their discrete counter parts on the finite element mesh. This section is organized as follows: We begin with a formal definition of lifted function and then describe the standard evolving surface mesh. Then, we introduce the lifted discrete material derivative. Afterwards, we present geometric perturbation error and sketch some proofs. After discussing the equivalence of discrete and continuous norm we list some bilinear form errors and sketch their proofs.

Lifting process

We recall that the time depended oriented distance function is introduced in (2.12). We say $\Gamma_h(t)$ can be compared with $\Gamma(t)$, if and only if we have $\Gamma_h(t) \subset \mathcal{N}(t)$ and $\Gamma_h(t)$ is a single covering of $\Gamma(t)$, i.e. there is a bijection between $p \in \Gamma_h(t)$ and $x \in \Gamma(t)$, which is given through (2.12),

$$p = x + d_\Gamma(p,t)\mathrm{n}(p,t).$$

We say that $x = x(p)$ is the **lift** of p and $p = p(x)$ is the **negative lift** of x. For a function $f\colon \Gamma_h(t) \to \mathbb{R}$ we define its **lift** $f^l\colon \Gamma(t) \to \mathbb{R}$ via

$$f^l(x) \coloneqq f(p), \tag{2.38}$$

where p is the negative lift of x. The **negative lift** of function is given by $(f^l)^{-l} = f$. We say $f \in S_h^l(t)$, if $f = \phi_h^l$ for some $\phi_h \in S_h(t)$.

A standard way to approximate $\Gamma(t)$ is to take an initial triangulation of $(x_i(0))_{i=1}^N \subset \Gamma(0)$, then to set

$$x_i(t) \coloneqq X(x_i(0), t) \tag{2.39}$$

and finally to assume that $x_h(t)$ does not degenerate.

From now on we assume (2.39) *unless otherwise stated.*

Lifted discrete dynamical parametrization

The **lifted discrete dynamical parametrization** is per definition the function X_h^l. It is an important additional dynamic parametrization of $\Gamma(t)$. Its material derivative is called **lifted discrete material derivative** $\partial_t^{X_h^l}$ and its velocity is called **lifted discrete velocity** $v_{X_h^l}$. Using the definitions (2.13) and (2.38) we observe that the lifted material derivative commutes in some sense with the lifting process:

$$\left(\partial_t^{X_h} f\right)^l = \partial_t^{X_h^l} f^l. \tag{2.40}$$

The following identities are corollaries from the previous section and commonly used: From (2.17) we deduce the following formula for the difference of the material derivative and the lifted material derivative:

$$\partial_t^X f(x,t) - \partial_t^{X_h^l} f(x,t) = \big(v_X(x,t) - v_{X_h^l}(x,t)\big) \cdot \nabla f(x,t). \tag{2.41}$$

From (2.20) we deduce the following product rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, m(f,g) = (\partial_t^{X_h^l} m)(f,g) + m(\partial_t^{X_h^l} f, g) + m(f, \partial_t^{X_h^l} g), \tag{2.42a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, a(f,g) = (\partial_t^{X_h^l} a)(f,g) + a(\partial_t^{X_h^l} f, g) + a(f, \partial_t^{X_h^l} g), \tag{2.42b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, a^*(f,g) = (\partial_t^{X_h^l} a^*)(f,g) + a^*(\partial_t^{X_h^l} f, g) + a^*(f, \partial_t^{X_h^l} g). \tag{2.42c}$$

Geometric perturbation estimates

We recall that $\mathrm{n}^{-l} = \mathrm{n}$. In this sense we may consider $\mathrm{pr}_{\mathrm{T}_x\Gamma(t)}$ as a function on $\Gamma_h(t)$.

We define $\delta_h = \delta_h(t) \colon \Gamma_h(t) \to \mathbb{R}$ via the requirement that for all $f_h \in L^1(\Gamma_{h,t})$ it holds

$$\int_{\Gamma(t)} f_h^l = \int_{\Gamma_h(t)} \delta_h f_h, \tag{2.43}$$

We collect some bounds.

$$\|d_{\Gamma(t)}\|_{L^\infty(\Gamma_{h,t})} \leq h^2 \,\mathrm{const}, \tag{2.44}$$

$$\|\mathrm{pr}_{\mathrm{T}_p\Gamma_h(t)}\, \mathrm{n}\|_{L^\infty(\Gamma_{h,t})} + \|\mathrm{n} - \mathrm{n}_h\|_{L^\infty(\Gamma_{h,t})} \leq h \,\mathrm{const}, \tag{2.45}$$

$$\frac{1}{\mathrm{const}} \leq \delta_h(p) \leq \mathrm{const} \tag{2.46}$$

$$\|1 - \delta_h\|_{L^\infty(\Gamma_{h,t})} \leq h^2 \,\mathrm{const}, \tag{2.47}$$

$$\|\mathrm{pr}_{\mathrm{T}_x\Gamma(t)} - \mathrm{pr}_{\mathrm{T}_x\Gamma(t)}\, \mathrm{pr}_{\mathrm{T}_p\Gamma_h(t)}\, \mathrm{pr}_{\mathrm{T}_x\Gamma(t)}\|_{L^\infty(\Gamma_{h,t})} \leq h^2 \,\mathrm{const}, \tag{2.48}$$

$$\|\partial_t^{X_h} d_{\Gamma(t)}\|_{L^\infty(\Gamma_{h,t})} \leq h^2 \operatorname{const}, \quad (2.49)$$

$$\|\partial_t^{X_h} (\operatorname{pr}_{T_p\Gamma_h(t)} \mathrm{n})\|_{L^\infty(\Gamma_{h,t})} \leq h \operatorname{const}, \quad (2.50)$$

$$\|\partial_t^{X_h} \delta_h\|_{L^\infty(\Gamma_{h,t})} \leq h^2 \operatorname{const}, \quad (2.51)$$

$$\|\partial_t^{X_h} (\operatorname{pr}_{T_x\Gamma(t)} - \operatorname{pr}_{T_x\Gamma(t)} \operatorname{pr}_{T_p\Gamma_h(t)} \operatorname{pr}_{T_x\Gamma(t)})\|_{L^\infty(\Gamma_{h,t})} \leq h^2 \operatorname{const}. \quad (2.52)$$

Bounds (2.44) – (2.51) have bee done by Dziuk and Elliott [34, 35]. (2.52) appears in [66]. We sketch all proofs. For (2.44) observe that after rotation we may consider $d_{\Gamma(t)}$ as a function on a straight element $E_i(t)$. Since on the nodes it vanishes, the interpolation of $d_{\Gamma(t)}$ is zero. For (2.45) take (2.2) into account. Since we have assumed that n_h is the third standard basis vector of $\mathbb{R}^3$, this shows that the first and second component of n are in $\mathcal{O}(h)$. Hence, we have

$$\|\operatorname{pr}_{T_p\Gamma_h(t)} \mathrm{n}\|_{L^\infty(\Gamma_{h,t})} \leq h \operatorname{const}.$$

To deduce (2.45) note that n has unit length and assume that $h < h_0$ is sufficiently small.

For (2.47) and (2.46) consider [34, (4.12) and below]. These proofs rely on (2.45).

For (2.48) observe that

$$\operatorname{pr}_{T_x\Gamma(t)} - \operatorname{pr}_{T_x\Gamma(t)} \operatorname{pr}_{T_p\Gamma_h(t)} \operatorname{pr}_{T_x\Gamma(t)} = \operatorname{pr}_{T_x\Gamma(t)} \mathrm{n}_h {}^{\mathrm{t}}\mathrm{n}_\mathrm{n} \operatorname{pr}_{T_x\Gamma(t)}.$$

The stated bound follows from

$$\|\operatorname{pr}_{T_x\Gamma(t)} \mathrm{n}_h\| = \|\operatorname{pr}_{T_x\Gamma(t)}(\mathrm{n}_h - \mathrm{n})\| \leq h \operatorname{const},$$

where we have used (2.45).

(2.49) and (2.50) follow from the same arguments as for (2.44) and (2.45), since $\partial_t^{X_h} d_{\Gamma(t)}$ vanishes on the nodes.

For (2.52) it suffices to show

$$\|\partial_t^{X_h} (\operatorname{pr}_{T_x\Gamma(t)} \mathrm{n}_h)\| \leq h \operatorname{const}.$$

Observe that

$$\begin{aligned}\partial_t^{X_h}(\mathrm{pr}_{\mathrm{T}_x\Gamma(t)}\,\mathrm{n}_h) &= \partial_t^{X_h}(\mathrm{pr}_{\mathrm{T}_p\Gamma_h(t)}\,\mathrm{pr}_{\mathrm{T}_x\Gamma(t)}\,\mathrm{n}_h) + \partial_t^{X_h}(\mathrm{n}_h\,{}^{\mathrm{t}}\mathrm{n}_h\,\mathrm{pr}_{\mathrm{T}_x\Gamma(t)}\,\mathrm{n}_h)\\ &= -\,\partial_t^{X_h}(\mathrm{pr}_{\mathrm{T}_p\Gamma_h(t)}\,\mathrm{n}\,{}^{\mathrm{t}}\mathrm{n}\mathrm{n}_h) + \partial_t^{X_h}(\mathrm{n}_h\,{}^{\mathrm{t}}\mathrm{n}_h\,\mathrm{pr}_{\mathrm{T}_x\Gamma(t)}\,\mathrm{n}_h).\end{aligned}$$

Use (2.45) and (2.50) and observe that

$$\partial_t^{X_h}\,\mathrm{pr}_{\mathrm{T}_x\Gamma(t)} = \partial_t^{X_h}\mathrm{n}\,{}^{\mathrm{t}}\mathrm{n} + \mathrm{n}\,{}^{\mathrm{t}}(\partial_t^{X_h}\mathrm{n}), \qquad \partial_t^{X_h}\mathrm{n}\cdot\mathrm{n} = 0.$$

Equivalence of norms

We want to compare Lebesgue and Sobolev norms on $\Gamma(t)$ and $\Gamma_h(t)$. For any $p \in [1,\infty]$ we claim

$$\|f^l\|_{L^p(\Gamma_t)}\frac{1}{\mathrm{const}} \leq \|f\|_{L^p(\Gamma_{h,t})} \leq \|f^l\|_{L^p(\Gamma_t)}\,\mathrm{const}\,. \tag{2.53}$$

The case $p = \infty$ is trivial. The other cases follow from (2.46). Using (2.12), (2.4) and the chain rule we deduce the formula

$$\nabla_h f(p) = \mathrm{pr}_{\mathrm{T}_p\Gamma_h}(\mathrm{pr}_{\mathrm{T}_x\Gamma} - d_X(p)\mathcal{H}(p))\nabla f(x). \tag{2.54}$$

Again using (2.46) we deduce

$$|f^l|_{W^{1,p}(\Gamma_t)}\frac{1}{\mathrm{const}} \leq |f|_{W^{1,p}(\Gamma_{h,t})} \leq |f^l|_{W^{1,p}(\Gamma_t)}\,\mathrm{const}\,. \tag{2.55}$$

Bilinear form bounds

For $f, g\colon \Gamma_h(t) \to \mathbb{R}$ and $p, q \in [1,\infty]$ with $p^{-1} + q^{-1} = 1$ we claim

$$|m(f^l, g^l) - m_h(f, g)| \leq h^2\,\|f^l\|_{L^p(\Gamma_t)}\,\|g^l\|_{L^q(\Gamma_t)}\,\mathrm{const}, \tag{2.56a}$$

$$|a(f^l, g^l) - a_h(f, g)| \leq h^2\,|f^l|_{W^{1,p}(\Gamma_t)}\,|g^l|_{W^{1,q}(\Gamma_t)}\,\mathrm{const}, \tag{2.56b}$$

$$|(\partial_t^{X_h^l} m)(f^l, g^l) - (\partial_t^{X_h} m_h)(f, g)| \leq h^2 \, \|f^l\|_{L^p(\Gamma_t)} \, \|g^l\|_{L^q(\Gamma_t)} \, \text{const}, \tag{2.56c}$$

$$|(\partial_t^{X_h^l} a)(f^l, g^l) - (\partial_t^{X_h} a_h)(f, g)| \leq h^2 \, |f^l|_{W^{1,p}(\Gamma_t)} \, |g^l|_{W^{1,q}(\Gamma_t)} \, \text{const}\,. \tag{2.56d}$$

All four inequalities use (2.43) and either (2.53) or (2.55). The first inequality is an immediate consequence of (2.47). For the second inequality we use (2.54). Then we have to take care of

$$\begin{aligned} \mathrm{pr}_{\mathrm{T}_x\Gamma(t)} - \mathrm{pr}_{\mathrm{T}_x\Gamma(t)} \left(I_3 - d_X(p)\mathcal{H}(p)\right) \mathrm{pr}_{\mathrm{T}_p\Gamma_h(t)} \times \\ \times \left(I_3 - d_X(p)\mathcal{H}(p)\right) \mathrm{pr}_{\mathrm{T}_x\Gamma(t)}, \end{aligned} \tag{2.57}$$

where $\times$ represents a line break for a too long product. This is easily controlled with (2.44) and (2.48). For the third inequality observe

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} F^l &= \int_{\Gamma(t)} \partial_t^{X_h^l} F^l + F^l \operatorname{div} v_{X_h^l}, \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} F^l &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h(t)} \delta_h F = \int_{\Gamma_h(t)} \partial_t^{X_h} \delta_h F + \delta_h \partial_t^{X_h} F + \delta_h F \operatorname{div}_h v_{X_h} \end{aligned}$$

The estimate follows using (2.47) and (2.49). For the forth and last inequality we use the same idea. To handle the discrete material derivative of (2.57) use (2.49) and (2.52).

2.4. Some finite element bounds

In this section we introduce bounds for the Lagrange finite element interpolation operator, the lifted discrete velocity and a Ritz map.

An interpolation operator

For $f \in H^2(\Gamma_t)$ we define the Lagrange finite element **interpolation operator** $\mathscr{I}_h f \in S_h(t)$ via the requirement $\mathscr{I}_h f(x_i) = f(x_i)$. The following estimate is well known:

$$\|f - \mathscr{I}_h^l f\|_{L^2(\Gamma_t)} + h\,|f - \mathscr{I}_h^l f|_{H^1(\Gamma_t)} \le h^2\,\mathrm{const}\,. \tag{2.58}$$

We remark that the constant above depends on const (2.22).

Using (2.30), (2.31) and (2.40) we easily see that the interpolation satisfies

$$\partial_t^{X_h} \mathscr{I}_h f = \mathscr{I}_h \partial_t^X f, \qquad \partial_t^{X_h^l} \mathscr{I}_h^l f = \mathscr{I}_h^l \partial_t^X f. \tag{2.59}$$

Having (2.17) and (2.60) in mind we say that the interpolation operator *essentially commutes* with the material derivative.

Bounds for the lifted discrete velocity

In general we have $v_{X_h}^l \neq v_{X_h^l}$, which implies that the following bounds are not trivially obtained:

$$\|v_X - v_{X_h^l}\|_{L^\infty(\Gamma_t)} + h\,|v_X - v_{X_h^l}|_{W^{1,\infty}(\Gamma_t)} \le h^2\,\mathrm{const}\,. \tag{2.60}$$

We also have higher-order material derivatives bounds

$$\|(\partial_t^{X^l})^k (v_X - v_{X_h^l})\|_{L^\infty(\Gamma_t)} + h\,|(\partial_t^{X^l})^k (v_X - v_{X_h^l})|_{W^{1,\infty}(\Gamma_t)} \le h^2\,\mathrm{const}, \tag{2.61}$$

where $k \in \mathbb{N}$. This can be found in [66, lemma 6.3].

A Ritz map

The **Ritz map** $\mathscr{R}_h\, H^1(\Gamma_t) \to S_h(t)$ is defined via the requirement that for all $\phi_h \in S_h(t)$ we have

$$a^*(f, \phi_h^l) = a_h^*(\mathscr{R}_h f, \phi_h). \tag{2.62}$$

The bounds

$$\|f - \mathscr{R}_h^l f\|_{L^2(\Gamma_t)} + h\,\|f - \mathscr{R}_h^l f\|_{H^1(\Gamma_t)} \le h^2\,\mathrm{const}, \tag{2.63}$$

are standard in the literature, cf. [63, 66]. We sketch how to prove them. The H^1-estimate follows from the pseudo Galerkin orthogonality

$$|a^*(f - \mathscr{R}_h^l f, \phi_h^l)| \leq h^2 \|f\|_{H^1(\Gamma_t)} \|\phi_h\|_{H^1(\Gamma_t)}, \tag{2.64}$$

which requires (2.56), and the interpolation H^1-bound (2.58). The L^2-bounds follow from an Aubin-Nitsche trick with a PDE a priori estimate.

In contrast to the interpolation operator, we know that in general the Ritz map does not commute with the material derivative. However, we still can prove for arbitrary higher-order material derivatives

$$\|(\partial_t^{X_h^l})^k (f - \mathscr{R}_h^l f)\|_{L^2(\Gamma_t)} + h \|(\partial_t^{X_h^l})^k (f - \mathscr{R}_h^l f)\|_{H^1(\Gamma_t)} \leq h^2 \,\text{const}, \tag{2.65}$$

where $l \in \mathbb{N}$. We sketch how to obtain the first-order material derivative bound. The basic idea is to take the time derivative of (2.62) to get the Galerkin-type equation

$$a^*(f, \phi_h^l) - a_h^*(\mathscr{R}_h f, \phi_h) = (\partial_t^{X_h} a_h^*)(\mathscr{R}_h f, \phi_h) - (\partial_t^{X_h^l} a^*)(f, \phi_h^l).$$

An H^1-bound can be obtained by using (2.59) and the H^1-bounds for the Ritz map. The L^2-bound uses an Aubin-Nitsche argument.

Chapter 3.

Objectives of this work

We present four different parabolic problems, which correspond to the articles [57, 56, 58, 55], respectively. A recently discovered unpublished result will be presented in chapter 5.

In all our problems it is assumed that the surface evolution and the solution of the partial differential equation (PDE) are sufficiently regular, i.e. we have enough spatial and temporal derivatives for our analysis.

3.1. An arbitrary Lagrangian-Eulerian evolving surface finite element method

PDE 3.1.1. Let $X \in C^1(0,T;C^2(\Gamma_0^2;\mathbb{R}^3))$ be an evolving surface, $u_0 \in H^2(\Gamma_0)$ an initial value and $f \in L^2(0,T;L^2(\Gamma_t))$ a forcing term. Find $u \in L^2(0,T;H^2(\Gamma_t))$, which satisfies

$$(\partial_t^X + \operatorname{div}(v_X) - \Delta)u = f, \quad u(0) = u_0.$$

We can approximate u by combining the ESFEM with the implicit Euler method. Convergence in L^2-based norms is known assuming we have a quasi uniform mesh; cf. [34, 35, 33, 37, 31] for the analysis and cf. (2.22) for the quasi uniform mesh condition. However, the standard algorithm, cf. (2.39), does not guarantee that the resulting finite element mesh stays regular. This may lead to a large const (2.22), which implies a bad approximation. Two natural questions arise:

1. Is there another evolution $\mathcal{A}$ compatible with X and reasonable computable such that the resulting const (2.22) is smaller on the same time interval?

2. Does the resulting scheme converge?

Question 1 is investigated by Kovács in [53]. Our aim is to answer question 2.

Let us make the notion of "compatible" and "resulting scheme" more precise.

> **Definition 3.1.2.** An evolving surface $\mathcal{A}$ is an **arbitrary Lagrangian-Eulerian (ALE) map** for X, if and only if for all t it holds $\mathcal{A}(\Gamma_0, t) = X(\Gamma_0, t)$.

As a consequence we can prove that the difference of the velocities, $(v_X - v_{\mathcal{A}})$, is a tangent vector. This observation implies that the solution of PDE 3.1.1 agrees with the solution of the next

> **PDE 3.1.3.** Let X, u_0 and f be like in PDE 3.1.1 and let $\mathcal{A}$ be an ALE map for X. Find $u \in L^2(0, T; H^2(\Gamma_t))$, which satisfies
>
> $$(\partial_t^{\mathcal{A}} + \operatorname{div}(v_{\mathcal{A}}) - \Delta)u + \operatorname{div}\big((v_X - v_{\mathcal{A}})u\big) = f, \quad u(0) = u_0.$$

The basic idea is to discretize PDE 3.1.3 instead of PDE 3.1.1.

We describe our computational method. Modify the finite element mesh movement, cf. (2.39), by using the nodes $(\mathcal{A}(x_i, t))_{i=1}^N$ instead of $(X(x_i, t))_{i=1}^N$. Denote the resulting parametrization with $\mathcal{A}_h$. Redefine $\Gamma_h(t) \coloneqq \mathcal{A}_h(\Gamma_{h,0}, t)$ and search for a numerical solution in the finite element space $S_h(\mathcal{A}_h)$. This modification is called the **arbitrary Lagrangian-Eulerian evolving surface finite element method (ALE-ESFEM)**.

The matrix vector formulation of the ALE-ESFEM reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mu_h) + Au_h + Bu_h = f_h, \quad u_h(0) = \mathscr{I}_h u_0 \tag{3.1}$$

where M, A, B and f_h are the mass matrix, stiffness matrix, ALE matrix and load vector, respectively. The novel ALE matrix is given by

$$B_{ij}(t) \coloneqq \int_{\Gamma_h(t)} \chi_i (v_{\mathcal{A}_h} - \mathscr{I}_h v_X) \cdot \nabla_h \chi_j,$$

where $(\chi_i) \subset S_h(\mathcal{A}_h)$ denotes the usual Lagrangian finite element basis. Note that the sign above is correct.

For the initial value problem (IVP) above we consider two different classes of higher-order time stepping schemes: Backwards difference formulas (BDF) and Runge-Kutta methods (RKM). Of course, we restrict ourselves to a certain class of RKM, which will contain the Radau IIa method. Both time stepping schemes are popular stiff integrators and contain the implicit Euler method as a special case. We start with the description of the BDF.

Let $\tau > 0$ be the time step size and, for simplicity, assume that $N \in \mathbb{N}$ exists with $T = \tau N$. Set $t_n \coloneqq n\tau$. The k-step BDF applied on (3.1) reads as

$$\sum_{j=0}^{k} \delta_j M_{n-j} u_{n-j}^{\mathrm{BDF}} + \tau A_n u_n^{\mathrm{BDF}} + \tau B_n u_n^{\mathrm{BDF}} = \tau f_n, \quad n \geq k \tag{3.2}$$

where $M_n \coloneqq M(t_n)$, $A_n \coloneqq A(t_n)$ etc. and the coefficients $(\delta_j)_{j=0}^k$ are given by $\sum_{j=0}^k \delta_j x^j = \sum_{\ell=1}^k \frac{1}{\ell}(1-x)^\ell$. To keep things simple, we assume that the initial steps $u_n^{\mathrm{BDF}} = \mathscr{I}_h u(t_n)$, for $n = 0, \ldots, k-1$, are given.

On the other hand, if we discretize (3.1) using a RKM with Butcher tableau

$$\begin{array}{c|c} (c_j)_{j=1}^s & (a_{ij})_{i,j=1}^s \\ \hline & (b_j)_{j=1}^s \end{array},$$

then the resulting scheme reads as

$$\begin{aligned} M_{n+1} u_{n+1}^{\mathrm{RKM}} &= M_n u_n^{\mathrm{RKM}} + \tau \sum_{j=1}^{s} b_j \dot{U}_{nj}, \\ M_{ni} U_{ni} &= M_n u_n^{\mathrm{RKM}} + \tau \sum_{j=1}^{s} a_{ij} \dot{U}_{nj}, \\ \dot{U}_{nj} &= f_{nj} - A_{nj} U_{nj} - B_{nj} U_{nj}, \end{aligned} \tag{3.3}$$

where $i = 1, \ldots, s$ and where we have set $M_{ni} \coloneqq M(t_n + \tau c_i)$, $A_{ni} \coloneqq A(t_n + \tau c_i)$ etc. In the above equations $\dot{U}_{nj}$ is an unknown and not the time derivative of U_{nj}. We will assume that our RKM satisfies the following

> **Definition 3.1.4.** A RKM of stage order $q \geq 1$ and classical order $p \geq 1$ is called **admissible**, if and only if it satisfy:
>
> (i) $p \geq q + 1$,
>
> (ii) (a_{ij}) is invertible,
>
> (iii) $b_j > 0$ and $(b_i a_{ij} - b_j a_{ji} - b_i b_j)_{i,j=1}^s$ is non-negative definite,
>
> (iv) $b_j = a_{sj}$ and $c_s = 1$.

In [49], property (iii) is called **algebraic stability**, which is stronger then A-stability, and property (iv) is called **stiff accuracy**.

The objective of [57] was the following

Theorem 3.1.5. *Let u be the solution of the PDE 3.1.3. For the ALE-ESFEM together with the k-step BDF method there exists h_0 and τ_0 such that for all $h < h_0$ and $\tau < \tau_0$ we have*

$$\begin{aligned} \|u(t) - (u_{h,n}^{\mathrm{BDF}})^l\|_{L^\infty(0,T;L^2(\Gamma_t))} &\leq (\tau^k + h^2)\,\mathrm{const}, \\ |u(t) - (u_{h,n}^{\mathrm{BDF}})^l|_{L^2(0,T;H^1(\Gamma_t))} &\leq (\tau^k + h)\,\mathrm{const}\,. \end{aligned}$$

For the ALE-ESFEM together with an admissible RKM there exists h_0 and τ_0 such that for all $h < h_0$ and $\tau < \tau_0$ we have

$$\begin{aligned} \|u(t) - (u_{h,n}^{\mathrm{RKM}})^l\|_{L^\infty(0,T;L^2(\Gamma_t))} &\leq (\tau^k + h^2)\,\mathrm{const}, \\ |u(t) - (u_{h,n}^{\mathrm{RKM}})^l|_{L^2(0,T;H^1(\Gamma_t))} &\leq (\tau^q + h)\,\mathrm{const}\,. \end{aligned}$$

3.2. Full discretization of a quasilinear problem on evolving surfaces

Many important evolving surface problems are nonlinear. It comes as a surprise that for quasilinear problems there was not much rigorous analysis done. The author wanted to fill this gap.

> **PDE 3.2.1.** Let $\mathscr{A}\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function that is bounded from above and below by some positive constant. For given evolving surface $X \in C^1(0,T;C^2(\Gamma_0^2;\mathbb{R}^3))$, initial Data $u_0 \in H^2(\Gamma_0)$, forcing term $f \in L^2(0,T;L^2(\Gamma_t))$ and $\mathscr{A}$ find $u \in L^2(0,T;H^2(\Gamma_t))$, which satisfies
>
> $$\partial_t^X u + \operatorname{div}(v_X)u - \operatorname{div}\big(\mathscr{A}(u)\nabla u\big) = f, \quad u(0) = u_0.$$

For the choice $\mathscr{A}(z) = 1$ we get the linear PDE 3.1.1 from the previous section. We want to consider a fully discrete scheme to approximate u. We use a standard ESFEM discretization. The matrix vector formulation reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mu_h) + A(u_h)u_h = f_h, \quad u_h(0) = \mathscr{I}_h u_0, \tag{3.4}$$

where the novel nonlinear stiffness matrix $A(u)$ is given by

$$A(u;t)_{ij} := \int_{\Gamma_h(t)} \mathscr{A}(u)\nabla_h \chi_i \cdot \nabla_h \chi_j.$$

We consider two different time discretization methods for the IVP (3.4): Linearly implicit BDF and fully implicit RKM. Fully implicit BDF are also covered in [56], but since the linearly implicit variant is computationally more interesting, we restrict ourselves to the latter method here.

Again, let $\tau > 0$ be the time step size and, for simplicity, assume that there exists $N \in \mathbb{N}$ such that $T = \tau N$. Set $t_n := n\tau$. The linearly implicit k-step BDF applied on (3.4) reads as

$$\sum_{j=0}^{k} \delta_j M_{n-j} u_{n-j}^{\mathrm{BDF}} + \tau A_n \Big(\sum_{j=0}^{k-1} \gamma_j u_{n-k+j}^{\mathrm{BDF}} \Big) u_n^{\mathrm{BDF}} = \tau f_n, \quad n \geq k, \tag{3.5}$$

where $M_n := M(t_n)$, $f_n := f_h(t_n)$ and the coefficients $(\gamma_j)_{j=0}^{k-1}$ and $(\delta_j)_{j=0}^{k}$ are given by

$$\sum_{j=0}^{k-1} \gamma_j x^j = x^k - (x-1)^k \quad \text{and} \quad \sum_{j=0}^{k} \delta_j x^j = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-x)^\ell,$$

respectively. The nonlinear stiffness matrix is given by

$$A_n(\phi_h) := \int_{\Gamma_h(t_n)} \mathscr{A}(\phi_h) \nabla_h \chi_i \cdot \nabla_h \chi_j.$$

To keep things simple, we assume that the initial steps $u_n^{\mathrm{BDF}} = \mathscr{I}_h u(t_n)$, for $n = 0, \dots, k-1$, are given. Note that u_n^{BDF} only requires us to solve a linear system.

On the other hand, if we discretize (3.4) using an admissible RKM, cf. definition 3.1.4, with Butcher tableau,

$$\begin{array}{c|c} (c_j)_{j=1}^s & (a_{ij})_{i,j=1}^s \\ \hline & (b_j)_{j=1}^s \end{array},$$

then the resulting scheme reads as

$$\begin{aligned} M_{n+1} u_{n+1}^{\mathrm{RKM}} &= M_n u_n^{\mathrm{RKM}} + \tau \sum_{j=1}^{s} b_j \dot{U}_{nj}, \\ M_{ni} U_{ni} &= M_n u_n^{\mathrm{RKM}} + \tau \sum_{j=1}^{s} a_{ij} \dot{U}_{nj}, \\ \dot{U}_{nj} &= f_{nj} - A_{nj}(U_{nj}) U_{nj}, \end{aligned} \tag{3.6}$$

where $i = 1, \dots, s$, $M_{ni} := M(t_n + \tau c_i)$ etc. Again, $\dot{U}_{nj}$ is an unknown and not the time derivative of U_{nj}.

The objective of [56] was the following

Theorem 3.2.2. *Let u be the solution of the PDE 3.2.1. For the ESFEM together with the linearly implicit k-step BDF method there exists h_0 and τ_0 such that for all $h < h_0$ and $\tau < \tau_0$ we have*

$$\|u(t) - (u_{h,n}^{\mathrm{BDF}})^l\|_{L^\infty(0,T;L^2(\Gamma_t))} \leq (\tau^k + h^2)\,\mathrm{const},$$
$$|u(t) - (u_{h,n}^{\mathrm{BDF}})^l|_{L^2(0,T;H^1(\Gamma_t))} \leq (\tau^k + h)\,\mathrm{const}\,.$$

For the ESFEM together with an admissible RKM there exists h_0 and τ_0 such that for all $h < h_0$ and $\tau < \tau_0$ we have

$$\|u(t) - (u_{h,n}^{\mathrm{RKM}})^l\|_{L^\infty(0,T;L^2(\Gamma_t))} \leq (\tau^k + h^2)\,\mathrm{const},$$
$$|u(t) - (u_{h,n}^{\mathrm{RKM}})^l|_{L^2(0,T;H^1(\Gamma_t))} \leq (\tau^q + h)\,\mathrm{const}\,.$$

3.3. Some maximum-norm error estimates

During the preparation of our last theorem, the author proved an optimal $W^{1,\infty}$-stability bound for our Ritz map. Since the techniques were already developed by the author, we asked for maximum-norm estimates for the ESFEM. Such error bounds are a technically non-trivial extension of the corresponding L^2-based bound for ESFEM and maximum-norm error bounds for FEM on Euclidean domains.

Consider the linear heat equation on evolving surface, PDE 3.1.1. We use the ESFEM without any further time stepping method.

> **Theorem 3.3.1.** *Let u be the solution of the PDE 3.1.1. Then for the ESFEM there exists h_0 such that for all $h < h_0$ we have the estimate*
>
> $$\|u - u_h^l\|_{L^\infty(0,T;L^\infty(\Gamma_t))} \leq h^2 \, |\log h|^4 \, \mathrm{const},$$
> $$\|u - u_h^l\|_{L^\infty(0,T;W^{1,\infty}(\Gamma_t))} \leq h \, |\log h|^4 \, \mathrm{const}\,.$$

This theorem has been shown in [58]. As a prerequisite we also had to prove its elliptic counterpart, which is by itself an interesting result.

> **Theorem 3.3.2.** *Let u be sufficiently regular. Then we have for the Ritz map $\mathscr{R}_h$* (2.62) *the following estimates*
>
> $$\|u - \mathscr{R}_h^l u\|_{L^\infty(\Gamma_t)} \leq h^2 \, |\log h|^{\frac{3}{2}} \, \mathrm{const},$$
> $$\|u - \mathscr{R}_h^l u\|_{W^{1,\infty}(\Gamma_t)} \leq h \, |\log h| \, \mathrm{const},$$
> $$\|\partial_t^{X_h^l}(u - \mathscr{R}_h^l u)\|_{L^\infty(\Gamma_t)} \leq h^2 \, |\log h|^3 \, \mathrm{const},$$
> $$\|\partial_t^{X_h^l}(u - \mathscr{R}_h^l u)\|_{W^{1,\infty}(\Gamma_t)} \leq h \, |\log h|^{\frac{5}{2}} \, \mathrm{const}\,.$$

We remark that the logarithmic order in the inequalities above, is certainly not optimal. Since for evolving surfaces there are very few results presented, we consider that as not essential.

3.4. A semilinear parabolic problem coupled with a regularized velocity law

A starting point for studying ALE-ESFEM is an article by Elliott and Styles [42]. In that paper they considered a problem which originates from mathematical biology. The task is to compute an unknown surface, whose evolution is coupled with the solution of a reaction diffusion equation on the surface. In a private communication with Prof. Madzvamuse during a workshop in Oberwolfach it was revealed that originally Elliott and Styles wanted to consider the coupled problem

$$\begin{aligned} (\partial_t^X + \operatorname{div}(v_X) - \Delta)u &= f, & u(0) &= u_0, \\ v_X &= g\mathrm{n}_X, & X(\Gamma_0, 0) &= \Gamma_0. \end{aligned} \tag{3.7}$$

The ESFEM approximation to this problem behaves badly. Their solution was to regularized the velocity law (3.7) to

$$v_X = g\mathrm{n}_X + \varepsilon\Delta X, \quad X(\Gamma_0, 0) = \Gamma_0, \tag{3.8}$$

where $\varepsilon > 0$ is a small parameter. The ESFEM discretization of that problem leads to satisfactory results.

Independently, Prof. Lubich and Buyang Li observed by a theoretical argument that it should not be possible to derive stability for the velocity law (3.7). The numerical analysis of (3.8) in two space dimensions is currently out of reach. Hence, Prof. Lubich suggested to study the following

PDE 3.4.1. For a given surface $\Gamma_0^2 \subset \mathbb{R}^3$, $\alpha > 0$, initial Data $u_0 \in H^2(\Gamma_0)$ and forcing terms $f, g \in C^1(\mathbb{R} \times \mathbb{R}^3)$ find an evolving surface $X \in C^1(0, T; C^2(\Gamma_0; \mathbb{R}^3))$, and a function $u \in$

$L^2(0,T;H^2(\Gamma_t))$, which satisfy

$$\begin{aligned}(\partial_t^X + \operatorname{div}(v_X) - \Delta)u &= f(u,\nabla u), & u(0) &= u_0,\\ v_X - \alpha\Delta v_X &= g(u,\nabla u)\mathrm{n}_{\Gamma(t)}, & X(\Gamma_0,0) &= \Gamma_0.\end{aligned}$$

In contrast to (3.8) we regularize with $-\alpha\Delta v_X$ on the left-hand side instead of $-\varepsilon\Delta X$ on the right-hand side.

We describe the computational method. We modify the standard ESFEM algorithm, cf. (2.39). We recall that $x_h(t) = (x_i(t))_{i=1}^N \in \mathbb{R}^N \otimes \mathbb{R}^3$ are the nodes of our approximation surface and $u_h(t) = (u_i(t))_{i=1}^N \in \mathbb{R}^N$ are the nodal values of our finite element function. $M(x_h)$, $A(x_h)$ are the mass and stiffness matrices on $\Gamma(x_h)$. We set

$$\begin{aligned}M_3(x_h) &\coloneqq M(x_h)\otimes I_3, & A_3(x_h) &\coloneqq A(x_h)\otimes I_3,\\ M^*(x_h) &\coloneqq M_3(x_h) + \alpha A_3(x_h).\end{aligned}$$

Since X is unknown we determine x_h and u_h via the following ODE system:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(M(x_h)u_h) + A(x_h)u_h &= f_h(x_h,u_h), \quad u_h(0) = \mathscr{I}_h u_0,\\ M^*(x_h)\frac{\mathrm{d}x_h}{\mathrm{d}t} &= g_h(x_h,u_h), \quad x_h(0) = x_{h,0},\end{aligned} \tag{3.9}$$

where $f_h(x_h,u_h) = (f_i(x_h,u_h))_{i=1}^N \in \mathbb{R}^N$ with

$$f_i(x_h,u_h) \coloneqq \int_{\Gamma(x_h)} f(u_h,\nabla_h u_h)\chi_i,$$

and $g_h(x_h,u_h) = (g_i(x_h,u_h))_{i=1}^N \in \mathbb{R}^N\otimes\mathbb{R}^3$ with

$$g_i(x_h,u_h) \coloneqq \int_{\Gamma(x_h)} g(u_h,\nabla_h u_h)\mathrm{n}_{\Gamma(x_h)}\chi_i.$$

In [55] we have shown stability and convergence for (3.9). We need some additional notation for the statement. If $y_h(t) = (y_i(t))_{i=1}^N \in$

$\mathbb{R}^N \otimes \mathbb{R}^3$ denotes some other admissible mesh, then we set $u_h[y_h] := \sum_{i=1}^N u_i \chi_i[y_h]$ and $x_h[y_h] := \sum_{i=1}^N x_i \chi_i[y_h]$, where $(\chi_i[y_h]) \subset S_h(y_h)$ denotes the usual Lagrange finite element basis. The main result is the following

Theorem 3.4.2. *Let X and u be the solution of the PDE 3.4.1. Assume that for an admissible initial mesh $(x_i)_{i=1}^N \subset \Gamma_0$, and then for every refinement, we have that the mesh $y_h(t) := (y_i(t))_{i=1}^N := (X(x_i, t))_{i=1}^N \subset \Gamma(t)$ stays admissible. Then, it holds that for the ESFEM of order $k \geq 2$ there exists a sufficiently small $h_0 > 0$ such that for all $h < h_0$ we have the estimates*

$$\begin{aligned}
\|u - u_h[y_h]^l\|_{L^\infty(0,T;L^2(\Gamma_t))} &\leq h^k \,\mathrm{const}, \\
\|u - u_h[y_h]^l\|_{L^2(0,T;H^1(\Gamma_t))} &\leq h^k \,\mathrm{const}, \\
\|\mathbb{1}_{\Gamma(t)} - x_h[y_h]^l\|_{L^\infty(0,T;H^1(\Gamma_t))} &\leq h^k \,\mathrm{const}, \\
\|v_X - \dot{x}_h[y_h]^l\|_{L^\infty(0,T;H^1(\Gamma_t))} &\leq h^k \,\mathrm{const},
\end{aligned}$$

where $\dot{x}_h := \frac{\mathrm{d}}{\mathrm{d}t} x_h$.

Chapter 4.

Summary and discussion

We will discuss the techniques and ideas developed to prove the theorems presented in the last chapter. For the convenience of the reader, we use dependency graphs in our discussion. The arrow tips in the graphs should be read as "depends on" and a highlighted black box represents a final theorem.

All numerical experiments, which have been coded by the author, have been written in the C++ programming language. The Dune-FEM library, [22], provided the basis for our ESFEM code. Algebraic computations have been done with SAGE [24].

4.1. On an arbitrary Lagrangian-Eulerian evolving surface finite element method

We sketch how to show convergence for some fully discrete ALE-ESFEM schemes, theorem 3.1.5. This is done in four steps. First, we recall the problem using bilinear form notation. Then, we control the spatial discretization error. In the last two steps, we consider time stability of backwards difference formulas and Runge-Kutta method. Afterwards, the author discusses the contributions made in this work. Finally, numerical experiments are discussed.

Summary

Let us recall the basic idea of the ALE-ESFEM mentioned in section 3.1. Using an ALE map $\mathcal{A}$ we are able to express u as the solution of the PDE 3.1.3, which differs from the original PDE 3.1.1. Using the mass and stiffness form notation (2.19) and (2.33) we may express the weak form of PDE 3.1.3 as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(u,\phi)\right) + a(u,\phi) + m(u,(v_{\mathcal{A}} - v_X)\cdot\nabla\phi) = m(f,\phi) + m(u,\partial_t^{\mathcal{A}}\phi). \tag{4.1}$$

For the precise construction of the ALE-ESFEM we refer to section 3.1. The finite element weak form reads as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_h(u_h,\phi_h)\right) + a_h(u_h,\phi_h) + m_h(u_h,(\mathscr{I}_h v_X - v_{\mathcal{A}_h})\cdot\nabla\phi) \\ = m_h(f_h,\phi_h) + m_h(u_h,\partial_t^{\mathcal{A}_h}\phi_h).\end{aligned} \tag{4.2}$$

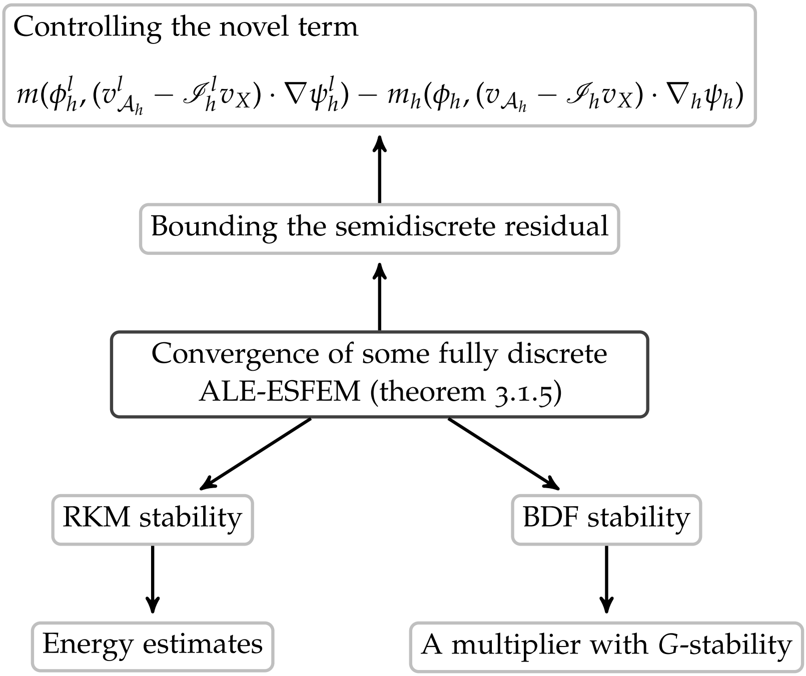
The corresponding matrix ODE system is formulated in (3.1). The schemes resulting from applying BDF or RKM on (3.1) are formulated in (3.2) and (3.3), respectively.

We will now sketch how to control the error. Regardless of time stepping method we split the error as

$$u(t_n) - u_n^l = [u(t_n) - \mathscr{R}_h^l u(t_n)] + [\mathscr{R}_h^l u(t_n) - u_n^l] =: \rho + \theta_n^l,$$

where $\mathscr{R}_h$ is the Ritz map (2.62).

We can bound ρ with (2.63). The main difficulty was to control θ_n. In graph 4.1 we display some steps necessary to achieve this goal. We start to discuss the gray frames above the black frame, which correspond to the spatial discretization error. Then, we proceed with the gray frames below the black frame, which correspond to the time stability.



Graph 4.1.: Some dependencies for theorem 3.1.5

Spatial discretization error

The semidiscrete residual $R_h \in S_h(\mathcal{A})$ is given via the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(m_h(\mathscr{R}_h u, \phi_h)\right) + a_h(u_h, \mathscr{R}_h u) + m_h(\mathscr{R}_h u, (\mathscr{I}_h v_X - v_{\mathcal{A}_h}) \cdot \nabla \phi)\\ &\quad = m_h(f^{-l}, \phi_h) + m_h(u_h, \partial_t^{\mathcal{A}_h} \phi_h) + m_h(R_h, \phi_h).\end{aligned} \tag{4.3}$$

Using the matrix vector notation we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\mathscr{R}_h u\right) + A\mathscr{R}_h u + B\mathscr{R}_h u = Mf_h + MR_h,$$

where have used our nodal value notation (2.29) (a finite element function and its nodal value vector are denoted by the same symbol). To show the bound

$$\|R_h\|_{H_h^{-1}(\Gamma_h(t))} \leq h^2 \,\mathrm{const}, \tag{4.4}$$

we take the difference of (4.3) and (4.1). The following ancillary bound is novel.

Lemma 4.1.1. *For linear finite elements $\phi_h, \psi_h \in S_h(\mathcal{A})$ we have the estimate*

$$\begin{aligned}&m(\phi_h^l, (v_{\mathcal{A}_h}^l - \mathscr{I}_h^l v_X) \cdot \nabla \psi_h^l) - m_h(\phi_h, (v_{\mathcal{A}_h} - \mathscr{I}_h v_X) \cdot \nabla_h \psi_h)\\ &\quad \leq h^2 \,\mathrm{const}\,.\end{aligned}$$

Such an estimate also appears in the work of Elliott and Venkataraman [43, lemma B.3]. Unfortunately with the arguments presented there we could only deduce a first order estimate in h.

Proof of lemma 4.1.1. We start with

$$\begin{aligned}&m_h(\phi_h, (v_{\mathcal{A}_h} - \mathscr{I}_h v_X) \cdot \nabla \psi_h)\\ &\quad = m(\phi_h^l, (v_{\mathcal{A}_h}^l - \mathscr{I}_h^l v_X) \cdot \frac{1}{\delta_h^l} \operatorname{pr}_{\mathrm{T}\Gamma_h}(I - d_{\mathcal{A}}\mathcal{H}) \operatorname{pr}_{\mathrm{T}\Gamma} \nabla \psi_h^l).\end{aligned}$$

Using (2.44) and (2.47) it suffices to prove the critical estimate

$$|(\mathrm{pr}_{\mathrm{T}\Gamma} - \mathrm{pr}_{\mathrm{T}\Gamma}\,\mathrm{pr}_{\mathrm{T}\Gamma_h})(v^l_{\mathcal{A}_h} - \mathscr{I}^l_h v_X)| \leq h^2 \,\mathrm{const}\,.$$

A quick calculation reveals

$$\mathrm{pr}_{\mathrm{T}\Gamma} - \mathrm{pr}_{\mathrm{T}\Gamma_h}\mathrm{pr}_{\mathrm{T}\Gamma} = (\mathrm{n}_h - \mathrm{n} + (\mathrm{n}_h - \mathrm{n})\cdot \mathrm{n}_h \mathrm{n})\,{}^{\mathrm{t}}\mathrm{n}_h.$$

Having (2.45) in mind it suffices to show

$$|\mathrm{n}_h \cdot (v^l_{\mathcal{A}_h} - \mathscr{I}^l_h v_X)| \leq h\,\mathrm{const},$$

which is equivalent to

$$|\mathscr{I}_h \mathrm{n} \cdot (v_{\mathcal{A}_h} - \mathscr{I}_h v_X)| \leq h\,\mathrm{const}\,.$$

Set $F_h := v_{\mathcal{A}_h} - \mathscr{I}_h v_X$ and note that $\mathscr{I}_h \mathrm{n} \cdot F_h$ vanishes on the nodes of Γ_h. Fix an element $E \subset \Gamma_h$ and assume for simplicity that E is h times the standard 2-simplex of $\Delta \subset \mathbb{R}^2 \subset \mathbb{R}^3$. Since we have linear finite elements, a straightforward calculation using Lagrange basis reveals

$$\begin{aligned}
&-\mathscr{I}_h \mathrm{n} \cdot (v_{\mathcal{A}_h} - \mathscr{I}_h v_X) = \\
&\quad (\mathrm{n}(h,0,0) - \mathrm{n}(0,h,0)) \cdot (F_h(h,0,0) - F_h(0,h,0))\frac{x}{h}\frac{y}{h} \\
&\quad + (\mathrm{n}(0,h,0) - \mathrm{n}(0)) \cdot (F_h(0,h,0) - F_h(0))\frac{y}{h}\left(1 - \frac{x}{h} - \frac{y}{h}\right) \\
&\quad + (\mathrm{n}(h,0,0) - \mathrm{n}(0)) \cdot (F_h(h,0,0) - F_h(0))\frac{x}{h}\left(1 - \frac{x}{h} - \frac{y}{h}\right).
\end{aligned}$$

A Taylor expansion argument concludes the proof. ■

Using the lifted product rule (2.42a), the definition of our Ritz map (2.62), the bilinear form bounds (2.56a), (2.56c), the Ritz map bounds (2.63), (2.65) and lemma 4.1.1 we deduce (4.4).

Time stability of backwards difference formulas

The perturbation defect δ_n^{BDF}, for $n \geq k$, is given via the equation

$$\sum_{j=0}^{k} \delta_j M_{n-j} \theta_{n-j}^{\mathrm{BDF}} + \tau A_n \theta_n^{\mathrm{BDF}} + \tau B_n \theta_n^{\mathrm{BDF}} = \tau f_n + R_n + \delta_n^{\mathrm{BDF}}. \tag{4.5}$$

A tedious calculation using Peano kernels shows that we have the bound

$$\|\delta_n^{\mathrm{BDF}}\|_{*,n}^2 \coloneqq \|\delta_n^{\mathrm{BDF}}\|_{*,t_n}^2 \leq \tau^{2k} \sum_{l=0}^{k} \int_0^T \|(\partial_t^{\mathcal{A}_h})^l \mathscr{R}_h u\|_{L^2(\Gamma_h(t))}^2 \,\mathrm{d}t \,\mathrm{const}\,.$$

Using the lift identity (2.40), the material derivative Ritz map bounds (2.65), the identity (2.41) and material derivative velocity bounds (2.61) we deduce that the integral term above is bounded.

We sketch how to derive the stability bound

$$|\theta_n^{\mathrm{BDF}}|_{M_n}^2 + \tau \sum_{j=k}^{n} |\theta_j^{\mathrm{BDF}}|_{A_j}^2 \leq \Big(\tau \sum_{j=k}^{n} \|\delta_j^{\mathrm{BDF}}\|_{*,j}^2 + \max_{0\leq i\leq k-1} |\theta_i^{\mathrm{BDF}}|_{M_i}^2\Big) \,\mathrm{const}\,.$$

We combine Dahlquist's G-stability theory [20] together with Nevanlinna's and Odeh's multiplier technique [69]. This means that for a sufficiently small $\eta \in (0,1)$ (depending on the method) there exists a positive definite matrix $(g_{ij})_{i,j=1}^k$ such that we have the crucial estimate

$$(\theta_n^{\mathrm{BDF}} - \eta\theta_{n-1}^{\mathrm{BDF}}) \cdot M_n \sum_{j=0}^{k} \delta_j \theta_{n-j}^{\mathrm{BDF}} \geq \sum_{i,j=1}^{k} g_{ij} \theta_{n-k+i}^{\mathrm{BDF}} \cdot M_n \theta_{n-k+j}^{\mathrm{BDF}} - \sum_{i,j=1}^{k} g_{ij} \theta_{n-k-1-i}^{\mathrm{BDF}} \cdot M_n \theta_{n-k-1-j}^{\mathrm{BDF}}.$$

We note that for the A-stable BDF 1 and BDF 2 we can even choose $\eta = 0$. The estimate above implies that we can test (4.5) with the multiplier $\theta_n^{\mathrm{BDF}} - \eta\theta_{n-1}^{\mathrm{BDF}}$. Differences of mass matrices are handled with (2.36a), while differences of stiffness matrices are handled with (2.36b). For the novel term B_n we readily deduce bound

$$|y \cdot B(t)z| \leq |z|_{M(t)}\, |y|_{A(t)} \,\mathrm{const}\,. \tag{4.6}$$

This in conjunction with some Young inequalities lead to the stability bound.

Time stability of Runge-Kutta methods

The perturbation defects δ_n^{RKM} and $\delta_{ni}^{\mathrm{RKM}}$ are defined via the equation system

$$\begin{aligned} M_{n+1}\theta_{n+1}^{\mathrm{RKM}} &= M_n\theta_n^{\mathrm{RKM}} + \tau\sum_{j=1}^{s} b_j\dot{\Theta}_{nj} + \delta_n^{\mathrm{RKM}}, \\ M_{ni}\Theta_{ni} &= M_n\theta_n^{\mathrm{RKM}} + \tau\sum_{j=1}^{s} a_{ij}\dot{\Theta}_{nj} + \delta_{ni}^{\mathrm{RKM}}, \\ \dot{\Theta}_{nj} &= M_{nj}R_{nj}, - A_{nj}\Theta_{nj} - B_{nj}\Theta_{nj}, \end{aligned} \tag{4.7}$$

where $i = 1, \dots, s$, $R_{nj} := R(t_{nj})$ etc. We remark that in this context $\dot{\Theta}_{nj}$ is a mere symbol and not the time derivative of Θ_{nj}. Using Peano kernels we can bound the defects with

$$\begin{aligned} \|\delta_n^{\mathrm{RKM}}\|_{*,n}^2 &\leq \tau^{2p}\sum_{l=0}^{p}\int_0^T \|(\partial_t^{\mathcal{A}_h})^l \mathscr{R}_h u\|_{L^2(\Gamma_h(t))}^2 \,\mathrm{d}t\,\mathrm{const}, \\ \|\delta_{ni}^{\mathrm{RKM}}\|_{*,n}^2 &\leq \tau^{2p}\sum_{l=0}^{p}\int_0^T \|(\partial_t^{\mathcal{A}_h})^l \mathscr{R}_h u\|_{L^2(\Gamma_h(t))}^2 \,\mathrm{d}t\,\mathrm{const}\,. \end{aligned}$$

With the same arguments as in the BDF case we deduce that the integral term above is bounded.

We sketch how to show the stability bound

$$
\begin{aligned}
&|\theta_n^{\mathrm{RKM}}|_{M_n}^2 + \tau \sum_{k=1}^{n} |\theta_k^{\mathrm{RKM}}|_{A_k}^2 \\
&\quad \leq C\Big\{ |\theta_0^{\mathrm{RKM}}|_{M_0}^2 + \tau \sum_{k=1}^{n-1} \sum_{i=1}^{s} \|M_{ki}R_{ki}\|_{*,ki}^2 + \tau \sum_{k=1}^{n} \left| \frac{\delta_k^{\mathrm{RKM}}}{\tau} \right|_{M_k}^2 \\
&\qquad + \tau \sum_{k=0}^{n-1} \sum_{i=1}^{s} \Big(|M_{ki}^{-1}\delta_{ki}^{\mathrm{RKM}}|_{M_{ki}}^2 + |M_{ki}^{-1}\delta_{ki}^{\mathrm{RKM}}|_{A_{ki}}^2 \Big) \Big\}.
\end{aligned}
$$

Test (4.7) with $\theta_{n+1}^{\mathrm{RKM}}$. Since the method is admissible we can use property (iii) from definition 3.1.4 This leads to the estimate

$$
\begin{aligned}
|\theta_{n+1}^{\mathrm{RKM}}|_{M_{n+1}}^2 &\leq (1 + \tau\,\mathrm{const}) + 2\tau \sum_{i=1}^{s} b_i \dot{\Theta}_{ni} \cdot M_{ni}^{-1}(M_{ni}\Theta_{ni} + \delta_{ni}) \\
&\qquad + \tau\, |\Theta_{n+1}|_{M_{n+1}}^2 + (1 + \tau\,\mathrm{const})\tau \left| \frac{\delta_{n+1}}{\tau} \right|_{M_{n+1}^{-1}}^2 .
\end{aligned}
$$

The novel term B is hidden in $\dot{\Theta}_{ni}$. We use the same matrix estimates as for the BDF case. Additionally, we need (2.36c).

Contribution

The contribution of the author can be summarized as:

- Giving a new proof for a bilinear form bound.
- Proving convergence for ALE-ESFEM with admissible RKM fully discrete schemes. In particular the important Radau IIa method is included as a time stepping method.
- Providing pictures and numerical experiments. More details in the next section.

The author read the first time about ALE-ESFEM in a computational paper of Elliott and Styles [42]. After reading an article of Formaggia and Nobile [45] the author understood how to formulate the ALE-ESFEM. The new proof for lemma 4.1.1 has been added by the author during the preparations of this thesis. The RKM stability proof was done by the author. He adapted a proof from Dziuk, Lubich and Mansour [37] and Mansour [66] to the ALE setting.

We discovered after submitting our article that independent from our work Elliott and Venkataraman formulated the same ALE-ESFEM as we did, cf. [43]. They proved error bounds for the semidiscrete ALE-ESFEM and for the fully discrete variant with BDF 2 as time stepping method. The last result is contained as a special case of our theorem 3.1.5.

We want to make clear in which points the analysis differs. In their analysis they use the Ritz projection R^h

$$a(R^h z, \phi_h^l) = a(z, \phi_h^l), \quad \text{for } \textstyle\int_\Gamma z = 0 \text{ and } \forall \phi_h \in S_h(\mathcal{A}_h),$$

which is not the Ritz map $\mathscr{R}_h$ (2.62). Further their stability analysis does not use Dahlquist G-stability or Nevanlinna and Odeh multiplier technique, which is the reason why higher-order BDF schemes could not be considered there.

Numerical experiments

Consider the following experiment: For $a(t) := 1 + 0.25\sin(2\pi t)$ we consider the following family of implicit given surfaces

$$\Gamma(t) := \{x \in \mathbb{R}^3 \mid a^{-1}(t)x_1^2 + x_2^2 + x_3^2 - 1 = 0\}, \quad t \in [0,1].$$

The dynamic parametrization for the PDE 3.1.1 is chosen such that the velocity has no tangential component. This evolution is not computed analytically but numerically with the same time stepping method as for the fully discrete scheme. On the other hand, we choose the ALE map for PDE 3.1.3 to be

$$\mathcal{A}(x,t) = \left(\sqrt{a(t)}x_1, x_2, x_3\right).$$

The movement of the corresponding meshes are illustrated in figure A.1 . For the PDE we calculated f such that $u(x,t) = e^{-6t}x_1x_2$ is the exact solution of the problem. The fully discrete scheme is ALE-ESFEM with BDF 1 or BDF 3. The sequence of meshes are chosen such that for the maximum element diameter we have $h_i \approx 2h_{i+1}$, $i = 1, \ldots, 4$. This is achieved by roughly doubling the number of nodes of the mesh. For the sequence of time steps we choose for BDF 1 $\tau_i \approx 4\tau_{i+1}$ and for BDF 3 $\tau_i \approx \sqrt[3]{4}\tau_{i+1}$, $i = 1, \ldots, 4$. The k-th experimental order of convergences, $k = 2, \ldots, 5$, for the errors E_{k-1} and E_k with mesh and time step sizes (h_{k-1}, τ_{k-1}) and resp. (h_k, τ_k) is given via the formula

$$EOC_k := \frac{\log\left(\frac{E_{k-1}}{E_k}\right)}{\log\left(\frac{h_{k-1}}{h_k}\right)}. \tag{4.8}$$

They are measured in table A.1 and table A.2. Some figures with convergence slopes are given in figure A.2, A.3, A.6, A.7.

The figure with the mesh movement has been plotted by the author. Forcing terms, error tables and error slopes for the BDF 1 and BDF 3 time stepping method have been computed by the author.

4.2. On a full discretization of a quasilinear problem on evolving surfaces

We sketch how to show convergence for some fully discrete ESFEM schemes for a quasilinear problem, theorem 3.1.5. This is done in five steps. First, we recall the problem using bilinear form notation. Then, we introduce a suitable Ritz map for the quasilinear problem. We analyze the new Ritz map and prove Sobolev maximum norm stability for this Ritz map. This is followed by a bound of the spatial discretization error. In the last two steps, we consider time stability of backwards difference formulas and admissible Runge-Kutta methods. Afterwards, a discussion of our contributions follows. This section is concluded with some numerical experiments.

Summary

The weak form of PDE 3.2.1 reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m(u,\phi)\right) + a(u;u,\phi) = m(f,\phi) + m(\partial_t^X u,\phi), \tag{4.9}$$

where the quasilinear form is given via

$$a(u;v,w) \coloneqq a(t;u,v,w) \coloneqq \int_{\Gamma(t)} \mathscr{A}(u)\nabla v \cdot \nabla w.$$

The ESFEM version reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_h(u_h,\phi_h)\right) + a_h(u_h;u_h,\phi_h) = m_h(f_h,\phi_h) + m_h(\partial_t^{X_h} u_h,\phi_h).,$$

where the discrete quasilinear form is given via

$$a_h(u;v,w) \coloneqq a_h(t;u,v,w) \coloneqq \int_{\Gamma_h(t)} \mathscr{A}(u)\nabla_h v \cdot \nabla_h w.$$

The corresponding matrix ODE system is formulated in (3.4). The schemes resulting from applying linearly implicit BDF or fully implicit RKM on (3.4) are formulated in (3.5) resp. (3.6).

We will now discuss how to control the error. To obtain optimal order error bounds in the L^2-norm we need to introduce a suitable Ritz map for our nonlinear problem.

Definition 4.2.1. For given $\psi, u \in L^\infty(0,T;H^1(\Gamma_t))$ we define $\mathscr{R}_h u := \mathscr{R}_h(\phi)u \in S_h(t)$ via the requirement that for all $\phi_h \in S_h(t)$ it holds

$$a_h^*(\psi^{-l};\mathscr{R}_h u, \phi_h) = a^*(\psi; u, \phi_h^l).$$

With this Ritz map we split regardless of time stepping method the error as

$$u(t_n) - u_n^l = [u(t_n) - \mathscr{R}_h^l u(t_n)] + [\mathscr{R}_h^l u(t_n) - u_n^l] =: \rho + \theta_n^l.$$

Generic constant

Compared to the numerical analysis of the preceding ALE-ESFEM section it is now essential that the generic constant depends on some Sobolev maximum norm of the exact solution. In our original work [56], we have led the generic constant depend on the $W^{2,\infty}$-norm of the exact solution. This made the presentation there easier. In the present work we will let the generic constant depend only on the $W^{1,\infty}$-norm of the exact solution.

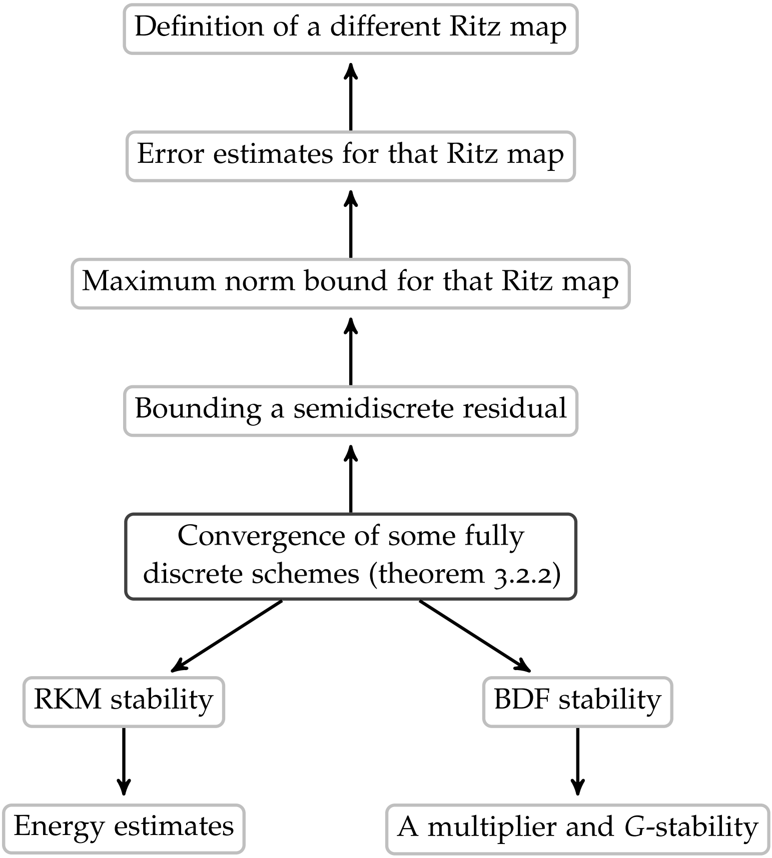
On the formal level we assume that u is an element of the set

$$\mathcal{S} := \Big\{ \phi \in L^\infty(0,T;H^1(\Gamma_t)) \,\Big|\, \|\phi\|_{L^\infty(0,T;W^{1,\infty}(\Gamma_t))} < \text{const} \Big\}. \qquad (4.10)$$

Then, we let the generic constant depend on const (4.10).

Error bounds for a Ritz map

Consider graph 4.2. We start do discuss the gray frames above the highlighted black frame, which corresponds to the spatial discretization errors.



Graph 4.2.: Some dependencies for theorem 3.2.2

We need the following bounds for ρ:

$$\|\rho\|_{L^2} + h\,\|\rho\|_{H^1} \le h^2\,\text{const}, \tag{4.11}$$

$$\|(\partial_t^{X_h^l})^m \rho\|_{L^2} + h\,\|(\partial_t^{X_h^l})^m \rho\|_{H^1} \le h^2\,\text{const}, \tag{4.12}$$

where $m \geq 1$. To derive the H^1-bounds in (4.11) we require a pseudo Galerkin orthogonality of the type

$$|a^*(\psi; u - \mathscr{R}_h^l u, \phi_h^l)| \le h^2\,\text{const}\,.$$

For this we needed the bilinear form bound

$$|a(\psi; u, \phi) - a_h(\psi^{-l}; u^{-l}, \phi^{-l})| \le h^2\,\text{const},$$

and the boundedness of $\mathscr{A}$. To derive the L^2-bound in (4.11) we employed a Aubin-Nitsche argument, which relies on the a priori estimate

$$\|u\|_{H^2} \le \|f\|_{L^2}\,\text{const}, \tag{4.13}$$

for the elliptic part of PDE 3.2.1,

$$u - \operatorname{div}(\mathscr{A}(\psi)\nabla u) = f. \tag{4.14}$$

For (4.12) we need to assume that

$$\partial_t^{X_h^l} \mathscr{A}(\psi) \in L^\infty(0, T; L^\infty(\Gamma_t))$$

holds. Then, we can extend a standard proof for (2.65).

A Sobolev maximum-norm bound for a Ritz map

We sketch how to prove

$$\|\mathscr{R}_h^l u\|_{W^{1,\infty}} \le \|u\|_{W^{1,\infty}}\,\text{const}\,. \tag{4.15}$$

A related estimate for the usual Ritz projection on the plain domain has been shown by Rannacher and Scott [74]. It is not a good strategy to repeat their proof with our Ritz map $\mathscr{R}_h$. The reason is that $\mathscr{R}_h$ is not a projection, $\mathscr{R}_h\mathscr{R}_h^l u \neq \mathscr{R}_h u$. We present a workaround. Define an auxiliary Ritz projection $\mathscr{R}_2 u = \mathscr{R}_2(\psi)u$ via

$$a^*(\psi;\mathscr{R}_2 u,\phi_h^l) = a^*(\psi;u,\phi_h^l), \quad \phi_h \in S_h(t),$$

where $\psi, u \in H^1(\Gamma_t)$. The technical part is to verify that the techniques shown in [74] extend to the evolving surface case:

$$\|\mathscr{R}_2 u\|_{W^{1,\infty}} \leq \|u\|_{W^{1,\infty}} \operatorname{const}. \tag{4.16}$$

The only serious obstacle are calculations with weight functions of the type

$$\sigma_z(x) := \sqrt{|x-z| + \kappa^2 h^2}, \tag{4.17}$$

where $\kappa > 0$ is a big number independent of h. We postpone that discussion to the next section, cf. section 4.3 (4.25).

With (4.16) we can deduce (4.15) as follows: Consider only one component of $\nabla\mathscr{R}_h^l u$ and denote it with $\partial\mathscr{R}_h^l u$. Assume that $\partial\mathscr{R}_h^l u$ takes its maximum on a lifted element $E \subset \Gamma$. Let $x \in \overline{E}$. Consider a regularized delta function δ_x for E, i.e. $\delta_x \in C_0^3(E)$ such that we have

$$m(\delta_x,\phi_h^l) = \phi_h^l(x), \quad \forall\phi_h \in S_h,$$
$$\|\delta_x\|_{L^p} \leq h^{-2(1-1/p)} \operatorname{const}.$$

Let $g \in H^1(\Gamma)$ be the solution of

$$a^*(\psi;g,\phi) = m(\delta,\partial\phi).$$

One easily gets the a priori estimate

$$\|g\|_{H^1} \leq h^{-1} \operatorname{const}.$$

Since $\partial\mathscr{R}_h^l u$ is constant on E, we calculate

$$\begin{aligned}\partial\mathscr{R}_h^l u &= m(\partial\mathscr{R}_h^l u, \delta_x) = a^*(\psi; \mathscr{R}_h^l u, g) = a^*(\psi; \mathscr{R}_h^l u, \mathscr{R}_2 g)\\ &= a^*(\psi; \mathscr{R}_h^l u, \mathscr{R}_2 g) - a_h^*(\psi^{-l}; \mathscr{R}_h u, \mathscr{R}_2^{-l} g) + a^*(\psi; \mathscr{R}_2 u, g)\\ &= a^*(\psi; \mathscr{R}_h^l u, \mathscr{R}_2 g) - a_h^*(\psi^{-l}; \mathscr{R}_h u, \mathscr{R}_2^{-l} g) + m(\partial\mathscr{R}_2 u, \delta_x).\end{aligned}$$

For the last term we use a Hölder estimate in conjunction with (4.16). For the bilinear form error we calculate

$$\begin{aligned}&|a^*(\psi; \mathscr{R}_h^l u, \mathscr{R}_2 g) - a_h^*(\psi^{-l}; \mathscr{R}_h u, \mathscr{R}_2^{-l} g)|\\ &\quad\leq h^2 \, \|\mathscr{R}_h^l u\|_{W^{1,\infty}} \, \|\mathscr{R}_2 g\|_{W^{1,1}} \,\text{const}\\ &\quad\leq h \, \|\mathscr{R}_h^l u\|_{W^{1,\infty}} \,\text{const}\end{aligned}$$

This shows

$$\|\nabla\mathscr{R}_h^l u\|_{L^\infty} \leq (h \, \|\mathscr{R}_h^l u\|_{W^{1,\infty}} + \|u\|_{W^{1,\infty}}) \,\text{const}$$

Repeat the argument for $\mathscr{R}_h^l u$ instead of $\partial\mathscr{R}_h^l u$ for a suitable choice of x in δ_x. For a sufficiently small $h < h_0$ we get (4.15).

Spatial discretization error

The semidiscrete residual R_h is defined via the equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(m_h(\mathscr{R}_h u, \phi_h)\right) + a_h(\mathscr{R}_h u; \mathscr{R}_h u, \phi_h)\\ &\quad= m_h(\mathscr{R}_h u, \partial_t^{X_h}\phi_h) + m_h(f^{-l}, \phi_h) + m_h(R_h, \phi_h)\end{aligned} \tag{4.18}$$

or using the matrix vector notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\mathscr{R}_h u\right) + A(\mathscr{R}_h u)\mathscr{R}_h u = f_h + MR_h.$$

We take the choice $\mathscr{R}_h(u)u = \mathscr{R}_h u$. We sketch how this implies the bound

$$\|R_h\|_{H_h^{-1}} \leq h^2 \,\text{const}\,. \tag{4.19}$$

Using the lifted product rule (2.42a), the bilinear form bounds (2.56a), (2.56c) and the new Ritz map bounds (4.11), (4.12) we can bound every term except for the stiffness form difference. For the latter the definition of $\mathscr{R}_h(u)$ to get

$$\begin{aligned} &a_h^*(\mathscr{R}_h u; \mathscr{R}_h u, \phi_h) - a^*(u; u, \phi_h^l) \\ &\quad = a_h^*(\mathscr{R}_h u; \mathscr{R}_h u, \phi_h) - a_h^*(u^{-l}; \mathscr{R}_h u, \phi_h) \\ &\quad = \int_{\Gamma_h} \big(\mathscr{A}(\mathscr{R}_h u) - \mathscr{A}(u^{-l})\big) \nabla_h \mathscr{R}_h u \cdot \nabla_h \phi_h. \end{aligned}$$

Using that $\mathscr{A}$ is Lipschitz continuous together with the Sobolev maximum norm bound (4.15) we can bound this term with an L^2-L^∞-L^2 Hölder-estimate. This finishes the proof. Obviously, const (4.19) depends on const (4.10).

Time stability of backwards difference formulas

The perturbation defect δ_n^{BDF}, for $n \geq k$, is given via the equation

$$\begin{aligned} &\sum_{j=0}^{k} \delta_j M_{n-j} \theta_{n-j}^{\mathrm{BDF}} \\ &\quad + \tau \left(A_n \Big(\sum_{j=0}^{k-1} \gamma_j \mathscr{R}_h u(t_{n-k+j}) \Big) - A_n \Big(\sum_{j=0}^{k-1} \gamma_j u_{n-k+j}^{\mathrm{BDF}} \Big) \right) \mathscr{R}_h u(t_n) \\ &\quad + \tau A_n \Big(\sum_{j=0}^{k-1} \gamma_j u_{n-k+j}^{\mathrm{BDF}} \Big) \theta_n^{\mathrm{BDF}} = M_n R_n. \end{aligned}$$

A tedious calculation using Peano kernels shows that we have the bound

$$\|\delta_n^{\mathrm{BDF}}\|_{*,n}^2 := \|\delta_n^{\mathrm{BDF}}\|_{*,t_n}^2 \leq \tau^{2k} \sum_{l=0}^{k} \int_0^T \|(\partial_t^{A_h})^l \mathscr{R}_h u\|_{L^2(\Gamma_h(t))}^2 \,\mathrm{d}t \,\mathrm{const}\,.$$

Using the lift identity (2.40), the new material derivative Ritz map bounds (4.12), the identity (2.41) and material derivative velocity bounds (2.61) we deduce that the integral term above is bounded.

We sketch how to derive the stability bound

$$|\theta_n^{\mathrm{BDF}}|_{M_n}^2 + \tau \sum_{j=k}^{n} |\theta_j^{\mathrm{BDF}}|_{A_j}^2 \leq \Big(\tau \sum_{j=k}^{n} \|\delta_j^{\mathrm{BDF}}\|_{*,j}^2 + \max_{0\leq i\leq k-1} |\theta_i^{\mathrm{BDF}}|_{M_i}^2\Big) \,\mathrm{const}\,.$$

Like in the previous section we us Dahlquist's G-stability theory together with Nevanlinna's and Odeh's multiplier technique, i.e. we can use the multiplier $\theta_n^{\mathrm{BDF}} - \eta\theta_{n-1}^{\mathrm{BDF}}$. Differences in the nonlinear term can be handled by using Lipschitz estimates and letting the generic constant depend on const (4.10). The nonlinear term can be changed to the usual stiffness matrix norm by using the lower bound of it.

Time stability of Runge-Kutta methods

The perturbation defects δ_n^{RKM} and $\delta_{ni}^{\mathrm{RKM}}$ are defined via the equation

$$\begin{aligned}
M_{n+1}\theta_{n+1}^{\mathrm{RKM}} &= M_n\theta_n^{\mathrm{RKM}} + \tau\sum_{j=1}^{s} b_j\dot{\Theta}_{nj} + \delta_n^{\mathrm{RKM}},\\
M_{ni}\Theta_{ni} &= M_n\theta_n^{\mathrm{RKM}} + \tau\sum_{j=1}^{s} a_{ij}\dot{\Theta}_{nj} + \delta_{ni}^{\mathrm{RKM}},\\
\dot{\Theta}_{nj} &= M_{nj}R_{nj} - \left(A_{nj}\left(\mathscr{R}_h u(t_{nj})\right) - A_{nj}(U_{nj})\right)\mathscr{R}_h u(t_{nj})\\
&\quad - A_{nj}(U_{nj})\Theta_{nj}^{\mathrm{RKM}},
\end{aligned}$$

where $i = 1,\ldots,s$, $R_{nj} \coloneqq R(t_{nj})$ etc. Using Peano kernels we can bound the defects with

$$\|\delta_n^{\mathrm{RKM}}\|_{*,n}^2 \leq \tau^{2p}\sum_{l=0}^{p}\int_0^T \|(\partial_t^{\mathcal{A}_h})^l\mathscr{R}_h u\|_{L^2(\Gamma_h(t))}^2 \,\mathrm{d}t\,\mathrm{const},$$

$$\|\delta_{ni}^{\mathrm{RKM}}\|_{*,n}^2 \leq \tau^{2p}\sum_{l=0}^{p}\int_0^T \|(\partial_t^{\mathcal{A}_h})^l\mathscr{R}_h u\|_{L^2(\Gamma_h(t))}^2 \,\mathrm{d}t\,\mathrm{const}\,.$$

With the same arguments as in the BDF case we deduce that the integral term above is bounded.

We sketch how to show the stability bound

$$
\begin{aligned}
&|\theta_n^{\mathrm{RKM}}|_{M_n}^2 + \tau \sum_{k=1}^{n} |\theta_k^{\mathrm{RKM}}|_{A_k}^2 \\
&\quad \le \Bigg(|\theta_0^{\mathrm{RKM}}|_{M_0}^2 + \tau \sum_{k=1}^{n-1} \sum_{i=1}^{s} \|M_{ki} R_{ki}\|_{*,ki}^2 + \tau \sum_{k=1}^{n} \Big| \frac{\delta_k^{\mathrm{RKM}}}{\tau} \Big|_{M_k}^2 \\
&\qquad + \tau \sum_{k=0}^{n-1} \sum_{i=1}^{s} \Big(|M_{ki}^{-1} \delta_{ki}^{\mathrm{RKM}}|_{M_i}^2 + |M_{ki}^{-1} \delta_{ki}^{\mathrm{RKM}}|_{A_{ki}}^2 \Big) \Bigg) \, \mathrm{const},
\end{aligned}
$$

Like in the ALE section, we test the perturbed equation with $\theta_{n+1}^{\mathrm{RKM}}$. Using again that the method is admissible we a similar bound like in the ALE section. The nonlinear difference is hidden in Θ_{ni}. Like in the BDF case, we can handle differences of the nonlinear term by using Lipschitz continuity and by letting the constant depend on const (4.10).

Contribution

The contribution of the author can be summarized as follows:

- Formulating together with the coauthor a suitable Ritz map for a quasilinear problem on evolving surfaces.
- Providing existence and an a priori bound for the PDE (4.14).
- Proving $W^{1,\infty}$-best-approximation bounds for our Ritz map.
- Providing all numerical experiments. More details in the next section.

The formulation of the Ritz map was done with the coauthor together. The coauthor extended the L^2-error bounds by using an appropriated PDE lemma proven by the author. Material derivative bounds were partly derived by the author. The $W^{1,\infty}$-best-approximation bounds for our Ritz map have been done by the author alone. The proof presented here is not presented in the article [56]. There the author used a simpler bound and let the generic constant depend on the $W^{2,\infty}$-norm of the exact solution.

Numerical experiments

The numerical experiments have been coded solely by the author in C++.

Consider the following experiments: Let $\Gamma_0 \subset \mathbb{R}^3$ be the unit sphere and let $[0,T] = [0,1]$. As an evolving surface we choose

$$X\colon \Gamma_0 \times [0,T] \to \mathbb{R}^3, \quad (x,t) \mapsto (a(t)x_1, x_2, x_3, t),$$

with $a(t) \coloneqq 1 + 0.25\sin(2\pi t)$. Set $\mathscr{A}(z) \coloneqq 1 - \frac{1}{2}e^{-z^2/4}$. For the PDE 3.2.1 we choose the forcing term f such that $u(x,t) = e^{-6t}x_1x_2$ is the exact solution. We use the implicit Euler method and the linearly implicit BDF 3 for time stepping. We already discussed in the last section how to choose the time steps, the meshes and what EOC means. In table B.1 we computed the EOC for the ESFEM with the implicit Euler method. For figure B.1 and figure B.2 we calculate the numerical solution until the end time $T = 1$ for different step sizes and meshes. The first mentioned figure is for the implicit Euler method and the second figure is for the linearly implicit BDF 3.

4.3. On some maximum-norm error estimates

In this section we discuss how to prove parabolic maximum norm bounds for the ESFEM, theorem 3.3.1, and how to prove maximum norm bounds for the Ritz map and its material derivative, theorem 3.3.2.

Summary

We consider a standard linear parabolic PDE on an evolving surface, PDE 3.1.1, and discretize that equation with the ESFEM. We split the error as follows:

$$u - u_h^l = (u - \mathscr{R}_h^l u) + (\mathscr{R}_h^l u - u_h^l) =: \rho + \theta_h^l,$$

where $\mathscr{R}_h$ is the Ritz map

$$a_h(\mathscr{R}_h u, \phi_h) = a(u, \phi_h^l), \quad \forall \phi_h \in S_h.$$

The semidiscrete residual $R_h(t)$ satisfies per definition for all $\phi_h \in S_h$ the equation

$$\begin{aligned} &m_h(t; \partial_t^X \mathscr{R}_h u, \phi_h) + (\partial_t^{X_h} m_h)(t; \mathscr{R}_h u, \phi_h) + a_h(t; \mathscr{R}_h u, \phi_h) \\ &\quad = m_h(t; f^{-l}, \phi_h) + m_h(t; R_h(t), \phi_h). \end{aligned}$$

As a consequence we have

$$m_h(t; \partial_t^X \theta_h, \phi_h) + (\partial_t^{X_h} m_h)(t; \theta_h, \phi_h) + a_h(t; \theta_h, \phi_h) = m_h(t; R_h(t), \phi_h).$$

If $E_h(t,s) \colon S_h(s) \to S_h(t)$ denotes the solution operator of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}(M u_h) + A u_h = 0,$$

that means $t \mapsto E_h(t,s)\phi_h$ solves the ODE with initial value ϕ_h at time s, then the variation of constant formula implies

$$\theta_h(t) = E_h(t,0)\theta_h(0) + \int_0^t E_h(t,s) R_h(s) \, \mathrm{d}s.$$

Hence, we have maximum-norm bounds,

$$\|u - u_h^l\|_{L^\infty(0,T;L^\infty(\Gamma_t))} + h\,\|u - u_h^l\|_{L^\infty(0,T;W^{1,\infty}(\Gamma_t))} \leq h^2\,|\log h|^p\,\mathrm{const}$$

for some power $p \geq 1$, if we can prove the estimates

$$\|\rho\|_{L^\infty(0,T;L^\infty(\Gamma_t))} + h\,\|\rho\|_{L^\infty(0,T;W^{1,\infty}(\Gamma_t))} \leq h^2\,|\log h|^p\,\mathrm{const}, \tag{4.20}$$

$$\|R_h\|_{L^\infty(0,T;L^\infty(\Gamma_{h,t}))} \leq h^2\,|\log h|^p\,\mathrm{const}, \tag{4.21}$$

$$\|E_h(t,0)\|_{L^\infty(0,T;L^\infty(\Gamma_{h,t}))} \leq |\log h|^p\,\mathrm{const}\,. \tag{4.22}$$

In addition inequality (4.21) calls for the estimate

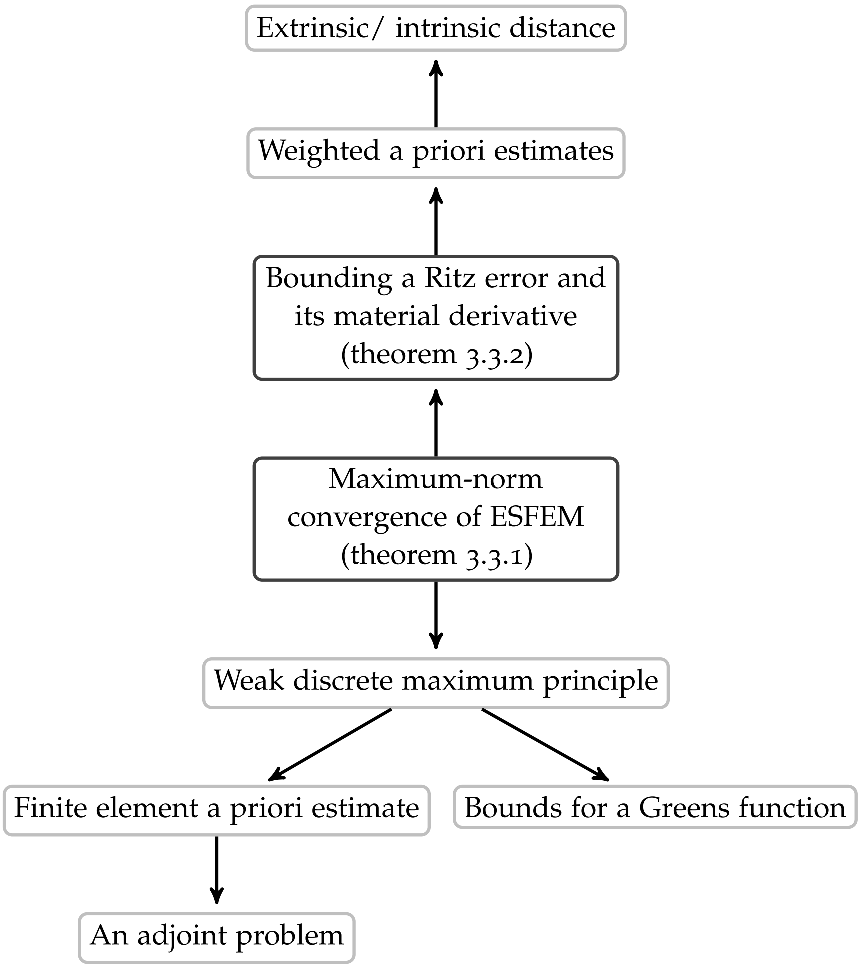
$$\|\partial_t^X \rho\|_{L^\infty(0,T;L^\infty(\Gamma_t))} \leq h^2\,|\log h|^p\,\mathrm{const}\,. \tag{4.23}$$

Consider graph 4.3. Next, we discus the connection between the maximum-norm with some weighted norms. This requires a comparison of an extrinsic distance with an intrinsic distance. Afterwards, we sketch how to proof maximum-norm Ritz error bounds, (4.20), in three steps. The second step is the difficult derivation of some weighted a priori estimates. Then, we sketch how this techniques enable us to obtain maximum-norm bound for the material derivative Ritz error, (4.23). Time stability is derived via the weak discrete maximum principle, (4.22). This is obtained by bounding the maximum norm of an elliptic finite element Greens function and by considering the adjoint dynamic of $E_h(t,s)$.

Extrinsic and intrinsic distance

For our moving surface $X\colon \Gamma_0 \times [0,T] \to \mathbb{R}^3$ we define the extrinsic distance on $\Gamma(t) = X(\Gamma_0, t)$ as the Euclidean distance between two points, $|x-y|$, and the intrinsic distance as the Riemannian distance between two points, $d_\Gamma(x,y)$, which means length of a length minimizing geodesic joining them.[1] For the analysis it is important

[1] The Hopf Rinow theorem implies that on a compact manifold without boundary two points can always be joint by a length minimizing geodesic.



Graph 4.3.: Some dependencies for theorem 3.3.1 and theorem 3.3.2

to consider weight functions of the type

$$\mu(x,y) = |x-y|^2 + \rho^2, \tag{4.24}$$

$$\tilde{\mu}(x,y) = \sqrt{|x-y|^2 + h^2}, \tag{4.25}$$

where $\rho^2 := \gamma h^2 |\log h|$ with a large number $\gamma > 0$, which we require to be independent of h. A natural question arise: "Is it beneficial to stay with the extrinsic distance or is it better to change everything to the intrinsic distance". Calculating the material derivative or the lift of some weight function is better done with the extrinsic distance. By some nontrivial argument we can show that for a function $f(r)$ with $r = d_\Gamma(x,y)$ we have the estimate

$$\int_{\Gamma(t)} f(d_\Gamma(x,y))\,\mathrm{d}x \leq \int_0^R r^m f(r)\,\mathrm{d}r\,\mathrm{const}\,.$$

This means that integral calculation via polar coordinates are feasible with the intrinsic distance. To have all the mentioned benefits without any trade-offs we show that both distances are equivalent.

The following inequality is trivial:

$$|x-y| \leq d_\Gamma(x,y).$$

The converse inequality,

$$d_\Gamma(x,y) \leq |x-y|\,\mathrm{const}\,, \quad \text{for const} \geq 1,$$

follows by some calculations in a local chart.

Some weight functions on evolving surfaces

With the weight function (4.24) we can establish a weak equivalence of the maximum-norm and some weighted norm. We step into the details. Set

$$\begin{aligned} \|u\|^2_{L^2,\alpha} &:= \int_\Gamma \mu^{-\alpha}\,|u|^2, \\ \|u\|^2_{H^1,\alpha} &:= \|u\|^2_{L^2,\alpha} + \|\nabla u\|^2_{L^2,\alpha}, \\ \|u\|^2_{H^2,\alpha} &:= \|u\|^2_{H^1,\alpha} + \|\nabla^2 u\|^2_{L^2,\alpha}. \end{aligned}$$

For simplicity assume that $\phi_h \in S_h$ takes its maximum (and also the maximum of its derivative) on $E \subset \Gamma_h(t)$. Using an inverse estimate we verify

$$\begin{aligned} \|\phi_h^l\|_{W^{1,\infty}(\Gamma_t)} &= \|\phi_h^l\|_{W^{1,\infty}(E)} \le \gamma\,|\log h|^{1/2} \|\phi_h^l\|_{H^1,1}\,\text{const}, \\ \|\phi_h^l\|_{L^\infty(\Gamma_t)} &= \|\phi_h^l\|_{L^\infty(E)} \le h\,|\log h|\rho^{-1}\|u\|_{L^2,2}\,\text{const}\,. \end{aligned}$$

On the other hand, we calculate for arbitrary $\phi \in H^1(\Gamma)$

$$\begin{aligned} \|\phi_h^l\|_{H^1,1} &\le \rho^{-1}\,\|\phi_h^l\|_{W^{1,\infty}(\Gamma_t)}\,\text{const}, \\ \|\phi_h^l\|_{L^2,2} &\le |\log \rho|^{1/2}\,\|\phi_h^l\|_{W^{1,\infty}(\Gamma_t)}\,\text{const}\,. \end{aligned}$$

This means that $\|\cdot\|_{W^{1,\infty}}$ and $\|\cdot\|_{H^1,1}$ are equivalent up to $|\log h|$. Further, $\|\cdot\|_{H^1,1}$ and $\|\cdot\|_{L^2,2}$ can be considered as equally strong norms. This is advantageously for the analysis. The upshot of this calculation is that we can formulate our maximum norm problem in a Hilbert space setting.

Maximum-norm bounds for a Ritz map

Lemma 4.3.1.

$$\begin{aligned} &\|u - \mathscr{R}_h^l u\|^2_{L^2,2} + \|u - \mathscr{R}_h^l u\|^2_{H^1,1} \\ &\qquad \le h^2\,|\log h|^{3/2}\,\|u\|^2_{W^{2,\infty}(\Gamma_t)}\,\text{const}\,. \end{aligned}$$

We sketch the proof in three steps.

First step for lemma 4.3.1. Let us introduce the notation e, e_1 and e_2 via

$$e = u - \mathscr{R}_h^l u = (u - \mathscr{I}_h^l u) + (\mathscr{I}_h^l u - \mathscr{R}_h^l u) = e_1 + e_2.$$

e_1 is nice, because lemma 4.3.1 with $\mathscr{I}_h^l$ instead of $\mathscr{R}_h^l$ is correct.[2] e_2 is a finite element function. We can exploit that fact for the following inequality

$$\|\mu^{-1}\phi_h^l - I_h(\mu^{-1}\phi_h^l)\|_{H^1,-1} \leq \left(\frac{h}{\rho} + h\right)(\|\phi_h^l\|_{L^2,2} + \|\nabla\phi_h^l\|_{L^2,1})\,\mathrm{const},$$

where we require $\phi_h \in S_h$. For a sufficiently small $h < h_0$ and sufficiently large $\gamma > \gamma_0$, cf. (4.24), we can make the first factor arbitrary small. Thus, using $e_2 = e - e_1$ and a triangle inequality we can perform helpful absorption arguments.

Our goal is to prove

$$\|e\|_{H^1,1}^2 \leq (h^2\,|\log h|\,\|u\|_{W^{2,\infty}}^2 + \|e\|_{L^2,2}^2)\,\mathrm{const}\,. \tag{4.26}$$

An elementary calculation leads to the bound

$$\|e\|_{H^1,1}^2 \leq a(e, \mu^{-1}e) + \|e\|_{L^2,2}^2\,\mathrm{const}\,.$$

We split the first term as

$$a(e, \mu^{-1}e) = a(e, \mu^{-1}e_1) + a(e, \mu^{-1}e_2 - \mathscr{I}_h^l(\mu^{-1}e_2)) + a(e, \mathscr{I}_h^l(\mu^{-1}e_2))$$

Using the definition of the Ritz map and the already mentioned arguments, we can handle all three terms. ■

[2] It is even correct with $|\log h|$ instead of $|\log h|^{3/2}$.

Lemma 4.3.2 (second step). *The solution u of*

$$-\Delta u + u = \mu^{-2} f,$$

satisfy the a priori estimate

$$\|u\|_{H^1}^2 \leq \rho^{-2} |\log \rho| \|f\|_{L^2,2}^2 \operatorname{const}.$$

Proof of lemma 4.3.2. By some complicated eigenvalue argument we can reduce the task into finding a lower bound for the Rayleigh-quotient

$$\inf_{u \neq 0} \frac{\|u\|_{H^1}^2}{\|u\|_{L^2,2}^2}.$$

Hence, we are done if we can prove the bound

$$\|u\|_{L^2,2}^2 \leq \rho^{-2} |\log \rho| \, \|u\|_{H^1}^2.$$

A Hölder inequality shows

$$\|u\|_{L^2,2}^2 \leq \|\mu\|_{L^{2p}}^2 \, \|u\|_{L^{2q}}^2,$$

with $p^{-1} + q^{-1} = 1$. Observe that

$$h^{1/|\log h|} = \operatorname{const}.$$

Using the quantitative Sobolev estimate

$$\|u\|_{L^{2q}} \leq q \, \|u\|_{H^1} \operatorname{const},$$

where the constant is independent of q, we can deduce the claim for the choice $q = \sqrt{|\log \rho|}$. ∎

Third step of lemma 4.3.1. Let $e := u - \mathscr{R}_h^l u$. It suffices to show show that for arbitrary $\delta > 0$ we have

$$\|e\|_{L^2,2}^2 \leq \delta \|e\|_{H^1,1}^2 + h^4 \, \|u\|_{W^{2,\infty}}^2 \operatorname{const}. \tag{4.27}$$

We employ a Aubin-Nitsche type argument. Let w be the solution of

$$w - \Delta w = \mu^{-2} e.$$

We calculate

$$\|e\|^2_{L^2,2} = a^*(e, w) = a^*(e, w - \mathscr{I}^l_h w) + a^*(e, \mathscr{I}^l_h w)$$

Using the a priori estimate

$$\|w\|_{H^2,-1} \leq (\|\mu^{-2} e\|_{L^2,-1} + \|w\|_{H^1}) \,\text{const}$$

together with lemma 4.3.2, a sufficiently small $h < h_0$ and sufficiently large $\gamma > \gamma_0$ concludes the proof. ■

We remark that with (4.15) we can obtain a log free $W^{1,\infty}$-bound.

Maximum-norm error bound for the material derivative of a Ritz map

Lemma 4.3.3.

$$\begin{aligned}
&\|\partial_t^{X_h^l}(u - \mathscr{R}^l_h)\|^2_{L^2,2} + \|\partial_t^{X_h^l}(u - \mathscr{R}^l_h)\|^2_{H^1,1} \\
&\quad \leq ch^2 \,|\log h|^4 (\|u\|^2_{W^{2,\infty}(\Gamma(t))} + \|\partial_t^X u\|^2_{W^{2,\infty}(\Gamma(t))}).
\end{aligned}$$

Proof. We introduce the notation $\partial_t e$, $\partial_t e_1$ and $\partial_t e_2$:

$$\begin{aligned}
\partial_t e = \partial_t^{X_h^l} u - \mathscr{R}^l_h u &= (\partial_t^{X_h^l} u - \mathscr{I}^l_h \partial_t^X u) + (\mathscr{I}^l_h \partial_t^X u - \partial_t^{X_h^l} \mathscr{R}^l_h u) \\
&= \partial_t e_1 + \partial_t e_2.
\end{aligned}$$

$\partial_t e_1$ is a good term, because lemma 4.3.2 with $\partial_t e_1$ instead of $\partial_t e$ is valid.[3] $\partial_t e_2$ is a finite element function and the same arguments as for e_2 in the proof of lemma 4.3.1 apply. For simplicity we let

[3] The estimate would be still correct, if we would replace $|\log h|^4$ with $|\log h|$.

from now on the generic constant depend on u. Then, $\partial_t e$ satisfy the inequality

$$|a^*(\partial_t e, \phi_h^l)| \leq (h^2 \|\partial_t e\|_{H^1,1} + h\,|\log h|^{1/2}) \|\phi_h^l\|_{H^1,-1} \,\mathrm{const},$$

which is our substitute for the usual Galerkin orthogonality.

Step 1: This is similar as the first step for lemma 4.3.1. We aim to show

$$\|\partial_t e\|^2_{H^1,1} \leq (h^2\,|\log h|^4 + \|\partial_t e\|^2_{L^2,2}) \,\mathrm{const}\,.$$

We start with

$$\|\partial_t e\|^2_{H^1,1} \leq a(\partial_t e, \mu^{-1}\partial_t e) + \|\partial_t e\|^2_{L^2,2} \,\mathrm{const},$$

and split the first term as follows

$$\begin{aligned} a(\partial_t e, \mu^{-1}\partial_t e) = a\big(\partial_t e, \mu^{-1}\partial_t e_1\big) + a\big(\partial_t e, \mu^{-1}\partial_t e_2 - \mathscr{I}_h^l(\mu^{-1}\partial_t e_2)\big) \\ + a\big(\partial_t e, \mathscr{I}_h^l(\mu^{-1}\partial_t e_2)\big). \end{aligned}$$

All three terms can be handled by the arguments mentioned above.

Step 2: We aim for the estimate

$$\|\partial_t e\|^2_{L^2,2} \leq \delta \|\partial_t e\|^2_{H^1,1} + h^2\,|\log h|^4 \,\mathrm{const},$$

where $\delta > 0$ is an arbitrary small number. Let w be the solution of

$$w - \Delta w = \mu^{-2}\partial_t e.$$

We calculate

$$\|\partial_t e\|^2_{L^2,2} = a^*(\partial_t e, w) = a^*(\partial_t e, w - \mathscr{I}_h^l w) + a^*(\partial_t e, \mathscr{I}_h^l w).$$

The first term can be handled like in lemma 4.3.1. The second term is surprisingly technical due to the weighted norms. We refer to the original source, lemma C.4.8. ∎

A Lebesgue projection

We want to discuss how to reach the weak discrete maximum principle. We require a suitable Lebesgue projection. The L^2-projection $\mathscr{P}_h \colon L^2\left(\Gamma_{h,t}\right) \to S_h$ is defined via the requirement that for all $\phi_h \in S_h(t)$ we have

$$m_h(\mathscr{P}_h f, \phi_h) = m_h(f, \phi_h). \tag{4.28}$$

The L^2-projection is stable in any L^p-norm, i.e. it holds for all $p \in [1, \infty]$

$$\|\mathscr{P}_h f\|_{L^p(\Gamma_{h,t})} \leq \|f\|_{L^p(\Gamma_{h,t})} \operatorname{const}.$$

Further, it satisfy the following **exponential decay property**: For disjoint $A_1, A_2 \subset \Gamma_h(t)$ with $\operatorname{supp} f \subset A_1$ we have

$$\|\mathscr{P}_h f\|_{L^2(A_2)} \leq e^{-\operatorname{dist}_h(A_1,A_2)h^{-1}\operatorname{const}} \|f\|_{L^2(A_1)} \operatorname{const},$$

where dist_h is the intrinsic distance of $\Gamma_h(t)$.

This difficult result was shown for Euclidean domain finite elements in [27]. By using definition (4.28) we can simply repeat their arguments. Stability bounds for the bounds for the L^2-projection will be useful to prove the parabolic finite element maximum principle.

A finite element delta function

The whole analysis is based on the finite element delta function $\delta_h^{t,x} \in S_h(t)$, where $x \in \Gamma_h(t)$, which satisfies per definition for all $\phi_h \in S_h(t)$

$$m_h(\delta_h^{t,x}, \phi_h) = \phi_h(x).$$

$\delta_h^{t,x}$ is compatible with the weight function $\sigma^x(y) = \sigma(x,y)$, cf. (4.25), i.e. it satisfies the bound

$$\|\sigma^x \delta_h^{t,x}\|_{L^2} \leq |\log h| \operatorname{const}. \tag{4.29}$$

The proof of this requires the Lebesgue projection with its stability and exponential decay property.

Two important finite element functions are related with $\delta_h^{t,x}$: An elliptic finite element Greens function and a parabolic finite element Greens function.

An elliptic finite element Greens function

The elliptic finite element Greens function $G_h^{t,x}$ satisfies per definition for all $\phi_h \in S_h(t)$

$$a_h^*(G_h^{t,x}, \phi_h) = \phi_h(x).$$

With the operator $T_h^{*,t}\colon S_h(t) \to S_h(t)$, which fulfills per definition for all $\phi_h \in S_h(t)$

$$m_h(u_h, \phi_h) = a_h^*(T_h^{*,t} u_h, \phi_h),$$

we have the identity $G_h^{t,x} = T_h^{*,t}\delta_h^{t,x}$.

To analyze $G_h^{t,x}$ we need its smooth counter part.

Theorem 4.3.4. *There exists an elliptic Greens function* $\mathbf{G}(t; x, y)$ *for* $\Gamma(t)$ *with the properties*

$$\begin{aligned} |\mathbf{G}(t; x, y)| &\leq \log(1 + |d_\Gamma(x, y)|)\,\mathrm{const}, \\ |\nabla^x \mathbf{G}(t; x, y)| &\leq \frac{1}{d_\Gamma(x, y)}\,\mathrm{const}, \\ \varphi(x, t) &= \frac{1}{V}\int_{\Gamma(t)} \varphi(y, t)\mathrm{d}y - \int_{\Gamma(t)} \mathbf{G}(t; x, y)\Delta_\Gamma \varphi(y, t)\mathrm{d}y, \end{aligned}$$

where $\varphi \in C^2(\Gamma(t))$ *and* V *denotes the* 2*-dimensional Hausdorff measure of* $\Gamma(t) \subset \mathbb{R}^3$.

Proof. First, we observe that the inverse function theorem can be extended in such a way that for a family of diffeomorphism (f_t), which depend smooth on t, the open neighborhood, on which f_t is a diffeomorphism, is independent of t. Then, we show that the injectivity radius of $\Gamma(t)$ can be bounded from below independent of t. With this we can repeat the construction of Aubin [5] to get our desired Greens function. ■

The existence of $\mathbf{G}(t;x,y)$ implies for arbitrary $\phi_h \in S_h$ the estimate

$$\|\phi_h^l\|_{L^\infty(\Gamma_t)} \leq |\log h|^{\frac{1}{2}} \, \|\phi_h^l\|_{H^1(\Gamma_t)} \, \mathrm{const}\,. \tag{4.30}$$

An immediate consequence is the following nice observation

$$\begin{aligned} \|G_h^{t,x}\|_{L^\infty}^2 &\leq \; |\log h| \, \|G_h^{t,x}\|_{H^1}^2 \, \mathrm{const} \\ &= \; |\log h| G_h^{t,x}(x) \, \mathrm{const} \\ &\leq \; \|G_h^{t,x}\|_{L^\infty} \, |\log h| \, \mathrm{const}, \end{aligned}$$

which gives us the bound

$$\|G_h^{t,x}\|_{L^\infty} \leq |\log h| \, \mathrm{const}\,. \tag{4.31}$$

An adjoint evolution operator

We already introduced the evolution operator $E_h(t,s)$. Its adjoint

$$E_h(t,s)^* \colon S_h(s) \to S_h(t),$$

satisfies per definition for all $\phi_h(s) \in S_h(s)$ and $\psi_h(t) \in S_h(t)$

$$m_h\big(t; E_h(t,s)\phi_h(s), \psi_h(t)\big) = m_h\big(t; \phi_h(s), E_h(t,s)^*\psi_h(t)\big).$$

A calculation shows that $E_h(t,s)^*$ is the solution operator of the ODE

$$M(s)\frac{\mathrm{d}}{\mathrm{d}s}u_h(s) - A(s)u_h(s) = 0. \tag{4.32}$$

The sign before $A(s)$ is correct, since the time variable s is going backwards in time. (4.32) comes with an important energy estimate. The solution u_h satisfy

$$\|u_h\|^2_{L^2(0,t;L^2(\Gamma_{h,s}))} \leq m_h(t; T_h^{*,t} u_h, u_h)\,\text{const}\,. \tag{4.33}$$

The proof of this estimates relies essentially on the matrix vector formulation of ODE (4.32).

A parabolic finite element Greens function

We calculate

$$u_h(x,t) = m(t; u_h, \delta_h^{t,x}) = m(t; E_h(t,0)u_h, \delta_h^{t,x}) = m(0; u_h, G^x(t,0)),$$

where $G^x(t,s) := E_h(t,s)^* \delta_h^{t,x}$ is the parabolic finite element Greens function. This means that the weak discrete maximum principle

$$\|u_h\|_{L^\infty(0,T;L^\infty(\Gamma_h(t)))} \leq |\log h|\ \|u_h\|_{L^\infty(\Gamma_h(0))},$$

follows if we have the bound

$$\|G^x(t,0)\|_{L^1(\Gamma_{h,0})} \leq |\log h|\,\text{const}\,.$$

We sketch the proof. We calculate

$$\begin{aligned}\|G^x(t,0)\|_{L^1} &\leq \|(\sigma^x)^{-1}\|_{L^2}\ \|\sigma^x G^x(t,0)\|_{L^2} \\ &\leq |\log h|^{1/2}\ \|\sigma^x G^x(t,0)\|_{L^2}\,\text{const}\,.\end{aligned}$$

By using a generalization of the L^2-projection for the ESFEM and by using bilinear form bounds we reach after a tedious calculation at

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}s}\ \|\sigma^x G^x(t,s)\|^2_{L^2(\Gamma_{h,s})} + \|\sigma^x \nabla\sigma G^x(t,s)\|^2_{L^2(\Gamma_{h,s})} \\ &\quad\leq (\|\sigma^x G^x(t,s)\|^2_{L^2(\Gamma_{h,s})} + \|G^x(t,s)\|^2_{L^2(\Gamma_{h,s})})\,\text{const}\,.\end{aligned}$$

With a backward Gronwall estimate we deduce

$$\begin{aligned} &\|\sigma^x G^x(t,0)\|^2_{L^2(\Gamma_{h,0})} \\ &\leq \Big(\|\sigma^x \delta_h^{t,x}\|^2_{L^2(\Gamma_{h,t})} \int_0^t \|G^x(t,s)\|^2_{L^2(\Gamma_{h,s})} \,\mathrm{d}s \Big) \,\mathrm{const}\,. \end{aligned} \tag{4.34}$$

We conclude the proof by a combination of (4.29), (4.33) and (4.31).

Contribution

The following list details the contributions made by the author:

- Providing all necessary tools to extend techniques based on weighted norms for evolving surfaces problems. This includes
  - equivalence of an extrinsic and an intrinsic distance together with some helpful bounds on its spatial derivative and material derivative,
  - providing all necessary calculations to transfer polar coordinate calculations on moving surfaces.
- Proving existence of an elliptic Greens function together with some bounds on its derivative. This has been investigated for stationary surfaces, but to our knowledge we provide the first proof for the evolving surface case.
- Giving a new proof for the weighted norm a priori estimate lemma 4.3.2.

  Unfortunately, the proof of Nitsche could not be extended to the evolving surface case. He explicitly uses that the smallest Eigenvalue of

  $$-\Delta_{\mathbb{R}^{d+1}} u = \lambda \mu^{-2} u, \quad \text{on } \Omega \subset \mathbb{R}^{d+1} \text{ with } u|_{\partial\Omega} = 0,$$

  are monotonic decreasing w.r.t. Ω, i.e. we have

  $$\tilde{\Omega} \supseteq \Omega \implies \lambda_{\min}(\tilde{\Omega}) \leq \lambda_{\min}(\Omega).$$

Obviously this argument cannot be extended for surfaces. The author wants to thank Buyang Li for teaching him a refined Sobolev bound, which was essential for the proof.

- Proving maximum-norm bounds for the Ritz error (4.20) and maximum-norm bounds for the material derivative of the Ritz error (4.23).

  In the initial phase of our article [58] we wanted to restrain ourselves to prove the weak discrete maximum principle (4.22) and then to search for an appropriate reference for (4.20). The state of art for maximum-norm estimates on surfaces at that time was an article of Demlow [23, corollary 4.6]. We want to make clear why that article does not imply (4.20):

  - Demlow considered an elliptic PDE on a stationary surfaces.
  - His Ritz projection (in his notation u_h, cf. equation (3.7)) is not identical to our Ritz map. The ancillary functional F, which may encode geometric errors resulting from the discrete approximation of the surface, is required to be in the factor space $H^1(\Gamma)/\mathbb{R}$.

  In addition, our sketch for the proof of (4.23) makes clear that (4.23) is not a simple corollary of (4.20).

  We want to remark that recently Körner [59] also derives error bound for a full discretization of a heat equation. In that work the surface is stationary and the error bounds are in $\mathcal{O}(|\log h| h + \sqrt{\tau})$. We suppose that a power of h is lost because of the absence of a bound like 4.23.

- Formulating the adjoint problem (4.32).

  In the original work of Schatz, Thomée and Wahlbin [75] they based their analysis on the semi-group corresponding to linear heat equation on a bounded domain. The author is not aware that a semi-group theory for evolving surface problems has been developed. The author wants to thank Prof. Lubich, whose intuition led to the discovery of this adjoint problem.

- Proving the energy estimate (4.33) for this adjoint problem.

  Since the adjoint problem did not appear in Schatz, Thomée and Wahlbin [75], the above mentioned estimate was not required. Actually, we do not have a heuristic argument why this estimate should be correct. After reaching (4.34) the author guessed that such an energy estimate should be correct.

- Verification of some technical weighted estimates and an extension of a L^2-projection for evolving surfaces. This has been done with the coauthor together.

- Providing all numerical experiments.

A numerical experiment

The numerical experiments have been coded solely by the author in C++.

Let $\Gamma_0 \subset \mathbb{R}^3$ be the unit sphere and let $[0, T] = [0, 1]$. As an evolving surface we choose

$$X \colon \Gamma_0 \times [0, T] \to \mathbb{R}^3, \quad (x, t) \mapsto (a(t)x_1, x_2, x_3, t),$$

with $a(t) := 1 + 0.25 \sin(2\pi t)$. For the PDE 3.1.1 we choose the forcing term f such that $u(x, t) = e^{-6t} x_1 x_2$ is the exact solution. We used a time stepping method with such a small time step such that the spatial error was clearly dominating. Cf. section 4.1 for the choice of meshes. For this problem we are considering the norms $\| \cdot \|_{L^\infty(L^\infty)}$ and $\| \cdot \|_{L^2(W^{1,\infty})}$. So, the errors in the EOC-formula (4.8) have to be taken w.r.t. that norms. A error table is given in table C.1.

4.4. On a semilinear parabolic problem coupled with a regularized velocity law

In this section we discuss how to analyze the coupled problem (3.9) and to prove convergence, theorem 3.4.2. For the convenience of the reader we recap and introduce some notation. Then, we discuss some evolving surface finite element matrix estimates and proceed with stability estimates. We sketch how to obtain the residual estimates and conclude with some numerical experiments.

Notation

Throughout this section we will assume that

- $x_h(t) := (x_h(t))_{i=1}^N$ and $y_h(t) := (y_i(t))_{i=1}^N$ with $x_h(t), y_h(t) \in \mathbb{R}^N \otimes \mathbb{R}^{d+1}$ represent the nodes of some admissible finite element meshes, with silently fixed element relations,
- $u_h(t) := (u_i(t))_{i=1}^N$, $w_h(t) := (w_i(t))_{i=1}^N$ and $z_h(t) := (z_i(t))_{i=1}^N$ with $u_h(t), w_h(t), z_h(t) \in \mathbb{R}^N$ are the nodal values of some scalar-valued finite element function,
- $\vec{u}_h(t) := (\vec{u}_i(t))_{i=1}^N$, $\vec{w}_h(t) := (\vec{w}_i(t))_{i=1}^N$ and $\vec{z}_h(t) := (\vec{z}_i(t))_{i=1}^N$ with $\vec{u}_h(t), \vec{w}_h(t), \vec{z}_h(t) \in \mathbb{R}^N \otimes \mathbb{R}^3$ are the nodal values of some vector-valued finite element function, where $u_h(t)$, $w_h(t)$ and $z_h(t)$ are not connected with $\vec{u}_h(t)$, $\vec{w}_h(t)$ and $\vec{z}_h(t)$, respectively,
- and that (y_h, w_h) are typically tied to something "close" to the exact solution while (x_h, u_h) are generically the solution of (3.9).

We define

$$
\begin{aligned}
|u_h|_{y_h}^2 &:= u_h \cdot M(y_h) u_h, \\
\|u_h\|_{y_h}^2 &:= u_h \cdot A(y_h) u_h,
\end{aligned}
$$

$$\|u_h\|^2_{*,y_h} := u_h \cdot M(y_h) \left(M + A\right)^{-1}(y_h) M(y_h) u_h.$$

Note that $|\cdot|_{y_h}$ and $\|\cdot\|_{*y_h}$ are norms, but $\|\cdot\|_{y_h}$ is only a seminorm. Further, we define the norms

$$\begin{aligned} \|x_h\|^2_{y_h} &:= x_h \cdot M^*(y_h) x_h, \\ \|x_h\|^2_{*,y_h} &:= x_h \cdot M(y_h) M^*(y_h)^{-1} M(y_h) x_h. \end{aligned}$$

Note that we have overloaded $\|\cdot\|_{y_h}$ and $\|\cdot\|_{*,y_h}$. Their meaning depend on the meaning of their arguments. If y_h represents some "good" nodes, then we write $|\cdot|$, $\|\cdot\|$ or $\|\cdot\|_*$ instead of $|\cdot|_{y_h}$, $\|\cdot\|_{y_h}$ or $\|\cdot\|_{*,y_h}$, respectively. Finally, we set $|x_h|_{W^{1,\infty}(\Gamma(y_h))} = |x_h|_{1,\infty}$ as the $W^{1,\infty}(\Gamma(y_h))$-seminorm of the corresponding finite element function.

Summary

We split the error as

$$\begin{aligned} u - u_h[y_h]^l &= (u - w_h[y_h]^l) + (w_h[y_h]^l - u_h[y_h]^l) \\ &= \rho_w + e_u[y_h]^l, \\ \mathbb{1}_\Gamma - x_h[y_h]^l &= (\mathbb{1}_\Gamma - y_h[y_h]^l) + (y_h[y_h]^l - x_h[y_h]^l) \\ &= \rho_y + e_x[y_h]^l, \\ v_X - \dot{x}_h[y_h]^l &= (v_X - \dot{y}_h[y_h]^l) + (\dot{y}_h[y_h]^l - \dot{x}_h[y_h]^l) \\ &= \rho_{\dot{y}} + e_{\dot{x}}[y_h]^l, \end{aligned}$$

and take the choice $y_i := X(x_i(0), t)$ and $w_h[y_h] := \mathscr{I}_h u[y_h]$. Since we are working with higher-order finite elements, we refer to Kovács [54] for the bounds on ρ_w, ρ_y and $\rho_{\dot{y}}$. Our task is derive stability estimates for e_u, e_x and $e_{\dot{x}}$. For this we observe that y_h and w_h satisfy the following perturbed equation system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(M(y_h)w_h) + A(y_h)w_h &= f_h(y_h, w_h) + \delta_w, \quad w_h(0) = u_h(0), \\ M^*(y_h)\frac{\mathrm{d}y_h}{\mathrm{d}t} &= g_h(y_h, w_h) + \delta_y, \quad y_h(0) = x_h(0), \end{aligned} \tag{4.35}$$

where δ_w and δ_y are some semidiscrete residuals. We claim for the perturbation residuals the following

Lemma 4.4.1. *For the choice $y_i := X(x_i(0), t)$ and $w_h = \mathscr{I}_h u[y_h]$ we have the bounds*

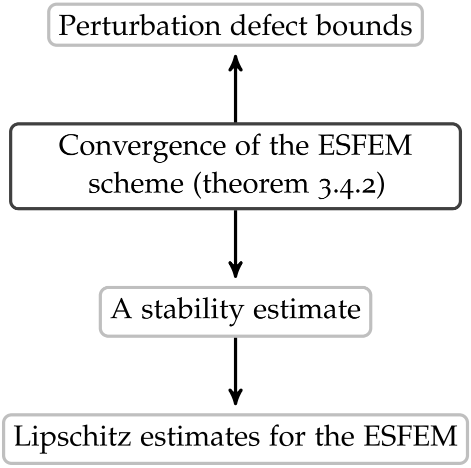
$$\|\delta_y\|_* \leq h^k \,\text{const}, \qquad \|\delta_w\|_* \leq h^k \,\text{const}\,.$$

A nice feature about our stability analysis is that the calculations are detached from the residual estimates. However, to derive the required stability bound on the whole time interval we need lemma 4.4.1.

Lemma 4.4.2. *Assume that lemma 4.4.1 holds for $k > 1$. Then, there exists $h_0 > 0$ sufficiently small such that for all $h < h_0$ and for all $t \in [0, T]$ it holds*

$$\begin{aligned} &|e_u|^2(t) + \int_0^T \|e_u\|^2(s)\,\mathrm{d}s + \|e_x\|^2(t) + \int_0^T \|e_{\dot{x}}\|^2(s)\,\mathrm{d}s \\ &\quad\leq \left(|e_u|^2(0) + \|e_x\|^2(0) + \int_0^T \|\delta_y\|_*^2(s) + \|\delta_w\|_*^2(s)\,\mathrm{d}s\right)\text{const}, \end{aligned} \tag{4.36}$$

Consider graph 4.4. We will first describe the finite element Lipschitz estimates in the unknown nodes. Then we will describe how to derive the stability estimates. Then, we continue with the residual estimates. We conclude the section with some numerical experiments.



Graph 4.4.: Some dependencies for theorem 3.4.2

Evolving surface finite element estimates

As a first step, we establish some conditionally equivalence of norms.

> **Lemma 4.4.3.** *If* $|e_x[y_h]|_{1,\infty} \leq \frac{1}{2}$, *then we have the bound*
>
> $$|u_h[y_h]|_{1,\infty} \frac{1}{\text{const}} \leq |u_h[x_h]|_{1,\infty} \leq |u_h[y_h]|_{1,\infty} \, \text{const} \, .$$

Proof. We use the discrete tangential Jacobian (2.26) with the chain rule (2.8) to deduce

$$\begin{aligned} \nabla_{\Gamma(y_h)} u_h[y_h] &= \nabla_{\Gamma(x_h)} u_h[x_h] \, \text{jac}_h(x_h[y_h]) \\ &= \nabla_{\Gamma(x_h)} u_h[x_h] \, (\mathbb{1} - \text{jac}_h(e_x[y_h])), \end{aligned}$$

Use the von Neumann series to get

$$\left\| (\mathbb{1} - \text{jac}_h(e_x[y_h]))^{-1} \right\|_2 \leq \frac{1}{1 - \|\text{jac}_h(e_x[y_h])\|_2},$$

where $\|\cdot\|_2$ denotes the operator norm induced by the Euclidean norm $|\cdot|$. Conclude the proof with bound (2.7). ∎

Such an conditionally equivalence statements also holds for mass norm $|\cdot|$ and stiffness seminorm $\|\cdot\|$. For our error analysis we require the following Lipschitz estimates.

Lemma 4.4.4. *Provided $|e_x[y_h]|_{1,\infty} \le \frac{1}{2}$ we have the following bound for scalar valued finite element functions*

$$|u_h \cdot (M(x_h) - M(y_h))w_h| \le |u_h|_{y_h} |w_h|_{y_h} |x_h - y_h|_{W^{1,\infty}(\Gamma(y_h))} \,\mathrm{const}, \tag{4.37a}$$

$$\left|u_h \cdot \frac{\mathrm{d}}{\mathrm{d}t} (M(x_h) - M(y_h))w_h\right| \le |u_h|_{y_h}(\|x_h - y_h\|_{y_h} + \|\dot{x}_h - \dot{y}_h\|_{y_h}) \, |w_h|_{y_h,\infty} \,\mathrm{const}, \tag{4.37b}$$

$$|u_h \cdot (A(x_h) - A(y_h))w_h| \le \|u_h\|_{y_h} \|w_h\|_{y_h} |x_h - y_h|_{W^{1,\infty}(\Gamma(y_h))} \,\mathrm{const}, \tag{4.37c}$$

$$|u_h \cdot (A(x_h) - A(y_h))w_h| \le \|u_h\|_{y_h} |w_h|_{W^{1,\infty}(\Gamma(y_h))} \|x_h - y_h\|_{y_h} \,\mathrm{const}, \tag{4.37d}$$

$$z_h \cdot |(f_h(y_h, w_h) - f_h(x_h, u_h))| \le |z_h|_{y_h} \left(\|y_h - x_h\|_{y_h} + |w_h - u_h|_{y_h} + \|w_h - u_h\|_{y_h}\right) \mathrm{const}, \tag{4.37e}$$

and the following bounds for vector valued finite elements functions

$$|\vec{u}_h \cdot (M^*(x_h) - M^*(y_h))\vec{w}_h| \le \|\vec{u}_h\|_{y_h} \|\vec{w}_h\|_{y_h} |x_h - y_h|_{W^{1,\infty}(\Gamma(y_h))} \,\mathrm{const}, \tag{4.38a}$$

$$|\vec{u}_h \cdot (M^*(x_h) - M^*(y_h))\vec{w}_h| \le \|\vec{u}_h\|_{y_h} \|\vec{w}_h\|_{W^{1,\infty}(\Gamma(y_h))} \|x_h - y_h\|_{y_h} \,\mathrm{const}, \tag{4.38b}$$

$$|\vec{z}_h \cdot (g_h(y_h, w_h) - g_h(x_h, u_h))| \tag{4.38c}$$

$$\leq |\vec{z}_h|_{y_h} \left(\|y_h - x_h\|_{y_h} + |w_h - u_h|_{y_h} + \|w_h - u_h\|_{y_h} \right) \text{const}\,.$$

The basic idea is to introduce the intermediate mesh Γ_θ, where $\theta \in [0,1]$ with the corresponding nodes $y_h + \theta(x_h - y_h)$. Then, proceed similar as for (2.36), by replacing t with θ, and use lemma 4.4.3.

A stability estimate

Without doubt the most important result in this section is the derivation of the stability estimates, lemma 4.4.2. We sketch the proof in three steps.

First step for lemma 4.4.2. We want to show that lemma 4.4.2 is correct on a feigned smaller time interval. The following idea is due to Buyang Li. For a given $h_0 > 0$ choose $t^* \in (0, T]$ maximal in the sense that the estimate

$$|e_{\dot{x}}|_{1,\infty}(t) \leq h^{1/2}, \quad \forall t \in [0, t^*]. \tag{4.39}$$

holds on the biggest possible time interval. Existence of t^* is clear, because we anyway assume $\dot{y}_h(0) = \dot{x}_h(0)$. ■

Lemma 4.4.5 (second step for lemma 4.4.2). *Bound* (4.39) *implies the stability bound* (4.36) *on the time interval* $(0, t^*)$*, i.e. replace* T *with* t^* *in the inequalities.* const (4.36) *is independent of* t^**.*

Proof. (a) Surface error bound: For $\varepsilon > 0$ arbitrary small we claim

$$\frac{\mathrm{d}}{\mathrm{d}t} \|e_x\|^2 + \|e_{\dot{x}}\|^2 \leq \varepsilon \|e_u\|^2 + \left(\|e_x\|^2 + |e_u|^2 + \|\delta_y\|_*^2 \right) \text{const}\,.$$

Calculate

$$\frac{\mathrm{d}}{\mathrm{d}t} \|e_x\|^2 \leq \left(\|e_x\|^2 + \|e_{\dot{x}}\|^2 \right) \text{const}\,.$$

Then, formulate the error equation for $e_{\dot{x}}$:

$$M^*(y_h)e_{\dot{x}} = (M^*(y_h) - M^*(x_h))\dot{x}_h + g_h(y_h, w_h) - g_h(x_h, u_h) + \delta_y .$$

Test with $e_{\dot{x}}$ and use (4.38a), (4.38b), (4.38c) and (4.39) to deduce

$$\|e_{\dot{x}}\|^2 \leq \varepsilon \|e_u\|^2 + (\|e_x\|^2 + |e_u|^2 + \|\delta_y\|_*^2) \operatorname{const} .$$

This implies (a).[4]

(b) Heat error: For $\varepsilon > 0$ we claim

$$\frac{\mathrm{d}}{\mathrm{d}t} |e_u|^2 + \|e_u\|^2 \leq \frac{\mathrm{d}}{\mathrm{d}t} \Big(e_u \cdot (M(y_h) - M(x_h))e_u \Big) + \varepsilon \|e_{\dot{x}}\|^2 + (|e_u|^2 + \|e_x\|^2 + \|\delta_w\|_*^2) \operatorname{const} .$$

Formulate the error equation for e_u:

$$\frac{\mathrm{d}}{\mathrm{d}t} (M(y_h)e_u) + A(y_h)e_u = \frac{\mathrm{d}}{\mathrm{d}t} \Big((M(y_h) - M(x_h))u_h \Big) + (A(y_h) - A(x_h))u_h + f_h(y_h) - f_h(x_h) + \delta_w .$$

Test with e_u, use (4.37a), (4.37b), (4.37c), (4.37d), (4.37e) and (4.39).

(C) Combination: Sum up (a) and (b) and absorb. Then, integrate the resulting bound from 0 to t^*. After absorbing a term on the right-hand side with (4.39) we get for all $t \in (0, t^*)$

$$\begin{aligned} &\|e_x\|^2(t) + \int_0^{t^*} \|e_{\dot{x}}\|^2(s) \, \mathrm{d}s + |e_u|^2(t) + \int_0^{t^*} \|e_u\|^2(s) \, \mathrm{d}s \\ &\leq \Big(\int_0^{t^*} \|e_x\|^2(s) + |e_u|^2(s) \, \mathrm{d}s + \|e_x\|^2(0) + |e_u|^2(0) \\ &\quad + \int_0^{t^*} \|\delta_y\|_*^2(s) + \|\delta_w\|_*^2(s) \, \mathrm{d}s \Big) \operatorname{const} . \end{aligned}$$

A Gronwall inequality finishes the proof. ■

[4] We remark that without the parameter α we would not be able to reach that inequality.

Third step for lemma 4.4.2. Assuming that the defect is in $\mathcal{O}(h^k)$, lemma 4.4.1, where $k \geq 2$ is the degree of the finite element space, a combination of (4.36) and an inverse estimate shows

$$|e_{\dot{x}}|_{1,\infty} \leq h^{k-1} \,\mathrm{const}\,.$$

Since t^* is maximal and the constant above is independent of t^* a sufficiently small h_0 implies $t^* = T$. ■

Residual estimates

The residual estimates do not use any new or unknown technique. We sketch the proofs. Denote by $Y_h(t) := \sum_{i=1}^N y_i(t)\chi[x_i(0)]$ the parametrization for $\Gamma(y_h(t))$. For the diffusion equation we calculate

$$\begin{aligned}
m_h(\delta_w, \phi_h) &= m_h(\partial_t^{Y_h} w_h, \phi_h) + (\partial_t^{Y_h} m_h)(w_h, \phi_h) \\
&\quad + a_h(w_h, \phi_h) - m_h(f(w_h, \nabla_h w_h), \phi_h) \\
&= \left(m_h(\partial_t^{Y_h} w_h, \phi_h) - m(\partial_t^{Y_h^l} u, \phi_h^l)\right) \\
&\quad + \left((\partial_t^{Y_h} m_h)(w_h, \phi_h) - (\partial_t^{Y_h^l} m)(u, \phi_h^l)\right) \\
&\quad + \left(a_h(w_h, \phi_h) - a(u, \phi_h^l)\right) \\
&\quad + \left(m(f(u, \nabla u), \phi_h) - m_h(f(w_h, \nabla_h w_h), \phi_h)\right) \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

$I_1, \ldots, I_4$ require higher-order version of the bilinear form bounds (2.56) and interpolation estimates (2.58). Both may be found in [54]. For I_1 we require in addition

$$\mathscr{I}_h^l \partial_t^X u - \partial_t^{Y_h^l} \mathscr{I}_h^l u = (v_{Y_h^l} - v_X) \cdot \nabla u,$$
$$\mathscr{I}_h^l \partial_t^X u = \partial_t^{Y_h^l} \mathscr{I}_h^l u.$$

For I_4 we need Lipschitz continuity of f, the identity

$$(\nabla_h w_h)^l = \mathrm{pr}_h (I - d_X \mathcal{H}) \nabla w_h^l,$$

where pr denotes the projection onto the tangent space and pr_h denotes the projection onto the discrete tangent space, and the bound

$$\|\mathrm{pr} - \mathrm{pr}_h \, \mathrm{pr}\| \leq h \, \mathrm{const}\,.$$

For the regularized velocity law observe that with $\dot{y}_h = \mathscr{I}_h v_X$ we have

$$\begin{aligned} m_h(\delta_y, \phi_h) &= m_h(\mathscr{I}_h v_X, \phi_h) + \alpha\, a_h(\mathscr{I}_h v_X, \phi_h) - m(g(w_h, \nabla_h w_h) \mathrm{n}_{\Gamma(y_h)}, \phi_h) \\ &= \big(m_h(\mathscr{I}_h v_X, \phi_h) - m(v_X, \phi_h^l)\big) \\ &\quad + \alpha\, \big(a_h(\mathscr{I}_h v_X, \phi_h) - a(v_X, \phi_h^l)\big) \\ &\quad + \big(m(g(u, \nabla u) \mathrm{n}_{\Gamma(t)}, \phi_h^l) - m_h(g(w_h, \nabla_h w_h) \mathrm{n}_{\Gamma(y_h)}, \phi_h)\big), \end{aligned}$$

where $\phi_h \in S_h(y_h; \mathbb{R}^3)$ is temporarily a vector-valued finite element function. There are no new hidden techniques, thus, the already mentioned arguments imply lemma 4.4.1.

Contribution

The contribution of the author can be summarized as follows:

- Discussing several matrix bounds with the authors. The proof presented here for lemma 4.4.3 has been done by the author.
- Discussing and calculating the important stability lemma 4.4.5.
- Prove stability for a regularized mean curvature flow.
- Providing all pictures and numerical experiments.

Numerical experiments

The numerical experiments have been coded solely by the author in C++.

We use in all our experiments linear finite elements instead of higher-order estimates, because they are easier to implement.[5] Also, we use a time stepping method with such a small time step that the spatial error was clearly dominating. The sequence of meshes are the same as in the previous numerical experiments.

We consider three different test cases. In our first test case, we calculate some EOC, cf. (4.8). This shall illustrate our convergence theorem 3.4.2. The PDE was

$$\begin{aligned}(\partial_t^X + \operatorname{div}(v_X) - \Delta)u &= f(t,x),\\ v_X - \alpha\Delta v_X &= \delta u \mathrm{n}_\Gamma + g(t,x)\mathrm{n}_\Gamma,\end{aligned}$$

with the parameters $T = 1$, $\alpha = 1$ and $\delta = 0.4$. The forcing terms f and g are such that $X(p,t) = r(t)p$ with

$$r(t) = \frac{r_0 r_K}{r_K e^{-kt} + r_0(1 - e^{-kt})}, \tag{4.40}$$

with the parameters $r_0 = 1$, $r_k = 2$ and $k = 0.5$, and $u(X,t) = X_1 X_2 e^{-6t}$ are the exact solution of the problem. We refer to table D.1 for the results. We observe that the surface and surface velocity errors behave like we would expect. Also, the $L^\infty(L^2)$-errors for u are as expected. For the $L^2(H^1)$-errors we calculate an EOC of 2. Our theory cannot explain that.

In our next test case, we want to compare quantitatively the velocity law (3.8) with our velocity law, cf. PDE 3.4.1. For this we considered the PDE

$$v_X - \alpha\Delta v_X - \beta\Delta X = g(t,x)\mathrm{n}_\Gamma.$$

The exact solution is again (4.40) with the common parameters $T = 2$, $r_0 = 1$, $r_K = 2$, $k = 0.5$. Our first experiment was done with $(\alpha,\beta) = (0,1)$ and the second with $(\alpha,\beta) = (1,0)$. In table D.2 we plot

[5] The subsequent chapter discusses convergence for the linear ESFEM.

the results. We observe that the $L^\infty(H^1)$-error of the mean curvature velocity is growing for our sequence of meshes. The author believes that it should not be possible to show convergence in this norm. For our regularized velocity we predicted convergence in this norm and also observe it. Additionally, every error calculated with our velocity law is smaller then the mean curvature corresponding one.

In our last test case, we want to compare qualitatively the two velocity laws. We consider the following PDE

$$\begin{aligned}(\partial_t^X + \operatorname{div}(v_X) - \Delta)u &= f_1(u,w),\\ (\partial_t^X + \operatorname{div}(v_X) - D_c\Delta)w &= f_2(u,w),\\ v_X - \alpha\Delta v_X - \beta\Delta X &= \delta u \mathrm{n}_\Gamma,\end{aligned}$$

where

$$f_1(u,w) = \gamma(a - u + u^2 w), \quad f_2(u,w) = \gamma(b - u^2 w),$$

with non-negative parameters $D_c, \gamma, a, b, \alpha, \beta$. This is a variation of a PDE, which appeared in Elliott and Styles [42] and Barreira, Elliott and Madzvamuse [6]. The common parameters are $D_c = 10$, $\gamma = 100$, $a = 0.1$, $b = 0.9$, $T = 5$. The generation of the initial values is a complicated task.[6] We take small perturbation around the steady state

$$\begin{pmatrix}\widetilde{u}_0\\ \widetilde{w}_0\end{pmatrix} = \begin{pmatrix} a + b + \varepsilon_1(x)\\ \frac{b}{(a+b)^2} + \varepsilon_2(x)\end{pmatrix},$$

where $\varepsilon_1(x), \varepsilon_2(x) \in [0, 0.01]$ take random values. Then, we apply the stationary surface finite element method and solve the problem until $\widetilde{T} = 5$. We take that result as initial values for the coupled problem, $u_0 = \widetilde{u}(\widetilde{T})$ and $w_0 = \widetilde{w}(\widetilde{T})$.

We remark that it appears that the actual values of the initial perturbation have little effect on the solution $(\widetilde{u}(\widetilde{T}), \widetilde{w}(\widetilde{T}))$. Further, we observed that the parameter γ influences the number of dots

[6] I want to thank Raquel Barreira for explaining me the procedure.

on the surface: a small value of γ generates few dots and big dots, while a huge value of γ generates many and smaller dots.

We computed the solution for $(\alpha_1, \beta_1) = (0, 0.01)$ and $(\alpha_2, \beta_2) = (0.01, 0)$. The result is displayed in figure D.1. Clearly, our new velocity law is competitive.

Implementation

The author wants to describe, which C++ technical obstacles appeared during the implementation of the code.

- Slow compilation times.

  Dune makes heavy use of C++ templates. We briefly describe that feature. For C++ (also for C and Fortran) there is a separation between compilation of code (translation into machine language) and execution of code.[7] C++ templates only executes during compilation, i.e. the resulting binary cannot tell, if the code used templates at all. This means

  - we can perform flow control (`if`, `else`, for), and hence theoretically every computation, during compile time,
  - and further perform inlining of short function, i.e. instead of calling a function, which is expensive if the body of the function is short, we directly include the function body in the binary.

  It is known that proper usage of templates can lead to amazingly fast code. However, our initial code, which we got from a Dune school, also used templates in places, where it does not give substantial benefits (templates are no panacea). This led to code, which took the compiler more then one minute to produce an executable, regardless how small our changes

[7] Matlab on the other hand uses a JIT compiler (just in time compiler). It is very difficult for the user to manipulate the compilation process.

were. This made testing and debugging a horror. Our solution was to break the big template file into multiple smaller translation units. There we specialized our template classes and functions. Afterwards, we could reduce the compilation time to less then 10 seconds (in many cases).

- The finite element mesh is a private member.

  Dune hides the actual values of its meshes from the users. Once a mesh is constructed from a dgf-file (Dune grid format) there is no way to change that values. This is not desirable for evolving surface problems. However, it is possible to give the corresponding Dune class an explicit parametrization. This parametrization does not change the underlining mesh. Instead Dune generates for every for-loop on the fly the new mesh via the given parametrization. To simulate an evolving surface mesh, we use a so called hash-map.[8] After constructing a reasonable efficient hash function, we could easily manipulate the image of our hash-map, which leads to our finite element mesh movement.

[8] A hash-map is something like a more complicated and efficient red-green balanced binary tree, where every node consist of a key value pair.

Chapter 5.

Almost best approximation for linear finite elements on an evolving surface driven by diffusion on the surface

In this section we present a novel result, which has not been submitted elsewhere.

5.1. Objectives

We consider PDE 3.4.1. In section 3.4 we stated the convergence theorem 3.4.2. There are three aspects, which the author considers as unnatural:

- The convergence rate in the L^2-norm is too low.
- It is not possible to state the result with the natural lift on the numerical computed surface.

- The analysis only works for a higher-order finite element method. The computationally import linear ESFEM case is not covered.

Using the notation introduce there we can formulate our main

Theorem 5.1.1. *Let X and u be the solution of the PDE 3.4.1. Assume that for an admissible initial mesh $(x_i)_{i=1}^N \subset \Gamma_0$, and then for every refinement, we have that the mesh $\widetilde{y}_h(t) \coloneqq (\widetilde{y}_i(t))_{i=1}^N \coloneqq (X(x_i,t))_{i=1}^N \subset \Gamma(t)$ stays admissible. Then, for every $\varepsilon > 0$ there exists for the linear ESFEM a sufficiently small $h_0 > 0$ such that for all $h < h_0$ we have the estimates*

$$\begin{aligned}
\|u - u_h[x_h]^l\|_{L^\infty(0,T;L^2(\Gamma_t))} &\le h^{2-\varepsilon}\,\text{const},\\
\|u - u_h[x_h]^l\|_{L^2(0,T;H^1(\Gamma_t))} &\le h^{1-\varepsilon}\,\text{const},\\
\|d_{\Gamma(t)}\|_{L^\infty(0,T;L^\infty(\Gamma(x_h)))} &\le h^{2-\varepsilon}\,\text{const},\\
\|v_X - \dot{x}_h[x_h]^l\|_{L^2(0,T;L^\infty(\Gamma_t))} &\le h^{2-\varepsilon}\,\text{const},
\end{aligned}$$

where $\dot{x}_h \coloneqq \frac{\mathrm{d}}{\mathrm{d}t} x_h$.

For simplicity we set from now on $\alpha = 1$ in PDE 3.4.1.

5.2. Preliminaries

We require a modified result from our previous work. Assume for an admissible mesh y_h that for $r > \frac{3}{4}$ we have the bound

$$\|d_{\Gamma(t)}\|_{L^\infty(\Gamma(y_h))} \le h^{r+\frac{1}{2}}\,\text{const}\,.$$

Then repeating the proofs for (2.45), (2.47) and (2.48) we get

$$\|\mathrm{pr}_{\mathrm{T}_p\Gamma_h(t)}\,\mathrm{n}\|_{L^\infty(\Gamma(y_h))} + \|\mathrm{n} - \mathrm{n}_h\|_{L^\infty(\Gamma(y_h))} \le h^{r-\frac{1}{2}}\,\text{const}, \tag{5.1a}$$

$$\|1 - \delta_h\|_{L^\infty(\Gamma(y_h))} \le h^{2r-1} \,\text{const}, \tag{5.1b}$$

$$\|\mathrm{pr}_{\mathrm{T}_x\Gamma} - \mathrm{pr}_{\mathrm{T}_x\Gamma}\,\mathrm{pr}_{\mathrm{T}_p\Gamma(y_h)}\,\mathrm{pr}_{\mathrm{T}_x\Gamma}\|_{L^\infty(\Gamma(y_h))} \le h^{2r-1} \,\text{const}\,. \tag{5.1c}$$

We require the following PDE a priori estimate for evolving surfaces.

Lemma 5.2.1. *For $d_1 \in L^3(\Gamma_t)$ and $d_2 \in W^{1,3}(\Gamma_t;\mathbb{R}^3)$ there exists an weak solution $u \in H^1(\Gamma_t)$ of the PDE*

$$u - \Delta u = d_1 + \operatorname{div}(d_2),$$

with the a priori estimate

$$\|u\|_{L^\infty(\Gamma_t)} \le (\|d_1\|_{L^3(\Gamma_t)} + \|d_2\|_{L^3(\Gamma_t)}) \,\text{const}\,.$$

Proof. The estimate

$$\|u\|_{W^{1,3}(\Gamma_t)} \le (\|d_1\|_{L^3(\Gamma_t)} + \|d_2\|_{L^3(\Gamma_t)}) \,\text{const}$$

has been shown in the Euclidean domain case in [25]. This can be extended by using a dynamic parametrization with local charts to the evolving surface case. A Sobolev estimate concludes the proof. ∎

Theorem 5.2.2. *For $\mathscr{R}_2\colon H^1(\Gamma_t;\mathbb{R}^3) \to S_h^l(y_h;\mathbb{R}^3)$, which satisfies per definition for all $\phi_h \in S_h(y_h;\mathbb{R}^3)$*

$$a^*(y_h;\mathscr{R}_2 u, \phi_h^l) = a^*(y_h; u, \phi_h^l),$$

it holds

$$\|u - \mathscr{R}_2 u\|_{L^\infty(\Gamma_t)} \le |\log h|\; \|u - \mathscr{I}_h^l u\|_{L^\infty(\Gamma_t)} \,\text{const}\,.$$

Proof. Such a result has been proven for the scalar valued Ritz projection on a Euclidean domain with mixed boundary conditions by Leykekhman and Li [61]. The theorem above follows by repeating their arguments. We sketch the steps. First, replace in that work every occurrence of $-\Delta$ with $\mathbb{1} - \Delta$ and every occurrence of $(\nabla \cdot, \nabla \cdot)$ with $a^*(\cdot, \cdot)$. The Poincaré lemma is not needed since $\|\cdot\|_{L^2} \leq \|\cdot\|_{H^1}$. PDE existence and a priori estimates are also correct in our case, since we have no boundary in contrast to the polygonal boundaries appearing in [61]. We want to extend [61, lemma 9]. The proof of this lemma is done in [61, section 4]. [61, lemma 10] states the existence of a Green's function with some auxiliary bounds. For the evolving surface case this has been established by us in theorem 4.3.4. A regularized delta function is then introduced together with a regularized Green's function, which also exists in the evolving surface case. In [61, lemma 11] the domain Ω is subdivided in disjoint radial symmetric subdomains. In [61, lemma 12] bounds for the regularized Green's function on such subdomains are proven. Such subdivision arguments were also present for the proof of our parabolic maximum norm theorem 3.3.1. In section 4.3 we showed that the intrinsic and extrinsic distance on $\Gamma(t)$ are equivalent and further showed how to calculate integrals with geodesic polar coordinates. This technique allows us to extend both lemmata to the evolving surface case. The last step is [61, section 4.1], which can be repeated without any obstacles. ■

5.3. Road map

Before stepping into the technical aspects we outline how to prove theorem 5.1.1. We split the error as follows.

$$\begin{aligned} u - u_h[x_h]^l &= (u - w_h[y_h]^l) + (w_h[y_h]^l - u_h[y_h]^l) \\ &\quad + (u_h[y_h]^l - u_h[x_h]^l) \end{aligned}$$

$$
\begin{aligned}
&= \rho_w + e_u[y_h]^l + \sigma_u, \\
\mathbb{1}_\Gamma - x_h[x_h]^l &= (\mathbb{1}_\Gamma - y_h[y_h]^l) + (y_h[y_h]^l - x_h[y_h]^l) \\
&\quad + (x_h[y_h]^l - x_h[x_h]^l) \\
&= \rho_y + e_x[y_h]^l + \sigma_x, \\
v_X - \dot{x}_h[y_h]^l &= (v_X - \dot{y}_h[y_h]^l) + (\dot{y}_h[y_h]^l - \dot{x}_h[y_h]^l) \\
&\quad + (\dot{x}_h[y_h]^l - \dot{x}_h[x_h]^l) \\
&= \rho_{\dot{y}} + e_{\dot{x}}[y_h]^l + \sigma_{\dot{x}}.
\end{aligned}
$$

Set $y_h(t)$ as the Ritz mesh, which we define later, and set $w_h[y_h] \coloneqq \mathscr{R}_h u[y_h]$, where $\mathscr{R}_h$ is our usual Ritz map (2.62), which is not $\mathscr{R}_2$. First, we prove the following fundamental lemma for the Ritz mesh.

Lemma 5.3.1. *For $h < h_0$ sufficiently small the Ritz mesh is an admissible mesh for $t \in [0, T]$. For any $\varepsilon > 0$ there exists an $h_0 > 0$ sufficiently small such that for all $h < h_0$ and for all $i = 1, \dots, N$ we have*

$$|\widetilde{y}_i(t) - y_i(t)| \leq h^{2-\varepsilon} \,\text{const}, \tag{5.2a}$$

$$\|d_{\Gamma(t)}\|_{L^\infty(\Gamma(y_h))} \leq h^{2-\frac{\varepsilon}{2}} \,\text{const}, \tag{5.2b}$$

$$\|v_X - v_{y_h}^l\|_{L^\infty(\Gamma_t)} \leq h^{2-\varepsilon} \,\text{const}\,. \tag{5.2c}$$

The Euclidean norm of ρ_y is the absolute value of $d_{\Gamma(t)}$. Hence, (5.2b) and (5.2c) imply the bound

$$\|\rho_y\|_{L^\infty(0,T;L^\infty(\Gamma_t))} + \|\rho_{\dot{y}}\|_{L^\infty(0,T;L^\infty(\Gamma_t))} \leq h^{2-\varepsilon} \,\text{const}\,.$$

The Ritz map bound

$$\|\rho_w\|_{L^2(\Gamma_t)} + h\,\|\rho_w\|_{H^1(\Gamma_t)} \leq h^{2-\varepsilon} \,\text{const},$$

are readily obtained by repeating the standard proofs with (5.2b) and (5.1).

The next step is to bound e_u, e_x and $e_{\dot{x}}$. We consider the perturbed error equation system (4.35) from section 4.4. We will show the following bounds for the semidiscrete Residuals δ_w and δ_y.

Lemma 5.3.2. *For the Ritz mesh $y_h(t)$ with the Ritz map $w_h = \mathscr{R}_h u[y_h]$ we have the bounds*

$$\|\delta_y\|_* \leq h^{2-\varepsilon}\,\text{const}, \qquad \|\delta_w\|_* \leq h^{2-\varepsilon}\,\text{const}\,.$$

Assume for the moment that the lemma above is correct. Then, we may apply stability lemma 4.4.2 to deduce

$$\|e_u[y_h]^l\|_{L^\infty(0,T;L^2(\Gamma_s))} + \|e_u[y_h]^l\|_{L^2(0,T;H^1(\Gamma_t))} \leq h^{2-\varepsilon}\,\text{const},$$
$$\|e_x[y_h]^l\|_{L^\infty(0,T;H^1(\Gamma_t))} + \|e_{\dot{x}}[y_h]^l\|_{L^2(0,T;H^1(\Gamma_t))} \leq h^{2-\varepsilon}\,\text{const}\,.$$

Use the forbidden Sobolev estimate,

$$\|\phi_h^l\|_{L^\infty(\Gamma_t)} \leq |\log h|^{\frac{1}{2}}\,\|\phi_h^l\|_{H^1(\Gamma_t)}\,\text{const},$$

cf. (4.30), to arrive at

$$\|e_u[y_h]^l\|_{L^\infty(0,T;L^2(\Gamma_s))} + \|e_u[y_h]^l\|_{L^2(0,T;H^1(\Gamma_t))} \leq h^{2-\varepsilon}\,\text{const}, \tag{5.3a}$$
$$\|e_x[y_h]^l\|_{L^\infty(0,T;L^\infty(\Gamma_t))} + \|e_{\dot{x}}[y_h]^l\|_{L^2(0,T;L^\infty(\Gamma_t))} \leq h^{2-\varepsilon}\,\text{const}, \tag{5.3b}$$

where we let ε be a little bit larger. Assume for the moment the following

Lemma 5.3.3. *It holds*

$$\|\sigma_u^l\|_{L^\infty(0,T;L^2(\Gamma_t))} \leq h^{2-\varepsilon}\,\text{const},$$
$$\|\sigma_x^l\|_{L^\infty(0,T;L^\infty(\Gamma_t))} + \|\sigma_{\dot{x}}^l\|_{L^2(0,T;L^\infty(\Gamma_t))} \leq h^{2-\varepsilon}\,\text{const}\,.$$

Use an inverse estimate to bound the lift-lift error σ_u in the norm $\|\cdot\|_{L^2(H^1)}$. This shows theorem 5.1.1.

It remains to show that the assumed lemmata are correct. We will first give a definition of the Ritz map and then prove the fundamental Ritz mesh lemma 5.3.1. Then, we will proceed with the residual estimate lemma 5.3.2. The final part consist of the proof of lift-lift error lemma 5.3.3.

5.4. A Ritz mesh

We start with a

> **Definition 5.4.1.** The **Ritz mesh** $y_h(t)$ of an evolving surface X is given via the requirement that for all $z_h \in \phi_h(y_h; \mathbb{R}^3)$ we have
>
> $$a_h^*(y_h; v_{y_h}, z_h) = a_h^*(y_h; v_X^{-l}, z_h)$$
>
> with $y_h(0) = \widetilde{y}_h(0)$.

Short-time existence of an admissible Ritz mesh $y_h(t)$ follows by the same argument as for the numerical solution $x_h(t)$ of (3.9): $y_h(t)$ is the solution of an ODE, where the right-hand side is Lipschitz in $y_h(t)$, cf. matrix bounds in lemma 4.4.4. Long-time existence of an admissible Ritz mesh $y_h(t)$ follows, if the fundamental lemma 5.3.1 is correct.

Proof of the fundamental Ritz mesh lemma

The proof is subdivided into four steps.

First step

Let $[0, \delta] \subseteq [0, T]$ be the maximal time interval such that the bound

$$\|d_{\Gamma(t)}\|_{L^\infty(\Gamma(y_h))} \leq h^{r+\frac{1}{2}}, \tag{5.4}$$

with $r > \frac{3}{4}$ is correct. We define $d_1 \in H^1(\Gamma_t)$ and $d_2 \in H^1(\Gamma_t; \mathbb{R}^3)$ as

$$\int_\Gamma -d_1 z_h^l - d_2 \cdot \nabla z_h^l = a^*(v_X - v_{y_h}^l, z_h^l).$$

Using (5.1) we calculate

$$\begin{aligned}
\|d_1\|_{L^3(\Gamma)} + \|d_2\|_{L^3(\Gamma)} &\le h^{2r-1} \|v_{y_h}^l - v_X\|_{W^{1,3}(\Gamma)} \operatorname{const} \\
&\le h^{2r-1}(\|v_{y_h}^l - \mathscr{I}_h^l v_X\|_{W^{1,3}(\Gamma)} \\
&\qquad + \|\mathscr{I}_h^l v_X - v_X\|_{W^{1,3}(\Gamma)}) \operatorname{const} \\
&\le h^{2r-1}(h^{-\frac{1}{3}} \|v_{y_h}^l - \mathscr{I}_h^l v_X\|_{L^\infty(\Gamma)} + h) \operatorname{const} \\
&\le (h^{2r-\frac{4}{3}} \|v_{y_h}^l - v_X\|_{L^\infty(\Gamma)} + h^{2r}) \operatorname{const}.
\end{aligned}$$

According to lemma 5.2.1 there exists $w \in H^1(\Gamma)$ such that

$$a^*(w, z) = \int_\Gamma d_1 z + d_2 \cdot \nabla z,$$

with the a priori estimate

$$\|w\|_{L^\infty} \le (\|d_1\|_{L^3} + \|d_2\|_{L^3}) \operatorname{const}.$$

Because of the opposite sign above we get the crucial identity

$$v_{y_h}^l = \mathscr{R}_2(v_X + w).$$

Using the almost best approximation property of the Ritz projection, theorem 5.2.2, we calculate

$$\begin{aligned}
\|v_X + w - v_{y_h}^l\|_{L^\infty(\Gamma)} &\le |\log h| \, \|v_X + w - \mathscr{I}_h^l(v_X + w)\|_{L^\infty(\Gamma)} \operatorname{const} \\
&\le |\log h|(\|w\|_{L^\infty(\Gamma)} + \|v_X - \mathscr{I}_h^l v_X\|_{L^\infty(\Gamma)}) \operatorname{const} \\
&\le (h^{2r-\frac{4}{3}} |\log h| \, \|v_{y_h}^l - v_X\|_{L^\infty(\Gamma)} \\
&\qquad + h^{2r} |\log h|) \operatorname{const},
\end{aligned}$$

On the other hand we have

$$\|v_X - v^l_{y_h}\|_{L^\infty(\Gamma)} \leq \|v_X + w - v^l_{y_h}\|_{L^\infty(\Gamma)} + \|w\|_{L^\infty(\Gamma)}.$$

An absorption argument, which requires $h < h_0$ sufficiently small, shows

$$\|v_X - v^l_{y_h}\|_{L^\infty(\Gamma)} \leq h^{2r} \, |\log h| \, \mathrm{const}\,. \tag{5.5}$$

Second step

We want to use (5.5) to show

$$\|d_{\Gamma(t)}\|_{L^\infty(\Gamma(y_h))} \leq h^{2r} \, |\log h| \, \mathrm{const}\,. \tag{5.6}$$

Recall the definition of the lifted velocity $v_{y^l_h}$, cf. text before (2.40). Let $y = y(t) \in \Gamma(y_h)$ flow according to the discrete mesh movement and denote by $p = p(t) \in \Gamma(t)$ its corresponding lift. A quick calculation using (2.12) reveals

$$\begin{aligned} v_{y^l_h}(p,t) = -\frac{\partial d_X(y,t)}{\partial t}\mathrm{n}(y,t) - d_{\Gamma(t)}(y,t)\frac{\partial \mathrm{n}(y,t)}{\partial t} \\ + (\mathrm{pr}_{\mathrm{T}_p\Gamma(t)} - d_{\Gamma(t)}(y,t)\mathcal{H}(y,t))v_{y_h}(y,t). \end{aligned} \tag{5.7}$$

Note that the first summand points in normal direction and the other terms are tangent vectors. Further, use (2.16) to get

$$-\frac{\partial d_X(y,t)}{\partial t} = v_X(p,t) \cdot \mathrm{n}(y,t). \tag{5.8}$$

Since $p - y$ points in normal direction we have

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,|p-y|^2 &= (v_{y^l_h}(p,t) - v_{y_h}(y,t)) \cdot (p-y) \\ &= (v_X(p,t) - v^l_{y_h}(p,t)) \cdot \mathrm{n}(y,t)\,\mathrm{n}(y,t) \cdot (p-y) \\ &\leq h^{4r}\,|\log h|^2 \,\mathrm{const} + \frac{1}{2}\,|p-y|^2. \end{aligned}$$

A Gronwall argument shows (5.6).

Third step

We want to show

$$|\widetilde{y}_i(t) - y_i(t)| \leq h^{2r}\, |\log h|\, \text{const}\,. \tag{5.9}$$

Denote by $p_i = p_i(t) \in \Gamma(t)$ the lift of y_i. According to (5.6), it suffices to show

$$|\widetilde{y}_i(t) - p_i(t)| \leq h^{2r}\, |\log h|\, \text{const}\,. \tag{5.10}$$

Using (5.5), (5.6), (5.7) and (5.8) we easily see

$$|v_X(p,t) - v_{y_h^l}(p,t)| \leq h^{2r}\, |\log h|\, \text{const}\,.$$

For the velocity we calculate

$$\begin{aligned} |v_X(\widetilde{y}_i,t) - v_{y_h^l}(p_i,t)| &\leq |v_X(\widetilde{y}_i,t) - v_X(p_i,t)| + |v_X(p_i,t) - v_{y_h^l}(p_i,t)| \\ &\leq (|\widetilde{y}_i - p_i| + h^{2r}\, |\log h|)\, \text{const}\,. \end{aligned}$$

Combine the bound above with the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, |\widetilde{y}_i - p_i|^2 \leq \frac{1}{2}\, |\widetilde{y}_i - p_i|^2 + \frac{1}{2}\, |v_X(\widetilde{y}_i(t),t) - v_{y_h^l}(p_i(t),t)|^2.$$

Use a Gronwall argument to deduce (5.10).

Fourth step

With a sufficiently small $h < h_0$ we can use (5.6) to show that (5.4) must be already correct on a larger time interval. The bound (5.9) guarantees that the mesh stays admissible. Lemma 5.3.1 follows from (5.5), (5.6) and (5.9).

5.5. Residual bounds

In this section we prove the semidiscrete residual bound lemma 5.3.2. We calculate for the diffusion equation defect δ_u using $\phi_h \in S_h(y_h)$

$$\begin{aligned} m_h(\delta_w, \phi_h) = {} & \big(m_h(\partial_t^{y_h} \mathscr{R}_h u, \phi_h) + (\partial_t^{y_h} m_h)(\mathscr{R}_h u, \phi_h) + a_h(\mathscr{R}_h u, \phi_h)\big) \\ & - \big(m(\partial_t^{y_h^l} u, \phi_h^l) + (\partial_t^{y_h^l} m)(u, \phi_h^l) + a(u, \phi_h^l)\big) \\ & + \big(m_h(f(\mathscr{R}_h u, \nabla_h \mathscr{R}_h u), \phi_h) - m(f(u, \nabla u), \phi_h^l)\big) \end{aligned}$$

All terms, except for the last nonlinear f term, can be handled with standard arguments. For the surface defect δ_y we calculate using $\vec{\phi}_h \in S_h(y_h; \mathbb{R}^3)$

$$\begin{aligned} m_h(\delta_y, \vec{\phi}_h) = {} & \big(a_h(v_{y_h}, \vec{\phi}_h) - m_h(g(\mathscr{R}_h u, \nabla_h \mathscr{R}_h u) \mathrm{n}_{\Gamma(y_h)}, \vec{\phi}_h)\big) \\ & - \big(a(v_X, \vec{\phi}_h^l) - m(g(u, \nabla u) \mathrm{n}_{\Gamma(t)}, \vec{\phi}_h^l)\big). \end{aligned}$$

Using definition 5.4.1 of our Ritz mesh it remains to bound the term with g.

A normal bound

The following bound is novel and critical

> **Lemma 5.5.1.** *Let y_h be an evolving discrete mesh, which satisfies bounds like in lemma 5.3.1. For all $f, g \in W^{1,\infty}(\Gamma_t)$ we have*
>
> $$\int_{\Gamma_t} (\mathrm{n}_{\Gamma(t)} - \mathrm{n}_{\Gamma(y_h)}) \cdot f \le h^{2-\varepsilon} \|f\|_{W^{1,1}(\Gamma_t)} \,\mathrm{const},$$
>
> $$\int_{\Gamma(t)} (\mathrm{pr}_{\mathrm{T}_x \Gamma(t)} - \mathrm{pr}_{\mathrm{T}_p \Gamma(y_h)}) f \cdot g \le h^{2-\varepsilon} \|f \cdot g\|_{W^{1,1}(\Gamma_t)} \,\mathrm{const}\,.$$

Proof. For the second inequality observe that

$$\begin{aligned} \mathrm{pr}_{\mathrm{T}_x \Gamma(t)} - \mathrm{pr}_{\mathrm{T}_p \Gamma(y_h)} = {} & (\mathrm{n}_{\Gamma(y_h)} - \mathrm{n}_{\Gamma(t)})^{\mathrm{t}} (\mathrm{n}_{\Gamma(y_h)} - \mathrm{n}_{\Gamma(t)}) \\ & + \mathrm{n}_{\Gamma(t)}{}^{\mathrm{t}} (\mathrm{n}_{\Gamma(y_h)} - \mathrm{n}_{\Gamma(t)}) + (\mathrm{n}_{\Gamma(y_h)} - \mathrm{n}_{\Gamma(t)})^{\mathrm{t}} \mathrm{n}_{\Gamma(t)}. \end{aligned}$$

A combination of the first bound with (5.1a) shows the second bound. Hence, it remains to show

$$\int_{\Gamma(t)} (\mathrm{n}_{\Gamma(t)} - \mathrm{n}_{\Gamma(y_h)}) \cdot f \leq h^{2-\varepsilon} \|f\|_{W^{1,1}(\Gamma_t)} \,\mathrm{const}\,.$$

The key idea is to use partial integration on the discrete normal vector $\mathrm{n}_{\Gamma(y_h)}$, which is not obvious. We explain how this is done. We parametrize $\Gamma(y_h)$ via the map

$$Y\colon \Gamma_0 \times [0,T] \to \mathbb{R}^3, \quad X_h(x^{-l}, t),$$

where X_h is the parametrization of $\Gamma(y_h)$ and x_0^{-l} is the negative lift of $x_0 \in \Gamma_0$ on $\Gamma(y_h(0))$. We note that Y is Lipschitz and hence in $W^{1,\infty}(\Gamma_0 \times [0,T])$. In particular, we can use partial integration. For the normal we have the formula

$$\mathrm{n}_{\Gamma(y_h)}(x^{-l}, t) = \frac{\partial_1 Y(x_0) \wedge \partial_2 Y(x_0)}{|\partial_1 Y(x_0) \wedge \partial_2 Y(x_0)|},$$

where $\wedge$ denotes the outer vector product on $\mathbb{R}^3$, $x \in \Gamma(t)$ with $X(x_0, t) = x$ and $Y(x_0, t) = x^{-l}$. The formula above is also correct for $\mathrm{n}_{\Gamma(t)}$ and X instead of $\mathrm{n}_{\Gamma(y_h)}$ and Y. We observe that the right-hand side above is Lipschitz continuous in the six variables $(\partial_1 Y, \partial_2 Y)$. Denote its first and second derivative w.r.t. this six variables with $\mathrm{Dn}(X)$ and $\mathrm{D^2n}(X)$, where the argument X means "insert the variables $(\partial_1 X, \partial_2 X)$". Using Taylor expansion we get

$$\begin{aligned} \mathrm{n}_{\Gamma(t)} - \mathrm{n}_{\Gamma(y_h)} &= \mathrm{Dn}(X)\,\big(\partial_1(X-Y), \partial_2(X-Y)\big) \\ &\quad + \int_0^1 \mathrm{D^2n}(\theta X + (1-\theta)Y)\,\big(\partial_1(X-Y), \partial_2(X-Y)\big)^2 \,\mathrm{d}\theta. \end{aligned} \tag{5.11}$$

A straightforward calculation shows $X - Y = \mathcal{O}(h^{2-\varepsilon})$ and $\partial_i(X - Y) = \mathcal{O}(h^{1-\varepsilon})$. Thus, the Taylor remainder is in $\mathcal{O}(h^{2-2\varepsilon})$. Use the local integral formula (2.9) and partial integration to conclude the proof. ■

Bounding the semilinear term

Lemma 5.5.2. *If $f, g \in C^2(\Gamma_t)$ then we have for linear finite elements*

$$\int_{\Gamma(t)} f(u, \nabla u)\phi_h - \int_{\Gamma(y_h)} f(\mathscr{R}_h u, \nabla_h \mathscr{R}_h u)\phi_h \\ \leq h^{2-\varepsilon} \|\phi_h\|_{H^1} \text{const}.$$

$$\int_{\Gamma(t)} g(u, \nabla u)\mathrm{n}_{\Gamma(t)} \cdot \vec{\phi}_h^l - \int_{\Gamma(y_h)} g(\mathscr{R}_h u, \nabla_h \mathscr{R}_h u)\mathrm{n}_{\Gamma(y_h)} \cdot \vec{\phi}_h \\ \leq h^{2-\varepsilon} \|\vec{\phi}_h\|_{H^1} \text{const}.$$

Proof. It suffices to show the second bound. We calculate

$$\begin{aligned}
&\int_{\Gamma(t)} g(u, \nabla u)\mathrm{n}_{\Gamma(t)} \cdot \vec{\phi}_h^l - \int_{\Gamma(y_h)} g(\mathscr{R}_h u, \nabla_h \mathscr{R}_h u)\mathrm{n}_{\Gamma(y_h)} \cdot \vec{\phi}_h \\
&\quad = \int_{\Gamma(t)} g(u, \nabla u)\mathrm{n}_{\Gamma(t)} \cdot \vec{\phi}_h^l - g(u, \mathrm{pr}_h \nabla \mathscr{R}_h^l u)\mathrm{n}_{\Gamma(y_h)} \cdot \vec{\phi}_h^l \\
&\qquad + \int_{\Gamma(t)} g(u, \mathrm{pr}_h \nabla \mathscr{R}_h^l u)\mathrm{n}_{\Gamma(y_h)} \cdot \vec{\phi}_h^l - \int_{\Gamma(y_h)} g(u^{-l}, \nabla_h \mathscr{R}_h u)\mathrm{n}_{\Gamma(y_h)} \cdot \vec{\phi}_h \\
&\qquad + \int_{\Gamma(y_h)} (g(u^{-l}, \nabla_h \mathscr{R}_h u) - g(\mathscr{R}_h u, \nabla_h \mathscr{R}_h u))\mathrm{n}_{\Gamma(y_h)} \cdot \vec{\phi}_h,
\end{aligned}$$

where $\mathrm{pr}_h = \mathrm{pr}_{\mathrm{T}_p\Gamma(x)}$ is the projection on the discrete tangent space. Using (5.1b) and the Lipschitz continuity of g we easily bound the second and third difference. We introduce notation (only for this proof!): We set $\mathrm{pr} = \mathrm{pr}_{\mathrm{T}_x\Gamma(t)}$, $g(u, \nabla u) = g(\nabla u)$, $\mathrm{n}_h = \mathrm{n}_{\Gamma(y_h)}$ and $u_h = \mathscr{R}_h u$. Denote by g_p and g_{pp} the Jacobian and Hessian of g, respectively, w.r.t. the argument ∇u. We calculate

$$\begin{aligned}
&\int g(\nabla u)\mathrm{n} - g(\mathrm{pr}_h \nabla u_h^l)\mathrm{n}_h \cdot \vec{\phi}_h \\
&\quad = \int (g(\nabla u) - g(\mathrm{pr}_h \nabla u))\mathrm{n} \cdot \vec{\phi}_h \\
&\qquad + \int (g(\mathrm{pr}_h \nabla u) - g(\mathrm{pr}_h \nabla u_h^l))\mathrm{n} \cdot \vec{\phi}_h
\end{aligned}$$

$$
\begin{aligned}
&+ \int (g(\mathrm{pr}_h \nabla u_h^l) - g(\nabla u))(\mathrm{n} - \mathrm{n}_h) \cdot \vec{\phi}_h \\
&+ \int g(u, \nabla u)(\mathrm{n} - \mathrm{n}_h) \cdot \vec{\phi}_h \\
= &\, I_1 + I_2 + I_3 + I_4
\end{aligned}
$$

I_3 and I_4 are easy. For I_3 use the standard normal bound (5.1a), Lipschitz continuity of g and the usual H^1-bound for our Ritz map. For I_4 we use our normal vector lemma 5.5.1.

For I_1 calculate

$$
\begin{aligned}
g(\nabla u) - g(\mathrm{pr}_h \nabla u) &= g_p(\nabla u)(\mathrm{pr} - \mathrm{pr}_h)\nabla u \\
&+ \int_0^1 g_{pp} \left(\nabla u + \theta(\mathrm{pr} - \mathrm{pr}_h)\nabla u\right) \left((\mathrm{pr} - \mathrm{pr}_h)\nabla u\right)^2 \mathrm{d}\theta.
\end{aligned}
$$

(5.1a) implies that the Taylor remainder is in $\mathcal{O}(h^{2-\varepsilon})$. Bound the first term with the normal vector lemma 5.5.1.

For I_2 calculate

$$
\begin{aligned}
g(\mathrm{pr}_h \nabla u) - g(\mathrm{pr}_h \nabla u_h^l) &= g_p(\mathrm{pr}_h \nabla u)\nabla(u - u_h^l) \\
&- g_p(\mathrm{pr}_h \nabla u)\mathrm{n}_h(\mathrm{n}_h - \mathrm{n}) \cdot \nabla(u - u_h^l) \\
&+ \int_0^1 g_{pp} \left(\mathrm{pr}_h \nabla \left(u + \theta(u_h - u)\right)\right) \left(\mathrm{pr}_h \nabla(u - u_h^l)\right)^2 \mathrm{d}\theta.
\end{aligned}
$$

The second and the Taylor remainder term are in $\mathcal{O}(h^{2-\varepsilon})$. For the first term we calculate

$$
\begin{aligned}
g_p(\mathrm{pr}_h \nabla u) &= g_p(\nabla u) \\
&+ \int_0^1 g_{pp} \left(\mathrm{pr}_h + \theta(\mathrm{pr} - \mathrm{pr}_h)\nabla u\right)(\mathrm{pr} - \mathrm{pr}_h)\nabla u \, \mathrm{d}\theta.
\end{aligned}
$$

Use the partial integration formula (2.10) for $\nabla(u - u_h^l)$ and the L^2-bound for the Ritz map to bound I_2. ■

5.6. Bounding the lift error

In this section we want to prove lemma 5.3.3. We start with an ancillary

Lemma 5.6.1. *For every $p \in \Gamma(t)$ there exists an unique $y \in \Gamma(y_h)$ and an unique $x \in \Gamma(x_h)$ such that the lift of y and x is equal to p. For x exists a unique $Y \in \Gamma(y_h)$ such that $x \in E_k(x_h)$, $Y \in E_k(y_h)$ and x, Y share the same standard simplex coordinates, cf.* (2.21). *For Y exists a unique lifted point $q \in \Gamma(t)$. It holds*

$$|p - q| \leq h^{2-\varepsilon} \operatorname{const}.$$

Proof. Existence of x, y, Y and q is obvious. Note that (5.3b) implies

$$|x - Y| \leq h^{2-\varepsilon} \operatorname{const}.$$

Using the definition of lift (2.12) we have

$$p - q = \left(x - d_{\Gamma(t)}(x)\mathrm{n}(x)\right) - \left(Y - d_{\Gamma(t)}(Y)\mathrm{n}(Y)\right).$$

The expression above is Lipschitz in x and Y. This gives us

$$|p - q| \leq |x - Y| \operatorname{const}.$$ ■

Proof of lemma 5.3.3. We calculate for σ_u

$$\begin{aligned} u_h[y_h]^l(p) - u_h[x_h]^l(p) &= u_h[y_h](y) - u_h[x_h](x) \\ &= u_h[y_h](y) - u_h[y_h](Y) \\ &= u_h[y_h]^l(p) - u_h[y_h]^l(q) \\ &= \int_0^1 \nabla u_h[y_h]^l \left(p + \theta(q-p)\right) \cdot (q - p) \, \mathrm{d}\theta, \end{aligned}$$

where we have extended $u_h[y_h]^l$ on $\mathcal{N}_t$ according to (2.1). Deduce with (5.3a) and an inverse estimate that $\|u_h[y_h]^l\|_{L^\infty(H^1)}$ is bounded. Using lemma 5.6.1 we deduce the first stated bound in lemma 5.3.3. The other bounds follow similarly. ■

5.7. Contribution

Buyang Li has shown the author the Ritz best approximation property and the PDE a priori estimate used in this section. Further, he discussed with the author, different kinds of Ritz meshes. Apart from that, all proofs in this chapter has been done solely by the author.

Appendix A.

Higher-order time discretization with arbitrary Lagrangian Eulerian finite elements on evolving surfaces

The content of this section is accepted for publication, cf. [57].

Abstract

A linear evolving surface partial differential equation is first discretized in space by an arbitrary Lagrangian Eulerian (ALE) evolving surface finite element method, and then in time either by a Runge-Kutta method, or by a backward difference formula. The ALE technique allows one to maintain the mesh regularity during the time integration, which is not possible in the original evolving surface finite element method. Stability and high order convergence of the full discretizations is shown, for algebraically stable and stiffly accurate Runge-Kutta methods, and for backward differentiation formulas of order less than 6. Numerical experiments are included, supporting the theoretical results.

A.1. Introduction

There are various approaches to solve parabolic problems on evolving surfaces. A starting point of the finite element approximation of (elliptic) surface partial differential equations is the paper of [29]. Later this theory was extended to general parabolic equations on stationary surfaces by [32]. They introduced the *evolving surface finite element method* (ESFEM) to discretize parabolic partial differential equations on moving surfaces, and shown H^1-error estimates, cf. [31]. They gave optimal order error estimates in the L^2-norm, see [35]. There is a review by [34], which also serves as a rich source of details and references.

Dziuk and Elliott also studied fully discrete methods, see e.g. [33]. The numerical analysis of convergence of full discretizations with higher-order time integrators was first studied by [37]. They proved optimal order convergence for the case of algebraically stable implicit Runge-Kutta methods, and [63] proved optimal convergence for backward differentiation formulas (BDF).

The ESFEM approach and convergence results were later extended to wave equations on evolving surfaces by [62] and [67]. A unified presentation of ESFEM for parabolic problems and wave equations is given in [66].

These results are for the Lagrangian case.

As it was pointed out by Dziuk and Elliott, *"A drawback of our method is the possibility of degenerating grids. The prescribed velocity may lead to the effect, that the triangulation $\Gamma_h(t)$ is distorted"*[1]. To resolve this problem [42] proposed an *arbitrary Lagrangian Eulerian* (ALE) ESFEM approach, which in contrast to the (pure Lagrangian) ESFEM method, allows the nodes of the triangulation to move with a velocity which may not be equal to the surface (or material)

[1]Quoted from Gerhard Dziuk and Charles M. Elliott from [31, Section 7.2]

velocity. They presented numerous examples where smaller errors can be achieved using a *better* mesh.

Arbitrary Lagrangian Eulerian FEM for moving domains were investigated by [45]. They also suggest some possible ways to define the new mesh if the movement of the boundary is given. [12, 13] proved stability and optimal order a-priori error estimates for discontinuous Galerkin time discrete Runge-Kutta-Radau methods of high order.

This paper extends the convergence results and techniques of [37] for the Runge-Kutta discretizations and of [63] for the backward differentiation formulas (both shown for the Lagrangian case), to the ALE framework.

[42] proposed a fully discrete ALE ESFEM algorithm to solve parabolic problems on evolving surfaces. [43] proved convergence results for this type of scheme and in addition prove convergence of fully discrete ALE ESFEM with second-order backward differentiation formulas. They also give numerous numerical experiments. The primary consideration of the present work is to prove convergence of ALE ESFEM with higher-order time discretizations. We use different techniques to achieve this and thus give a new proof for the convergence of the fully discrete method suggested by [42].

We prove stability and convergence of these higher-order time discretizations classes, and also their convergence as a full discretization for evolving surface linear parabolic PDEs when coupled with the arbitrary Lagrangian Eulerian evolving surface finite element method as a space discretization. The stability results do not require a time step restriction by powers of the mesh size, i.e. no CFL-type condition is required.

First, the stability of stiffly accurate algebraically stable implicit Runge-Kutta methods (having the Radau IIA methods in mind) is shown using energy estimates and the algebraic stability as a key property, using some of the basic ideas appeared in [64] for quasilinear parabolic problems.

Second, we show stability for the k-step backward differentiation formulas up to order five. Because of the lack of A-stability of the BDF methods of order greater than two, our proof requires a different technique than [43], namely we used G-stability results of [20], and multiplier techniques of [69]. Therefore, we handle all BDF ($k = 1, 2, \dots, 5$) methods at once.

For the fully discrete convergence results, in both cases, the study of the error of a generalized Ritz map, and also for the error in its material derivatives, plays an important role.

In the presentation we focus on the main differences compared to the previous results, and put less emphasis on those parts where minor modifications of the cited proofs are sufficient. In most cases the Lagrangian proof can be repeated in the ALE case, these are therefore omitted.

Our convergence estimates for BDF 1 and BDF 2 match the ones achieved with a different technique in [43].

This paper is organised as follows. In Section A.2 we formulate the considered evolving surface parabolic problem, and describe the concept of arbitrary Lagrangian Eulerian methods together with other basic notions. The ALE weak formulation of the problem is also given. In Section A.3 we define the mesh approximating our moving surface and derive the semidiscrete version of the ALE weak form, which is equivalent to a system of ODEs. Then we recall some properties of the evolving matrices, and some estimates of bilinear forms. We also prove the analogous estimate for the new term appearing in the ALE formulation. The definition of the used generalized Ritz map is also given here. In Section A.4 we prove stability of high order Runge-Kutta (R-K) methods applied to the ALE ESFEM semidiscrete problem and the same results for the BDF methods. Section A.5 contains the main results of this paper: convergence of the fully discrete methods, ALE ESFEM together with R-K or BDF method, having a high order convergence both in time. Finally, in Section A.6 we present numerical experiments, to illustrate our theoretical results.

A.2. The arbitrary Lagrangian Eulerian approach for evolving surface PDEs

In the following we consider a smooth evolving closed hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$, $0 \leqq t \leq T$, with $m \leq 3$, which moves with a given smooth velocity v. Let $\partial^{\bullet} u = \partial_t u + v \cdot \nabla u$ denote the material derivative of u. Define the tangential gradient ∇_Γ by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \mathrm{n}\mathrm{n}$, where n is the unit normal and denote by $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ the Laplace-Beltrami.

We consider the following linear problem derived in [31]:

$$\begin{cases} \partial^{\bullet} u(x,t) + u(x,t)\nabla_{\Gamma(t)} \cdot v(x,t) & \\ \qquad\qquad -\Delta_{\Gamma(t)} u(x,t) = f(x,t) & \text{on } \Gamma(t), \\ \qquad\qquad\qquad u(x,0) = u_0(x) & \text{on } \Gamma(0). \end{cases} \tag{A.1}$$

Basic and detailed references on evolving surface PDEs are [31, 34, 35] and [66]. We are working in the same framework as these references.

For simplicity reasons we set in all sections $f = 0$, since the extension of our results to the inhomogeneous case are straightforward.

An important tool is the Green's formula (on closed surfaces), which takes the form

$$\int_\Gamma \nabla_\Gamma z \cdot \nabla_\Gamma \varphi = - \int_\Gamma (\Delta_\Gamma z) \varphi .$$

We use Sobolev spaces on surfaces: For a smooth surface Γ we define

$$H^1(\Gamma) = \{ \varphi \in L^2(\Gamma) \mid \nabla_\Gamma \varphi \in L^2(\Gamma)^{m+1} \},$$

and analogously $H^k(\Gamma)$ for $k \in \mathbb{N}$ [31, section 2.1]. Finally, $\mathcal{G}_T$ denotes the space-time manifold, i.e. $\mathcal{G}_T := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}$. We assume that $\mathcal{G}_T \subset \mathbb{R}^{m+2}$ is a smooth hypersurface (with boundary $\partial\mathcal{G}_T = (\Gamma(0) \times \{0\}) \cup (\Gamma(T) \times \{T\})$).

The weak formulation of this problem reads as

Definition A.2.1 (weak solution, [31, definition 4.1]). A function $u \in H^1(\mathcal{G}_T)$ is called a *weak solution* of (A.1), if for almost every $t \in [0,T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)}u \cdot \nabla_{\Gamma(t)}\varphi = \int_{\Gamma(t)} u\partial^\bullet\varphi \tag{A.2}$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(\,\cdot\,,0) = u_0$.

For suitable u_0 existence and uniqueness results for (A.2) were obtained by [31, theorem 4.4] and in a more abstract framework in [3, theorem 3.6] (both works consider inhomogeneous problems).

A.2.1. The ALE map and ALE velocity

We assume that for each $t \in [0,T]$, $T > 0$, $\Gamma^m(t) \subset \mathbb{R}^{m+1}$ is a closed surface. We call a subset $\Gamma^m \subset \mathbb{R}^{m+1}$ a *closed surface*, if Γ is an oriented compact submanifold of codimension 1 without boundary. Moreover we assume $m = 1, 2$ or 3 and that $\Gamma \in C^\infty$, evolving smoothly, cf. [34]. We assume that there exists a smooth map $\mathrm{n}\colon \mathcal{G}_T \to \mathbb{R}^{m+1}$ such that for each t the restriction

$$\mathrm{n}(\,\cdot\,,t)\colon \Gamma(t) \to \mathbb{R}^{m+1}$$

is the smooth normal field on $\Gamma(t)$.

Now we shortly recall the surface description by diffeomorphic parametrization, also used by [31], and by [13]. An other important representation of the surface is based on a signed distance function. For this we refer to [31].

We assume that there exists a smooth map

$$\Phi\colon \Gamma(0) \times [0,T] \to \mathbb{R}^{m+1}$$

which we call a *dynamical system* or *diffeomorphic parametrization* satisfying that

$$\Phi_t\colon \Gamma(0) \to \Gamma(t), \qquad \Phi_t(y) := \Phi(y,t)$$

is a diffeomorphism for every $t \in [0,T]$. (Φ_t) is called the *flow* of Φ. We observe:

- If $F\colon U \subset \mathbb{R}^m \to \Gamma(0)$ is a smooth parametrization of $\Gamma(0)$ then $F_t := \Phi_t \circ F$ is a smooth parametrization of $\Gamma(t)$, hence the name diffeomorphic parametrization.
- If we interpret $\Gamma(0) \times [0,T] \subset \mathbb{R}^{m+2}$ as a hypersurface, then Φ gives rise to a diffeomorphism

$$\tilde{\Phi}\colon \Gamma(0) \times [0,T] \to \mathcal{G}_T, \quad \tilde{\Phi}(y,t) := \big(\Phi_t(y), t\big).$$

The dynamical system Φ defines a (special) vector field v and (special) time derivative $\partial^\bullet$ as follows: First, consider the differential equation (for Φ)

$$\partial_t \Phi(\,\cdot\,,t) = v\left(\Phi(\,\cdot\,,t), t\right), \quad \Phi(\,\cdot\,,0) = \mathbb{1}. \tag{A.3}$$

The unique vector field v is called the *velocity of the surface evolution*, or the *material velocity*. We assume, that the material velocity is the same velocity as in problem (A.1). It has the normal component v^{N}. Second, the derivative $\partial^\bullet$ is defined as follows (see e.g. [31, Section 2].2 or [13, Section 1]): for smooth $f\colon \mathcal{G}_T \to \mathbb{R}$ and $x \in \Gamma(t)$, such that $y \in \Gamma(0)$ for which $\Phi_t(y) = x$, the *material derivative* is defined as

$$\partial^\bullet f(x,t) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{(y,t)} f \circ \tilde{\Phi}. \tag{A.4}$$

Suppose that f has a smooth extension $\bar{f}$ onto an open neighbourhood of $\Gamma(t)$, then by the chain rule the following identity for the material derivative holds:

$$\partial^\bullet f(x,t) = \left.\frac{\partial \bar{f}}{\partial t}\right|_{(x,t)} + v(x,t)\cdot\nabla\bar{f}(x,t),$$

which is clearly independent of the extension by (A.4). In section 2.3 [34] has shown how to use the oriented distance function to construct an extension $\bar{f}$.

Remark A.2.2. An evolving surface $\Gamma(t)$ generally posses many different dynamical systems. Consider for example the (constant) evolving surface $\Gamma(t) = \Gamma(0) = S^m \subset \mathbb{R}^{m+1}$ with the two (different) dynamical system $\Phi(x,t) = x$ and $\Psi(x,t) = \alpha(t)x$, where $\alpha\colon [0,T] \to O(m+1)$ is a smooth curve in the orthogonal matrices.

Definition A.2.3. Let $\mathcal{A} \neq \Phi$ be any other dynamical system for $\Gamma(t)$. It is called an *arbitrary Lagrangian Eulerian map* (ALE map). The associated velocity will be denoted by w, which we refer as the *ALE velocity* and finally $\partial^{\mathcal{A}}$ denotes the ALE material derivative.

One can show that for all $t \in [0,T]$ and $x \in \Gamma(t)$

$$v(x,t) - w(x,t) \quad \text{is a tangential vector.} \tag{A.5}$$

The formula for the differentiation of a parameter-dependent surface integral played a decisive role in the analysis of evolving surface problems. In the following lemma we will state its ALE version, together with the connection between the material derivative and ALE material derivative.

Lemma A.2.4. *Let $\Gamma(t)$ be an evolving surface and f be a function defined in $\mathcal{G}_T$, such that all the following quantities exist.*

(a) (Leibniz formula [31]/ Reynolds transport identity [13]) There holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} f = \int_{\Gamma(t)} \partial^{\mathcal{A}} f + f\,\nabla_{\Gamma(t)}\cdot w. \tag{A.6}$$

(b) There also holds

$$\partial^{\mathcal{A}} f = \partial^{\bullet} f + (w - v)\cdot\nabla_{\Gamma} f. \tag{A.7}$$

Proof. At first we prove (b): consider an extension $\bar{f}$ of f. Use the chain rule for $\partial^{\mathcal{A}} f$ and $\partial^{\bullet} f$ and note the identity (cf. (A.5))

$$\big(w(\,\cdot\,,t) - v(\,\cdot\,,t)\big)\cdot\nabla\bar{f}(\,\cdot\,,t) = \big((w(\,\cdot\,,t) - v(\,\cdot\,,t)\big)\cdot\nabla_{\Gamma} f(\,\cdot\,,t).$$

To prove (a) use the original Leibniz formula from [31]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma} f = \int_{\Gamma} \partial^{\bullet} f + f\,\nabla_{\Gamma}\cdot v.$$

Now use (b) and Greens identity for surfaces to complete the proof. ∎

A.2.2. Weak formulation

Now we have everything at our hands to derive the ALE version of the weak form of the evolving surface PDE (A.1).

Lemma A.2.5 (ALE weak solution). *The arbitrary Lagrangian Eulerian weak solution for an evolving surface partial differential equa-*

tion is a function $u \in H^1(\mathcal{G}_T)$, if for almost every $t \in [0, T]$

$$\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi + \int_{\Gamma(t)} u(w - v) \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u \partial^{\mathcal{A}} \varphi$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(\,\cdot\,, 0) = u_0$. If u solves equation (A.2) *then u is an ALE weak solution.*

Proof. We start by substituting the material derivative by the ALE material derivative in (A.2). Now using the relation (A.7), connecting the different material derivatives (cf. (A.5)), i.e. by putting

$$\partial^{\bullet} \varphi = \partial^{\mathcal{A}} \varphi + (v - w) \cdot \nabla_{\Gamma} \varphi$$

into (A.2), and rearranging the terms, we get the desired formulation. ∎

A.3. Semidiscretization: ALE evolving surface finite element method

This section is devoted to the spatial semidiscretization of the parabolic moving surface PDE with the ALE version of the evolving surface finite element method. The ESFEM was developed by [31]. In the original case the nodes were moving only with the material velocity along the surface, which could lead to degenerated meshes. One can maintain the good properties of the initial mesh by having additional tangential velocity.

The ALE ESFEM discretization will lead to a system of ordinary differential equations (ODEs) with time dependent matrices. We will prove basic properties of those matrices, which will be one of our main tools to prove stability of time discretizations and convergence of full discretizations. We will also recall the lifting operator and its properties introduced by [31], which enables us to compare functions from the discrete and continuous surface.

A.3.1. Basic notations

First, the initial surface $\Gamma(0)$ is approximated by a triangulated one denoted by $\Gamma_h(0)$, which is given as

$$\Gamma_h(0) := \bigcup_{E(0)\in\mathcal{T}_h(0)} E(0).$$

Let $a_i(0)$, ($i = 1, 2, \ldots, N$), denote the initial nodes lying on the initial continuous surface. Now the nodes are evolved with respect to the ALE map $\mathcal{A}$, i.e. $a_i(t) := \mathcal{A}\left(a_i(0), t\right)$. Obviously they remain on the continuous surface $\Gamma(t)$ for all t. Therefore the smooth surface $\Gamma(t)$ is approximated by the triangulated one denoted by $\Gamma_h(t)$, which is given as

$$\Gamma_h(t) := \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ form an admissible triangulation $\mathcal{T}_h(t)$ with h denoting the maximum diameter. Admissible triangulations were introduced in [31] section 5.1: every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by ch with $c > 0$ and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$.

The discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)} f := \nabla f - \nabla f \cdot \mathrm{n}_h \mathrm{n}_h = \mathrm{pr}_h(\nabla f),$$

understood in a element-wise sense, with n_h denoting the normal to $\Gamma_h(t)$ and $\mathrm{pr}_h := I - \mathrm{n}_h \mathrm{n}_h^T$.

For every $t \in [0, T]$ we define the finite element subspace

$$S_h(t) := \left\{ \phi_h \in C\left(\Gamma_h(t)\right) \,\middle|\, \phi_h|_E \text{ is linear, for all } E \in \mathcal{T}_h(t) \right\}$$

The piecewise linear moving basis functions χ_j are defined by $\chi_j\,(a_i(t), t) = \delta_{ij}$ for all $i, j = 1, 2, \ldots, N$, and hence

$$S_h(t) = \operatorname{span}\{\chi_1(\,\cdot\,, t), \chi_2(\,\cdot\,, t), \ldots, \chi_N(\,\cdot\,, t)\}.$$

We continue with the definition of the interpolated velocities on the discrete surface $\Gamma_h(t)$:

$$\begin{aligned} V_h(\,\cdot\,, t) &= \sum_{j=1}^{N} v\,(a_j(t), t)\chi_j(\,\cdot\,, t), \\ W_h(\,\cdot\,, t) &= \sum_{j=1}^{N} w\,(a_j(t), t)\chi_j(\,\cdot\,, t) \end{aligned} \tag{A.8}$$

are the discrete velocity, and the discrete ALE velocity, respectively. The discrete material derivative, and its ALE version is given by

$$\partial_h^\bullet \phi_h = \partial_t \phi_h + V_h \cdot \nabla \phi_h, \quad \partial_h^{\mathcal{A}} \phi_h = \partial_t \phi_h + W_h \cdot \nabla \phi_h,$$

where $\partial_t \phi_h(x, t_0)$ and $\nabla \phi_h(x, t_0)$ is meant in the following sense: Denote by $\mathcal{G}_h \coloneqq \bigcup_{t\in[0,T]} \Gamma_h(t) \times \{t\} \subset \mathbb{R}^{m+2}$ the discrete time space manifold and for simplicity assume that the coefficients of $\phi_h\colon \mathcal{G}_h \to \mathbb{R}$ w.r. to the standard finite element basis are smooth in t. Assume that x is lying in the interior of $E(t_0) \subset \Gamma_h(t_0)$ and denote by $E(t)$ the evolution of $E(t_0)$. Finally set $\mathcal{E} \coloneqq \bigcup_{t\in[0,T]} E(t) \times \{t\}$. For the restricted function $\phi_h|_{\mathcal{E}}$ there exists an smooth extension $\overline{\phi_h}$ on a $(m+2)$-dimensional neighborhood of $\mathcal{E}$. We set $\partial_t \phi_h = \partial_t \overline{\phi_h}$ and $\nabla \phi_h = \nabla \overline{\phi_h}$. A straightforward calculation shows that $\partial_h^\bullet \phi_h$ and $\partial_h^{\mathcal{A}} \phi_h$ are well-defined.

In the ALE setting the key *transport property* is the following

$$\partial_h^{\mathcal{A}} \chi_k = 0 \qquad \text{for} \quad k = 1, 2, \ldots, N. \tag{A.9}$$

It can be shown as its non-ALE version, see [31, proposition 5.4].

The spatially discrete ALE problem for evolving surfaces is formulated in

Problem A.3.1 (Semidiscretization in space). Find $U_h \in S_h(t)$ so that

$$\int_{\Gamma_h(t)} U_h\phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)} U_h \cdot \nabla_{\Gamma_h}\phi_h \int_{\Gamma_h(t)} U_h(W_h - V_h)\cdot\nabla_{\Gamma_h}\phi_h$$
$$= \int_{\Gamma_h(t)} U_h \partial_h^A \phi_h, \quad (\forall \phi_h \in S_h(t)), \quad \text{(A.10)}$$

with the initial condition $U_h(\,\cdot\,,0) = U_h^0 \in S_h(0)$ is a sufficient approximation of u_0.

A.3.2. The ODE system

The ODE form of the above problem can be derived by setting

$$U_h(\,\cdot\,,t) = \sum_{j=1}^{N} \alpha_j(t)\chi_j(\,\cdot\,,t) \quad \text{(A.11)}$$

and testing by $\phi_h = \chi_k$ for $k = 1,2,\dots,N$ in (A.10) and using the transport property for evolving surfaces (A.9).

Proposition A.3.2 (ODE system for evolving surfaces). *The spatially semidiscrete problem is equivalent to the ODE system for the vector $\alpha(t) = (\alpha_j(t)) \in \mathbb{R}^N$, representing $U_h(\,\cdot\,,t)$,*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\alpha(t)\big) + A(t)\alpha(t) + B(t)\alpha(t) = 0 \\ \alpha(0) = \alpha_0 \end{cases} \quad \text{(A.12)}$$

where $M(t)$ and A are the evolving mass and stiffness matrices defined by

$$M(t)_{kj} = \int_{\Gamma_h(t)} \chi_j\chi_k, \quad A(t)_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\chi_j \cdot \nabla_{\Gamma_h(t)}\chi_k,$$

and the evolving matrix $B(t)$ is given by

$$B(t)_{kj} = \int_{\Gamma_h(t)} \chi_j (W_h - V_h) \cdot \nabla_{\Gamma_h(t)} \chi_k \tag{A.13}$$

The proof of this proposition is analogous to the corresponding one in [37, section 3].

Remark A.3.3. In the original ESFEM setting there was no direct involvement of velocities, but in the ALE formulation there is. We remark here that since the normal components of the continuous ALE and material velocity are equal, during computations one can work only with the difference of the two discrete velocities. We keep the above formulation to leave the presentation plain and simple.

A.3.3. Lifting process

In the following we define the so called *lift operator*, which was introduced by [29] and further investigated by [31, 35]. The lift operator can be interpreted as a geometric projection: it projects a finite element function $\phi_h \colon \Gamma_h(t) \to \mathbb{R}$ on the discrete surface $\Gamma_h(t)$ onto a function $\phi_h^l \colon \Gamma(t) \to \mathbb{R}$ on the smooth surface $\Gamma(t)$. Therefore it is crucial for our error estimates.

We assume that there exists an open bounded set $U(t) \subset \mathbb{R}^{m+1}$ such that $\partial U(t) = \Gamma(t)$. The *oriented distance function d* is defined as

$$\mathbb{R}^{m+1} \times [0, T] \to \mathbb{R}, \quad d(x,t) \coloneqq \begin{cases} -\operatorname{dist}(x, \Gamma(t)), & x \in U(t), \\ \operatorname{dist}(x, \Gamma(t)), & \text{else}. \end{cases}$$

For $\mu > 0$ we define $\mathcal{N}(t)_\mu \coloneqq \{x \in \mathbb{R}^{m+1} \mid \operatorname{dist}(x, \Gamma(t)) < \mu\}$. Clearly $\mathcal{N}(t)_\mu$ is an open neighbourhood of $\Gamma(t)$. In [46] lemma 14.16 it is shown the following important regularity result about d.

Lemma A.3.4. *Let $U(t) \subset \mathbb{R}^{m+1}$ be bounded and $\Gamma(t) \in C^k$ for $k \geq 2$. Then there exists a positive constant μ depending on U such that $d \in C^k(\mathcal{N}(t)_\mu)$.*

The same reference also mentions that μ^{-1} bounds the principal curvatures of $\Gamma(t)$.

In the following we recall the lift operator from [29, equation (2)]. For each $x \in \Gamma(t)_\mu$ there exists a unique $p = p(x,t) \in \Gamma(t)$ such that $|x - p| = \operatorname{dist}(x, \Gamma(t))$, then x and p are related by the important equation:

$$x = p + \mathrm{n}(p,t)d(x,t). \tag{A.14}$$

We assume that $\Gamma_h(t) \subset \mathcal{N}(t)$. The *lift operator* $\mathcal{L}$ maps a continuous function $\phi_h \colon \Gamma_h(t) \to \mathbb{R}$ onto a function $\mathcal{L}(\phi_h) \colon \Gamma(t) \to \mathbb{R}$ as follows: for every $x \in \Gamma_h(t)$ exists via equation (A.14) a unique $p = p(x,t)$. We set pointwise

$$\mathcal{L}(\eta_h)(p,t) \coloneqq \eta_h^l(p,t) \coloneqq \eta_h(x,t).$$

$\mathcal{L}(\phi_h) \colon \Gamma \to \mathbb{R}$ is continuous. If ϕ_h has weak derivatives then $\mathcal{L}(\phi_h)$ also has weak derivatives.

Finally, we have the lifted finite element space

$$S_h^l(t) \coloneqq \{\varphi_h = \phi_h^l \mid \phi_h \in S_h(t)\}.$$

A.3.4. Properties of the evolving matrices

Clearly the evolving stiffness matrix is symmetric, positive semi-definite and the mass matrix is symmetric, positive definite. Through the paper we will work with the norm and semi-norm introduced by [37]:

$$\begin{aligned} |z(t)|_{M(t)} &= \|Z_h\|_{L^2(\Gamma_h(t))}, \\ |z(t)|_{A(t)} &= \|\nabla_{\Gamma_h} Z_h\|_{L^2(\Gamma_h(t))}, \end{aligned} \tag{A.15}$$

for arbitrary $z(t) \in \mathbb{R}^N$, where $Z_h(\,\cdot\,, t) = \sum_{j=1}^{N} z_j(t)\chi_j(\,\cdot\,, t)$.

A very important lemma in our analysis is the following.

Lemma A.3.5 ([37, lemma 4.1] and [63, lemma 2.2]). *There are constants μ, κ, depending on $\|\nabla_\Gamma \cdot w\|_{L^\infty(\Gamma(t))}$, but independent of h, such that*

$$z^T\,(M(s) - M(t))y \leq (e^{\mu(s-t)} - 1)\,|z|_{M(t)}\,|y|_{M(t)} \tag{A.16}$$

$$z^T\,(M^{-1}(s) - M^{-1}(t))y \leq (e^{\mu(s-t)} - 1)\,|z|_{M^{-1}(t)}\,|y|_{M^{-1}(t)} \tag{A.17}$$

$$z^T\,(A(s) - A(t))y \leq (e^{\kappa(s-t)} - 1)\,|z|_{A(t)}\,|y|_{A(t)} \tag{A.18}$$

for all $y, z \in \mathbb{R}^N$ and $s, t \in [0, T]$.

We will use this lemma with s close to t (usually, $t = s + k\tau$ for some positive integer k independent of the time step τ). Hence, $(e^{\mu(s-t)} - 1) \leq 2\mu(s-t)$ holds. In particular for $y = z$ we have

$$|z|^2_{M(s)} \leq (1 + 2\mu(t-s))\,|z|^2_{M(t)}, \tag{A.19}$$

$$|z|^2_{A(s)} \leq (1 + 2\kappa(t-s))\,|z|^2_{A(t)}. \tag{A.20}$$

The following technical lemma will play a crucial role in this paper, while handling the nonsymmetric term.

Lemma A.3.6. *Let $y, z \in \mathbb{R}^N$ and $t \in [0, T]$ be arbitrary, then*

$$|\langle B(t)z\,|\,y\rangle| \leq c_{\mathcal{A}}\,|z|_{M(t)}\,|y|_{A(t)},$$

where the constant $c_{\mathcal{A}} > 0$ depends only on the velocity difference $w - v$.

Proof. Using the definition of the matrix $B(t)$ (see (A.13)) we can write

$$
\begin{aligned}
|\langle B(t)z \mid y\rangle| &= \left|\int_{\Gamma_h} Z_h(W_h - V_h)\cdot \nabla_{\Gamma_h} Y_h\right| \\
&\leq \|W_h - V_h\|_{L^\infty(\Gamma_h(t))} \int_{\Gamma_h} |Z_h|\,|\nabla_{\Gamma_h} Y_h|.
\end{aligned}
$$

For a first order finite element function $\varphi_h \in S_h(t)$ it holds

$$\|\varphi_h\|_{L^\infty(\Gamma_h)} = |\varphi_h(p)|$$

for an appropriate node $p \in \Gamma_h(t)$. Hence using (A.8) we can estimate as

$$\|W_h - V_h\|_{L^\infty(\Gamma_h(t))} \leq (m+1)\,\|w - v\|_{L^\infty(\Gamma_h(t))}. \tag{A.21}$$

Now apply the Cauchy-Schwarz inequality and using the equivalence of norms over the discrete and continuous surface (cf. [31], Lemma 5.2) to obtain the stated result. ■

A.3.5. Interpolation estimates

Let $I_h\colon C(\Gamma(t)) \to S_h^l(t)$ be the lifted Lagrange interpolation operator, where $C(\Gamma(t))$ denotes the space of continuous functions on $\Gamma(t)$; cf. [31] for further details on the interpolation operator. The following interpolation estimate holds.

Lemma A.3.7. *For $m \leq 3$ and $p \in \{2, \infty\}$ there exists a constant $c > 0$ independent of h and t such that for $u \in W^{2,p}(\Gamma(t))$:*

$$
\begin{aligned}
&\|u - I_h u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma(u - I_h u)\|_{L^p(\Gamma(t))} \\
&\qquad \leq ch^2 \left(\|\nabla_\Gamma^2 u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma u\|_{L^p(\Gamma(t))}\right).
\end{aligned}
$$

Proof. Since $m \leq 3$ and $\Gamma(t)$ is smooth and compact, a Sobolev embedding theorem, cf. [5, theorem 2].20, implies $W^{2,p}(\Gamma(t)) \subset C(\Gamma(t))$. Hence $I_h u$ is well defined.

The estimate for the case $p = 2$ is stated in [31, lemma 5.3]. On the reference element a interpolation estimate for the case $p = \infty$ was shown in [76, theorem 3.1]. Using the estimates appearing in the proof of [29, lemma 3] and combining these with standard estimates of the reference element technique we obtain the stated result. ■

A.3.6. Discrete geometric estimates

We recall some notions using the lifting process from [29], [31] and [66] using the notation of the last reference. By δ_h we denote the quotient between the continuous and discrete surface measures, $\mathrm{d}A$ and $\mathrm{d}A_h$, defined as $\delta_h \,\mathrm{d}A_h = \mathrm{d}A$. Further, we recall that

$$\mathrm{pr} \coloneqq I - \mathrm{n}\mathrm{n}^T \quad \text{and} \quad \mathrm{pr}_h = I - \mathrm{n}_h\mathrm{n}_h^T,$$

are the projections onto the tangent spaces of Γ and Γ_h. Finally $\mathcal{H}$ ($\mathcal{H}_{ij} = \frac{\partial \mathrm{n}_i}{\partial x_j}$) is the (extended) Weingarten map. For these quantities we recall some results from [31, 35] and [66], having the exact same proofs for the ALE case.

Lemma A.3.8. *Assume that $\Gamma_h(t)$ and $\Gamma(t)$ satisfy the above, and $\Gamma(t)$ is C^ℓ in time, then we have the estimates*

$$\begin{aligned} \|d_X\|_{L^\infty(\Gamma_h)} &\leq ch^2, & \|\mathrm{n} - \mathrm{n}_h\|_{L^\infty(\Gamma_h)} &\leq ch, \\ \|1 - \delta_h\|_{L^\infty(\Gamma_h)} &\leq ch^2, & \|(\partial_h^{\mathcal{A}})^{(\ell)} d\|_{L^\infty(\Gamma_h)} &\leq ch^2, \\ \|\mathrm{pr} - \mathrm{pr}\,\mathrm{pr}_h\,\mathrm{pr}\|_{L^\infty(\Gamma_h)} &\leq ch^2, \end{aligned}$$

where $(\partial_h^\bullet)^{(\ell)}$ denotes the ℓ-th discrete ALE material derivative.

Proof. The first three inequalities have been proven in [35] lemma 5.4. The fourth inequality for $\ell \geq 1$ is presented in [66] lemma 6.1. The last inequality has been shown in [31] lemma 5.1. ■

A.3.7. Bilinear forms and their properties

We use the time dependent bilinear forms defined as in [35] section 3.3. For $z, \varphi \in H^1(\Gamma(t))$ we set

$$
\begin{aligned}
a(z,\varphi) &= \int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, \\
m(z,\varphi) &= \int_{\Gamma(t)} z\varphi, \\
g(w;z,\varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot w) z\varphi, \\
b(w;z,\varphi) &= \int_{\Gamma(t)} \mathcal{B}(w)\nabla_\Gamma z \cdot \nabla_\Gamma \varphi,
\end{aligned}
$$

and for $Z_h, \phi_h \in S_h(t)$ we define their discrete analogs as

$$
\begin{aligned}
a_h(Z_h,\phi_h) &= \sum_{E\in\mathcal{T}_h} \int_E \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\
m_h(Z_h,\phi_h) &= \int_{\Gamma_h(t)} Z_h\phi_h \\
g_h(W_h;Z_h,\phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot W_h) Z_h\phi_h, \\
b_h(W_h;Z_h,\phi_h) &= \sum_{E\in\mathcal{T}_h} \int_E \mathcal{B}_h(W_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h,
\end{aligned}
$$

where the discrete tangential gradients are understood in a piecewise sense, and with the matrices

$$
\begin{aligned}
\mathcal{B}(w)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot w) - \big((\nabla_\Gamma)_i w_j + (\nabla_\Gamma)_j w_i\big), \\
\mathcal{B}_h(W_h)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot W_h) - \big((\nabla_{\Gamma_h})_i (W_h)_j + (\nabla_{\Gamma_h})_j (W_h)_i\big),
\end{aligned}
$$

where $i, j = 1, 2, \ldots, n$.

Following [35] the ALE velocity of lifted material points is defined as follows: Denote by $\mathcal{L}_0 \colon \Gamma_h(0) \to \Gamma(0)$ the Lift for the initial surface and denote by $\mathcal{L}^t \colon \Gamma_h(t) \to \Gamma(t)$ the lift at time t, cf. equation (A.14). In a straightforward way the ALE dynamical system $\mathcal{A}$ on $\Gamma(t)$

defines a discrete ALE dynamical system $\mathcal{A}_h$ on $\Gamma_h(t)$. $\mathcal{A}_h$ can be interpreted as the interpolation of $\mathcal{A}$. It holds

$$\frac{\mathrm{d}\mathcal{A}_h}{\mathrm{d}t}(x_h^0, t) = W_h(\mathcal{A}_h(x_h^0, t), t).$$

Define

$$\mathcal{A}_h^l\colon \Gamma_0 \times [0,T] \to \mathbb{R}^{m+1}, \quad (x_0, t) \mapsto \mathcal{L}^t\left(\mathcal{A}_h\left(\mathcal{L}_0^{-1}(x_0), t\right)\right).$$

Obviously it holds $\mathcal{A}_h^l(\Gamma_0, t) = \Gamma(t)$. We note that $\mathcal{A}_h^l$ is just curved element wise smooth. Analogous to equation (A.3) we define the corresponding velocity $\Gamma(t) \to \mathbb{R}^{m+1},\ x \mapsto w_h(x,t)$ via

$$w_h(\mathcal{A}_h^l(x_0, t), t) := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_h^l(x_0, t). \tag{A.22}$$

Again like in section A.2.1 the map

$$\widetilde{\mathcal{A}_h^l}\colon \Gamma_0 \times [0,T] \to \mathcal{G}_T$$

is bijective and now analogously to equation (A.4) we define the corresponding discrete ALE material derivative for function on the smooth surface as

$$\partial_h^{\mathcal{A}} f(x,t) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{(\widetilde{\mathcal{A}_h^l})^{-1}(x,t)} f \circ \widetilde{\mathcal{A}_h^l}.$$

If $\bar{f}$ denotes an extension of f on an open neighborhood of $\mathcal{G}_T$ then we have

$$\partial_h^{\mathcal{A}} f(x,t) = \left.\frac{\partial \bar{f}}{\partial t}\right|_{(x,t)} + w_h(x,t)\cdot\nabla\bar{f}(x,t).$$

Lemma A.3.9. *There exists $c > 0$ independent of h such that*

$$\|w - w_h\|_{L^\infty(\mathcal{G}_T)} \leq ch^2.$$

Proof. The proof has been done in [35] lemma 5.6. For the convenience of the reader we recap the main arguments. Applying the chain rule at the right-hand side of (A.22) leads to

$$w_h(x,t) = I_h w(x,t) - d_X\left((\mathcal{L}^t)^{-1}(x),t\right)\left(\mathcal{H}(x,t)I_h w(x,t) + \frac{\partial \mathrm{n}}{\partial t}(x,t)\right) - \mathrm{n}(x,t)\left(\frac{\partial d_X}{\partial t}\left((\mathcal{L}^t)^{-1}(x),t\right) + \mathrm{n}(x,t)\cdot I_h w(x,t)\right).$$

Since

$$w_h(x,t)\cdot \mathrm{n}(x,t) = w(x,t)\cdot \mathrm{n}(x,t)$$

and

$$\frac{\partial \mathrm{n}}{\partial t}(x,t)\cdot \mathrm{n}(x,t) = 0$$

it follows that multiplying the equation above by $\mathrm{n}(x,t)$ yields

$$\frac{\partial d_X}{\partial t}\left((\mathcal{L})^{-1}(x),t\right) = w(x,t)\cdot \mathrm{n}(x,t).$$

The claim now follows by lemma A.3.8 and lemma A.3.7. ∎

Lemma A.3.10 ([35, lemma 4.2], [43, lemma 3.8]). *For lifted element functions $z_h, \varphi_h \in S_h^l(t)$ with discrete ALE material derivatives $\partial_h^{\mathcal{A}} z_h, \partial_h^{\mathcal{A}} \varphi_h \in S_h^l(t)$ we have:*

$$\frac{\mathrm{d}}{\mathrm{d}t} m(z_h,\varphi_h) = m(\partial_h^{\mathcal{A}} z_h,\varphi_h) + m(z_h,\partial_h^{\mathcal{A}}\varphi_h) + g(w_h; z_h,\varphi_h),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} a(z_h,\varphi_h) = a(\partial_h^{\mathcal{A}} z_h,\varphi_h) + a(z_h,\partial_h^{\mathcal{A}}\varphi_h) + b(w_h; z_h,\varphi_h).$$

Versions of this lemma with continuous non-ALE material derivatives, or discrete bilinear forms are also true, see e.g. [66] lemma 6.4.

We will need the following estimates between the continuous and discrete bilinear forms.

Lemma A.3.11 ([35], [43]). *For arbitrary $Z_h, \phi_h \in S_h(t)$, with corresponding lifts $z_h, \varphi_h \in S_h^l(t)$ we have the bound*

$$
\begin{aligned}
&|m(z_h, \varphi_h) - m_h(Z_h, \phi_h)| \leq ch^2 \|z_h\|_{L^2(\Gamma(t))} \|\varphi_h\|_{L^2(\Gamma(t))},\\
&|a(z_h, \varphi_h) - a_h(Z_h, \phi_h)| \leq ch^2 \|\varphi_h\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))},\\
&|g(w_h; z_h, \varphi_h) - g_h(W_h; Z_h, \phi_h)| \leq ch^2 \|z_h\|_{L^2(\Gamma(t))} \|\varphi_h\|_{L^2(\Gamma(t))},\\
&|m(z_h, (w-v)\cdot\nabla_\Gamma \varphi_h) - m_h(Z_h, (W_h - V_h)\cdot \nabla_{\Gamma_h}\phi_h)|\\
&\quad \leq ch^2 \|z_h\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}.
\end{aligned}
$$

Proof. For the first three inequalities we refer to [35] lemma 5.5. For the last inequality observe that

$$
\begin{aligned}
&\left| m(z_h, (w-v)\cdot\nabla_\Gamma \varphi_h) - m_h(Z_h, (W_h - V_h)\cdot \nabla_{\Gamma_h}\phi_h)\right|\\
\leq &\left| m(z_h, ((w-v) - (W_h^l - V_h^l))\cdot\nabla_\Gamma \varphi_h)\right|\\
&+ \left| m(z_h, (W_h^l - V_h^l)\cdot\nabla_\Gamma \varphi_h) - m_h(Z_h, (W_h - V_h)\cdot \nabla_{\Gamma_h}\phi_h)\right|\\
\leq & ch^2 \|z_h\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}\\
&+ \left| m(z_h, (W_h^l - V_h^l)\cdot\nabla_\Gamma \varphi_h) - m_h(Z_h, (W_h - V_h)\cdot \nabla_{\Gamma_h}\phi_h)\right|,
\end{aligned}
$$

where we have used lemma A.3.7 for the last inequality. The inequality

$$
\begin{aligned}
&\left| m(z_h, (W_h^l - V_h^l)\cdot\nabla_\Gamma \varphi_h) - m_h(Z_h, (W_h - V_h)\cdot \nabla_{\Gamma_h}\phi_h)\right|\\
&\quad \leq ch^2 \|z_h\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}
\end{aligned}
$$

follows from [43, lemma B.3]. For the convenience of the reader we recap the arguments. It holds

$$\begin{aligned}\int_{\Gamma(t)} z_h(W_h^l - V_h^l) \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}A - \int_{\Gamma_h(t)} Z_h(W_h - V_h) \cdot \nabla_{\Gamma_h} \phi_h \,\mathrm{d}A_h \\ = \int_{\Gamma(t)} z_h(W_h^l - V_h^l) \cdot (\mathrm{pr} - (\delta_h^l)^{-1} \mathrm{pr}_h (I - d\mathcal{H}) \,\mathrm{pr}) \nabla_\Gamma \varphi_h \,\mathrm{d}A.\end{aligned}$$

Using lemma A.3.8 we estimate as

$$\begin{aligned}&|\mathrm{pr} - (\delta_h^l)^{-1} \mathrm{pr}_h (I - d\mathcal{H}) \,\mathrm{pr}| \\ &\qquad \leq ch^2 + |(I - \mathrm{pr}_h(I - d\mathcal{H}))\,\mathrm{pr}| \leq ch^2,\end{aligned}$$

which implies the claim. ■

A.3.8. The Ritz map

We use nearly the same Ritz map introduced in [62] definition 8.1, but for the parabolic case a much simpler version suffices:

Definition A.3.12. For a given $z \in H^1(\Gamma(t))$ there is a unique $\widetilde{\mathcal{P}_h} z \in S_h(t)$ such that for all $\phi_h \in S_h(t)$, with the corresponding lift $\varphi_h = \phi_h^l$, we have

$$a_h^*(\widetilde{\mathcal{P}_h} z, \phi_h) = a^*(z, \varphi_h), \tag{A.23}$$

where $a^* := a + m$ and $a_h^* := a_h + m_h$, to make the forms a and a_h positive definite. Then $\mathcal{P}_h z \in S_h^l(t)$ is defined as the lift of $\widetilde{\mathcal{P}_h} z$, i.e. $\mathcal{P}_h z = (\widetilde{\mathcal{P}_h} z)^l$.

Remark A.3.13. The Ritz map in (A.23) is a simplified version of the Ritz map considered in [66] definition 7.1 and [62] definition 8.1. The Ritz map in the first reference is actually a more general one then (A.23), since for the choice $\zeta \equiv 0$ there, we obtain our Ritz map, see the proof below.

A different Ritz projection has been used [35] and [43] appendix C. In these articles a Ritz projection is defined via

$$a(\mathcal{P}_h z, \varphi_h) = a(z, \varphi_h), \quad \forall \varphi_h \in S_h^l(t)$$

and

$$\int_{\Gamma(t)} \mathcal{P}_h z \, \mathrm{d}A = \int_{\Gamma(t)} z \, \mathrm{d}A = 0.$$

More recently in [40, section 3].6, a different Ritz map is defined via

$$a_h(\widetilde{\mathcal{P}}_h z, \phi_h) = a(z, \phi_h^l), \quad \forall \phi_h \in S_h(t)$$

and

$$\int_{\Gamma_h(t)} \widetilde{\mathcal{P}}_h z \, \mathrm{d}A_h = \int_{\Gamma(t)} z \, \mathrm{d}A.$$

Lemma A.3.14. *The Ritz map satisfies the bounds, for $0 \leq t \leq T$ and $h \leq h_0$ with a sufficiently small h_0,*

$$\begin{aligned}
&\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))} + h \|\nabla_\Gamma (z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} \leq ch^2 \|z\|_{H^2(\Gamma(t))}, \\
&\|(\partial_h^{\mathcal{A}})^{(\ell)}(z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} + h \|\nabla_\Gamma ((\partial_h^{\mathcal{A}})^{(\ell)}(z - \mathcal{P}_h z))\|_{L^2(\Gamma(t))} \\
&\quad \leq c_\ell h^2 \sum_{j=0}^{\ell} \|(\partial^{\mathcal{A}})^{(j)} z\|_{H^2(\Gamma(t))},
\end{aligned}$$

where the constants c and c_ℓ are independent of h and $t \in [0, T]$.

Proof. [66] has defined a Ritz map as follows. For given $\zeta \in H^1(\Gamma(t))$ he defined $\widetilde{\mathcal{P}_h} \colon H^1(\Gamma(t)) \to S_h(t)$ via the equation

$$a_h^*(\widetilde{\mathcal{P}}_h z, \phi_h) = a^*(z, \phi_h^l) + m(\zeta, (v_h - v) \cdot \nabla_\Gamma \phi_h^l) \quad \forall \phi_h \in S_h(t),$$

where v_h plays no role in our setting. Nevertheless, since the proof includes the case $\zeta \equiv 0$ our claim follows from [66, theorem 7.2 and 7.3]. ∎

A.4. Stability

A.4.1. Stability of implicit Runge-Kutta methods

We consider an s-stage implicit Runge-Kutta method (R-K) for the time discretization of the ODE system (A.12), coming from the ALE ESFEM space discretization of the parabolic evolving surface PDE.

In the following we extend the stability result for R-K methods of [37], Lemma 7.1, to the case of ALE evolving surface finite element method. Apart form the properties of the ALE ESFEM the proof is based on the energy estimation techniques of [64, theorem 1].1.

For the convenience of the reader we recall the method: for simplicity, we assume equidistant time steps $t_n := n\tau$, with step size τ. Our results can be straightforwardly extended to the case of nonuniform time steps. The s-stage implicit Runge-Kutta method, defined by the given Butcher tableau.

$$\begin{array}{c|c} (c_i) & (a_{ij}) \\ \hline & (b_i) \end{array}, \qquad \text{for} \quad i,j = 1,2,\dots,s,$$

applied to the system (A.12):

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left(M(t)\alpha(t)\right) + A(t)\alpha(t) + B(t)\alpha(t) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \alpha(0) = \alpha_0, \end{cases}$$

where we have set $U_h^0 = \sum_{j=1}^N \alpha_{0,j}\chi_j(\,\cdot\,,0)$, reads as

$$M_{ni}\alpha_{ni} = M_n\alpha_n + \tau\sum_{j=1}^{s} a_{ij}\dot{\alpha}_{nj}, \quad i = 1,2,\dots,s, \tag{A.24a}$$

$$M_{n+1}\alpha_{n+1} = M_n\alpha_n + \tau\sum_{i=1}^{s} b_i\dot{\alpha}_{ni}, \tag{A.24b}$$

where the internal stages satisfy

$$0 = \dot{\alpha}_{ni} + B_{ni}\alpha_{ni} + A_{ni}\alpha_{ni}, \quad i = 1,2,\dots,s, \tag{A.24c}$$

with $A_{ni} := A(t_n + c_i\tau)$, $B_{ni} := B(t_n + c_i\tau)$, $M_{ni} := M(t_n + c_i\tau)$ and $M_{n+1} := M(t_{n+1})$. Here $\dot{\alpha}_{ni}$ is not a derivative but a suggestive notation.

We recall that $U_h(\,\cdot\,, t) = \sum_{j=1}^N \alpha_j(t)\chi_j(\,\cdot\,, t)$ for the semidiscrete case from section A.3.2, and for the fully discrete case we define $U_h^n = \sum_{j=1}^N \alpha_{n,j}\chi_j(\,\cdot\,, t_n)$.

Assumption A.4.1. We assume that:

- The method has stage order $q \geq 1$ and classical order $p \geq q+1$.
- The coefficient matrix (a_{ij}) is invertible; the inverse will be denoted by upper indices (a^{ij}).
- The method is *algebraically stable*, i.e. $b_j > 0\ \forall j$ and the following matrix is positive semi-definite:

$$(b_i a_{ij} - b_j a_{ji} - b_i b_j)_{i,j=1}^s. \tag{A.25}$$

- The method is *stiffly accurate*, i.e. for $j = 1, 2, \dots, s$ it holds

$$b_j = a_{sj}, \qquad \text{and} \qquad c_s = 1. \tag{A.26}$$

Instead of (A.12), let us consider the following perturbed equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\widetilde{\alpha}(t)\big) + A(t)\widetilde{\alpha}(t) + B(t)\widetilde{\alpha}(t) = M(t)r(t), \\ \widetilde{\alpha}(0) = \widetilde{\alpha}_0. \end{cases} \tag{A.27}$$

The substitution of the true solution $\widetilde{\alpha}(t)$ of the perturbed problem into the R-K method, yields the defects Δ_{ni} and δ_{ni}, by setting $e_n = \alpha_n - \widetilde{\alpha}(t_n)$, $E_{ni} = \alpha_{ni} - \widetilde{\alpha}(t_n + c_i\tau)$ and $\dot{E}_{ni} = \dot{\alpha}_{ni} - \dot{\widetilde{\alpha}}(t_n + c_i\tau)$, again $\dot{E}_{ni}$ is not a derivative. Then by subtraction the following *error equations* hold:

$$M_{ni}E_{ni} = M_n e_n + \tau \sum_{j=1}^{s} a_{ij}\dot{E}_{nj} - \Delta_{ni}, \tag{A.28a}$$

$$M_{n+1}e_{n+1} = M_n e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni} - \delta_{n+1}, \tag{A.28b}$$

where the internal stages satisfy:

$$\dot{E}_{ni} + A_{ni}E_{ni} + B_{ni}E_{ni} = -M_{ni}r_{ni}, \tag{A.28c}$$

with $r_{ni} \coloneqq r(t_n + c_i\tau)$ and $i = 1, 2, \dots, n$.

Similar to [37], lemma 7.1 or [66], lemma 3.1, we present a stability estimate (such that the choice of τ is independent of h) for the above class of Runge-Kutta methods. Since the method (A.24) and the error equation (A.28) both involve only matrices and vectors, we first establish this stability estimate in terms of nodal error vectors with corresponding time-dependent norms (A.15). Using (A.15) this estimate can be translated into L^2- and H^1-norms of the corresponding finite element error functions. This result will be related to the norms of U_h, through the error, later in theorem A.5.2 and A.5.3.

Lemma A.4.2. *For an s-stage implicit Runge-Kutta method satisfying Assumption A.4.1, there exists a* $\tau_0 > 0$*, depending only on the constants* μ *and* κ*, such that for* $\tau \leq \tau_0$ *and* $t_n = n\tau \leq T$*, that the error* e_n *is bounded by*

$$|e_n|^2_{M_n} + \tau \sum_{k=1}^{n} |e_k|^2_{A_k} \leq C\Big\{ |e_0|^2_{M_0} + \tau \sum_{k=1}^{n-1} \sum_{i=1}^{s} \|M_{ki}r_{ki}\|^2_{*,ki}$$

$$+\tau \sum_{k=1}^{n} \left| \frac{\delta_k}{\tau} \right|_{M_k}^2 + \tau \sum_{k=0}^{n-1} \sum_{i=1}^{s} \Big(|M_{ki}^{-1} \Delta_{ki}|_{M_{ki}}^2 + |M_{ki}^{-1} \Delta_{ki}|_{A_{ki}}^2 \Big) \Big\},$$

where $\|w\|_{*,k}^2 = w^T \left(A(t_k) + M(t_k)\right)^{-1} w$. *The constant* C *is independent of* h, τ *and* n, *but depends on* μ, κ, T *and on the norm of the difference of the velocities. The constant* τ_0 *depends on the ALE velocity, see lemma A.3.5.*

Proof. (a) By using (A.28a) – (A.28c) and algebraic stability (A.25) the following inequality holds for the ALE setting:

$$|e_{n+1}|_{M_{n+1}}^2 \leq (1 + 2\mu\tau) \, |e_n|_{M_n}^2 + 2\tau \sum_{i=1}^{s} b_i \, \langle \dot{E}_{ni} \mid M_{n+1}^{-1} \mid M_{ni} E_{ni} + \Delta_{ni} \rangle$$
$$+ \tau \, |E_{n+1}|_{M_{n+1}}^2 + (1 + 3\tau)\tau \left| \frac{\delta_{n+1}}{\tau} \right|_{M_{n+1}^{-1}}^2 . \qquad \text{(A.29)}$$

We want to estimate the second term on the right-hand side of (A.29). Obviously the equation

$$\langle \dot{E}_{ni} \mid M_{n+1}^{-1} \mid M_{ni} E_{ni} + \Delta_{ni} \rangle = \langle \dot{E}_{ni} \mid M_{ni}^{-1} \mid M_{ni} E_{ni} + \Delta_{ni} \rangle$$
$$+ \langle \dot{E}_{ni} \mid M_{n+1}^{-1} - M_{ni}^{-1} \mid M_{ni} E_{ni} + \Delta_{ni} \rangle \quad \text{(A.30)}$$

holds. The second term on the right-hand side of (A.30) can be estimated by (cf. [66] lemma 3.1, (3.14)):

$$\langle \dot{E}_{ni} \mid M_{n+1}^{-1} - M_{ni}^{-1} \mid M_{ni} E_{ni} + \Delta_{ni} \rangle$$
$$\leq C \Big\{ |e_n|_{M_n}^2 + \sum_{j=1}^{s} |E_{nj}|_{M_{nj}}^2 + |\Delta_{nj}|_{M_{nj}^{-1}}^2 \Big\}. \quad \text{(A.31)}$$

(b) We have to modify the estimation of the first term on the right-hand side of (A.30). Using the definition of internal stages (A.28c), we have

$$\begin{aligned}\langle \dot{E}_{ni} \mid M_{ni}^{-1} \mid M_{ni}E_{ni} + \Delta_{ni}\rangle = & -|E_{ni}|_{A_{ni}}^2 - \langle M_{ni}r_{ni} \mid E_{ni} + M_{ni}^{-1}\Delta_{ni}\rangle \\ & - \langle E_{ni} \mid A_{ni} \mid M_{ni}^{-1}\Delta_{ni}\rangle \\ & - \langle B_{ni}E_{ni} \mid E_{ni} + M_{ni}^{-1}\Delta_{ni}\rangle. \end{aligned} \tag{A.32}$$

The last term can be estimated by Lemma A.3.6 as

$$\begin{aligned}|\langle B_{ni}E_{ni} \mid E_{ni} + M_{ni}^{-1}\Delta_{ni}\rangle| &\leq |\langle B_{ni}E_{ni} \mid E_{ni}\rangle| + |\langle B_{ni}E_{ni} \mid M_{ni}^{-1}\Delta_{ni}\rangle| \\ &\leq C\,|E_{ni}|_{M_{ni}}\,|E_{ni}|_{A_{ni}} + C\,|E_{ni}|_{M_{ni}}\,|M_{ni}^{-1}\Delta_{ni}|_{A_{ni}} \\ &\leq C\,|E_{ni}|_{M_{ni}}^2 + \frac{1}{4}\,|E_{ni}|_{A_{ni}}^2 + C\,|E_{ni}|_{M_{ni}}^2 + C\,|M_{ni}^{-1}\Delta_{ni}|_{A_{ni}}^2. \end{aligned} \tag{A.33}$$

While the other terms can be estimated by the following inequality (shown in [66] lemma 3.1):

$$\begin{aligned} & -|E_{ni}|_{A_{ni}}^2 + |\langle M_{ni}r_{ni} \mid E_{ni} + M_{ni}^{-1}\Delta_{ni}\rangle| + |\langle E_{ni} \mid A_{ni} \mid M_{ni}^{-1}\Delta_{ni}\rangle| \\ & \qquad \leq -\frac{1}{2}\,|E_{ni}|_{A_{ni}}^2 + \frac{1}{4}\,|E_{ni}|_{M_{ni}}^2 \\ & \qquad\quad + C(|M_{ni}^{-1}\Delta_{ni}|_{M_{ni}}^2 + |M_{ni}^{-1}\Delta_{ni}|_{A_{ni}}^2). \end{aligned} \tag{A.34}$$

We continue to estimate the right-hand side of (A.32) with (A.33), (A.34) and arrive to

$$\begin{aligned}\langle \dot{E}_{ni} \mid M_{n+1}^{-1} \mid M_{ni}E_{ni} + \Delta_{ni}\rangle \leq & -\frac{1}{4}\,|E_{ni}|_{A_{ni}}^2 + C(|E_{ni}|_{M_{ni}}^2 \\ & + |M_{ni}^{-1}\Delta_{ni}|_{M_{ni}}^2 + |M_{ni}^{-1}\Delta_{ni}|_{A_{ni}}{}^2). \end{aligned} \tag{A.35}$$

(c) Now we return to the main inequality (A.29), consider equation

(A.32) and plug in the inequalities (A.31) and (A.35) to get

$$
\begin{aligned}
&|e_{n+1}|^2_{M_{n+1}} - |e_n|^2_{M_n} + \frac{1}{4}\tau \sum_{i=1}^{s} b_i \, |E_{ni}|^2_{A_{ni}} \\
&\leq C\tau \Big\{ |e_n|^2_{M_n} + \sum_{j=1}^{s} |E_{nj}|^2_{M_{nj}} + \|M_{nj} r_{nj}\|^2_{*,nj} \\
&\quad + \sum_{j=1}^{s} (|M_{nj}^{-1}\Delta_{nj}|^2_{M_{nj}} + |M_{nj}^{-1}\Delta_{nj}|^2_{A_{nj}}) + \left| \frac{\delta_{n+1}}{\tau} \right|^2_{M^{-1}_{n+1}} \Big\}.
\end{aligned} \tag{A.36}
$$

(d) Next we estimate $|E_{nj}|^2_{M_{nj}}$, in [66] lemma 3.1 one can find the estimate:

$$
|E_{ni}|^2_{M_{ni}} \leq C\Big(|e_n|^2_{M_n} + \tau \sum_{j=1}^{s} a_{ij} \, \langle \dot{E}_{nj} \mid E_{ni} \rangle + |M_{ni}^{-1}\Delta_{ni}|^2_{M_{ni}} \Big). \tag{A.37}
$$

We have to estimate $\langle \dot{E}_{nj} \mid E_{ni} \rangle$, with equation (A.28c) we get

$$
\langle \dot{E}_{nj} \mid E_{ni} \rangle = - \langle E_{nj} \mid A_{nj} \mid E_{ni} \rangle - \langle M_{nj} r_{nj} \mid E_{ni} \rangle - \langle B_{nj} E_{nj} \mid E_{ni} \rangle. \tag{A.38}
$$

The following inequalities can be shown easily using Young's-inequality (ε will be chosen later) and Cauchy-Schwarz inequality:

$$
\begin{aligned}
&- \langle E_{nj} \mid A_{nj} \mid E_{ni} \rangle \leq C(\kappa)(|E_{nj}|^2_{A_{nj}} + |E_{ni}|^2_{A_{ni}}), \\
&- \langle B_{nj} E_{nj} \mid E_{ni} \rangle \leq \varepsilon \, |E_{nj}|^2_{M_{nj}} + \frac{1}{4\varepsilon} C(\kappa) \, |E_{ni}|^2_{A_{ni}} \\
&- \langle M_{nj} r_{nj} \mid E_{ni} \rangle \leq C(\mu, \kappa) \Big(\frac{1}{4\varepsilon} \, \|M_{nj} r_{nj}\|^2_{*,nj} + \varepsilon (|E_{ni}|^2_{M_{ni}} + |E_{ni}|^2_{A_{ni}}) \Big).
\end{aligned}
$$

Using the above three inequalities to estimate (A.38), we get

$$
\begin{aligned}
\langle \dot{E}_{nj} \mid E_{ni} \rangle &\leq C(\mu, \kappa) \Big(\varepsilon \, |E_{ni}|^2_{M_{ni}} + C(\varepsilon) \, |E_{ni}|^2_{A_{ni}} \Big) \\
&\quad + |E_{nj}|^2_{A_{nj}} + C(\varepsilon) \, \|M_{nj} r_{nj}\|^2_{*,nj}.
\end{aligned} \tag{A.39}
$$

Using this for a sufficiently small ε (independent of τ) we can proceed by estimating (A.37) further as

$$|E_{ni}|^2_{M_{ni}} \leq C\Big(|e_n|^2_{M_n} + \tau \sum_{j=1}^{s} a_{ij} \big(|E_{nj}|^2_{A_{nj}} + \|M_{nj} r_{nj}\|^2_{*,nj} \big) + |M_{ni}^{-1}\Delta_{ni}|^2_{M_{ni}} \Big).$$

(e) Now for a sufficiently small τ we can use the above inequality to estimate (A.36) to

$$|e_{n+1}|^2_{M_{n+1}} - |e_n|^2_{M_n} + \frac{1}{8}\tau \sum_{i=1}^{s} b_i \, |E_{ni}|^2_{A_{ni}} \leq C\tau \Big\{ |e_n|^2_{M_n} + \sum_{i=1}^{s} \|M_{ni} r_{ni}\|^2_{*,ni} + \sum_{i=1}^{s} \big(|M_{ni}^{-1}\Delta_{ni}|^2_{M_{ni}} + |M_{ni}^{-1}\Delta_{ni}|^2_{A_{ni}} \big) + \Big| \frac{\delta_{n+1}}{\tau} \Big|^2_{M_{n+1}^{-1}} \Big\}.$$

Summing up over n and applying a discrete Gronwall inequality yields the desired result. ■

A.4.2. Stability of Backward Difference Formulas

We apply a backward difference formula (BDF) as a temporal discretization to the ODE system (A.12), coming from the ALE ESFEM space discretization of the parabolic evolving surface PDE.

In the following we extend the stability result for BDF methods of [63], Lemma 4.1 to the case of ALE evolving surface finite element method. Apart from the properties of the ALE ESFEM the proof is based on the G-stability theory of [20] and the multiplier technique of [69]. We will prove that the fully discrete method is stable for the k-step BDF methods up to order five. Again the stability holds without a CFL-type condition.

We recall the k-step BDF method, applied to the ODE system (A.12), with step size $\tau > 0$ and given starting values $\alpha_{n-k}, \dots, \alpha_{n-1}$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A(t_n)\alpha_n + B(t_n)\alpha_n = 0, \quad (n \geq k), \qquad \text{(A.40)}$$

where the coefficients of the method is given by $\delta(\zeta) = \sum_{j=0}^{k}\delta_j\zeta^j = \sum_{\ell=1}^{k}\frac{1}{\ell}(1-\zeta)^\ell$, while the initial values are $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$. Again U_h and α is related through (A.11). The method is known to be 0-stable for $k \leq 6$ (but not A-stable for $k \geq 3$) and have order k, for more details we refer to [49] chapter V.

Instead of (A.12), let us consider again the perturbed problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\tilde{\alpha}(t)\big) + A(t)\tilde{\alpha}(t) + B(t)\tilde{\alpha}(t) = M(t)r(t) \\ \tilde{\alpha}(0) = \tilde{\alpha}_0. \end{cases} \qquad \text{(A.41)}$$

By substituting the true solution $\tilde{\alpha}(t)$ of the perturbed problem into the BDF method (A.40), we obtain

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\tilde{\alpha}_{n-j} + A(t_n)\tilde{\alpha}_n + B(t_n)\tilde{\alpha}_n = -d_n, \qquad (n \geq k).$$

Then by introducing the error $e_n = \alpha_n - \tilde{\alpha}(t_n)$, multiplying by τ, and by subtraction we have the error equation

$$\sum_{j=0}^{k}\delta_j M_{n-j}e_{n-j} + \tau A_n e_n + \tau B_n e_n = \tau d_n, \qquad (n \geq k). \qquad \text{(A.42)}$$

We recall two important preliminary results.

Lemma A.4.3 ([20]). *Let $\delta(\zeta)$ and $\mu(\zeta)$ be polynomials of degree at most k (at least one of them of exact degree k) that have no common divisor. Let $\langle \cdot \mid \cdot \rangle$ be an inner product on $\mathbb{R}^N$ with associated norm $\|\cdot\|$. If*

$$\operatorname{Re}\frac{\delta(\zeta)}{\mu(\zeta)} > 0, \qquad \text{for} \quad |\zeta| < 1,$$

then there exists a symmetric positive definite matrix $G = (g_{ij}) \in \mathbb{R}^{k\times k}$ and real $\gamma_0, \dots, \gamma_k$ such that for all $v_0, \dots, v_k \in \mathbb{R}^N$

$$\left\langle \sum_{i=0}^{k} \delta_i v_{k-i} \,\middle|\, \sum_{i=0}^{k} \mu_i v_{k-i} \right\rangle = \sum_{i,j=1}^{k} g_{ij} \langle v_i \mid v_j \rangle - \sum_{i,j=1}^{k} g_{ij} \langle v_{i-1} \mid v_{j-1} \rangle + \left| \sum_{i=0}^{k} \gamma_i v_i \right|^2$$

holds.

Together with this result, the case $\mu(\zeta) = 1 - \eta\zeta$ will play an important role:

Lemma A.4.4 ([69]). *If $k \leq 5$, then there exists $0 \leq \eta < 1$ such that for $\delta(\zeta) = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-\zeta)^\ell$,*

$$\operatorname{Re}\frac{\delta(\zeta)}{1-\eta\zeta} > 0, \qquad \text{for} \quad |\zeta| < 1.$$

The smallest possible values of η is found to be η = 0, 0, 0.0836, 0.2878, 0.8160 for $k = 1, 2, \dots, 5$, respectively.

We now state and prove the analogous stability result for the BDF methods. Again using (A.15) this estimate can be translated into L^2- and H^1-norms of the corresponding finite element error functions, see later in theorem A.5.4.

Lemma A.4.5. *For a k-step BDF method with $k \leq 5$ there exists a $\tau_0 > 0$, depending only on the constants μ and κ, such that for $\tau \leq \tau_0$ and $t_n = n\tau \leq T$, that the error e_n is bounded by*

$$|e_n|_{M_n}^2 + \tau \sum_{j=k}^{n} |e_j|_{A_j}^2 \leq C\tau \sum_{j=k}^{n} \|d_j\|_{*,j}^2 + C \max_{0\leq i\leq k-1} |e_i|_{M_i}^2$$

where $\|w\|_{*,k}{}^2 = w^T \left(A(t_k) + M(t_k)\right)^{-1} w$. The constant C is independent of h, τ and n, but depends on μ, κ, T and on the norm of the difference of the velocities. The constant τ_0 depends on the ALE velocity, see lemma A.3.5.

Proof. Our proof follows the one of lemma 4.1 in [63].

(a) The starting point of the proof is the following reformulation of the error equation (A.42)

$$M_n \sum_{j=0}^{k} \delta_j e_{n-j} + \tau A_n e_n + \tau B_n e_n = \tau d_n + \sum_{j=1}^{k} \delta_j (M_n - M_{n-j}) e_{n-j}$$

and using a modified energy estimate. We multiply both sides by $e_n - \eta e_{n-1}$, for $n \geq k+1$, which gives us:

$$I_n + II_n = III_n + IV_n - V_n,$$

where

$$\begin{aligned}
I_n &= \left\langle \sum_{j=0}^{k} \delta_j e_{n-j} \,\middle|\, M_n \,\middle|\, e_n - \eta e_{n-1} \right\rangle, \\
II_n &= \tau \left\langle e_n \,|\, A_n \,|\, e_n - \eta e_{n-1} \right\rangle, \\
III_n &= \tau \left\langle d_n \,|\, e_n - \eta e_{n-1} \right\rangle, \\
IV_n &= \sum_{j=1}^{k} \left\langle e_{n-j} \,|\, M_n - M_{n-j} \,|\, e_n - \eta e_{n-1} \right\rangle,
\end{aligned}$$

$$V_n = \tau \langle e_n \mid B_n \mid e_n - \eta e_{n-1} \rangle.$$

(b) The estimates of I_n, II_n, III_n and IV_n are the same as in the proof of [63, lemma 4.1]. For the convenience of the reader we repeat them:

$$\begin{aligned} I_n &\geq \|E_n\|_{G,n}^2 - \|E_{n-1}\|_{G,n}^2, \\ \frac{II_n}{\tau} &\geq \frac{2-\eta}{2} |e_n|_{A_n}^2 - c\eta |e_{n-1}|_{A_{n-1}}^2, \\ \frac{|III_n|}{\tau} &\leq c\frac{1}{1-\eta} \|d_n\|_{*,n}^2 + \frac{1-\eta}{2}(\varepsilon |e_n|_{A_n}^2 + |e_n|_{M_n}^2) \\ &\quad + (1-\eta)c(|e_{n-1}|_{A_{n-1}}^2 + |e_{n-1}|_{M_{n-1}}^2), \\ \frac{|IV_n|}{\tau} &\leq c(\|E_n\|_{G,n}^2 + \|E_{n-1}\|_{G,n}^2). \end{aligned}$$

We note that during the estimation of III_n we used Young's inequality with sufficiently small (τ independent) ε.

The nonsymmetric term V_n is estimated using Lemma A.3.6 and Young's inequality (with sufficiently small ε, independent of τ):

$$\begin{aligned} |V_n| &\leq C\tau |e_n|_{M_n}(|e_n|_{A_n} + \eta |e_{n-1}|_{A_{n-1}}) \\ &= C\tau |e_n|_{M_n} |e_n|_{A_n} + C\eta\tau |e_n|_{M_n} |e_{n-1}|_{A_{n-1}} \\ &\leq \tau C\frac{1}{\varepsilon} |e_n|_{M_n}^2 + \varepsilon\tau |e_n|_{A_n}^2 + \tau C\frac{1}{\varepsilon} |e_n|_{M_n}^2 + \varepsilon\eta^2\tau |e_{n-1}|_{A_{n-1}}^2. \end{aligned}$$

(c) Combining all estimates, choosing a sufficiently small ε (independently of τ), and summing up gives, for $\tau \leq \tau_0$ and for $k \geq n+1$:

$$\begin{aligned} \|E_n\|_{G,n}^2 + (1-\eta)\frac{\tau}{8} \sum_{j=k+1}^{n} |e_j|_{A_j}^2 \leq C\tau \sum_{j=k}^{n-1} \|E_j\|_{G,j}^2 \\ + C\tau \sum_{j=k+1}^{n} \|d_j\|_{*,t_j}^2 + C\eta^2\tau |e_k|_{A_k}^2, \end{aligned} \tag{A.43}$$

where $E_n = (e_n, \dots, e_{n-k+1})$, and the

$$\|E_n\|_{G,n}^2 \coloneqq \sum_{i,j=1}^{k} g_{ij} \, \langle e_{n-k+1} \,|\, M_n \,|\, e_{n-k+j} \rangle.$$

(d) To achieve the stated result we have to estimate the extra term $|e_k|_{M_k}^2 + \tau \, |e_k|_{A_k}^2$. For that we take the inner product of the error equation for $n = k$ with e_k to obtain

$$\delta_0 \, |e_k|_{M_k}^2 + \tau \, |e_k|_{A_k}^2 = \tau \, \langle d_k \,|\, e_k \rangle - \sum_{j=1}^{k} \delta_j \, \langle M_{k-j} e_{k-j} \,|\, e_k \rangle + \tau \, |\langle e_k \,|\, B_k \,|\, e_k \rangle|.$$

Then the use of Lemma A.3.6 and Young's inequality (again with sufficiently small ε) and (A.16), yields

$$|e_k|_{M_k}^2 + \tau \, |e_k|_{A_k}^2 \leq C\tau \, \|d_k\|_{*,k}^2 + C \max_{0 \leq i \leq k-1} |e_i|_{M_i}^2.$$

Similarly as in [63, lemma 4.1] using the discrete Gronwall inequality for (A.43) and the above estimate concludes the result. ■

A.5. Error bounds for the fully discrete solutions

We start by connecting the stability results of the previous section with the continuous solution of the parabolic problem. Then using the Ritz map of u we will show the convergence of the error, which — together with the stability results — leads us to our main results. We will prove that the full discretizations, ALE evolving surface finite element method coupled with Runge-Kutta or BDF methods converges. The convergence does not require a bound on τ in terms of h.

A.5.1. Bound of the semidiscrete residual

Before turning to the fully discrete problem we show error bounds for the semidiscretization.

Since the stability analysis only uses the matrix-vector formulation (A.27), (A.41), but not the semidiscrete weak form, we follow [63, Section 5], using the Ritz map, to define the finite element residual,

$$R_h(\,\cdot\,,t) = \sum_{j=1}^{N} r_j(t)\chi_j(\,\cdot\,,t) \in S_h(t),$$

by duality pointwise in time, as follows. Let

$$\int_{\Gamma_h(t)} R_h(\,\cdot\,,t)\phi_h = L_t(\phi_h), \quad \forall \phi_h \in S_h(t), \tag{A.44}$$

where, for a fixed $t \in [0,T]$, the linear functional $L_t : S_h(t) \to \mathbb{R}$ is defined as follows: for a given finite element function

$$\phi_h = \sum_{j=1}^{N} c_j\chi_j(\,\cdot\,,t) \in S_h(t),$$

define the temporal extension $\varphi_h(s) \in S_h(s)$ as the finite element function with the same nodal values

$$\varphi_h(s) = \sum_{j=1}^{N} c_j\chi_j(\,\cdot\,,s) \in S_h(s), \quad (s \in [0,T]).$$

Then, $\partial_h^{\mathcal{A}}\varphi_h(s) = 0$ for all s, by the transport property (A.9) of the basis functions. We now define

$$\begin{aligned} L_t(\phi_h) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} \widetilde{\mathcal{P}}_h u(\,\cdot\,,t)\varphi_h(\,\cdot\,,t) + \int_{\Gamma_h(t)} \nabla_{\Gamma_h}(\widetilde{\mathcal{P}}_h u)(\,\cdot\,,t) \\ \cdot \nabla_{\Gamma_h}\varphi_h(\,\cdot\,,t) + \int_{\Gamma_h(t)} (\widetilde{\mathcal{P}}_h u)(\,\cdot\,,t)(W_h - V_h)(\,\cdot\,,t) \cdot \nabla_{\Gamma_h}\varphi_h(\,\cdot\,,t) \end{aligned}$$

and determine the residual $R_h(\,\cdot\,,t)$ by (A.44).

The above construction yields the following linear ODE system with the vector $r(t) = (r_j(t)) \in \mathbb{R}^N$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\tilde{\alpha}(t)\big) + A(t)\tilde{\alpha}(t) + B(t)\tilde{\alpha}(t) = M(t)r(t),$$

which is the perturbed ODE system (A.27) and (A.41).

We show second order error bounds for the residual R_h using the bounds on the Ritz map.

Theorem A.5.1 (Bound of the semidiscrete residual). *Let u, the solution of the parabolic problem, be smooth. Then there exists a constant $C > 0$ and $h_0 > 0$, such that for all $h \leq h_0$ and $t \in [0, T]$, the finite element residual R_h of the Ritz map is bounded by*

$$\|R_h(\,\cdot\,, t)\|_{H_h^{-1}(\Gamma_h(t))} \leq Ch^2,$$

where the constant C is independent of h and t, but depends on T and on the solution u. The H_h^{-1}-norm of R_h is defined as

$$\|R_h(\,\cdot\,, t)\|_{H_h^{-1}(\Gamma_h(t))} := \sup_{\phi_h \neq 0} \frac{\langle R_h(\,\cdot\,, t), \phi_h\rangle_{L^2(\Gamma_h(t))}}{\|\phi_h\|_{H^1(\Gamma_h(t))}},$$

where $\phi_h \in S_h(t)$ is fix in time.

Proof. (a) We start by applying the discrete ALE transport property to the residual equation (A.44) and using the definition of L_t, for $\widetilde{\mathcal{P}}_h u \in S_h(t)$:

$$\begin{aligned} m_h(R_h, \phi_h) = m_h(\partial_h^A \widetilde{\mathcal{P}}_h u, \varphi_h) + a_h(\widetilde{\mathcal{P}}_h u, \varphi_h) + g_h(W_h; \widetilde{\mathcal{P}}_h u, \varphi_h) \\ + m_h(\widetilde{\mathcal{P}}_h u, (W_h - V_h) \cdot \nabla_{\Gamma_h} \varphi_h). \end{aligned}$$

(b) We continue by the transport property with discrete ALE material derivatives from Lemma A.3.10, but for the ALE weak form (from Lemma A.2.5)

$$0 = m(\partial_h^{\mathcal{A}} u, \varphi_h^l) + a(u, \varphi_h^l) + g(w_h; u, \varphi_h^l) + m(u, (w - v) \cdot \nabla_\Gamma \varphi_h^l).$$

(c) Subtraction of the two equations, and using the definition of the Ritz map (A.23), we obtain the following expression for the residual:

$$\begin{aligned} m_h(R_h, \phi_h) = {} & m_h(\partial_h^{\mathcal{A}} \widetilde{\mathcal{P}}_h u, \varphi_h) - m(\partial_h^{\mathcal{A}} u, \varphi_h^l) \\ & + g_h(W_h; \widetilde{\mathcal{P}}_h u, \varphi_h) - g(w_h; u, \varphi_h^l) \\ & - (m_h(\widetilde{\mathcal{P}}_h u, \varphi_h) - m(u, \varphi_h^l)) \\ & + m_h(\widetilde{\mathcal{P}}_h u, (W_h - V_h) \cdot \nabla_{\Gamma_h} \varphi_h) - m(u, (w - v) \cdot \nabla_\Gamma \varphi_h^l). \end{aligned}$$

(d) We estimate these pairs separately, we show the basic idea by using the nonsymmetric term: We aim to use lemma A.3.8 and the error estimate for the Ritz map, Lemma A.3.14, namely we estimate as

$$\begin{aligned} & m_h(\widetilde{\mathcal{P}}_h u, (W_h - V_h) \cdot \nabla_{\Gamma_h} \varphi_h) - m(\mathcal{P}_h u, (w - v) \cdot \nabla_\Gamma \varphi_h^l) \\ & + m(\mathcal{P}_h u - u, (w - v) \cdot \nabla_\Gamma \varphi_h^l) \leq Ch^2 \, \|\varphi_h^l\|_{H^1(\Gamma(t))}. \end{aligned}$$

The other pairs can be estimated in the same way: by Lemma A.3.11 and the errors in the Ritz map (in fact they can be bounded by $Ch^2 \, \|\varphi_h\|_{L^2(\Gamma(t))}$). ■

A.5.2. Error bounds

The direct application of the stability lemmata for Runge-Kutta methods and BDF methods (Lemma A.4.2 and Lemma A.4.5, respectively) gives error estimates between the projection $\widetilde{\mathcal{P}}_h u(\,\cdot\,, t_n)$ and the fully discrete solution U_h^n (ALE ESFEM combined with a temporal discretization), i.e.

$$U_h^n := \sum_{j=1}^N \alpha_j^n \chi_j(\,\cdot\,, t_n) \in S_h(t),$$

where the vectors α^n are generated, either by an s-stage implicit Runge-Kutta method (A.24), or by a BDF method of order k (A.40).

Implicit Runge-Kutta methods

Now we can prove the analogous error estimation result from [37, theorem 8.1] ([66, theorem 5.1]).

Theorem A.5.2. *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method as space discretization of the parabolic problem* (A.1) *with time discretization by an s-stage implicit Runge-Kutta method satisfying Assumption A.4.1. Assume that the solution u and the surface* $\Gamma(t)$ *is smooth. Then there exists* $\tau_0 > 0$*, independent of h, but depending on the ALE velocity (see lemma A.3.5), such that for* $\tau \leq \tau_0$*, for the error*

$$E_h^n = U_h^n - \widetilde{\mathcal{P}}_h u(\,\cdot\,, t_n)$$

the following estimate holds for $t_n = n\tau \leq T$*:*

$$\begin{aligned} \|E_h^n\|_{L^2(\Gamma_h(t_n))} &+ \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|_{L^2(\Gamma_h(t_j))}^2\Big)^{\frac{1}{2}} \\ &\leq C\tilde{\beta}_{h,q}\tau^{q+1} + C\Big(\tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \|R_h(\,\cdot\,, t_k + c_i\tau)\|_{H_h^{-1}(\Gamma_h(t_k+c_i\tau))}^2\Big)^{\frac{1}{2}} \\ &\quad + C\,\|E_h^0\|_{L^2(\Gamma_h(t_0))}, \end{aligned}$$

where the constant C is independent of h and τ*, but depends on T, and we have*

$$\tilde{\beta}_{h,q}^2 = \int_0^T \sum_{\ell=1}^{q+2} \|(\partial_h^{\mathcal{A}})^{(\ell)}(\widetilde{\mathcal{P}}_h u)(\,\cdot\,, t)\|_{L^2(\Gamma_h(t))}$$

$$+\sum_{\ell=1}^{q+1} \|\nabla_{\Gamma_h(t)}(\partial_h^{\mathcal{A}})^{(\ell)}(\widetilde{\mathcal{P}}_h u)(\,\cdot\,,t)\|^{L^2(\Gamma_h(t))}\,\mathrm{d}t.$$

Later on in the proofs we will use the existence of high order material derivatives of $\widetilde{\mathcal{P}}_h u$. This follows as a combination of the assumed regularity of the evolution of $\Gamma(t)$ and the assumed regularity of the exact solution u.

The version with the classical order p from [37] Theorem 8.2 (or [66, Theorem 5.2]) also holds in the ALE case, if the stronger regularity conditions are satisfied:

$$\left| M(t)^{-1} \frac{\mathrm{d}^{(k_j,\dots,k_1)}}{\mathrm{d}t^{(k_j,\dots,k_1)}} \Big(A(t)M(t)^{-1}\Big) \frac{\mathrm{d}^{\tilde{k}-1}}{\mathrm{d}t^{\tilde{k}-1}} \Big(M(t)\tilde{\alpha}(t)\Big) \right|_{M(t)} \leq \gamma,$$

$$\left| M(t)^{-1} \frac{\mathrm{d}^{(k_j,\dots,k_1)}}{\mathrm{d}t^{(k_j,\dots,k_1)}} \Big(A(t)M(t)^{-1}\Big) \frac{\mathrm{d}^{\tilde{k}-1}}{\mathrm{d}t^{\tilde{k}-1}} \Big(M(t)\tilde{\alpha}(t)\Big) \right|_{A(t)} \leq \gamma,$$

for all $k_j \geq 1$ and $\tilde{k} \geq q+1$ with $k_1 + \cdots + k_j + \tilde{k} \leq p+1$, where

$$\frac{\mathrm{d}^{(k_j,\dots,k_1)} f}{\mathrm{d}t^{(k_j,\dots,k_1)}} := \frac{\mathrm{d}^{k_j-1} f}{\mathrm{d}t^{k_j-1}} \cdots \frac{\mathrm{d}^{k_1-1} f}{\mathrm{d}t^{k_1-1}}$$

Theorem A.5.3. *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method as space discretization of the parabolic problem* (A.1)*, with time discretization by an s-stage implicit Runge-Kutta method satisfying Assumption A.4.1 with* $p > q+1$*. Assuming the above regularity conditions. There exists* $\tau_0 > 0$ *independent of h, but depending on the ALE velocity (see lemma A.3.5), such that for* $\tau \leq \tau_0$*, for the error* $E_h^n = U_h^n - \widetilde{\mathcal{P}}_h u(\,\cdot\,,t_n)$ *the* *following estimate holds for* $t_n = n\tau \le T$:

$$\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|_{L^2(\Gamma_h(t_j))}^2\Big)^{\frac{1}{2}}$$
$$\le C_0\tau^p + C\Big(\tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \|R_h(\,\cdot\,, t_k + c_i\tau)\|_{H_h^{-1}(\Gamma_h(t_k+c_i\tau))}^2\Big)^{\frac{1}{2}}$$
$$+ C\,\|E_h^0\|_{L^2(\Gamma_h(t_0))},$$

where the constant C_0 *is independent of* h *and* τ*, but depends on* T *and* γ*.*

Proof of theorem A.5.2 and A.5.3. The proofs of the two theorem above is a combination of our previous results, especially the stability lemma, lemma A.4.2, and the relation

$$\|M_n r_n\|_{*,n} = \|R_h(\,\cdot\,, t_n)\|_{H_h^{-1}(\Gamma_h(t_n))},$$

cf. [66] (5.5). Otherwise they are the same as the proof in [37, section 8] or see [66, theorem 5.1, 5.2]. The h and τ independency holds since the used stability lemma is also independent of them. ■

Backward differentiation formulas

We prove the analogous result of [63, theorem 5.1] ([66, theorem 5.3]).

Theorem A.5.4. *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method as space discretization of the parabolic problem* (A.1) *with time discretization by a k-step backward difference formula of order* $k \le 5$*. Assume that the solution* u *and the surface* $\Gamma(t)$ *is smooth. Then there exists* $\tau_0 > 0$*, independent of* h*, but depending on the ALE velocity (see lemma A.3.5), such that for* $\tau \le \tau_0$*, for the error* $E_h^n = U_h^n - \widetilde{\mathcal{P}}_h u(\,\cdot\,, t_n)$ *the following estimate* *holds for $t_n = n\tau \leq T$:*

$$\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\Big)^{\frac{1}{2}}$$
$$\leq C\tilde{\beta}_{h,k}\tau^k + \Big(\tau \sum_{j=1}^{n} \|R_h(\,\cdot\,,t_j)\|^2_{H_h^{-1}(\Gamma_h(t_j))}\Big)^{\frac{1}{2}}$$
$$+ C \max_{0\leq i\leq k-1} \|E_h^i\|_{L^2(\Gamma_h(t_i))},$$

where the constant C is independent of h and τ, but depends on T, and we have

$$\tilde{\beta}_{h,k}^2 = \int_0^T \sum_{\ell=1}^{k+1} \|(\partial_h^{\mathcal{A}})^{(\ell)}(\widetilde{\mathcal{P}}_h u)(\,\cdot\,,t)\|_{L^2(\Gamma_h(t))}\,\mathrm{d}t.$$

Proof. The proof of this theorem relays on the corresponding h and τ independent stability result, i.e. lemma A.4.5. Otherwise we follow the proof of [63, theorem 5.1], or [66, theorem 5.3]. ■

Remark A.5.5. The quantities $\widetilde{\beta}_{h,q}^2$ and $\widetilde{\beta}_{h,k}^2$ from theorem A.5.2 and theorem A.5.4 require existence of higher order discrete ALE material derivatives of the Ritz projection of u and further that they are bounded w.r. to the L^2 resp. H^1 norm. The existence of higher order material derivatives can be seen as follows: Rewrite equation (A.23) as a matrix vector equation for the coefficients of $\widetilde{\mathcal{P}}_h u$. Discrete ALE material derivatives corresponds to usual time derivatives for the coefficients of $\widetilde{\mathcal{P}}_h u$. Hence if we assume that the ALE dynamical system is smooth and that the exact solution is smooth, then it follows that higher order discrete ALE material derivatives of u exists. The boundedness of them follow from lemma A.3.14.

A.5.3. Error of the full ALE discretizations

We compare the lifted fully discrete numerical solution $u_h^n := (U_h^n)^l$ with the exact solution $u(\,\cdot\,, t_n)$ of the evolving surface PDE (A.1), where $U_h^n = \sum_{j=1}^N \alpha_j^n \chi_j(\,\cdot\,, t_n)$, where the vectors α^n are generated by the Runge–Kutta (A.24) or the BDF method (A.40).

Now we state and prove the main results of this paper.

Theorem A.5.6 (ALE ESFEM and R–K). *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method as space discretization of the parabolic problem* (A.1) *with time discretization by an s-stage implicit Runge–Kutta method satisfying Assumption A.4.1. Let u be a smooth solution of the problem, as in theorem A.5.2 and A.5.3, and assume that the initial value is approximated as*

$$\|u_h^0 - (\mathcal{P}_h u)(\,\cdot\,, 0)\|_{L^2(\Gamma(0))} \leq C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leq h_0$ and $\tau \leq \tau_0$, the following error estimate holds for $t_n = n\tau \leq T$:

$$\|u_h^n - u(\,\cdot\,, t_n)\|_{L^2(\Gamma(t_n))} + h\left(\tau \sum_{j=1}^n |u_h^j - u(\,\cdot\,, t_j)|^2_{H^1(\Gamma(t_j))}\right)^{\frac{1}{2}}$$
$$\leq C\left(\tau^{q+1} + h^2\right).$$

The constant C is independent of h, τ and n, but depends on T and on the solution u.

Assuming that we have more regularity: conditions of Theorem A.5.3 are additionally satisfied, then we have p instead of $q + 1$.

The analogous statement with BDF time discretization reads as follows.

Theorem A.5.7 (ALE ESFEM and BDF). *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method as space discretization of the parabolic problem* (A.1) *with time discretization by a k-step backward difference formula of order* $k \leq 5$*. Let u be a smooth solution of the problem, as in theorem A.5.4, and assume that the starting values are satisfying*

$$\max_{0\leq i\leq k-1} \|u_h^i - (\mathcal{P}_h u)(\,\cdot\,,t_i)\|_{L^2(\Gamma(0))} \leq C_0 h^2.$$

Then there exists $h_0 > 0$ *and* $\tau_0 > 0$*, such that for* $h \leq h_0$ *and* $\tau \leq \tau_0$*, the following error estimate holds for* $t_n = n\tau \leq T$*:*

$$\|u_h^n - u(\,\cdot\,,t_n)\|_{L^2(\Gamma(t_n))} + h\left(\tau\sum_{j=1}^{n} |u_h^j - u(\,\cdot\,,t_j)|^2_{H^1(\Gamma(t_j))}\right)^{\frac{1}{2}}$$
$$\leq C\,(\tau^k + h^2).$$

The constant C is independent of h, n and n, but depends on T and on the smooth solution u.

Proof of theorem A.5.6–A.5.7. The global error is decomposed into two parts

$$u_h^n - u(\,\cdot\,,t_n) = \Big(u_h^n - (\mathcal{P}_h u)(\,\cdot\,,t_n)\Big) + \Big((\mathcal{P}_h u)(\,\cdot\,,t_n) - u(\,\cdot\,,t_n)\Big),$$

and the terms are estimated by previous results.

The first term is estimated by a combination of the theorems and lemmas form the previous sections, in particular the convergence results for Runge–Kutta or BDF methods: theorem A.5.2, A.5.3 or A.5.4, respectively, together with the residual bound theorem A.5.1, and the errors for the Ritz map and for its material derivatives, lemma A.3.14.

The second part is estimated again by the error estimates for the Ritz map, lemma A.3.14.

The h and τ independency holds, since all our previous results are shown to be independent of these quantities, therefore this property is preserved. The constant τ_0 depends on the ALE velocity, see lemma A.3.5. ■

A.6. Numerical experiments

We present numerical experiments for an evolving surface parabolic problem discretized by the original and the ALE evolving surface finite elements coupled with various time discretizations. The fully discrete methods were implemented in Matlab and DUNE [22], while the initial triangulations were generated using distmesh from [71].

The ESFEM and the ALE ESFEM case were integrated by identical codes, except the involvement of the nonsymmetric B matrix and the evolution of the surface. The ODE system giving the surface movement (see (A.45) below) was solved by the exact same time discretization method as the PDE problem itself (with the same step size), while in one experiment the ALE map is given (see (A.46)). To illustrate our theoretical results we choose two problems which were intensively investigated in the literature before, see [37, 63, 43] and [6]. Specially for ALE experiments see [42] and [43]. For all experiments the material velocity equals the normal velocity.

Observed order of convergence: With the aid of the first experiment we will present experimental order of convergences (EOCs). We choose a problem which was presented before in, e.g. [37].

Namely the surface is given by

$$\Gamma(t) = \{x \in \mathbb{R}^3 \mid a(t)^{-1}x_1^2 + x_2^2 + x_3^2 - 1 = 0\},$$

where $a(t) = 1 + 0.25\sin(2\pi t)$. The problem is considered over the time interval $[0, 1]$. The right-hand side f was computed as to have $u(x,t) = e^{-6t}x_1x_2$ as the true solution of the problem (A.1).

The normal velocity is given by the above distance function, cf. [31, section 2]. The ALE velocity is chosen to be

$$w_1(x,t) = \frac{0.25\pi\cos(2\pi t)}{1 + 0.25\sin(2\pi t)}x_1, \quad w_2(x,t) = 0, \quad w_3(x,t) = 0.$$

Discretization in space is always done with ALE ESFEM. Discretization in time is done with BDF 1 and BDF 3. For $k = 1, \dots, n$ let $(\mathcal{T}_k(t))$ and (τ_k) be a series of triangulations and time steps. In general we choose $2h_k \approx h_{k-1}$. For BDF 1 we choose $4\tau_k = \tau_{k-1}$ with initial step $\tau_1 = 0.1$ and for BDF 3 we choose $\sqrt[3]{4}\tau_k = \tau_{k-1}$ with initial step $\tau_1 = 0.01$. By e_k we denote the error corresponding to the mesh $\mathcal{T}_k(t)$ and stepsize τ_k. Then the EOCs are given as

$$EOC_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \quad (k = 2, 3, \dots, n).$$

In table A.1 and table A.2 we report on the EOCs, for the ALE ESFEM with backward Euler method (BDF 1) and BDF 3, respectively, corresponding to the norm and seminorm

$$L^\infty(L^2): \qquad \max_{1\le n\le N} \|u_h^n - u(\,\cdot\,, t_n)\|_{L^2(\Gamma(t_n))},$$

$$L^2(H^1): \qquad \Big(\tau\sum_{n=1}^{N} \|\nabla_{\Gamma(t_n)}(u_h^n - u(\,\cdot\,, t_n))\|_{L^2(\Gamma(t_n))}\Big)^{1/2}.$$

The results for BDF 1 have already independently been reported in [43]. The non-ALE data for the same example can be found in [37, 66].

Comparison of ALE and non-ALE methods: We consider the evolving surface parabolic PDE (A.1) over the closed surface $\Gamma(t)$ given by

level	dof	$L^\infty(L^2)$	EOCs	$L^2(H^1)$	EOCs
1	126	0.02455766	–	0.05203599	–
2	516	0.00753037	1.7053	0.01689990	1.6224
3	2070	0.00201268	1.9036	0.00583376	1.5345
4	8208	0.00051164	1.9759	0.00282697	1.0451
4	32682	0.00012858	1.9923	0.00141542	0.9980

Table A.1.: Errors and EOCs for BDF 1 in the $L^\infty(L^2)$ and $L^2(H^1)$ norms for the ALE case

level	dof	$L^\infty(L^2)$	EOCs	$L^2(H^1)$	EOCs
1	126	0.00917003	–	0.02266929	–
2	516	0.00246862	1.8932	0.00977487	1.2136
3	2070	0.00061587	2.0030	0.00442116	1.1447
4	8208	0.00015516	1.9889	0.00210023	1.0739
5	32682	0.00003929	1.9815	0.00098204	1.0967

Table A.2.: Errors and EOCs for BDF 3 in the $L^\infty(L^2)$ and $L^2(H^1)$ norms for the ALE case

the zero level set of the distance function

$$d(x,t) \coloneqq x_1^2 + x_2^2 + K(t)^2 G\Big(\frac{x_3^2}{L(t)^2}\Big) - K(t)^2,$$

i.e.,

$$\Gamma(t) \coloneqq \{x \in \mathbb{R}^3 \mid d(x,t) = 0\}.$$

Here the functions G, L and K are given as

$$\begin{aligned} G(s) &= 200s\Big(s - \frac{199}{200}\Big), \\ L(t) &= 1 + 0.2\sin(4\pi t), \\ K(t) &= 0.1 + 0.05\sin(2\pi t). \end{aligned}$$

The velocity v is the normal velocity of the surface defined by the differential equation (formulated for the nodes):

$$\frac{\mathrm{d}}{\mathrm{d}t} a_j = V_j \mathrm{n}_j, \tag{A.45}$$

where,

$$V_j = \frac{-\partial_t d(a_j, t)}{|\nabla d(a_j, t)|}, \quad \mathrm{n}_j = \frac{\nabla d(a_j, t)}{|\nabla d(a_j, t)|}.$$

The right-hand side f is chosen as to have the function $u(x,t) = e^{-6t} x_1 x_2$ to be the true solution of (A.1).

Finally we give the applied ALE movement (from [42] and [43]):

$$\begin{aligned} &(a_i(t))_1 = (a_0(t))_1 \frac{K(t)}{K(0)}, \quad (a_i(t))_2 = (a_0(t))_2 \frac{K(t)}{K(0)}, \\ &(a_i(t))_3 = (a_0(t))_3 \frac{L(t)}{L(0)}, \end{aligned} \tag{A.46}$$

hence $d(a_i(t), t) = 0$ for every $t \in [0, T]$, for $i = 1, 2, \ldots, N$.

The discrete surfaces evolved with normal and ALE velocities shown in Figure A.1, for time $t = 0, 0.2, 0.4, 0.6$. In the following we compare the ALE and non-ALE methods with three spatial refinements, and integrate the evolving surface PDE with various time discretizations, with a time step τ, until $T = 0.6$. We set $e_h(\,\cdot\,, t) := u_h(\,\cdot\,, T) - u(\,\cdot\,, T)$ $(T = n\tau)$. and compute the following norm and seminorm of it

$$|e_h|_M := \|e_h(\,\cdot\,, T)\|_{L^2(\Gamma(T))}, \qquad |e_h|_A := \|\nabla_\Gamma e_h(\,\cdot\,, T)\|_{L^2(\Gamma(T))}.$$

The following plots show the above error norms at time $T = 0.6$ (left M-norm, right A-seminorm) plotted against the time step size τ (on logarithmic scale), different error curves are representing different spatial discretizations.

In the experiments we used three different time discretizations. The convergence in time can be seen (note the reference line). For

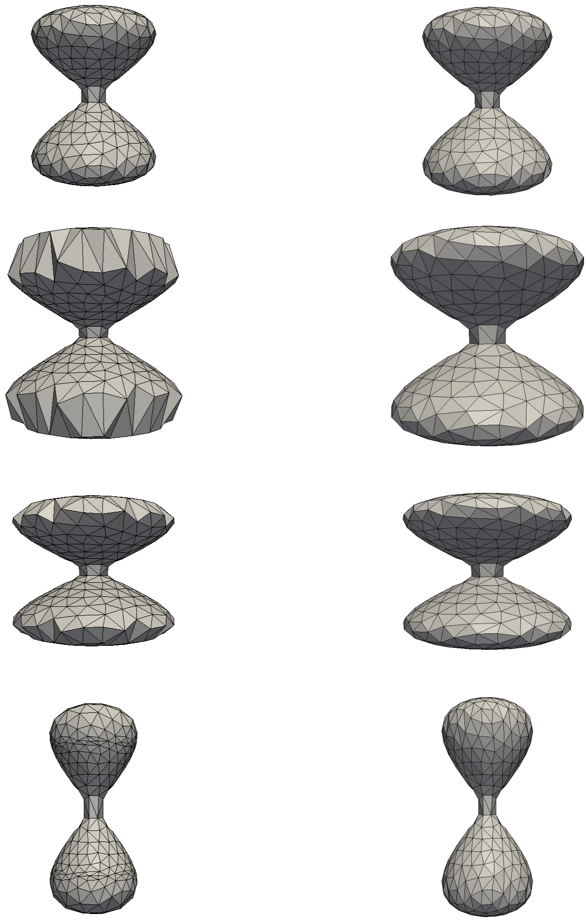


Figure A.1.: Meshes with 376 nodes. Left: normal movement, with Radau IIA method ($s = 3$). Right: ALE movement.

sufficiently small time steps τ the spatial error is dominating, in agreement with the theoretical results. The figures show that the errors in the ALE ESFEM are significantly smaller than for the non-ALE case.

Figure A.2 and A.3 show the errors obtained by the backward Euler method coupled with the two different spatial discretizations.

The following plots, Figure A.4 and A.5, show the same norms but they are made by the five order Radau IIA method ($s = 3$) as a time integrator. The last two figures, Figure A.6 and A.7, show the results obtained by the three step BDF method.

In the case of BDF methods with non-ALE ESFEM, for bigger values of τ, the surface itself (but not the PDE) is evolved with smaller time steps, due to difficulties within the time integration of the surface.

Acknowledgement

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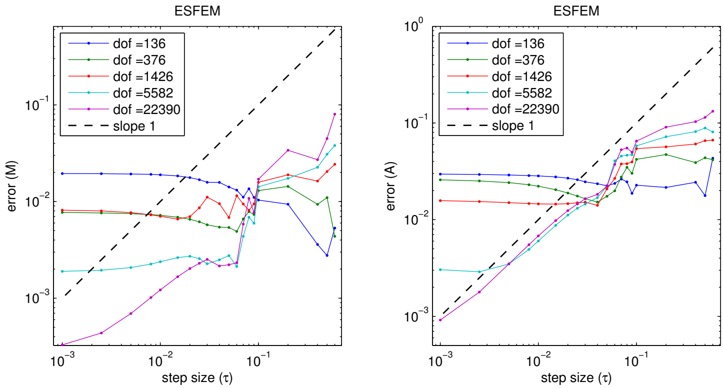

Figure A.2.: Errors of the ESFEM and the implicit Euler method

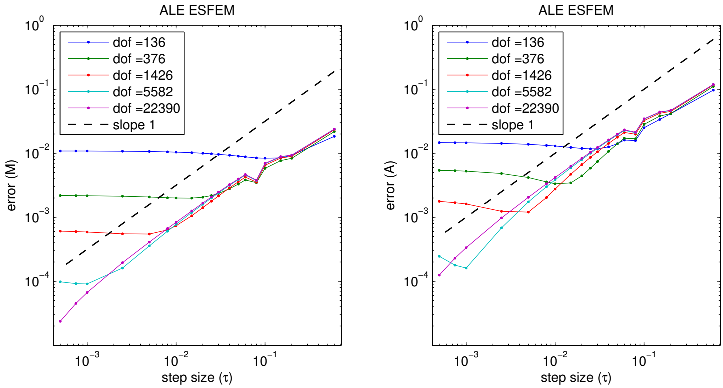

Figure A.3.: Errors of the ALE ESFEM and the implicit Euler method

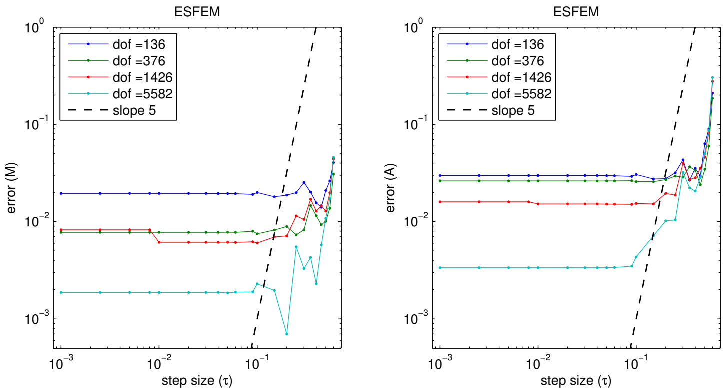

Figure A.4.: Errors of the ESFEM and the three stage Radau IIA method

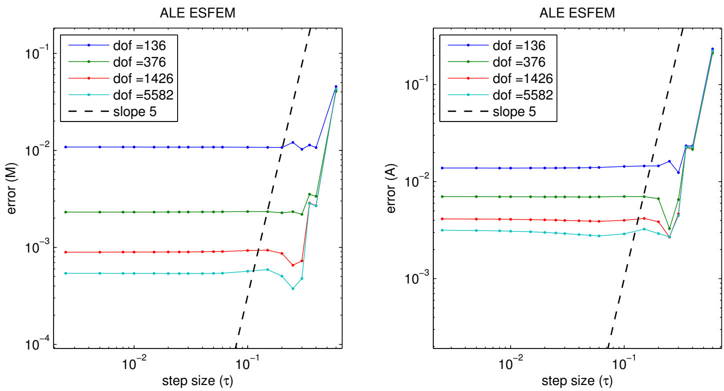

Figure A.5.: Errors of the ALE ESFEM and the three stage Radau IIA method

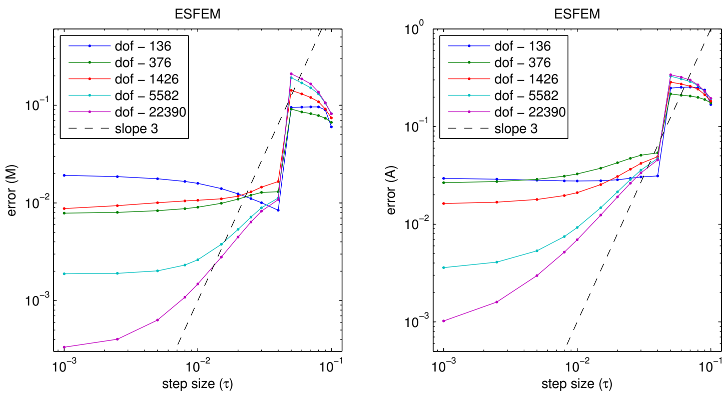

Figure A.6.: Errors of the ESFEM and the BDF3 method

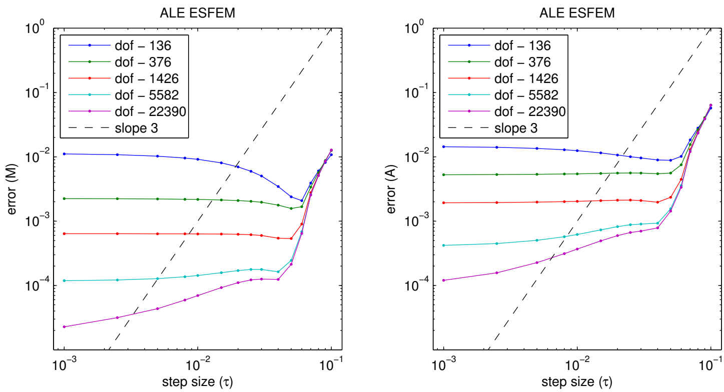

Figure A.7.: Errors of the ALE ESFEM and the BDF3 method

Appendix B.

Error analysis for full discretizations of quasilinear parabolic problems on evolving surfaces

The content of this chapter is published as [56].

Abstract

Convergence results are shown for full discretizations of quasilinear parabolic partial differential equations on evolving surfaces. As a semidiscretization in space the evolving surface finite element method is considered, using a regularity result of a generalized Ritz map, optimal order error estimates for the spatial discretization is shown. Combining this with the stability results for Runge-Kutta and BDF time integrators, we obtain convergence results for the fully discrete problems.

B.1. Introduction

In this paper we show convergence of full discretizations of quasilinear parabolic partial differential equations on evolving surfaces. As a spatial discretization we consider the evolving surface finite element method. The resulting system of ordinary differential equations is discretized, either with an algebraically stable Runge-Kutta method, or with an implicit or linearly implicit backward differentiation formulae.

To our knowledge [40] is the only work on error analysis for nonlinear problems on evolving surfaces. They give semidiscrete error bounds for the Cahn-Hilliard equation. The authors are not aware of fully discrete error estimates published in the literature.

We show convergence results for full discretizations of quasilinear parabolic problems on evolving surfaces with prescribed velocity. We prove unconditional stability and higher-order convergence results for Runge-Kutta and BDF methods. We show convergence as a full discretization when coupled with the ESFEM method as a space discretization for quasilinear problems. Similarly to the linear case the stability analysis relies on energy estimates and multiplier techniques.

First, we generalize some geometric perturbation estimates to the quasilinear setting. We define a *generalized Ritz map* for quasilinear operators, and use it to show optimal order error estimates for the spatial discretization. For deriving the optimal order L^2-error bounds of the Ritz map we will use a similar argument as Wheeler in [78], and elliptic regularity for evolving surfaces. A further important point of the analysis is the required *regularity* of the generalized Ritz map. This will be used together with the assumed Lipschitz-type estimate for the nonlinearity, analogously as in [26, 64, 2].

We show stability and convergence results for the case of stiffly accurate algebraically stable implicit Runge-Kutta methods (having the Radau IIA methods in mind), and for an implicit and linearly implicit k-step backward differentiation formulae up to order five. These results are relying on the techniques used in [64, 37] and [2, 63]. By combining the results for the spatial semidiscretization with stability and convergence estimates we show high-order convergence bounds for the fully discrete approximation.

A starting point of the finite element approximation to (elliptic) surface partial differential equations is the paper of Dziuk [29]. Various convergence results for space discretizations of linear parabolic problems using the evolving surface finite element method (ESFEM) were shown in [31, 35], a fully discrete scheme was analysed in [33]. These results are surveyed in [34].

The convergence analysis of full discretizations with higher-order time integrators within the ESFEM setting for linear problems were shown: for algebraically stable Runge-Kutta methods in [37]; for backward differentiation formulae (BDF) in [63]. The ESFEM approach and convergence results were later extended to wave equations on evolving surfaces, see [62].

A unified presentation of ESFEM and time discretizations for parabolic problems and wave equations can be found in [66].

A great number of real-life phenomena are modeled by nonlinear parabolic problems on evolving surfaces. Apart from general quasilinear problems on moving surfaces, see e.g. example 3.5 in [32], more specific applications are the nonlinear models: diffusion induced grain boundary motion [16, 44, 50, 21, 42]; Allen-Cahn and Cahn-Hilliard equations on evolving surfaces [15, 39, 40, 41, 19]; modeling solid tumor growth [17, 42]; pattern formation modeled by reaction-diffusion equations [60, 65]; image processing [52]; Ginzburg-Landau model for superconductivity [28].

A number of nonlinear problems, in a general setting, were collected by Dziuk and Elliott in [31, 32, 34], also see the references therein. A great number of nonlinear problems with numerical experiments were presented in the literature, see e.g. the above references, in particular [31, 32, 42, 21].

The paper is organized in the following way: In section B.2 we formulate our problem and detail our assumptions. In section B.3 we recall the evolving surface finite element method, together with some of its important properties and estimates. We introduce the generalized Ritz map, and show optimal order error estimates for the residual, using the crucial $W^{1,\infty}$ regularity estimate mentioned above. section B.4 covers the stability results and error estimates for Runge-Kutta and for implicit and linearly implicit BDF methods. section B.5 is devoted to the error bounds of the semidiscrete residual, which then leads to error estimates for the fully discretized problem. In section B.6 we briefly discuss how our results can be extended to semilinear problems, and to the case where the upper and lower bounds of the elliptic part are depending on the norm of the solution. Numerical results are presented in section B.7 to illustrate our theoretical results.

B.2. The problem and assumptions

Let us consider a smooth evolving compact hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$ ($m \leq 2$), $0 \leq t \leq T$, which moves with a given smooth velocity v. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of the function u, where ∇_Γ is the tangential gradient given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu\nu$, with unit normal ν. We are sharing the setting of [31, 35]. We consider the following quasilinear problem for $u = u(x,t)$:

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \nabla_{\Gamma(t)} \cdot \Big(\mathcal{A}(u)\nabla_{\Gamma(t)} u\Big) &= f, \quad \text{on } \Gamma(t), \\ u(\,\cdot\,,0) &= u_0, \quad \text{on } \Gamma(0), \end{aligned} \tag{B.1}$$

where $\mathcal{A}\colon \mathbb{R} \to \mathbb{R}$ is sufficiently smooth function. For simplicity we set $f = 0$, but all our results hold with an non-vanishing f as well.

Remark B.2.1. The results of the paper can be generalized to the case of a sufficiently smooth matrix valued diffusion coefficient $\mathcal{A}(x,t,u) : T_x\Gamma(t) \to T_x\Gamma(t)$. The proofs are similar to the ones presented here, except they are more technical and lengthy, therefore they are not presented here.

The abstract setting of this quasilinear evolving surface PDE is a suitable combination of [64, section 1] and [3, section 2.3]: Let $H(t)$ and $V(t)$ be real and separable Hilbert spaces (with norms $\|\cdot\|_{H(t)}$, $\|\cdot\|_{V(t)}$, respectively) such that $V(t)$ is densely and continuously embedded into $H(t)$, and the norm of the dual space of $V(t)$ is denoted by $\|\cdot\|_{V(t)'}$. The dual space of $H(t)$ is identified with itself, and the duality $\langle\cdot\,,\cdot\rangle_t$ between $V(t)'$ and $V(t)$ coincides on $H(t) \times V(t)$ with the scalar product of $H(t)$, for all $t \in [0,T]$.

The problem casts the following nonlinear operator:

$$\langle A\,(u)v, w\rangle_t = \int_{\Gamma(t)} \mathcal{A}(u)\nabla_\Gamma v \cdot \nabla_\Gamma w.$$

We assume that A satisfies the following three conditions:
The bilinear form associated to the operator $A\,(u)\colon V(t) \to V(t)'$ is *elliptic* with $\boldsymbol{m} > 0$

$$\langle A\,(u)w, w\rangle_t \geq \boldsymbol{m}\,\|w\|^2_{V(t)}, \quad (w \in V(t)), \tag{B.2}$$

uniformly in $u \in V(t)$ and for all $t \in [0,T]$. It is *bounded* with $\boldsymbol{M} > 0$

$$|\langle A\,(u)v, w\rangle_t| \leq \boldsymbol{M}\,\|v\|_{V(t)}\,\|w\|_{V(t)}, \quad (v, w \in V(t)), \tag{B.3}$$

uniformly in $u \in V(t)$ and for all $t \in [0,T]$. We further assume that there is a subset $\mathcal{S}(t) \subset V(t)$ such that the following *Lipschitz-type* estimate holds: for every $\delta > 0$ there exists $L = L(\delta, (\mathcal{S}(t))_{0\le t\le T})$ such that

$$\|(A(w_1) - A(w_2))u\|_{V(t)'} \le \delta \|w_1 - w_2\|_{V(t)} + L \|w_1 - w_2\|_{H(t)}, \tag{B.4}$$

for $u \in \mathcal{S}(t)$, $w_1, w_2 \in V(t)$, $0 \le t \le T$.

The above conditions were also used to prove error estimates using energy techniques in [77] and in [64, 26], or more recently in [2].

The weak formulation uses Sobolev spaces on surfaces: For a sufficiently smooth surface Γ we define

$$H^1(\Gamma) = \{\eta \in L^2(\Gamma) \mid \nabla_\Gamma \eta \in L^2(\Gamma)^{m+1}\},$$

and analogously $H^k(\Gamma)$ for $k \in \mathbb{N}$ and $W^{k,p}(\Gamma)$ for $k \in \mathbb{N}, p \in [1, \infty]$, cf. [31, section 2.1]. Finally, $\mathcal{G}_T = \cup_{t\in[0,T]}\Gamma(t) \times \{t\}$ denotes the space-time manifold.

The weak problem corresponding to (B.1) can be formulated by choosing the setting: $V(t) = H^1(\Gamma(t))$ and $H(t) = L^2(\Gamma(t))$, and the operator:

$$\langle A(u)v, w\rangle_t = \int_{\Gamma(t)} \mathcal{A}(u)\nabla_\Gamma v \cdot \nabla_\Gamma w.$$

Assumption B.2.2. The coefficient function $\mathcal{A}\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

(a) It is bounded, and Lipschitz-bounded with constant ℓ.

(b) The function $\mathcal{A}(s) \ge \boldsymbol{m} > 0$ for arbitrary $s \in \mathbb{R}$.

Throughout the paper we use the following subspace of $V(t)$, for $r > 0$,

$$\mathcal{S}(t) \coloneqq \mathcal{S}(t, r) \coloneqq \{u \in H^2(\Gamma(t)) \mid \|u\|_{W^{2,\infty}(\Gamma(t))} \le r\},$$

i.e. $W^{2,\infty}(\Gamma(t))$ functions with norm less then r.

Then the following proposition easily follows.

Proposition B.2.3. *Under Assumption B.2.2 and $u \in \mathcal{S}(t)$ $(0 \le t \le T)$ the above operator A satisfies the conditions* (B.2), (B.3) *and* (B.4) *(with $\delta = 0$), they possibly depend on $\mathcal{S}(t, r)$.*

Proof. The first two conditions (B.2) and (B.3) follow from (a) and (b). Condition (B.4) holds, since for $u \in \mathcal{S}(t)$, $w_1, w_2 \in H^1(\Gamma(t))$ and any $z \in H^1(\Gamma(t))$, we have

$$\left|\Big\langle (A\,(w_1) - A\,(w_2))u, z\Big\rangle_t\right| = \left|\int_{\Gamma(t)} \Big(\mathcal{A}(w_1) - \mathcal{A}(w_2)\Big)\nabla_\Gamma u \cdot \nabla_\Gamma z\right|$$
$$\le c\ell\, \|w_1 - w_2\|_{L^2(\Gamma(t))} r\, \|z\|_{H^1(\Gamma(t))},$$

hence $L = c\ell r$, where the constant ℓ is from Assumption B.2.2 (a). ■

Definition B.2.4 (Weak form). A function $u \in H^1(\mathcal{G}_T)$ is called a *weak solution* of (B.1), if for almost every $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{A}(u)\nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi \qquad \text{(B.5)}$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(\,\cdot\,, 0) = u_0$.

B.3. Spatial semidiscretization: evolving surface finite elements

As a spatial semidiscretization we use the evolving surface finite element method introduced by Dziuk and Elliott in [31]. We shortly recall some basic notations and definitions from [31], for more details the reader is referred to Dziuk and Elliott [29, 35, 34].

B.3.1. Basic notations

The smooth surface $\Gamma(t)$ is approximated by a triangulated one denoted by $\Gamma_h(t)$, whose vertices $a_i(t)$, $i = 1, 2, \ldots, N$, are sitting on the surface, given as

$$\Gamma_h(t) = \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ are forming an admissible triangulation $\mathcal{T}_h(t)$, with h denoting the maximum diameter. Admissible triangulations were introduced in [31, section 5.1]: $\Gamma(t)$ is a uniform triangulation, i.e. every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by ch with $c > 0$, and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$. Then the discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h}\phi := \nabla\phi - \nabla\phi \cdot \mathrm{n}_h \mathrm{n}_h,$$

understood in a piecewise sense, with n_h denoting the normal to $\Gamma_h(t)$ (see [31]).

For every $t \in [0, T]$ we define the finite element subspace $S_h(t)$ spanned by the continuous, piecewise linear evolving basis functions χ_j, satisfying $\chi_j\left((a_i(t), t\right) = \delta_{ij}$ for all $i, j = 1, 2, \ldots, N$, therefore

$$S_h(t) = \mathrm{span}\{\chi_1(\,\cdot\,, t), \chi_2(\,\cdot\,, t), \ldots, \chi_N(\,\cdot\,, t)\}.$$

We interpolate the surface velocity on the discrete surface using the basis functions and denote it with V_h. Then the discrete material derivative is given by

$$\partial_h^\bullet \phi_h = \partial_t \phi_h + V_h \cdot \nabla\phi_h, \quad (\phi_h \in S_h(t)).$$

The key *transport property* derived in [31, proposition 5.4], is the following

$$\partial_h^\bullet \chi_k = 0, \quad \text{for } k = 1, 2, \ldots, N. \tag{B.6}$$

The spatially discrete quasilinear problem for evolving surfaces is formulated in

Problem B.3.1 (Semidiscretization in space). Find $U_h \in S_h(t)$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} U_h\phi_h + \int_{\Gamma_h(t)} \mathcal{A}(U_h)\nabla_{\Gamma_h}U_h \cdot \nabla_{\Gamma_h}\phi_h \\ = \int_{\Gamma_h(t)} U_h\partial_h^\bullet\phi_h, \quad (\forall\phi_h \in S_h(t)),\end{aligned} \tag{B.7}$$

with the initial condition $U_h(\,\cdot\,,0) = U_h^0 \in S_h(0)$ being a suitable approximation to u_0.

We postpone existence and uniqueness of (B.7) to the next subsection.

B.3.2. The ODE system

The ODE form of the above problem can be derived by setting

$$U_h(\,\cdot\,,t) = \sum_{j=1}^{N}\alpha_j(t)\chi(\,\cdot\,,t)$$

into (B.7), testing with $\phi_h = \chi_j$ and using the transport property (B.6).

Proposition B.3.2 (quasilinear ODE system). *The spatially semi-discrete problem* (B.7) *is equivalent to the following nonlinear ODE system for the vector $\alpha(t) = (\alpha_j(t)) \in \mathbb{R}^N$, collecting the nodal values of $U_h(\,\cdot\,,t)$:*

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\Big(M(t)\alpha(t)\Big) + A\,(\alpha(t))\alpha(t) = 0, \\ \alpha(0) = \alpha_0,\end{cases} \tag{B.8}$$

where the evolving mass matrix $M(t)$ and a nonlinear stiffness matrix

$A(\alpha(t))$ are defined as

$$M(t)_{kj} = \int_{\Gamma_h(t)} \chi_j \chi_k,$$
$$A(\alpha(t))_{kj} = \int_{\Gamma_h(t)} \mathcal{A}(U_h) \nabla_{\Gamma_h} \chi_j \cdot \nabla_{\Gamma_h} \chi_k,$$

for $\alpha(t)$ defining $U_h = \sum_{j=1}^N \alpha_j(t) \chi_j(\cdot, t)$.

The proof of this proposition is analogous to the corresponding one in [37].

Existence and uniqueness of (B.8) and hence of (B.7) can be shown as follows. Since $\mathcal{A}(u)$ is Lipschitz continuous in u, we deduce that $A(\alpha)\alpha$ is Lipschitz continuous is α. Then since $M(t)$ is invertible, the existence and uniqueness of $\alpha(t)$ for the system (B.8) follows from the Picard-Lindelöf theorem.

Time discretizations

We briefly introduce the time discretizations applied to the above ODE system (B.8). However, more details can be found in section B.4.

We use algebraically stable s-stage implicit Runge-Kutta methods, defined by its Butcher tableau, with step size $\tau > 0$:

$$\begin{aligned} M_{ni}\alpha_{ni} &= M_n\alpha_n + \tau \sum_{j=1}^{s} a_{ij}\dot{\alpha}_{nj}, \qquad &&\text{for} \quad i = 1, 2, \dots, s, \\ M_{n+1}\alpha_{n+1} &= M_n\alpha_n + \tau \sum_{i=1}^{s} b_i\dot{\alpha}_{ni}, \\ 0 &= \dot{\alpha}_{ni} + A(\alpha_{ni})\alpha_{ni} &&\text{for} \quad i = 1, 2, \dots, s, \end{aligned}$$

with $M_{ni} := M(t_n + c_i\tau)$ and $M_{n+1} := M(t_{n+1})$.

We also use k-step BDF methods with step size $\tau > 0$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A\,(\alpha_n)\alpha_n = 0, \quad (n \geq k),$$

where the coefficients of the method are given by

$$\delta(\zeta) = \sum_{\ell=1}^{k}\frac{1}{\ell}(1-\zeta)^{\ell}.$$

Similarly, we also consider their linearly implicit modification, using the polynomial $\gamma(\zeta) = \zeta^k - (\zeta-1)^{k-1}$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A\left(\sum_{j=1}^{k}\gamma_j\alpha_{n-j}\right)\alpha_n = 0, \quad (n \geq k).$$

B.3.3. Discrete Sobolev norm estimates

Through the paper we will work with the norm and semi-norm introduced in [37]. We denote these discrete Sobolev-type norms as

$$|z(t)|_{M(t)} := \|Z_h\|_{L^2(\Gamma_h(t))}, \qquad |z(t)|_{\boldsymbol{A}(t)} := \|\nabla_{\Gamma_h} Z_h\|_{L^2(\Gamma_h(t))}, \quad \text{(B.10)}$$

for arbitrary $z(t) \in \mathbb{R}^N$, where $Z_h(\,\cdot\,,t) = \sum_{j=1}^{N} z_j(t)\chi_j(\,\cdot\,,t)$, further by $M(t)$ we mean the above mass matrix and by $\boldsymbol{A}(t)$ we mean the linear (but time dependent) stiffness matrix:

$$\boldsymbol{A}(t)_{kj} := \int_{\Gamma_h(t)} \nabla_{\Gamma_h}\chi_j \cdot \nabla_{\Gamma_h}\chi_k.$$

A very important lemma in our analysis is the following:

Lemma B.3.3 ([37, lemma 4.1]). *There are constants μ, κ (independent of h) such that*

$$z^T(M(s) - M(t))y \leq (e^{\mu(s-t)} - 1)\,|z|_{M(t)}\,|y|_{M(t)},$$
$$z^T(M^{-1}(s) - M^{-1}(t))y \leq (e^{\mu(s-t)} - 1)\,|z|_{M^{-1}(t)}\,|y|_{M^{-1}(t)},$$

$$z^T(\boldsymbol{A}(s) - \boldsymbol{A}(t))y \le (e^{\kappa(s-t)} - 1)\,|z|_{\boldsymbol{A}(t)}\,|y|_{\boldsymbol{A}(t)}$$

for all $y, z \in \mathbb{R}^N$ and $s, t \in [0, T]$.

B.3.4. Lifting process and approximation results

In the following we recall the so called *lift operator*, which was introduced in [29] and further investigated in [31, 35]. The lift operator projects a finite element function on the discrete surface onto a function on the smooth surface.

that $\partial U(t) = \Gamma(t)$. The *oriented distance function* d_X is [46] in lemma 14.16 have shown the following important regularity result about d_X.

Using the *oriented distance function* d_X ([31, section 2.1]), for a continuous function $\phi_h \colon \Gamma_h(t) \to \mathbb{R}$ its lift is define as

$$\eta_h^l(p, t) := \phi_h(x, t), \qquad x \in \Gamma(t),$$

where for every $x \in \Gamma_h(t)$ the value $p = p(x, t) \in \Gamma(t)$ is uniquely defined via $x = p + \mathrm{n}(p, t) d_X(x, t)$. By η^{-l} we mean the function whose lift is η.

We now recall some notions using the lifting process from [29, 31] and [66]. We have the lifted finite element space

$$S_h^l(t) := \{\varphi_h = \phi_h^l \mid \phi_h \in S_h(t)\}.$$

By δ_h we denote the quotient between the continuous and discrete surface measures, $\mathrm{d}A$ and $\mathrm{d}A_h$, defined as $\delta_h \,\mathrm{d}A_h = \mathrm{d}A$. Further, we recall that

$$\mathrm{pr} := (\delta_{ij} - \nu_i \nu_j)_{i,j=1}^{m+1} \quad \text{and} \quad \mathrm{pr}_h := (\delta_{ij} - \nu_{h,i}\nu_{h,j})_{i,j=1}^{m+1}$$

are the projections onto the tangent spaces of Γ and Γ_h. Further, from [35], we recall the notation

$$Q_h = \frac{1}{\delta_h}(I - d\mathcal{H})\,\mathrm{pr}\,\mathrm{pr}_h\,\mathrm{pr}(I - d\mathcal{H}),$$

where $\mathcal{H}$ ($\mathcal{H}_{ij} = \frac{\partial \mathrm{n}_i}{\partial x_j}$) is the (extended) Weingarten map. For these quantities we recall some results from [31, lemma 5.1], [35, lemma 5.4] and [66, lemma 6.1].

Lemma B.3.4. *Assume that $\Gamma_h(t)$ and $\Gamma(t)$ is from the above setting, then we have the estimates:*

$$\begin{aligned} \|d\|_{L^\infty(\Gamma_h(t))} &\leq ch^2, & \|\nu_j\|_{L^\infty(\Gamma_h(t))} &\leq ch, \\ \|1-\delta_h\|_{L^\infty(\Gamma_h(t))} &\leq ch^2, & \|\partial_h^\bullet d\|_{L^\infty(\Gamma_h(t))} &\leq ch^2, \\ \|\mathrm{pr} - Q_h\|_{L^\infty(\Gamma_h(t))} &\leq ch^2, & \|\mathrm{pr}(\partial_h^\bullet Q_h)\,\mathrm{pr}\|_{L^\infty(\Gamma_h(t))} &\leq ch^2, \end{aligned}$$

with constants depending on $\mathcal{G}_T$, but not on t.

Lemma B.3.5. *For $1 \leq p \leq \infty$ there exists constants $c_1, c_2 > 0$ independent of t and h such that the for all $u_h \in W^{1,p}(\Gamma_h(t))$ it holds that $u_h^l \in W^{1,p}(\Gamma(t))$ with the estimates*

$$c_1 \|u_h\|_{W^{1,p}(\Gamma_h(t))} \leq \|u_h^l\|_{W^{1,p}(\Gamma(t))} \leq c_2 \|u_h\|_{W^{1,p}(\Gamma_h(t))}.$$

Proof. The proofs follows easily from the relation

$$\nabla_{\Gamma_h} u_h = \mathrm{pr}_h (I - d\mathcal{H}) \nabla_\Gamma u_h^l,$$

cf. [29, lemma 3]. ■

B.3.5. Bilinear forms and their estimates

Apart from the ξ dependence, we use the time dependent bilinear forms defined in [35]: for arbitrary $z, \varphi, \xi \in H^1(\Gamma)$, $\xi \in \mathcal{S}(t)$ we set

$$
\begin{aligned}
m(z,\varphi) &= \int_{\Gamma(t)} z\varphi, \\
a(\xi; z,\varphi) &= \int_{\Gamma(t)} \mathcal{A}(\xi)\nabla_\Gamma z \cdot \nabla_\Gamma \varphi, \\
g(v; z,\varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot v) z\varphi, \\
b(\xi; v; z,\varphi) &= \int_{\Gamma(t)} \mathcal{B}(\xi; v)\nabla_\Gamma z \cdot \nabla_\Gamma \varphi,
\end{aligned}
$$

and for their discrete analogs for $Z_h, \phi_h, \xi_h \in S_h$ we set

$$
\begin{aligned}
m_h(Z_h,\phi_h) &= \int_{\Gamma_h(t)} Z_h\phi_h \\
a_h(\xi_h; Z_h,\phi_h) &= \int_{\Gamma_h(t)} \mathcal{A}(\xi_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\
g_h(V_h; Z_h,\phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot V_h) Z_h\phi_h, \\
b_h(\xi_h; V_h; Z_h,\phi_h) &= \int_{\Gamma_h(t)} \mathcal{B}_h(\xi_h; V_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h,
\end{aligned}
$$

where the discrete tangential gradients are understood in a piecewise sense, and with the tensors given as

$$
\begin{aligned}
\mathcal{B}(\xi; v)_{ij} &= \partial^\bullet(\mathcal{A}(\xi)) + \nabla_\Gamma \cdot v\mathcal{A}(\xi) - 2\mathcal{A}(\xi)\mathcal{D}(v), \\
\mathcal{B}_h(\xi_h; V_h)_{ij} &= \partial_h^\bullet(\mathcal{A}(\xi_h)) + \nabla_{\Gamma_h} \cdot V_h\mathcal{A}(\xi_h) - 2\mathcal{A}(\xi_h)\mathcal{D}_h(V_h),
\end{aligned}
$$

for $i, j = 1, 2, \dots, m+1$, with

$$
\begin{aligned}
\mathcal{D}(v)_{ij} &= \frac{1}{2}\big((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i\big), \\
\mathcal{D}_h(V_h)_{ij} &= \frac{1}{2}\big((\nabla_{\Gamma_h})_i (V_h)_j + (\nabla_{\Gamma_h})_j (V_h)_i\big),
\end{aligned}
$$

for $i, j = 1, 2, \dots, m+1$. For more details see [35, lemma 2.1] (and the references in the proof), or [34, lemma 5.2].

We will also use the transport lemma (note that $\partial_h^\bullet z_h = \partial_t z_h + v_h \cdot \nabla z_h$ for a $z_h \in S_h^l(t)$):

Lemma B.3.6. *For arbitrary $\xi_h^l \in S_h^l(t)$ and $z_h,\ \varphi_h,\ \partial_h^\bullet z_h,\ \partial_h^\bullet \varphi_h \in S_h^l(t)$ we have:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m(z_h, \varphi_h) &= m(\partial_h^\bullet z_h, \varphi_h) + m(z_h, \partial_h^\bullet \varphi_h) + g(v_h; z_h, \varphi_h),\\ \frac{\mathrm{d}}{\mathrm{d}t} a(\xi_h^l; z_h, \varphi_h) &= a(\xi_h^l; \partial_h^\bullet z_h, \varphi_h) + a(\xi_h^l; z_h, \partial_h^\bullet \varphi_h)\\ &\quad + b(\xi_h^l; v_h; z_h, \varphi_h),\end{aligned}$$

where v_h is the velocity of the surface, see [35, definition 4.9].

Proof. This lemma can be shown analogously as [35, lemma 4.2], therefore the proof is omitted. ■

Versions of this lemma with continuous material derivatives, or discrete bilinear forms are also true.

Lemma B.3.7 (Geometric perturbation errors). *For any $\xi \in \mathcal{S}(t)$, and $Z_h, \phi_h \in S_h(t)$ with corresponding lifts $z_h, \varphi_h \in S_h^l(t)$ we have the following bounds*

$$\begin{aligned}|m(z_h, \varphi_h) - m_h(Z_h, \phi_h)| &\leq ch^2 \|z_h\|_{L^2} \|\varphi_h\|_{L^2},\\ |a(\xi; z_h, \varphi_h) - a_h(\xi^{-l}; Z_h, \phi_h)| &\leq ch^2 |z_h|_{H^1} |\varphi_h|_{H^1},\\ |g(v_h; z_h, \varphi_h) - g_h(V_h; Z_h, \phi_h)| &\leq ch^2 \|z_h\|_{L^2} \|\varphi_h\|_{L^2},\\ |b(\xi; v_h; z_h, \varphi_h) - b_h(\xi^{-l}; V_h; Z_h, \phi_h)| &\leq ch^2 |z_h|_{H^1} |\varphi_h|_{H^1}.\end{aligned}$$

Proof. The first estimate was proved in [35, lemma 5.5] and the third in [62, lemma 7.5].

The proof of the second estimate is similar to the linear case found in [34, lemma 4.7]. For the convenience of the reader we recap the arguments. Using

$$Q_h = \frac{1}{\delta_h}(I - d\mathcal{H})\,\mathrm{pr}\,\mathrm{pr}_h\,\mathrm{pr}(I - d\mathcal{H})$$

we obtain

$$\mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h = \delta_h \mathcal{A}(\xi^{-l}) Q_h \nabla_\Gamma z_h(p,\,\cdot\,) \cdot \nabla_\Gamma \varphi_h(p,\,\cdot\,). \quad \text{(B.11)}$$

Similarly as in [35, lemma 5.5], the boundedness (proposition B.2.3) and the geometric estimate $\|\mathrm{pr} - Q_h\|_{L^\infty(\Gamma_h)} \leq ch^2$ provides the estimate

$$\begin{aligned}
&|a(\xi; z_h, \varphi_h) - a_h(\xi^{-l}; Z_h, \phi_h)| \\
&\quad = \Big| \int_{\Gamma(t)} \mathcal{A}(\xi)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}A - \int_{\Gamma_h(t)} \mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h \,\mathrm{d}A_h \Big| \\
&\quad = \Big| \int_{\Gamma(t)} \mathcal{A}(\xi)(\,\mathrm{pr} - Q_h)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \,\mathrm{d}A \Big| \\
&\quad \leq \boldsymbol{M} \,\|(\mathrm{pr} - Q_h)\nabla_\Gamma z_h\|_{L^2(\Gamma(t))} \,\|Q_h\|_{L^\infty(\Gamma_h(t))} \,\|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))} \\
&\quad \leq \boldsymbol{M} ch^2 \,\|\nabla_\Gamma z_h\|_{L^2(\Gamma(t))} \,\|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}.
\end{aligned}$$

To prove the fourth estimate we follow [62]: starting with the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} \mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \mathcal{A}(\xi) Q_h^l \nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h$$

then the transport lemma (lemma B.3.6 above) yields

$$\begin{aligned}
&\int_{\Gamma_h(t)} \mathcal{A}(\xi^{-l})\nabla_{\Gamma_h}\partial_h^\bullet Z_h \cdot \nabla_{\Gamma_h}\phi_h + \int_{\Gamma_h(t)} \mathcal{A}^{-l}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\partial_h^\bullet \phi_h \\
&\quad + \int_{\Gamma_h(t)} \mathcal{B}_h(\xi^{-l}; V_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h = \int_{\Gamma(t)} \mathcal{A}(\xi) Q_h^l \nabla_\Gamma \partial_h^\bullet z_h \cdot \nabla_\Gamma \varphi_h \\
&\qquad + \int_{\Gamma(t)} \nabla_\Gamma z_h \cdot \mathcal{A}(\xi) Q_h^l \nabla_\Gamma \partial_h^\bullet \varphi_h + \int_{\Gamma(t)} \mathcal{B}(\xi; v_h) Q_h^l \nabla_\Gamma z_h \\
&\qquad \cdot \nabla_\Gamma \varphi_h + \int_{\Gamma(t)} \mathcal{A}(\xi)\partial_h^\bullet(Q_h^l)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h.
\end{aligned}$$

Therefore using that the lift of $\partial_h^\bullet Z_h$ is $\partial_h^\bullet z_h$, (B.11) and lemma B.3.4 provides

$$\begin{aligned}
&|b_h(\xi^{-l}; V_h; Z_h, \phi_h) - b(\xi; v_h; Z_h, \phi_h)| \\
&= \left| \int_{\Gamma(t)} \mathcal{A}(\xi) \partial_h^\bullet (Q_h^l) \nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \right| \\
&\quad + \left| \int_{\Gamma(t)} \mathcal{B}(\xi; v_h) \, (Q_h^l - I) \nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \right| \\
&\leq ch^2 \, \|\nabla_\Gamma z_h\|_{L^2(\Gamma(t))} \, \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))},
\end{aligned}$$

where the last estimates follow from lemma B.3.4, similarly as in [62, theorem 7.5]. ■

B.3.6. Interpolation estimates

By $I_h \colon H^1(\Gamma(t)) \to S_h^l(t)$ we denote the finite element interpolation operator, having the error estimate below.

> **Lemma B.3.8.** *For $m \leq 3$, there exists a constant $c > 0$ independent of h and t such that for $u \in H^2(\Gamma(t))$:*
>
> $$\|u - I_h u\|_{L^2(\Gamma(t))} + h \, |u - I_h u|_{H^1(\Gamma(t))} \leq ch^2 \, \|u\|_{H^2(\Gamma(t))}$$
>
> *Furthermore, if $u \in W^{2,\infty}(\Gamma(t))$, it also satisfies*
>
> $$|u - I_h u|_{W^{1,\infty}(\Gamma(t))} \leq ch \, \|u\|_{W^{2,\infty}(\Gamma(t))},$$
>
> *where $c > 0$ is also independent of h and t.*

Proof. The first inequality was shown in [29]. The dimension restriction is especially discussed in [34, lemma 4.3].

The analogue of the second estimate for a reference element were shown in [76, theorem 3.1]. Denote by $E_h(t) \subset \Gamma_h(t)$ an arbitrary element and denote by $E(t) \subset \Gamma(t)$ the lift of this triangle.

$$\begin{aligned}\|\nabla_\Gamma(u - I_h u)\|_{L^\infty(E(t))} &\leq c\,\|\nabla_{\Gamma_h}(u^{-l} - I_h u^{-l})\|_{L^\infty(E_h(t))}\\ &\leq c\frac{1}{h}\,\|\nabla_{\mathbb{R}^2}(\hat{u} - I_h\hat{u})\|_{L^\infty(E_0)}\\ &\leq c\frac{1}{h}\,\|\nabla^2_{\mathbb{R}^2}\hat{u}\|_{L^\infty(E_0)}\\ &\leq c\frac{1}{h}h^2\,\|\nabla^2_{\Gamma_h}u^{-l}\|_{L^\infty(E_h(t))}\\ &\leq ch\,\|u\|_{W^{2,\infty}(E(t))},\end{aligned}$$

where $E_0 \subset \mathbb{R}^2$ is the standard unit simplex, $\hat{u}\colon E_0 \to \mathbb{R}$ is the representation of $u^{-l}|_{E_h(t)}$ on E_0 w.r.t. a suitable affine linear transformation and $\nabla^2_{\mathbb{R}^2}\hat{u}$ denote the usual Hessian of $\hat{u}$. For the first and the last inequality we have used, that the discrete and continuous norms are equivalent. The intermediate steps uses the uniformity of the triangulation together with standard estimates for the pullback, cf. [14] or [4, section 10.3]. ∎

B.3.7. The Ritz map for nonlinear problems on evolving surfaces

Ritz maps for quasilinear PDEs on stationary domains were investigated by Wheeler in [78]. We generalize this idea for the case of quasilinear evolving surface PDEs. We define a generalized Ritz map for quasilinear elliptic operators, for the linear case see [62].

By combining the above definitions we set the following.

Definition B.3.9 (Ritz map). For a given $z \in H^1(\Gamma(t))$ and a given function $\xi\colon \Gamma(t) \to \mathbb{R}$ there is a unique $\widetilde{\mathcal{P}}_h z \in S_h(t)$ such that for all $\phi_h \in S_h(t)$, with the corresponding lift $\varphi_h = \phi_h^l$, we have

$$a_h^*(\xi^{-l}; \widetilde{\mathcal{P}}_h z, \phi_h) = a^*(\xi; z, \varphi_h), \tag{B.12}$$

where

$$a^*(\xi; z, \varphi_h) := a(\xi; z, \varphi_h) + m(z, \varphi_h)$$

and

$$a_h^*(\xi^{-l}; z^{-l}, \phi_h) := a_h(\xi^{-l}; z^{-l}, \phi_h) + m_h(z^{-l}, \phi_h),$$

to make the forms $a^*(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ and $a_h^*(\,\cdot\,;\,\cdot\,,\,\cdot\,)$ positive definite. Then $\mathcal{P}_h z \in S_h^l(t)$ is defined as the lift of $\widetilde{\mathcal{P}_h} z$, i.e. $\mathcal{P}_h z = (\widetilde{\mathcal{P}_h} z)^l$.

We recall here that by ξ^{-l} we mean a function (living on the discrete surface) whose lift is ξ.

Galerkin orthogonality does not hold in this case, just up to a small defect:

Lemma B.3.10 (pseudo Galerkin orthogonality). *For any given $\xi \in \mathcal{S}(t)$ there holds, that for every $z \in H^1(\Gamma(t))$ and $\varphi_h \in S_h^l(t)$*

$$|a^*(\xi; z - \mathcal{P}_h z, \varphi_h)| \leq ch^2 \, \|\mathcal{P}_h z\|_{H^1(\Gamma(t))} \, \|\varphi_h\|_{H^1(\Gamma(t))}, \qquad \text{(B.13)}$$

where c is independent of ξ, h and t.

Proof. Using the definition of the Ritz map we get

$$\begin{aligned} |a^*(\xi; z - \mathcal{P}_h z, \varphi_h)| &= |a_h^*(\xi^{-l}; \widetilde{\mathcal{P}_h} z, \phi_h) - a^*(\xi; \mathcal{P}_h z, \varphi_h)| \\ &\leq \boldsymbol{M} ch^2 \, \|\mathcal{P}_h z\|_{H^1(\Gamma(t))} \, \|\varphi_h\|_{H^1(\Gamma(t))}, \end{aligned}$$

where we used lemma B.3.7. ■

Error bounds for the Ritz map and for its material derivatives

In this section we prove error estimates for the Ritz map (B.12) and also for its material derivatives, the analogous results for the linear case can be found in [35, section 6], [66, section 7]. The ξ independency of the estimates requires extra care, previous results, e.g. the ones cited above, or [62, section 8], are not applicable.

Theorem B.3.11. *The error in the Ritz map satisfies the bound, for arbitrary $\xi \in \mathcal{S}(t)$ and $0 \le t \le T$ and $h \le h_0$ with sufficiently small h_0,*

$$\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))} + h \|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} \le ch^2 \|z\|_{H^2(\Gamma(t))}.$$

where the constant c is independent of ξ, h and t (but depends on $\boldsymbol{m}$ and $\boldsymbol{M}$).

Proof. (a) We first prove the gradient estimate.

Starting by the ellipticity of the form a and the non-negativity of the form m, then using the estimate (B.13) we have:

$$\begin{aligned}
\boldsymbol{m} \|z - \mathcal{P}_h z\|^2_{H^1(\Gamma(t))} &\le a^*(\xi; z - \mathcal{P}_h z, z - \mathcal{P}_h z) \\
&= a^*(\xi; z - \mathcal{P}_h z, z - I_h z) + a^*(\xi; z - \mathcal{P}_h z, I_h z - \mathcal{P}_h z) \\
&\le \boldsymbol{M} \|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} \|z - I_h z\|_{H^1(\Gamma(t))} \\
&\quad + ch^2 \|\mathcal{P}_h z\|_{H^1(\Gamma(t))} \|I_h z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} \\
&\le \boldsymbol{M} ch \|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} \|z\|_{H^2(\Gamma(t))} \\
&\quad + ch^2 \Big(2 \|z - \mathcal{P}_h z\|^2_{H^1(\Gamma(t))} + \|z\|^2_{H^1(\Gamma(t))} + ch^2 \|z\|^2_{H^2(\Gamma(t))} \Big),
\end{aligned}$$

using the interpolation error, and for the second term we used the estimate

$$\begin{aligned}
\|\mathcal{P}_h z\|_{H^1(\Gamma(t))} \|I_h z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} &\le \Big(\|\mathcal{P}_h z - z\|_{H^1(\Gamma(t))} + \|z\|_{H^1(\Gamma(t))} \Big) \\
&\quad \Big(\|I_h z - z\|_{H^1(\Gamma(t))} + \|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} \Big) \\
&\le 2 \|z - \mathcal{P}_h z\|^2_{H^1(\Gamma(t))} + \|z\|^2_{H^1(\Gamma(t))} + ch^2 \|z\|^2_{H^2(\Gamma(t))}.
\end{aligned}$$

Now using Young's and Cauchy-Schwarz inequality, and for sufficiently small (but ξ independent) h we have the gradient estimate

$$\|z - \mathcal{P}_h z\|^2_{H^1(\Gamma(t))} \le \frac{1}{\boldsymbol{m}} \boldsymbol{M} ch^2 \|z\|_{H^2(\Gamma(t))}.$$

(b) The L^2-estimate follows from the Aubin-Nitsche trick. Let us consider the problem

$$-\nabla_\Gamma \cdot (\mathcal{A}(\xi)\nabla_\Gamma w) + w = z - \mathcal{P}_h z \qquad \text{on} \quad \Gamma(t),$$

then by elliptic theory, cf. theorem B.8.1, we have the estimate, for the solution $w \in H^2(\Gamma(t))$

$$\|w\|_{H^2(\Gamma(t))} \leq c \, \|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))},$$

where c is independent of t and ξ. By testing the elliptic weak problem with $z - \mathcal{P}_h z$ we have

$$\begin{aligned}
\|z - \mathcal{P}_h z\|^2_{L^2(\Gamma(t))} &= a^*(\xi; z - \mathcal{P}_h z, w) \\
&= a^*(\xi; z - \mathcal{P}_h z, w - I_h w) + a^*(\xi; z - \mathcal{P}_h z, I_h w) \\
&\leq \boldsymbol{M} \, \|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))} \, \|w - I_h w\|_{H^1(\Gamma(t))} \\
&\quad + ch^2 \, \|\mathcal{P}_h z\|_{H^1(\Gamma(t))} \, \|I_h w\|_{H^1(\Gamma(t))}.
\end{aligned}$$

Then the estimates of the interpolation error and combination of the above results yields

$$\begin{aligned}
\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))} \frac{1}{c} \, \|w\|_{H^2(\Gamma(t))} &\leq \|z - \mathcal{P}_h z\|^2_{L^2(\Gamma(t))} \\
&\leq \boldsymbol{M} c h^2 \, \|z\|_{H^2(\Gamma(t))} \, \|w\|_{H^2(\Gamma(t))},
\end{aligned}$$

which completes the proof of the first assertion. ■

Lemma B.3.12. *For $k \geq 0$ it holds there exists a constant $c = c(k) > 0$ independent of t and h such that*

$$\|(\partial_h^\bullet)^{(k)}(v - v_h)\|_{L^\infty(\Gamma(t))} + h \, \|\nabla_\Gamma (\partial_h^\bullet)^{(k)}(v - v_h)\|_{L^\infty(\Gamma(t))} \leq ch^2.$$

We need to control higher-order material derivatives of the error $(\partial_h^\bullet)^{(k)}(v - v_h)$, because we want to show error estimates for higher-order material derivatives of our Ritz map.

A proof of this lemma can be found in Mansour [66, lemma 6.3].

Theorem B.3.13. *Assume that $\xi \in \mathcal{S}(t)$ and in addition that for $k \geq 1$ it holds $(\partial_h^\bullet)^{(k)}(\mathcal{A}(\xi)) \in L^\infty(\mathcal{G}_T)$. The error in the material derivatives of the Ritz map satisfies the following bounds, for $0 \leq t \leq T$ and $h \leq h_0$ with sufficiently small h_0,*

$$\|(\partial_h^\bullet)^{(k)}(z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} + h \,\|\nabla_\Gamma (\partial_h^\bullet)^{(k)}(z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} \leq \boldsymbol{M} c_k h^2 \sum_{j=1}^{k} \|(\partial_h^\bullet)^{(j)} z\|_{H^2(\Gamma(t))}.$$

The constant $c_k > 0$ is independent of ξ and h (but depends on α and $\boldsymbol{M}$).

Proof. The proof is a modification of [66, theorem 7.3].

For $k = 1$: (a) We start by taking the time derivative of the definition of the Ritz map (B.12), use the transport properties (lemma B.3.6), and use the definition of the Ritz map once more, we arrive at

$$a^*(\xi; \partial_h^\bullet z, \varphi_h) = -b(\xi; v_h; z, \varphi_h) - g(v_h; z, \varphi_h) + a_h^*(\xi^{-l}; \partial_h^\bullet \widetilde{\mathcal{P}}_h z, \phi_h) + b_h(\xi^{-l}; V_h; \widetilde{\mathcal{P}}_h z, \phi_h) + g_h(V_h; \widetilde{\mathcal{P}}_h z, \phi_h).$$

Then we obtain

$$a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \varphi_h) = -b(\xi; v_h; z - \mathcal{P}_h z, \varphi_h) - g(v_h; z - \mathcal{P}_h z, \varphi_h) + F_1(\varphi_h), \tag{B.14}$$

where

$$F_1(\varphi_h) = \big(a_h^*(\xi^{-l}; \partial_h^\bullet \widetilde{\mathcal{P}}_h z, \phi_h) - a^*(\xi; \partial_h^\bullet \mathcal{P}_h z, \varphi_h)\big) + \big(b_h(\xi^{-l}; V_h; \widetilde{\mathcal{P}}_h z, \phi_h) - b(\xi; v_h; \mathcal{P}_h z, \varphi_h)\big) + \big(g_h(V_h; \widetilde{\mathcal{P}}_h z, \phi_h) - g(v_h; \mathcal{P}_h z, \varphi_h)\big).$$

Using the geometric estimates of lemma B.3.7 F_1 can be estimated as

$$|F_1(\varphi_h)| \leq c\boldsymbol{M} h^2 \big(\|\partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma(t))} + \|\mathcal{P}_h z\|_{H^1(\Gamma(t))}\big) \, \|\varphi_h\|_{H^1(\Gamma(t))}.$$

Then using $\partial_h^\bullet \mathcal{P}_h z$ as a test function in (B.14), and using the error estimates of the Ritz map, together with the estimates above, with $h \leq h_0$ independent of ξ, we have

$$\|\partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma(t))} \leq \boldsymbol{M} c \, \|\partial^\bullet z\|_{H^1(\Gamma(t))} + \boldsymbol{M} ch \, \|z\|_{H^2(\Gamma(t))}.$$

Combining all the previous estimates and using Young's inequality, Cauchy-Schwarz inequality, for sufficiently small (ξ independent) $h \leq h_0$, we obtain

$$\begin{aligned} &a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \varphi_h) \\ &\qquad \leq \boldsymbol{M} ch \left(\|z\|_{H^2(\Gamma(t))} + h \, \|\partial^\bullet z\|_{H^1(\Gamma(t))} \right) \|\varphi_h\|_{H^1(\Gamma(t))}. \end{aligned}$$

Then as in the previous proof we have

$$\begin{aligned} &\boldsymbol{m} \, \|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma(t))}^2 \\ &\quad \leq a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z) \\ &\quad = a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \partial_h^\bullet z - I_h \partial^\bullet z) \\ &\qquad + a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z) \\ &\quad \leq \boldsymbol{M} \, \|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma(t))} \, \|\partial_h^\bullet z - I_h \partial^\bullet z\|_{H^1(\Gamma(t))} \\ &\qquad + \boldsymbol{M} ch \Big(\|z\|_{H^2(\Gamma(t))} + h \, \|\partial^\bullet z\|_{H^1(\Gamma(t))} \Big) \|I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma(t))}. \end{aligned}$$

Then the interpolation estimates, Young's inequality, absorption using $h \leq h_0$, yields the gradient estimate.

(b) The L^2-estimate again follows from the Aubin-Nitsche trick. Let us now consider the problem

$$-\nabla_\Gamma \cdot (\mathcal{A}(\xi) \nabla_\Gamma w) + w = \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \quad \text{on } \Gamma(t),$$

together with the elliptic estimate (cf. theorem B.8.1), for the solution $w \in H^2(\Gamma(t))$

$$\|w\|_{H^2(\Gamma(t))} \leq c \, \|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{L^2(\Gamma(t))},$$

again, c is independent of t and ξ.

Then a similar calculation as [35, theorem 6.2], [66, theorem 7.3] provides the L^2-norm estimate.

For $k > 1$ the proof is analogous. ■

Remark B.3.14. If $\xi \in W^{k,\infty}(\Gamma(t))$ and $\mathcal{A} \in W^{k,\infty}(\mathbb{R})$ then it holds that $(\partial_h^\bullet)^{(k)}\mathcal{A}(\xi) \in L^\infty(\Gamma(t))$. For the convenience of the reader we give a proof for $k = 2$. It holds

$$(\partial_h^\bullet)^{(2)}(\mathcal{A}(\xi)) = \partial_h^\bullet(\mathcal{A}'(\xi)\partial_h^\bullet\xi) = \mathcal{A}''(\xi)(\partial_h^\bullet\xi)^2 + \mathcal{A}'(\xi)(\partial_h^\bullet)^{(2)}\xi.$$

We have the identity

$$\partial_h^\bullet\xi = \partial^\bullet\xi + (v_h - v) \cdot \nabla_\Gamma\xi.$$

For the second derivative we calculate

$$\begin{aligned}(\partial_h^\bullet)^{(2)}\xi = (\partial_h^\bullet)^{(2)}\xi + (v_h - v) \cdot \nabla_\Gamma\partial_h^\bullet\xi + \partial_h^\bullet(v_h - v)\\ \cdot \nabla_\Gamma\xi + (v_h - v) \cdot \partial_h^\bullet\nabla_\Gamma\xi + \nabla_\Gamma^2\xi(v_h - v)^2.\end{aligned}$$

Using lemma B.3.12 the claim follows.

Regularity of the Ritz map

The following technical result will play an important role in showing optimal bounds of the semidiscrete residual.

> **Lemma B.3.15.** *For $m \leq 2$, there exists a constant $c > 0$ independent of h and t such that for a function $u \in W^{2,\infty}(\Gamma(t))$ for all $t \in [0, T]$, the following estimate holds*
>
> $$\|\nabla_\Gamma\mathcal{P}_h u\|_{L^\infty(\Gamma(t))} \leq c\,\|u\|_{W^{2,\infty}(\Gamma(t))}.$$

Proof. Using the triangle inequality we start to estimate as

$$\begin{aligned}\|\nabla_\Gamma\mathcal{P}_h u\|_{L^\infty(\Gamma(t))} \leq{}& \|\nabla_\Gamma(\mathcal{P}_h u - I_h u)\|_{L^\infty(\Gamma(t))}\\ &+ \|\nabla_\Gamma(I_h u - u)\|_{L^\infty(\Gamma(t))} + \|\nabla_\Gamma u\|_{L^\infty(\Gamma(t))}.\end{aligned}$$

The last term is harmless. The second term is estimated using lemma B.3.8. For the first term, using the inverse estimate, error estimates for the Ritz map and for the interpolation operator we obtain

$$\begin{aligned}
\|\nabla_\Gamma(\mathcal{P}_h u - I_h u)\|_{L^\infty(\Gamma(t))} &\leq ch^{-m/2} \|\nabla_\Gamma(\mathcal{P}_h u - I_h u)\|_{L^2(\Gamma(t))} \\
&\leq ch^{-m/2}\Big(\|\nabla_\Gamma(\mathcal{P}_h u - u)\|_{L^2(\Gamma(t))} \\
&\qquad + \|\nabla_\Gamma(u - I_h u)\|_{L^2(\Gamma(t))}\Big) \\
&\leq ch^{-m/2} h \|u\|_{H^2(\Gamma(t))} \\
&\leq c \|u\|_{W^{2,\infty}(\Gamma(t))}. \qquad \blacksquare
\end{aligned}$$

Remark B.3.16. In fact the assumption $u \in W^{1,\infty}(\Gamma(t))$ is sufficient to show the stronger bound

$$\|\nabla_\Gamma \mathcal{P}_h u\|_{L^\infty(\Gamma(t))} \leq c \|u\|_{W^{1,\infty}(\Gamma(t))}.$$

However, the proof requires more sophisticated arguments, which are beyond the scope of this article, cf. [72].

B.4. Time discretizations: stability

B.4.1. Runge-Kutta methods

We consider an s-stage algebraically stable implicit Runge-Kutta (R-K) method for the time discretization of the ODE system (B.8), coming from the ESFEM space discretization of the quasilinear parabolic evolving surface PDE.

In the following we extend the stability result for R-K methods of [37, lemma 7.1], to the case of quasilinear problems. Apart form the properties of the ESFEM the proof is based on the energy estimation techniques, see Lubich and Ostermann [64, theorem 1.1]. Generally on Runge-Kutta methods we refer to [49].

For the convenience of the reader we recall the method: for simplicity, we assume equidistant time steps $t_n := n\tau$, with step size τ. Our results can be straightforwardly extended to the case of nonuniform time steps. The s-stage implicit Runge-Kutta method, defined by the given Butcher tableau

$$\begin{array}{c|c} (c_i) & (a_{ij}) \\ \hline & (b_i) \end{array}, \quad \text{for } i,j = 1,2,\dots,s,$$

applied to the system (B.8), reads as

$$\begin{aligned} M_{ni}\alpha_{ni} &= M_n\alpha_n + \tau\sum_{j=1}^{s} a_{ij}\dot{\alpha}_{nj}, \quad \text{for } i = 1,2,\dots,s, \\ M_{n+1}\alpha_{n+1} &= M_n\alpha_n + \tau\sum_{i=1}^{s} b_i\dot{\alpha}_{ni}, \end{aligned}$$

where the internal stages satisfy

$$0 = \dot{\alpha}_{ni} + A\,(\alpha_{ni})\alpha_{ni}, \quad \text{for } i = 1,2,\dots,s,$$

with $M_{ni} := M(t_n + c_i\tau)$ and $M_{n+1} := M(t_{n+1})$. Here $\dot{\alpha}_{ni}$ is not a derivative but a suggestive notation.

We recall that the fully discrete solution is $U_h^n = \sum_{j=1}^{N} \alpha_{n,j}\chi_j(\,\cdot\,, t_n)$. Existence and uniqueness of the Runge-Kutta solution can be obtained analogously to [48, theorem 7.2].

For the R-K method we make the following assumptions:

Assumption B.4.1.
- The method has stage order $q \geq 1$ and classical order $p \geq q+1$.
- The coefficient matrix (a_{ij}) is invertible.
- The method is *algebraically stable*, i.e. $b_j > 0$ for $j = 1,2,\dots,s$ and the following matrix is positive semi-definite:

$$(b_i a_{ij} + b_j a_{ji} - b_i b_j)_{i,j=1}^{s}.$$

- The method is *stiffly accurate*, i.e. $b_j = a_{sj}$, and $c_s = 1$ for $j = 1, 2, \ldots, s$.

Instead of (B.8), let us consider the following perturbed version of the equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(M(t)\widetilde{\alpha}(t)) + A(\widetilde{\alpha}(t))\widetilde{\alpha}(t) = M(t)r(t) \\ \widetilde{\alpha}(0) = \widetilde{\alpha}_0. \end{cases} \tag{B.15}$$

The substitution of the true solution $\widetilde{\alpha}(t)$ of the perturbed problem into the R-K method, yields the defects Δ_{ni} and δ_{ni}, by setting $e_n = \alpha_n - \widetilde{\alpha}(t_n)$, $E_{ni} = \alpha_{ni} - \widetilde{\alpha}(t_n + c_i\tau)$ and $\dot{E}_{ni} = \dot{\alpha}_{ni} - \dot{\widetilde{\alpha}}(t_n + c_i\tau)$, then by subtraction the following *error equations* hold:

$$M_{ni}E_{ni} = M_n e_n + \tau \sum_{j=1}^{s} a_{ij}\dot{E}_{nj} - \Delta_{ni} \tag{B.16a}$$

$$M_{n+1}e_{n+1} = M_n e_n + \tau \sum_{i=1}^{s} b_i\dot{E}_{ni} - \delta_{n+1}, \tag{B.16b}$$

where the internal stages satisfy

$$\dot{E}_{ni} + A\,(\alpha_{ni})E_{ni} = -(A\,(\alpha_{ni}) - A\,(\widetilde{\alpha}_{ni}))\widetilde{\alpha}_{ni} - M_{ni}r_{ni}, \tag{B.16c}$$

where $i = 1, \ldots, s$ and $r_{ni} := r(t_n + c_i\tau)$.

Now we state one of the key lemmas of this paper, which provide unconditional stability for the above class of Runge-Kutta methods.

> **Lemma B.4.2.** *For an s-stage implicit Runge-Kutta method satisfying Assumption B.4.1. If the equation* (B.5) *has a solution in* $\mathcal{S}(t)$ *for* $0 \leq t \leq T$*. Then there exists a* $\tau_0 > 0$*, such that for* $\tau \leq \tau_0$ *and*

$t_n = n\tau \leq T$, that the error e_n is bounded by

$$\begin{aligned} |e_n|^2_{M_n} + \tau \sum_{k=1}^{n} |e_k|^2_{\boldsymbol{A}_k} \\ \leq C\bigg(|e_0|^2_{M_0} + \tau \sum_{k=1}^{n-1} \sum_{i=1}^{s} \|M_{ki} r_{ki}\|^2_{*,ki} + \tau \sum_{k=1}^{n} \Big|\frac{\delta_k}{\tau}\Big|^2_{M_k} \\ + C\tau \sum_{k=0}^{n-1} \sum_{i=1}^{s} \Big(|M_{ki}^{-1}\Delta_{ki}|^2_{M_i} + |M_{ki}^{-1}\Delta_{ki}|^2_{\boldsymbol{A}_{ki}} \Big) \bigg), \end{aligned} \tag{B.17}$$

where $\|w\|^2_{*,k} := w^T \left(\boldsymbol{A}(t_k) + M(t_k)\right)^{-1} w$. The constant C is independent of h, τ and n (but depends on $\boldsymbol{m}$, $\boldsymbol{M}$, L, μ, κ and T).

Proof. The combination of proofs of theorem 1.1 from [64] and of lemma 7.1 from [37] (or [66, lemma 3.1]) suffices, therefore it is omitted here. To be precise, the proof of this result is more closely related to [37]. Except the estimates involving the (nonlinear) internal stages, see [64].

(a) We start as in the cited papers, i.e. to be able to benefit from algebraic stability, we write

$$\begin{aligned} |M_{n+1}e_{n+1}|^2_{M_{n+1}^{-1}} = &\Big| M_n e_n + \tau \sum_{j=1}^{s} b_j \dot{E}_{nj} \Big|^2_{M_{n+1}^{-1}} \\ &- 2 \Big\langle M_n e_n + \tau \sum_{j=1}^{s} b_j \dot{E}_{nj} \Big| M_{n+1}^{-1} \Big| \delta_{n+1} \Big\rangle + |\delta_{n+1}|^2_{M_{n+1}^{-1}}, \end{aligned}$$

and by expressing $M_n e_n$ from the Runge-Kutta method (B.16a), for the first term, we obtain

$$\begin{aligned} &\Big| M_n e_n + \tau \sum_{j=1}^{s} b_j \dot{E}_{nj} \Big|_{M_{n+1}^{-1}} \\ &\quad = |M_n e_n|^2_{M_{n+1}^{-1}} + 2\tau \sum_{j=1}^{s} b_j \left\langle \dot{E}_{nj} \mid M_{n+1}^{-1} \mid M_{nj}E_{nj} + \Delta_{nj} \right\rangle \end{aligned}$$

$$+ \tau^2 \sum_{i=1}^{s} \sum_{j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle \dot{E}_{ni} \mid M_{n+1}^{-1} \mid \dot{E}_{nj} \rangle,$$

where the last term is non-positive by algebraic stability. The middle term is rewritten as

$$\begin{aligned} \langle \dot{E}_{nj} \mid M_{n+1}^{-1} \mid M_{nj} E_{nj} + \Delta_{nj} \rangle &= \langle \dot{E}_{nj} \mid M_{nj}^{-1} \mid M_{nj} E_{nj} + \Delta_{nj} \rangle \\ &\quad + \langle \dot{E}_{nj} \mid M_{n+1}^{-1} - M_{nj}^{-1} \mid M_{nj} E_{nj} + \Delta_{nj} \rangle. \qquad \text{(B.18)} \end{aligned}$$

All the terms in the above equations can be estimated identically as in the mentioned proofs, except the first term in (B.18).

(b) To estimate this term, including the nonlinearity, we use proposition B.2.3 (i.e. the inequalities (B.2), (B.3) and (B.4)), like in [64]. Using (B.16c), the internal stages, give

$$\begin{aligned} &\langle \dot{E}_{nj} \mid M_{nj}^{-1} \mid M_{nj} E_{nj} + \Delta_{nj} \rangle \\ &\quad = \langle \dot{E}_{nj} \mid E_{nj} \rangle + \langle \dot{E}_{nj} \mid M_{nj}^{-1} \Delta_{nj} \rangle \\ &\quad = - \langle A\,(\alpha_{nj}) E_{nj} \mid E_{nj} \rangle - \langle A\,(\alpha_{nj}) E_{nj} \mid M_{nj}^{-1} \mid \Delta_{nj} \rangle \\ &\qquad - \langle (A\,(\alpha_{nj}) - A\,(\widetilde{\alpha}_{nj})) \widetilde{\alpha}_{nj} \mid E_{nj} + M_{nj}^{-1} \Delta_{nj} \rangle \\ &\qquad - \langle M_{nj}\, r_{nj} \mid E_{nj} + M_{nj}^{-1} \Delta_{nj} \rangle \end{aligned}$$

Using the results of proposition B.2.3 and that $\widetilde{\alpha}_{nj} = u(\,\cdot\,, t_n + c_j \tau)$ is assumed to be in $\mathcal{S}(t_n + c_j \tau)$, we can estimate as follows (using Cauchy-Schwarz and Young's inequality)

$$\begin{aligned} |\langle \dot{E}_{nj} \mid M_{nj}^{-1} \mid M_{nj} E_{nj} + \Delta_{nj} \rangle| &\leq -\boldsymbol{m}\, |E_{nj}|_{\boldsymbol{A}_{nj}}^2 + \boldsymbol{M}\, |E_{nj}|_{\boldsymbol{A}_{nj}}\, |M_{nj}^{-1} \Delta_{nj}|_{\boldsymbol{A}_{nj}} \\ &\quad + L\, |E_{nj}|_{M_{nj}}\, |E_{nj} + M_{nj}^{-1} \Delta_{nj}|_{\boldsymbol{A}_{nj}} \\ &\quad + |\langle M_{nj} r_{nj} \mid E_{nj} + M_{nj}^{-1} \Delta_{nj} \rangle| \\ &\leq -\frac{\alpha}{4}\, |E_{nj}|_{\boldsymbol{A}_{nj}}^2 + C\, |M_{nj}^{-1} \Delta_{nj}|_{\boldsymbol{A}_{nj}}^2 \\ &\quad + C\, |M_{nj}^{-1} \Delta_{nj}|_{M_j}^2 + C\, |E_{nj}|_{M_j}^2 \\ &\quad + C\, \|M_{nj} r_{nj}\|_{*,nj}^2. \end{aligned}$$

Since the right-hand side of this estimate is the same as in the cited proofs, it can be finished in the exact same way as in the mentioned references. ■

Then, using the above stability results, the error bounds are following analogously as in [37, theorem 8.1] (or [66, theorem 5.1]).

Theorem B.4.3. *Consider the quasilinear parabolic problem* (B.1), *having a solution in $\mathcal{S}(t)$ for $0 \le t \le T$. Couple the evolving surface finite element method as space discretization with time discretization by an s-stage implicit Runge-Kutta method satisfying Assumption B.4.1. Assume that the Ritz map of the solution has continuous discrete material derivatives up to order $q+2$. Then there exists $\tau_0 > 0$, independent of h, such that for $\tau \le \tau_0$, for the error $E_h^n = U_h^n - \mathcal{P}_h u(\,\cdot\,, t_n)$ the following estimate holds for $t_n = n\tau \le T$:*

$$\begin{aligned}
&\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\Big)^{\frac12} \\
&\quad \le C\tilde{\beta}_{h,q}\tau^{q+1} + C\left(\tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \|R_h(\,\cdot\,, t_k + c_i\tau)\|^2_{H_h^{-1}(\Gamma_h(t_k+c_i\tau))}\right)^{\frac12} \\
&\qquad + C\,\|E_h^0\|_{L^2(\Gamma_h(0))},
\end{aligned}$$

where the constant C is independent of h, τ and n (but depends on $\boldsymbol{m}$, $\boldsymbol{M}$, L, μ, κ and T). Furthermore

$$\begin{aligned}
\tilde{\beta}_{h,q}^2 = &\int_0^T \sum_{\ell=1}^{q+2} \|(\partial_h^\bullet)^{(\ell)}(\mathcal{P}_h u)(\,\cdot\,, t)\|_{L^2(\Gamma_h(t))}\,\mathrm{d}t \\
&+ \int_0^T \sum_{\ell=1}^{q+1} \|\nabla_{\Gamma_h(t)}(\partial_h^\bullet)^{(\ell)}(\mathcal{P}_h u)(\,\cdot\,, t)\|_{L^2(\Gamma_h(t))}\,\mathrm{d}t.
\end{aligned}$$

The H_h^{-1}-norm of R_h is defined as

$$\|R_h(\,\cdot\,,t)\|_{H_h^{-1}(\Gamma_h(t))} := \sup_{0\neq\phi_h\in S_h(t)} \frac{\langle R_h(\,\cdot\,,t),\phi_h\rangle_{L^{22}(\Gamma_h(t))}}{\|\phi_h\|_{H^1(\Gamma_h(t))}}. \quad \text{(B.19)}$$

Proof. Our proof is similar to [66, theorem 5.1].

We estimate the terms of the right-hand side of (B.17). At first we connect $\|\cdot\|_{*,k}$ and $\|\cdot\|_{H_h^{-1}(\Gamma_h(t_k))}$:

$$\begin{aligned}
\|Mr\|_* &= \left(r^T M\,(\boldsymbol{A}+M)^{-1}Mr\right)^{1/2} = \|(\boldsymbol{A}+M)^{-1/2}Mr\|_2 \\
&= \sup_{0\neq w\in\mathbb{R}^N} \frac{r^T(\boldsymbol{A}+M)^{-1/2}w}{w^Tw} = \sup_{0\neq z\in\mathbb{R}^N} \frac{r^TMz}{(z^T(\boldsymbol{A}+M)z)^{1/2}} \\
&= \sup_{0\neq\phi_h\in S_h} \frac{\langle R_h,\phi_h\rangle_{L^2(\Gamma_h)}}{\|\phi_h\|_{H^1(\Gamma_h)}} = \|R_h\|_{H_h^{-1}(\Gamma_h)}.
\end{aligned}$$

By Taylor expansion, the definition of stage and classical order, and with the bounded Peano kernels K and K_i, the defects satisfy

$$\begin{aligned}
\delta_{n+1} &= \tau^{q+1}\int_{t_n}^{t_{n+1}} K\Big(\frac{t-t_n}{\tau}\Big)(M\widetilde{\alpha})^{(q+2)}(t)\,\mathrm{d}t, \\
\Delta_{ni} &= \tau^{q}\int_{t_n}^{t_{n+1}} K_i\Big(\frac{t-t_n}{\tau}\Big)(M\widetilde{\alpha})^{(q+1)}(t)\,\mathrm{d}t,
\end{aligned}$$

hence, by a simple but lengthy calculation (cf. [66]) the following bound is obtained:

$$\tau\sum_{k=1}^{n}\Big|\frac{\delta_k}{\tau}\Big|_{M_k}^2 + C\tau\sum_{k=0}^{n-1}\sum_{i=1}^{s}\Big(|M_{ki}^{-1}\Delta_{ki}|_{M_i}^2 + |M_{ki}^{-1}\Delta_{ki}|_{\boldsymbol{A}_{ki}}^2\Big) \leq C\widetilde{\beta}_{h,q}^2(\tau^{q+1})^2,$$

and therefore, by inserting everything into (B.17), the proof is completed. ■

B.4.2. Backward differentiation formulae

We apply a k-step backward difference formula (BDF) for $k \leq 5$ as a discretization to the ODE system (B.8), coming from the ESFEM space discretization of the quasilinear parabolic evolving surface PDE. Both implicit and linearly implicit methods are discussed.

In the following we extend the stability result for BDF methods of [63, lemma 4.1], to the case quasilinear problems. Apart from the properties of the ESFEM the proof is based on Dahlquist's G-stability theory [20] and on the multiplier technique of Nevanlinna and Odeh [69].

We recall the k-step BDF method for (B.8) with step size $\tau > 0$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A\,(\alpha_n)\alpha_n = 0, \qquad (n \geq k), \tag{B.20}$$

where the coefficients of the method are given by $\delta(\zeta) = \sum_{j=0}^{k}\delta_j\zeta^j = \sum_{\ell=1}^{k}\frac{1}{\ell}(1-\zeta)^\ell$, while the starting values are $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$. The method is known to be 0-stable for $k \leq 6$ and have order k (for more details, see [49, chapter V]).

The linearly implicit modification is, using the polynomial $\gamma(\zeta) = \sum_{j=1}^{k}\gamma_j\zeta^j = \zeta^k - (\zeta-1)^{k-1}$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A\Big(\sum_{j=1}^{k}\gamma_j\alpha_{n-j}\Big)\alpha_n = 0, \qquad (n \geq k). \tag{B.21}$$

For more details we refer to [2], in particular for existence and uniqueness of the BDF solution see section 3.1 in [2].

Instead of (B.8) let us consider again the perturbed problem (B.15). By substituting the true solution $\widetilde{\alpha}(t)$ of the perturbed problem into the BDF method (B.20), we obtain

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\widetilde{\alpha}_{n-j} + A\,(\widetilde{\alpha}_n)\widetilde{\alpha}_n = -d_n, \qquad (n \geq k).$$

By introducing the error $e_n = \alpha_n - \widetilde{\alpha}(t_n)$, multiplying by τ, and by subtraction we have the error equation

$$\sum_{j=0}^{k} \delta_j M_{n-j} e_{n-j} + \tau A(\alpha_n) e_n + \tau\big(A(\alpha_n) - A(\widetilde{\alpha}_n)\big)\widetilde{\alpha}_n = \tau d_n, \quad (n \geq k). \tag{B.22}$$

In the linearly implicit case we obtain:

$$\sum_{j=0}^{k} \delta_j M_{n-j} e_{n-j} + \tau A\Big(\sum_{j=1}^{k} \gamma_j \alpha_{n-j}\Big) e_n$$
$$+ \tau\left(A\Big(\sum_{j=1}^{k} \gamma_j \alpha_{n-j}\Big) - A\Big(\sum_{j=1}^{k} \gamma_j \widetilde{\alpha}_{n-j}\Big)\right)\widetilde{\alpha}_n = \tau \hat{d}_n, \quad (n \geq k),$$

where $\hat{d}_n$ have similar properties as d_n, therefore it will be also denoted by d_n.

The stability results for BDF methods are the following.

Lemma B.4.4. *For a k-step implicit or linearly implicit BDF method with $k \leq 5$ there exists a $\tau_0 > 0$, such that for $\tau \leq \tau_0$ and $t_n = n\tau \leq T$, that the error e_n is bounded by*

$$|e_n|_{M_n}^2 + \tau \sum_{j=k}^{n} |e_j|_{\boldsymbol{A}_j}^2 \leq C\tau \sum_{j=k}^{n} \|d_j\|_{*,j}^2 + C \max_{0 \leq i \leq k-1} |e_i|_{M_i}^2$$

where $\|w\|_{*,k}^2 = w^T\left(\boldsymbol{A}(t_k) + M(t_k)\right)^{-1} w$. The constant C is independent of h, τ and n (but depends on $\boldsymbol{m}$, $\boldsymbol{M}$, L, μ, κ and T).

Proof. The proof follows the proof of lemma 4.1 from [63], and [2] section 6, using G-stability from [20] and multiplier techniques from [69]. Except in those terms where the nonlinearity appears, see theorem 1 in [2].

(a) The starting point of the proof is the following reformulation of the error equation (B.22):

$$
\begin{aligned}
M_n \sum_{j=0}^{k} \delta_j e_{n-j} + \tau A(\alpha_n) e_n + \tau \left(A(\alpha_n) - A(\widetilde{\alpha}_n)\right)\widetilde{\alpha}_n p \\
= \tau d_n + \sum_{j=1}^{k} \delta_j \left(M_n - M_{n-j}\right) e_{n-j}
\end{aligned}
$$

and using a modified energy estimate. Following [69], we multiply both sides with the multiplier $e_n - \eta e_{n-1}$, where the smallest possible values of η is found to be $\eta = 0,\ 0,\ 0.0836,\ 0.2878,\ 0.8160$ for $k = 1, 2, \ldots, 5$, respectively, cf. [69]. This gives us, for $n \geq k+1$:

$$
I_n + II_n^1 + II_n^2 = III_n + IV_n,
$$

where

$$
\begin{aligned}
I_n &= \left\langle \sum_{j=0}^{k} \delta_j e_{n-j} \,\middle|\, M_n \,\middle|\, e_n - \eta e_{n-1} \right\rangle, \\
II_n^1 &= \tau \left\langle e_n \,|\, A(\alpha_n) \,|\, e_n - \eta e_{n-1} \right\rangle, \\
II_n^2 &= \tau \left\langle \left(A(\alpha_n) - A(\widetilde{\alpha}_n)\right)\widetilde{\alpha}_n \,|\, e_n - \eta e_{n-1} \right\rangle, \\
III_n &= \tau \left\langle d_n \,|\, e_n - \eta e_{n-1} \right\rangle, \\
IV_n &= \sum_{j=1}^{k} \left\langle e_{n-j} \,|\, M_n - M_{n-j} \,|\, e_n - \eta e_{n-1} \right\rangle.
\end{aligned}
$$

We only have to estimate these terms in a suitable way.

(b) We start by bounding the nonlinear terms. First, we will estimate II_n^1 from below using (B.2) and lemma B.3.3:

$$
\begin{aligned}
\tau^{-1} II_n^1 &= \left\langle e_n \,|\, A(\alpha_n) \,|\, e_n \right\rangle - \eta \left| \left\langle e_n \,|\, A(\alpha_n) \,|\, e_{n-1} \right\rangle \right| \\
&\geq \boldsymbol{m} \, |e_n|_{\boldsymbol{A}_n}^2 - \boldsymbol{M} \ \eta \, |e_n|_{\boldsymbol{A}_n} |e_{n-1}|_{\boldsymbol{A}_n} \\
&\geq \left(\boldsymbol{m} - \frac{\boldsymbol{m}}{4} \eta \right) |e_n|_{\boldsymbol{A}_n}^2 - \frac{1}{\boldsymbol{m}} \boldsymbol{M}^2 \eta (1 + 2\kappa\tau) \, |e_{n-1}|_{\boldsymbol{A}_{n-1}}^2 .
\end{aligned}
$$

The other term is estimated using (B.4), Young's inequality, and again by lemma B.3.3:

$$\begin{aligned}
\tau^{-1} II_n^2 &\geq -\left|\left\langle ((A(\alpha_n) - A(\tilde{\alpha}_n))\tilde{\alpha}_n \,\middle|\, e_n - \eta e_{n-1} \right\rangle\right| \\
&\geq -L\,|e_n|_{M_n}\left(|e_n|_{\boldsymbol{A}_n} + \eta\,|e_{n-1}|_{\boldsymbol{A}_n}\right) \\
&\geq -\frac{\boldsymbol{m}}{4}\,|e_n|^2_{\boldsymbol{A}_n} - \frac{1}{\boldsymbol{m}}L^2\,|e_n|^2_{M_n} \\
&\quad - \frac{L}{2}\eta\,|e_n|^2_{M_n} + \frac{L}{2}\eta(1+2\kappa\tau)\,|e_{n-1}|^2_{\boldsymbol{A}_{n-1}}.
\end{aligned}$$

Combined, and using that $0 \leq \eta < 1$, we have

$$\begin{aligned}
II_n^1 + II_n^2 \geq \tau\frac{1}{2}\boldsymbol{m}\,|e_n|^2_{\boldsymbol{A}_n} &- \tau\eta\left(\frac{1}{\boldsymbol{m}}L^2 + \frac{L}{2}\right)|e_n|^2_{M_n} \\
&- \tau\eta\left(\frac{1}{\boldsymbol{m}}\boldsymbol{M}^2 + \frac{L}{2}\right)(1+2\kappa\tau)\,|e_{n-1}|^2_{\boldsymbol{A}_{n-1}}
\end{aligned}$$

The estimations of I_n, III_n and IV_n are the same as in the proof in [63], with G-stability of [20] as the main tool.

(c) Combining all estimates and summing up gives, for $\tau \leq \tau_0$ and for $n \geq k+1$:

$$|E_n|^2_{G,n} + \frac{\boldsymbol{m}}{4}\tau\sum_{j=k+1}^{n}|e_j|^2_{\boldsymbol{A}_j} \leq C\tau\sum_{j=k}^{n-1}|E_j|^2_{G,j} + C\tau\sum_{j=k+1}^{n}\|d_j\|^2_{*,j} + C\eta\tau\,|e_k|^2_{\boldsymbol{A}_k},$$

where $E_n = (e_n, \ldots, e_{n-k+1})$, and

$$\|E_n\|^2_{G,n} := \sum_{i,j=1}^{k} g_{ij}\,\langle e_{n-k+i} \mid M_n \mid e_{n-k+j}\rangle.$$

This is the same inequality as in [63], hence we can also proceed with the discrete Gronwall inequality.

(d) To achieve the stated result we have to estimate the extra term $C\,(\,|e_k|^2_{M_k} + \tau\,|e_k|^2_{\boldsymbol{A}_k})$. For that we take the inner product of the error equation for $n = k$ with e_k to obtain Similarly as for II_n^i, use the properties of operator A and lemma B.3.3, yields

$$|e_k|^2_{M_k} + \tau\rho\, |e_k|^2_{\boldsymbol{A}_k} \leq C\tau\, \|d_k\|^2_{*,k} + C \max_{0\leq i\leq k-1} |e_i|^2_{M_i}.$$

The insertion of this completes the proof.

The result follows from analogous arguments for linearly implicit methods, cf. [2, section 6]. ■

Again, using the above stability results, the error bounds are following analogously as in [63, theorem 5.1] (or [66, theorem 5.3]).

Theorem B.4.5. *Consider the quasilinear parabolic problem* (B.1), *having a solution in $\mathcal{S}(t)$ for $0 \leq t \leq T$. Couple the evolving surface finite element method as space discretization with time discretization by a k-step implicit or linearly implicit backward difference formula of order $k \leq 5$. Assume that the Ritz map of the solution has continuous discrete material derivatives up to order $k+1$. Then there exists $\tau_0 > 0$, independent of h, such that for $\tau \leq \tau_0$, for the error $E_h^n = U_h^n - \mathcal{P}_h u(\,\cdot\,, t_n)$ the following estimate holds for $t_n = n\tau \leq T$:*

$$\begin{aligned} \|E_h^n\|_{L^2(\Gamma_h(t_n))} &+ \Big(\tau \sum_{j=1}^n \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\Big)^{\frac{1}{2}} \\ &\leq C\tilde{\beta}_{h,k}\tau^k + \Big(\tau \sum_{j=1}^n \|R_h(\,\cdot\,, t_j)\|^2_{H_h^{-1}(\Gamma_h(t_j))}\Big)^{\frac{1}{2}} \\ &\quad + C \max_{0\leq i\leq k-1} \|E_h^i\|_{L^2(\Gamma_h(t_i))}, \end{aligned}$$

where the constant C is independent of h, n and τ (but depends on $\boldsymbol{m}$, $\boldsymbol{M}$, L, μ, κ and T). Furthermore

$$\tilde{\beta}_{h,k}^2 = \int_0^T \sum_{\ell=1}^{k+1} \|(\partial_h^\bullet)^{(\ell)}(\mathcal{P}_h u)(\,\cdot\,, t)\|_{L^2(\Gamma_h(t))}\, \mathrm{d}t.$$

Proof. The proof of this result is analogous to that of theorem B.4.3, it uses the norm identity, and bounded Peano kernels. For details see the above references. ∎

B.5. Error bounds for the fully discrete solutions

We follow the approach of [63, section 5] by defining the FEM residual $R_h(\,\cdot\,,t) = \sum_{j=1}^{N} r_j(t)\chi_j(\,\cdot\,,t) \in S_h(t)$ as

$$\int_{\Gamma_h} R_h\phi_h = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h} \widetilde{\mathcal{P}}_h u\phi_h + \int_{\Gamma_h} \mathcal{A}(\widetilde{\mathcal{P}}_h u)\nabla_\Gamma(\widetilde{\mathcal{P}}_h u)\cdot\nabla_\Gamma\phi_h - \int_{\Gamma_h} (\widetilde{\mathcal{P}}_h u)\partial_h^\bullet\phi_h, \tag{B.23}$$

where $\phi_h \in S_h(t)$, and the Ritz map of the true solution u is given as

$$\widetilde{\mathcal{P}}_h u(\,\cdot\,,t) = \sum_{j=1}^{N} \widetilde{\alpha}_j(t)\chi_j(\,\cdot\,,t).$$

The above problem is equivalent to the ODE system with the vector $r(t) = (r_j(t)) \in \mathbb{R}^N$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\widetilde{\alpha}(t)\big) + A(\widetilde{\alpha}(t))\widetilde{\alpha}(t) = M(t)r(t),$$

which is the perturbed ODE system (B.15).

B.5.1. Bound of the semidiscrete residual

We now show the optimal second order estimate of the residual R_h.

Theorem B.5.1. *Let u, the solution of the parabolic problem, be in $\mathcal{S}(t)$ for $0 \le t \le T$. Then there exists a constant $C > 0$ and $h_0 > 0$, such that for all $h \le h_0$ and $t \in [0,T]$, the finite element residual R_h*

of the Ritz map is bounded as

$$\|R_h\|_{H^{-1}(\Gamma_h(t))} \leq ch^2.$$

Proof. (a) We start by applying the discrete transport property to the residual equation (B.23)

$$\begin{aligned} m_h(R_h, \phi_h) &= \frac{\mathrm{d}}{\mathrm{d}t} m_h(\widetilde{\mathcal{P}}_h u, \phi_h) + a_h(\widetilde{\mathcal{P}}_h u; \widetilde{\mathcal{P}}_h u, \phi_h) - m_h(\widetilde{\mathcal{P}}_h u, \partial_h^\bullet \phi_h) \\ &= m_h(\partial_h^\bullet \widetilde{\mathcal{P}}_h u, \phi_h) + a_h(\widetilde{\mathcal{P}}_h u; \widetilde{\mathcal{P}}_h u, \phi_h) + g_h(V_h; \widetilde{\mathcal{P}}_h u, \phi_h). \end{aligned}$$

(b) We continue by the transport property with discrete material derivatives from lemma B.3.6, but for the weak form, with $\varphi := \varphi_h = \phi_h^l$:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} m(u, \varphi_h) + a(u; u, \varphi_h) - m(u, \partial^\bullet \varphi_h) \\ &= m(\partial_h^\bullet u, \varphi_h) + a(u; u, \varphi_h) + g(v_h; u, \varphi_h) + m(u, \partial_h^\bullet \varphi_h - \partial^\bullet \varphi_h). \end{aligned}$$

(c) Subtraction of the two equations, using the definition of the Ritz map with $\xi = u$ in (B.12), i.e.

$$a_h^*(u^{-l}; \widetilde{\mathcal{P}}_h u, \phi_h) = a^*(u; u, \varphi_h),$$

and using that

$$\partial_h^\bullet \varphi_h - \partial_h^\bullet \varphi_h = (v_h - v) \cdot \nabla_\Gamma \varphi_h$$

holds, we obtain

$$\begin{aligned} m_h(R_h, \phi_h) = {} & m_h(\partial_h^\bullet \widetilde{\mathcal{P}}_h u, \phi_h) - m(\partial_h^\bullet u, \varphi_h) \\ & + g_h(V_h; \widetilde{\mathcal{P}}_h u, \phi_h) - g(v_h; u, \varphi_h) \\ & + a_h^*(\widetilde{\mathcal{P}}_h u; \widetilde{\mathcal{P}}_h u, \phi_h) - a_h^*(u^{-l}; \widetilde{\mathcal{P}}_h u, \phi_h) \\ & + m(u, \varphi_h) - m_h(\widetilde{\mathcal{P}}_h u, \phi_h) \\ & + m(u, (v_h - v) \cdot \nabla_\Gamma \varphi_h). \end{aligned}$$

All the pairs can be easily estimated separately as $ch^2 \|\varphi_h\|_{L^2(\Gamma(t))}$, by combining the estimates of lemma B.3.7, and theorem B.3.11 and B.3.13, except the third, and the last term.

The term containing the velocity difference $(v_h - v)$ can be estimated, using $|v_h - v| + h|\nabla_\Gamma(v_h - v)| \leq ch^2$ from [35, lemma 5.6], as $ch^2 \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}$. The nonlinear terms are rewritten as:

$$
\begin{aligned}
&a_h^*(\widetilde{\mathcal{P}_h}u; \widetilde{\mathcal{P}_h}u, \phi_h) - a_h^*(u^{-l}; \widetilde{\mathcal{P}_h}u, \phi_h) \\
&\qquad = a_h^*(\widetilde{\mathcal{P}_h}u; \widetilde{\mathcal{P}_h}u, \phi_h) - a^*(\mathcal{P}_h u; \mathcal{P}_h u, \varphi_h) \\
&\qquad\quad + a^*(\mathcal{P}_h u; \mathcal{P}_h u, \varphi_h) - a^*(u; \mathcal{P}_h u, \varphi_h) \\
&\qquad\quad + a^*(u; \mathcal{P}_h u, \varphi_h) - a_h^*(u^{-l}; \widetilde{\mathcal{P}_h}u, \phi_h)
\end{aligned}
$$

For the first and the third term lemma B.3.7 provides an upper bound $ch^2 \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}$ (similarly like before).

Finally, using lemma B.3.15 we obtain, similarly to (B.4), that the second term can be bounded as

$$
\begin{aligned}
&|a^*(\mathcal{P}_h u; \mathcal{P}_h u, \varphi_h) - a^*(u; \mathcal{P}_h u, \varphi_h)| \\
&= \Big| \int_{\Gamma(t)} \big(\mathcal{A}(\mathcal{P}_h u) - \mathcal{A}(u)\big) \nabla_\Gamma \mathcal{P}_h u \cdot \nabla_\Gamma \varphi_h \Big| \\
&\leq c\ell \|\mathcal{P}_h u - u\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \mathcal{P}_h u\|_{L^\infty(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))} \\
&\leq c\ell \|\mathcal{P}_h u - u\|_{L^2(\Gamma(t))} \, c\, r \; \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))} \\
&\leq c\ell r\, h^2 \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}.
\end{aligned}
$$

Therefore, by (B.19), and using the equivalence of norms [29] ($\phi_h^l = \varphi_h$), we have

$$
\begin{aligned}
\|R_h(\,\cdot\,, t)\|_{H_h^{-1}(\Gamma_h(t))} &= \sup_{0 \neq \phi_h \in S_h(t)} \frac{m_h(R_h(\,\cdot\,, t), \phi_h)}{\|\phi_h\|_{H^1(\Gamma_h(t))}} \\
&\leq ch^2 \frac{\|\varphi_h\|_{H^1(\Gamma(t))}}{\|\phi_h\|_{H^1(\Gamma_h(t))}} \leq ch^2. \qquad \blacksquare
\end{aligned}
$$

B.5.2. Error estimates for the full discretizations

We compare the lifted fully discrete numerical solution $u_h^n := (U_h^n)^l$ with the exact solution $u(\,\cdot\,,t_n)$ of the evolving surface PDE (B.1), where $U_h^n = \sum_{j=1}^N \alpha_j^n \chi_j(\,\cdot\,,t)$, where the vectors α^n are generated by a Runge-Kutta or a BDF method.

Theorem B.5.2 (ESFEM and R-K). *Consider the evolving surface finite element method as space discretization of the quasilinear parabolic problem* (B.1)*, with time discretization by an s-stage implicit Runge-Kutta method satisfying Assumption B.4.1. Let u be a sufficiently smooth solution of the problem, which satisfies* $u(\,\cdot\,,t) \in \mathcal{S}(t)$ $(0 \le t \le T)$*, and assume that the initial value is approximated as*

$$\|u_h^0 - (\mathcal{P}_h u)(\,\cdot\,,0)\|_{L^2(\Gamma(0))} \le C_0 h^2.$$

Then there exists $h_0 > 0$ *and* $\tau_0 > 0$*, such that for* $h \le h_0$ *and* $\tau \le \tau_0$*, the following error estimate holds for* $t_n = n\tau \le T$*:*

$$\|u_h^n - u(\,\cdot\,,t_n)\|_{L^2(\Gamma(t_n))} + h\left(\tau \sum_{j=1}^n |u_h^j - u(\,\cdot\,,t_j)|^2_{H^1(\Gamma(t_j))}\right)^{\frac{1}{2}}$$
$$\le C\left(\tau^{q+1} + h^2\right).$$

The constant C is independent of h, τ *and n, but depends on* $\boldsymbol{m}$*,* $\boldsymbol{M}$ *and L, from* (B.2)*,* (B.3) *and* (B.4)*, on* μ*,* κ*, from lemma B.3.3, and on T.*

Theorem B.5.3 (ESFEM and BDF). *Consider the evolving surface finite element method as space discretization of the quasilinear parabolic problem* (B.1)*, with time discretization by a k-step implicit or linearly implicit backward difference formula of order* $k \le 5$*. Let u be a sufficiently smooth solution of the problem, which satisfies*

$u(\,\cdot\,,t) \in \mathcal{S}(t)$ ($0 \le t \le T$), and assume that the starting values are satisfying

$$\max_{0\le i\le k-1} \|u_h^i - (\mathcal{P}_h u)(\,\cdot\,,t_i)\|_{L^2(\Gamma(0))} \le C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \le h_0$ and $\tau \le \tau_0$, the following error estimate holds for $t_n = n\tau \le T$:

$$\|u_h^n - u(\,\cdot\,,t_n)\|_{L^2(\Gamma(t_n))} + h\left(\tau\sum_{j=1}^n |u_h^j - u(\,\cdot\,,t_j)|^2_{H^1(\Gamma(t_j))}\right)^{\frac{1}{2}}$$
$$\le C\,(\tau^k + h^2).$$

The constant C is independent of h, τ and n, but depends on $\boldsymbol{m}$, $\boldsymbol{M}$ and L, from (B.2), (B.3) *and* (B.4)*, on μ, κ, from lemma B.3.3, and on T.*

Proof of theorem B.5.2-B.5.3. The global error is decomposed into two parts:

$$u_h^n - u(\,\cdot\,,t_n) = \Big(u_h^n - (\mathcal{P}_h u)(\,\cdot\,,t_n)\Big) + \Big((\mathcal{P}_h u)(\,\cdot\,,t_n) - u(\,\cdot\,,t_n)\Big),$$

and the terms are estimated by previous results.

The first one is estimated by our results for Runge-Kutta or BDF methods: theorem B.4.3 or B.4.5, respectively, together with the residual bound theorem B.5.1, and by the Ritz error estimates theorem B.3.11 and B.3.13.

The second term is estimated by the error estimates for the Ritz map (theorem B.3.11 and B.3.13). ■

B.6. Semilinear problems extension

The presented results, in particular theorem B.5.2 and B.5.3, can be generalized to semilinear problems. Convergence results for BDF method were already shown for semilinear problems in [2]. For the analogous results for Runge-Kutta methods follow [64, remark 1.1]. Problems fitting into this framework can be found in the references given in the introduction.

Following remark 1.1 from [64], the inhomogeneity $f(t)$ in the evolving surface PDE (B.1) can be replaced by $f(t,u)$ satisfying a local Lipschitz condition (similar to (B.4)): for every $\delta > 0$ there exists $L = L(\delta, r)$ such that

$$\begin{aligned}\|f(t,w_1) - f(t,w_2)\|_{V(t)'} &\leq \delta \, \|w_1 - w_2\|_{V(t)} \\ &\quad + L \, \|w_1 - w_2\|_{H(t)}, \quad (0 \leq t \leq T)\end{aligned}$$

holds for arbitrary $w_1, w_2 \in V(t)$ with $\|w_1\|_{V(t)}, \|w_2\|_{V(t)} \leq r$, uniformly in t. Such a condition can be satisfied by using the same $\mathcal{S}$ set as for quasilinear problems.

To be precise: In this case the bilinear form $a(t;\cdot,\cdot)$ is not depending on ξ, it reduces to the case presented in [35]. Therefore, Section B.3 here would reduce to recall results mainly from [31, 35]. There is no ξ dependency in the definition of the generalized Ritz map, hence it is the one appeared in [62, 66], together with the error bounds presented there. The regularity result of the Ritz map is still needed from section B.3.7.

The stability estimates for the Runge-Kutta and BDF methods are needed to be revised in a straightforward way, cf. [64] and [2], respectively. To give more insight we give some details in the case of BDF methods. Runge-Kutta methods can be handled in a similar way.

The error equation for the semilinear problem reads as

$$\sum_{j=0}^{k} \delta_j M_{n-j} e_{n-j} + \tau A_n e_n = \tau\big(f(t_n, \alpha_n) - f(t_n, \widetilde{\alpha}_n)\big) + \tau d_n, \quad (n \geq k).$$

After testing with the multiplier $e_n - \eta e_{n-1}$ we obtain

$$I_n + II_n = III_n + IV_n + V_n.$$

The new nonlinear term is now estimated as

$$\begin{aligned}
\tau^{-1} |V_n| &= |\langle f(t_n, \alpha_n) - f(t_n, \widetilde{\alpha}_n) \,|\, e_n - \eta e_{n-1} \rangle| \\
&\le \|f(t, \alpha_n) - f(t, \widetilde{\alpha}_n)\|_{H_h^{-1}(\Gamma_h(t))} |e_n - \eta e_{n-1}|_{\boldsymbol{A}_n} \\
&\le (\delta |e_n|_{\boldsymbol{A}_n} + L |e_n|_{M_n}) (|e_n|_{\boldsymbol{A}_n} + \eta |e_{n-1}|_{\boldsymbol{A}_n}) \\
&\le 2\delta |e_n|^2_{\boldsymbol{A}_n} + C\eta |e_{n-1}|^2_{\boldsymbol{A}_n} + C |e_n|^2_{M_n}.
\end{aligned}$$

The other terms are either estimated as before, or in a much simple way, for instance in the case of II_n which is now linear, cf. [63].

B.7. Numerical experiments

We present a numerical experiment for an evolving surface quasilinear parabolic problem discretized by evolving surface finite elements coupled with the backward Euler method as a time integrator. The fully discrete methods were implemented in DUNE-FEM [22], while the initial triangulations were generated using DistMesh [71].

The evolving surface is given by

$$\Gamma(t) = \{x \in \mathbb{R}^3 \mid a(t)^{-1} x_1^2 + x_2^2 + x_3^2 - 1 = 0\},$$

where $a(t) = 1 + 0.25 \sin(2\pi t)$, see e.g. [31, 37, 66]. The problem is considered over the time interval $[0, 1]$. We consider the problem with the nonlinearity $\mathcal{A}(u) = 1 - \frac{1}{2} e^{-u^2/4}$. This satisfies the conditions in assumption B.2.2, since it has lower bound $1/2$, upper bound 1, and its derivative $\mathcal{A}'(u) = \frac{u}{4} e^{-u^2/4}$ is also bounded, hence $\mathcal{A}$ is Lipschitz continuous. The right-hand side f is computed as to have $u(x, t) = e^{-6t} x_1 x_2$ as the true solution of the quasilinear problem

$$\begin{cases}
\partial_h^\bullet u + u \nabla_{\Gamma(t)} \cdot v - \nabla_{\Gamma(t)} \cdot \Big(\mathcal{A}(u) \nabla_{\Gamma(t)} u \Big) = f & \text{on } \Gamma(t), \\
u(\,\cdot\,, 0) = u_0 & \text{on } \Gamma(0).
\end{cases}$$

level	dof	$L^\infty(L^2)$	EOCs	$L^2(H^1)$	EOCs
1	126	0.07121892	-	0.1404349	-
2	516	0.02077452	1.78	0.0404614	1.80
3	2070	0.00540906	1.94	0.0111377	1.86
4	8208	0.00136755	1.98	0.0033538	1.73
5	32682	0.00034289	2.00	0.0011904	1.49

Table B.1.: Errors and EOCs in the $L^\infty(L^2)$ and $L^2(H^1)$ norms

The time integrations require the solution of a nonlinear system at every timestep. As it is usual for Runge-Kutta methods, we used the simplified Newton iterations, cf. [49, section IV.8].

Let $(\mathcal{T}_k(t))_{k=1,2,\dots,n}$ and $(\tau_k)_{k=1,2,\dots,n}$ be a series of triangulations and timesteps, respectively, such that $2h_k \approx h_{k-1}$ and $4\tau_k = \tau_{k-1}$, with $\tau_1 = 0.1$. By e_k we denote the error corresponding to the mesh $\mathcal{T}_k(t)$ and step size τ_k. Then the experimental order of convergences (EOCs) are given as

$$EOC_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \qquad (k = 2, 3, \dots, n).$$

In table B.1 we report on the EOCs, for the ESFEM coupled with backward Euler method, corresponding to the norm and seminorm

$$\begin{aligned} L^\infty(L^2): &\qquad \max_{1\leq n\leq N} \|u_h^n - u(\,\cdot\,, t_n)\|_{L^2(\Gamma(t_n))}, \\ L^2(H^1): &\qquad \Big(\tau \sum_{n=1}^{N} \|\nabla_{\Gamma(t_n)}(u_h^n - u(\,\cdot\,, t_n))\|_{L^2(\Gamma(t_n))}{}^2\Big)^{1/2}. \end{aligned}$$

We computed the numerical solution using the backward Euler method coupled with ESFEM for four different meshes and a series of time steps, until the final time $T = 1$. Then we computed the errors in the discrete norm and seminorm, cf. (B.10), these error

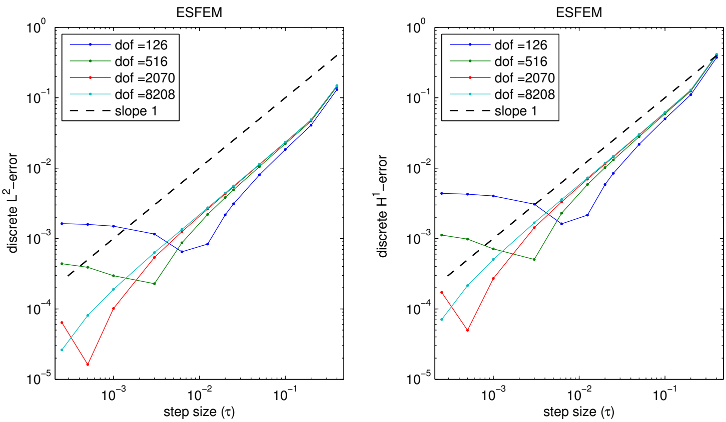

Figure B.1.: Errors of the ESFEM and the backward Euler method at time $T = 1$

curves are displayed in figure B.1. The convergence in time can be seen (note the reference line), while for sufficiently small τ the spatial error is dominating, in agreement with the theoretical results. Figure B.2 shows a similar plot: the errors here were obtained by the three step linearly implicit BDF method coupled with ESFEM for five different meshes and a series of time steps. Again the results are matching with the theoretical ones. We note that, for this example, no significant difference appeared between the fully implicit and linearly implicit BDF methods.

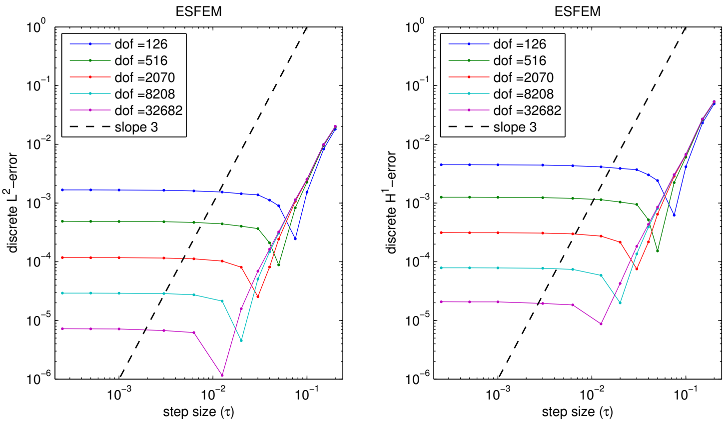

Figure B.2.: Errors of the ESFEM and the 3 step linearly implicit BDF method at time $T = 1$

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B.8. A priori estimates

The result presented here gives regularity result, with a t independent constant, for the elliptic problems appeared in the proofs of the errors in the Ritz map.

Theorem B.8.1 (Elliptic regularity for evolving surfaces). *Let $\Gamma(t)$ be an evolving surface, fix a $t \in [0, T]$ and a function $\xi\colon \Gamma(t) \to \mathbb{R}$.*

(i) Let $f \in H^{-1}(\Gamma(t))$ and

$$L(u) := -\nabla_\Gamma \cdot \big(\mathcal{A}(\xi)\nabla_\Gamma u\big) + u. \tag{B.24}$$

Then there exists a weak solution $u \in H^1(\Gamma(t))$ of the problem

$$L(u) = f \tag{B.25}$$

with the estimate

$$\|u\|_{H^1(\Gamma(t))} \leq c\, \|f\|_{H^{-1}(\Gamma(t))}, \tag{B.26}$$

where the constant above is independent of t.

(ii) Let $L(u)$ *be* (B.24)*, let* $f \in L^2(\Gamma(t))$ *and let* $u \in H^1(\Gamma(t))$ *be a weak solution of* (B.25)*. Then* u *is a strong solution of* (B.25)*, i.e.* u *solves* (B.25) *almost everywhere and there exists a constant* $c > 0$ *independent of* t *and* u *such that*

$$\|u\|_{H^2(\Gamma(t))} \leq c\big(\|u\|_{L^2(\Gamma(t))} + \|f\|_{L^2(\Gamma(t))}\big).$$

Proof. For (i): The Lax-Milgram lemma shows the existence of the weak solution u. Because the coercivity and boundedness constants (B.2) and (B.3) are independent of t, the constant in (B.26) also not depends on t. For (ii): Basically we consider pullback of the operator L to $\Gamma(0)$, rewrite it in a local chart and then apply the corresponding results of [46].

By assumption there exists a diffeomorphic parametrization of our evolving surface $\Gamma(t)$, i.e. we have a smooth map

$$\Phi\colon \Gamma(0) \times [0, T] \to \mathbb{R}^{m+1}$$

such that

$$\Phi_t\colon \Gamma(0) \to \mathbb{R}^{m+1}, \quad \Phi_t(x) \coloneqq \Phi(x, t)$$

is an injective immersion which is a homeomorphism onto its image with $\Phi_t(\Gamma(0)) = \Gamma(t)$. Because $\Gamma(0)$ is compact, there exists a finite atlas

$$\Big(\varphi_n(0)\colon U_n(0) \subset \Gamma(0) \to \mathbb{R}^m\Big)_{n=1}^{k}$$

such that $\varphi_n(U_n(0)) \subset \mathbb{R}^m$ is bounded and a finite family of compact sets $(V_n(0))_{n=1}^k$ with $V_n(0) \subset U_n(0)$, and $\bigcup_{n=1}^k V_n(0) = \Gamma(0)$. Using the properties of the diffeomorphic parametrization the new collections,

$$V_n(t) \coloneqq \Phi_t(V_n(0)), \quad U_n(t) \coloneqq \Phi_t(U_n(0)), \quad \varphi_n(t) \coloneqq \varphi_n(0) \circ \Phi_t^{-1},$$

still have the same properties. Now consider the following standard formulae of Riemannian geometry [38]:

$$\nabla_\Gamma h(x, t) = \sum_{i,j=1}^{m} g_n^{ij}(x, t) \frac{\partial(h \circ \varphi_n(t)^{-1})}{\partial x^i} \frac{\partial(\varphi_n(t)^{-1})}{\partial x^j},$$

where

$$g_{ij,n}(x,t) := \frac{\partial(\varphi_n(t)^{-1})}{\partial x^i} \cdot \frac{\partial(\varphi_n(t)^{-1})}{\partial x^j}\bigg|_x$$

is the first fundamental form and $g_n^{ij}(x,t)$ are entries of the inverse matrix of $g_n := (g_{ij,n})$, and

$$\nabla_\Gamma \cdot X = \sum_{i,j=1}^{m} \frac{1}{\sqrt{g_n}} \frac{\partial}{\partial x^i} (\sqrt{g_n} g_n^{ij} X_j)$$

where X is a smooth tangent vector field with $X_j = X \cdot \frac{\partial(\varphi(t)^{-1})}{\partial x^j}$ and $\sqrt{g_n} := \sqrt{\det(g_n)}$. It is straightforward to calculate that

$$\begin{aligned}\Big(-\nabla_\Gamma \cdot \mathcal{A}\nabla_\Gamma u + u\Big) \circ \varphi_n(t)^{-1}(x) = &\sum_{i,j=1}^{m} a_{ij,n}(x,t) \frac{\partial^2(u \circ \varphi_n(t)^{-1})}{\partial x^i \partial x^j} \\ &+ \sum_{i=1}^{m} b_{i,n}(x,t) \frac{\partial(u \circ \varphi_n(t)^{-1})}{\partial x^i} \\ &+ c_n(x,t)\, u \circ \varphi_n(t)^{-1}\end{aligned}$$

for appropriate $a_{ij,n} \in W^{1,\infty}(U_n(t))$ and $b_{i,n}, c_n \in L^\infty(U_n(t))$ where $a_{ij,n}$ represents a uniform elliptic matrix. Observe that the assumptions (B.2), (B.3) and (B.4) implies that the function above can be bounded independently of t. Now [46, theorem 8.8] states that, if $u \circ \varphi_n(t)^{-1}$ is the H^1-weak solution of (B.25), then it must be a strong solution as well.

For the estimate in (ii) observe that [46, theorem 9.11] gives us for $V_n(t)$ in particular the estimate

$$\begin{aligned}\|u \circ \varphi_n(t)^{-1}\|_{H^2(V_n'(t))} \leq c(&\|u \circ \varphi_n(t)^{-1}\|_{L^2(U_n'(t))} \\ &+ \|f \circ \varphi_n(t)^{-1}\|_{L^2(U_n'(t))}), \quad \text{(B.27)}\end{aligned}$$

where $V_n' := \varphi_n(t)(V_n(t))$ and $U_n' := \varphi_n(t)(U_n(t))$ are obviously independent of t. Thus the constant above is independent of t. Then theorem 3.41 in [1] shows that

$$\|u\|_{H^2(V_n(t))} \leq c(t)\, \|u \circ \varphi_n(t)^{-1}\|_{H^2(V_n'(t))} \leq c\, \|u \circ \varphi_n(t)^{-1}\|_{H^2(V_n'(t))},$$

where the constant in the middle depends continuously on t, hence the last constant is independent of t. A similar estimate holds for the right-hand side of (B.27). An easy calculation finishes the proof for (ii). ∎

Appendix C.

Maximum norm stability and error estimates for the evolving surface finite element method

The content of this chapter is based on [55].

Abstract

We show convergence in the natural L^∞- and $W^{1,\infty}$-norm for a semidiscretization with linear finite elements of a linear parabolic partial differential equations on evolving surfaces. To prove this we show error estimates for a Ritz map, error estimates for the material derivative of a Ritz map and a weak discrete maximum principle.

C.1. Introduction

Many important problems can be modeled by partial differential equations (PDEs) on evolving surfaces. Examples for such equations are given in material sciences, fluid mechanics and biophysics [44, 51, 41]. The basic linear parabolic PDE on a moving surface is

$$\partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u = f \quad \text{on } \Gamma(t).$$

Here the velocity v is explicitly given and we seek to compute a numerical approximation to the exact solution u. Dziuk and Elliott [32] introduced the evolving surface finite element method (ESFEM) to solve this problem. Error estimates for the semidiscretization with piecewise linear finite elements in the L^2- and H^1-norm are given in [35, 31].

The aim of this work is to give error bounds for the semidiscretization with linear finite elements in the L^∞- and $W^{1,\infty}$-norm. Such estimates are of interest for nonlinear parabolic PDEs on evolving surfaces and if the velocity v is not explicitly given, but depends on the exact solution u. Example of such problems are given in [21, 44, 8, 42, 17] and the references therein. The treatment of such more general equations are beyond the scope of this paper.

Our convergence proof for the semidiscretization of the linear heat equation on evolving surfaces relies on three main results. In our first result we give some error bounds in the L^∞- and $W^{1,\infty}$-norm for a suitable Ritz map. We give a proof based on Nitsche's weighted norm technique [70].

Our second result gives bounds in the L^∞- and $W^{1,\infty}$-norm for our material derivative Ritz map error, since the time derivative does not commute with our Ritz map.

Our third result extends a weak finite element maximum principle, which is due to Schatz, Thomée and Wahlbin [75], to the evolving surface case. In [75] they use basic properties of the semigroup corresponding to the linear heat equation on a bounded domain. Since there is no semigroup theory for the linear heat equation on evolving surfaces we are going to use a different approach.

The layout of the paper is as follows. We begin in section C.2 by fixing some notation and introducing the most basic notion. In the first three subsection of section C.3 we quickly develop the evolving surface finite element method (ESFEM) and recall basic results and estimates. In the following three subsection we introduce a surface version of Nitsche's weighted norms and finish with an L^2-projection. In section C.4 we give error bound in the maximum norm for our Ritz map. In section C.5 we derive a weak ESFEM maximum principle. In section C.6 we give error bounds for the semi discretization of the linear heat equation on evolving surfaces in the L^∞- and $W^{1,\infty}$-norm. In section C.7 we present the results of a numerical experiment. We gather technical details for calculations with our weight functions in appendix C.9.

C.2. A parabolic problem on evolving surfaces

Let us consider a smooth evolving closed hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$ (our main focus is on the case $m = 2$, but some of our results hold for more general cases), $0 \leq t \leq T$, which moves with a given smooth velocity v. More precise we assume that there exists a smooth dynamical system $\Phi\colon \Gamma_0 \times [0,T] \to \mathbb{R}^{m+1}$, such that for each $t \in [0,T]$ the map $\Phi_t \coloneqq \Phi(\,\cdot\,,t)$ is an embedding. We define $\Gamma(t) \coloneqq \Phi_t(\Gamma_0)$ and define the velocity v via the equation $\partial_t\Phi(x,t) = v(\Phi(x,t),t)$. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of the function u. The tangential gradient is given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu\nu$, where ν is the unit normal and finally we define the Laplace-Beltrami operator via $\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u$. This article shares the setting of Dziuk and Elliott [31, 35], and [66].

We consider the following linear problem derived in [31, section 3]:

$$\begin{cases} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u = f & \text{on } \Gamma(t), \\ u(\,\cdot\,,0) = u_0 & \text{on } \Gamma(0). \end{cases} \tag{C.1}$$

We use Sobolev spaces on surfaces: For a sufficiently smooth surface Γ and $1 \leq p \leq \infty$ we define

$$W^{1,p}(\Gamma) = \{\eta \in L^p(\Gamma) \mid \nabla_\Gamma \eta \in L^p(\Gamma)^{m+1}\},$$

and analogously $W^{k,p}(\Gamma)$ for $k \in \mathbb{N}$ [31, section 2.1]. We set $H^k(\Gamma) = W^{k,2}(\Gamma)$. Finally, $\mathcal{G}_T$ denotes the space-time manifold, i.e. $\mathcal{G}_T \coloneqq \cup_{t\in[0,T]}\Gamma(t) \times \{t\}$.

If $f = 0$ then a weak formulation of this problem reads as follows.

Definition C.2.1 (weak solution, [31] Definition 4.1). A function $u \in H^1(\mathcal{G}_T)$ is called a *weak solution* of (C.1), if for almost every $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(.,0) = u_0$.

For suitable f and u_0 existence and uniqueness results, for the strong and the weak problem, were obtained in [31, section 4].

Throughout this article we assume that f and u_0 a such regular that $u \in W^{3,\infty}(\mathcal{G}_T)$. Furthermore we set for simplicity reasons in all sections $f = 0$, since the extension of our results to the inhomogeneous case are straightforward.

C.3. Preliminaries

We give a summary of this section. In section C.3.1 we introduce the ESFEM, which is due to Dziuk and Elliott [31]. In section C.3.2 we recall the lifting process, which originates in Dziuk [29]. In section C.3.3 we collect important results from Dziuk and Elliott [35] and sometimes state them in a slightly more general fashion.

In section C.3.4 we introduce weighted norms, which are due to Nitsche [70], and give connections to the L^∞-norm. In section C.3.5 we give interpolation estimates in the L^2-, L^∞- and weighted norms and further give some special interpolation estimates in weighted norms. The latter two were first stated in Nitsche [70]. In section C.3.6 we introduce an L^2-projection, give a stability bound in L^p-norms and finish with a error estimate with respect to a different weight function. The basic reference for this is Douglas, Dupont, Wahlbin [27] and Schatz, Thomée, Wahlbin [75].

C.3.1. Semidiscretization with the evolving surface finite element method

The smooth surface $\Gamma(t)$ is approximated by a triangulated one denoted by $\Gamma_h(t)$, whose vertices $a_j(t) = \Phi(a_j(0), t)$ are sitting on the surface for all time, such that

$$\Gamma_h(t) = \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ are forming an admissible triangulation $\mathcal{T}_h(t)$, with h denoting the maximum diameter. Admissible triangulations were introduced in [31, section 5.1]: Every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by ch with $c > 0$, and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$. The discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)}\phi := \nabla\phi - \nabla\phi \cdot \nu_h\nu_h,$$

understood in a piecewise sense, with ν_h denoting the normal to $\Gamma_h(t)$ (see [31]).

For every $t \in [0, T]$ we define the finite element subspace $S_h(t)$ spanned by the continuous, piecewise linear evolving basis functions χ_j, satisfying

$$\chi_j(a_i(t), t) = \delta_{ij} \quad \text{for all } i, j = 1, 2, \dots, N,$$

therefore

$$S_h(t) = \operatorname{span}\{\chi_1(\,\cdot\,, t), \chi_2(\,\cdot\,, t), \dots, \chi_N(\,\cdot\,, t)\}.$$

We interpolate the dynamical system Φ by $\Phi_h\colon \Gamma_h(0) \to \mathbb{R}^{m+1}$, the discrete dynamical system of $\Gamma_h(t)$. This defines a discrete surface velocity V_h via $\partial_t \Phi_h(y_h, t) = V_h(\Phi_h(y_h, t), t)$. Then the discrete material derivative is given by

$$\partial_h^\bullet \phi_h = \partial_t \phi_h + V_h \cdot \nabla \phi_h \qquad (\phi_h \in S_h(t)).$$

The key *transport property* derived in [31, Proposition 5.4], is the following

$$\partial_h^\bullet \chi_k = 0 \qquad \text{for} \quad k = 1, 2, \dots, N. \tag{C.2}$$

The spatially discrete problem for evolving surfaces is: Find $U_h \in S_h(t)$ such that for all $\phi_h \in S_h(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h(t)} U_h \phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h = \int_{\Gamma_h(t)} U_h \partial_h^\bullet \phi_h, \tag{C.3}$$

with the initial condition $U_h(\,\cdot\,, 0) = U_h^0 \in S_h(0)$ being a sufficient approximation to u_0.

C.3.2. Lifts

In the following we recall the so called *lift operator*, which was introduced in [29] and further investigated in [31, 35]. The lift operator projects a finite element function on the discrete surface onto a function on the smooth surface.

Using the *oriented distance function d* ([31, section 2.1]), for a continuous function $\eta_h\colon \Gamma_h(t) \to \mathbb{R}$ its lift is define as

$$\eta_h^l(x^l, t) := \eta_h(x, t), \qquad x \in \Gamma(t),$$

where for every $x \in \Gamma_h(t)$ the value $x^l = x^l(x,t) \in \Gamma(t)$ is uniquely defined via $x = x^l + \nu(x^l,t)d_X(x,t)$. This notation for x^l will also be used later on. By η^{-l} we mean the function whose lift is η, and by E_h^l we mean the lift of the triangle E_h.

The following pointwise estimate was shown in the proof of lemma 3 from Dziuk [29]:

$$\frac{1}{c}\,|\nabla_\Gamma \eta_h^l(x^l)| \leq |\nabla_{\Gamma_h}\eta_h(x)| \leq c\,|\nabla_\Gamma \eta_h^l(x^l)|. \tag{C.4}$$

We now recall some notions using the lifting process from [29, 31]. We have the lifted finite element space

$$S_h^l(t) := \{\varphi_h = \phi_h^l \mid \phi_h \in S_h\}.$$

By δ_h we denote the quotient between the continuous and discrete surface measures, $\mathrm{d}A$ and $\mathrm{d}A_h$, defined as $\delta_h\,\mathrm{d}A_h = \mathrm{d}A$. For these quantities we recall some results from [31, Lemma 5.1], [35, Lemma 5.4] and [66, Lemma 6.1].

Lemma C.3.1. *We have the estimates*

$$\|d\|_{L^\infty(\Gamma_h(t))} \leq ch^2, \quad \|1-\delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2,$$

with constants independent of t and h.

C.3.3. Geometric estimates and bilinear forms

Let us denote by $\Phi_h^l\colon \Gamma_0 \times [0,T] \to \mathbb{R}^{m+1}$ the lift of Φ_h. We define the velocity v_h via the formula $\partial_t\Phi_h^l(x,t) = v_h(\Phi_h^l(x,t),t)$. Then the discrete material derivative on $\Gamma(t)$ is given by

$$\partial_h^\bullet u = \partial_t u + v_h \cdot \nabla u,$$

which satisfies the following relations, cf. [35]:

$$\partial^\bullet u = \partial_h^\bullet u + (v_h - v) \cdot \nabla_\Gamma u, \tag{C.5}$$

$$\|v - v_h\|_{L^\infty(\Gamma(t))} + h \|v - v_h\|_{W^{1,\infty}(\Gamma(t))} \le ch^2 \|v\|_{W^{2,\infty}(\Gamma(t))}, \tag{C.6}$$

We use the time dependent bilinear forms defined in [35, Section 3.3]: For $z, \varphi \in H^1(\Gamma(t))$ we set

$$\begin{aligned}
a(t; z, \varphi) &= \int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, \\
m(t; z, \varphi) &= \int_{\Gamma(t)} z\varphi, \\
g(t; v; z, \varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot v) z\varphi, \\
b(t; v; z, \varphi) &= \int_{\Gamma(t)} \mathcal{B}(v) \nabla_\Gamma z \cdot \nabla_\Gamma \varphi,
\end{aligned}$$

and for $Z_h, \phi_h \in H^1\big(\Gamma_h(t)\big)$ we set

$$\begin{aligned}
a_h(t; Z_h, \phi_h) &= \sum_{E \in \mathcal{T}_h} \int_E \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\
m_h(t; Z_h, \phi_h) &= \int_{\Gamma_h(t)} Z_h \phi_h \\
g_h(t; V_h; Z_h, \phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot V_h) Z_h \phi_h, \\
b_h(t; V_h; Z_h, \phi_h) &= \sum_{E \in \mathcal{T}_h} \int_E \mathcal{B}_h(V_h) \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h,
\end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense, and with the matrices

$$\begin{aligned}
\mathcal{B}(v)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot v) - \big((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i\big), \\
\mathcal{B}_h(V_h)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot V_h) - \big((\nabla_{\Gamma_h})_i (V_h)_j + (\nabla_{\Gamma_h})_j (V_h)_i\big),
\end{aligned}$$

where $i, j = 1, 2, \dots, m+1$.

First, we state a lemma for the time derivatives of the bilinear and then state bounds of the geometric perturbation errors in the bilinear forms. Important and often used results are the bounds of the geometric perturbation errors in the bilinear forms.

Lemma C.3.2 (Discrete transport property). *For $z, \varphi \in H^1(\Gamma(t))$ we have*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m(z,\varphi) &= m(\partial_h^\bullet z, \varphi) + m(z, \partial_h^\bullet \varphi) + g(v_h; z, \varphi), \\ \frac{\mathrm{d}}{\mathrm{d}t} a(z,\varphi) &= a(\partial_h^\bullet z, \varphi) + a(z, \partial_h^\bullet \varphi) + b(v_h; z, \varphi). \end{aligned} \tag{C.7}$$

Similarly for $Z_h, \phi_h \in H^1(\Gamma_h(t))$ we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m_h(Z_h,\phi_h) &= m_h(\partial^\bullet Z_h, \phi_h) + m_h(Z_h, \partial^\bullet \phi_h) + g_h(V_h; Z_h, \phi_h), \\ \frac{\mathrm{d}}{\mathrm{d}t} a_h(Z_h,\phi_h) &= a_h(\partial^\bullet Z_h, \phi_h) + a_h(Z_h, \partial^\bullet \phi_h) + b_h(V_h; Z_h, \phi_h). \end{aligned} \tag{C.8}$$

Lemma C.3.3. *For all $1 \leq p, q \leq \infty$, that are conjugate, $p^{-1} + q^{-1} = 1$, and for arbitrary $Z_h \in L^p(\Gamma_h(t))$ and $\phi_h \in L^q(\Gamma_h(t))$, with corresponding lifts $z_h \in L^p(\Gamma(t))$ and $\varphi_h \in L^q(\Gamma(t))$ we have the following estimates:*

$$\begin{aligned} |m(z_h,\varphi_h) - m_h(Z_h,\phi_h)| &\leq ch^2 \|z_h\|_{L^p(\Gamma(t))} \|\varphi_h\|_{L^q(\Gamma(t))}, \\ |a(z_h,\varphi_h) - a_h(Z_h,\phi_h)| &\leq ch^2 |z_h|_{W^{1,p}(\Gamma(t))} |\varphi_h|_{W^{1,q}(\Gamma(t))}, \\ |g(v_h; z_h,\varphi_h) - g_h(V_h; Z_h,\phi_h)| &\leq ch^2 \|z_h\|_{L^p(\Gamma(t))} \|\varphi_h\|_{L^q(\Gamma(t))}, \\ |b(v_h; z_h,\varphi_h) - b_h(V_h; Z_h,\phi_h)| &\leq ch^2 |z_h|_{W^{1,p}(\Gamma(t))} |\varphi_h|_{W^{1,q}(\Gamma(t))}, \end{aligned}$$

where the constant $c > 0$ is independent from $t \in [0, T]$ and the *mesh width h.*

Proof of lemma C.3.2 and C.3.3. These geometric estimates were established for the case $p = q = 2$ in [35, Lemma 5.5] and [62, Lemma 7.5]. To show the estimates for general p and q, the same proof apply, except the last step where we use a Hölder inequality. ∎

C.3.4. Weighted norms and basic estimates

Similarly, as in the works of Nitsche [70], weighted Sobolev norms and their properties play a very important and central role. In this section we recall some basic results for them.

Definition C.3.4 (Weight function). For $\gamma > 0$ sufficiently big but independent of t and h we set

$$\rho\colon [0,\infty) \to [0,\infty), \quad \rho^2 \coloneqq \rho^2(h) \coloneqq \gamma h^2 \,|\log h|.$$

We define a weight function $\mu = \mu(t;\,\cdot\,)\colon \Gamma(t) \to \mathbb{R}$ via the formula

$$\mu(x) \coloneqq \mu(x,y) \coloneqq |x-y|^2 + \rho^2 \quad \forall x \in \Gamma(t). \tag{C.9}$$

The actual choice of γ is going to be clear from the proofs.

Definition C.3.5 (Weighted norms, [70, section 2]). Let μ be a weight function and $\alpha \in \mathbb{R}$. We define the norms

$$\begin{aligned}
\|u\|_{L^2,\alpha}^2 &= \int_\Gamma \mu^{-\alpha}\,|u|^2,\\
\|u\|_{H^1,\alpha}^2 &= \|u\|_{L^2,\alpha}^2 + \|\nabla_\Gamma u\|_{L^2,\alpha}^2,\\
\|u\|_{H^2,\alpha}^2 &= \|u\|_{H^1,\alpha}^2 + \|\nabla_\Gamma^2 u\|_{L^2,\alpha}^2.
\end{aligned}$$

Lemma C.3.6. *Let* $\dim \Gamma(t) = 2$. *Let* $\phi_h \in S_h(t)$ *with corresponding lift* $\varphi_h \in S_h^l(t)$. *Then there exist constants* $c > 0$ *independent of* t, h *and* γ *such that*

$$\|\varphi_h\|_{L^\infty(\Gamma(t))} \leq ch\,|\log h|\,\|\varphi_h\|_{L^2,2}, \tag{C.10}$$

$$\|\varphi_h\|_{W^{1,\infty}(\Gamma(t))} \leq c\gamma\,|\log h|^{1/2}\|\varphi_h\|_{H^1,1}. \tag{C.11}$$

Proof. There is a point $y_{0,h} \in E_0 \subset \Gamma_h(t)$ such that

$$\|\phi_h\|_{W^{1,\infty}(\Gamma_h(t))} = |\phi_h(y_{0,h})| + |\nabla_{\Gamma_h}\phi_h(y_{0,h})| = \|\phi_h\|_{W^{1,\infty}(E_0)}.$$

Note that on E_0 the estimate $\mu_h(x_h) \leq c\rho^2$ holds for $h < h_0$, h_0 sufficiently small. Then the second bound yields from using inverse inequality (lemma C.3.13) and (C.54). The bound (C.10) is proved using similar arguments. ∎

Lemma C.3.7. *Let* $\dim \Gamma(t) = 2$. *Let* $u\colon \Gamma(t) \to \mathbb{R}$ *be a sufficiently smooth function. Then the following estimates hold, with a sufficiently small* $h_0 > 0$,

$$\|u\|_{L^2,2} \leq c\rho^{-1}\,\|u\|_{L^\infty(\Gamma(t))}, \tag{C.12}$$

$$\|u\|_{H^1,1} \leq c\,|\log\rho|^{1/2}\,\|u\|_{W^{1,\infty}(\Gamma(t))}, \tag{C.13}$$

for $0 < h < h_0$, *where the constant* $c = c(h_0) > 0$ *is independent of* t, h *and* γ.

Proof. For $\alpha = 1$ or 2 we obviously have

$$\|u\|_{L^2,\alpha} \leq \|u\|_{L^\infty(\Gamma(t))}\,\|\mu^{-\alpha}\|^{\frac{1}{2}}_{L^1(\Gamma(t))}.$$

Then a straightforward calculation, using appendix C.9 shows both estimates. ∎

Naturally, there is a weighted version of the Cauchy-Schwarz inequality, namely we have

$$\begin{aligned} |a^*(z_h, \varphi_h)| &\le \|z_h\|_{H^1,\alpha} \|\varphi_h\|_{H^1,-\alpha}, \\ |a_h^*(Z_h, \phi_h)| &\le c\|z_h\|_{H^1,\alpha} \|\varphi_h\|_{H^1,-\alpha}, \end{aligned} \tag{C.14}$$

and similarly for the bilinear forms g and b. Furthermore, this yields a weighted version of the geometric errors of the bilinear forms (lemma C.3.3).

Lemma C.3.8. *The following estimates hold, with a constant $c > 0$ independent of t, h and γ,*

$$|a^*(z_h^l, \phi_h^l) - a_h^*(Z_h, \phi_h)| \le ch^2 \|z_h^l\|_{H^1,\alpha} \|\phi_h^l\|_{H^1,-\alpha}, \tag{C.15}$$

$$\begin{aligned} &|(g + b)(v_h; z_h^l, \phi_h^l) - (g_h + b_h)(V_h; Z_h, \phi_h)| \\ &\quad \le ch^2 \|z_h^l\|_{H^1,\alpha} \|\phi_h^l\|_{H^1,-\alpha}. \end{aligned} \tag{C.16}$$

Lemma C.3.9. *(i) Derivatives of μ^{-1} are bounded as*

$$|\nabla_\Gamma \mu^{-1}| \le 2\mu^{-1,5}, \quad |\Delta_\Gamma \mu^{-1}| \le c\mu^{-2} \tag{C.17}$$

with $c > 0$ independent of t, h and γ.

(ii) For arbitrary $u \in H^1(\Gamma(t))$ the following norm inequalities hold:

$$\|\mu^{-1} u\|_{H^1,-1} \le c(\|u\|_{L^2,2} + \|u\|_{H^1,1}), \tag{C.18}$$

$$\|\mu^{-2} u\|_{L^2,-1} \le \rho^{-1} \|u\|_{L^2,2}. \tag{C.19}$$

Proof. (i): The first esitmate follows from

$$|\nabla_\Gamma \mu^{-1}| \le |\nabla \mu^{-1}| \le \frac{2\,|x - y|}{\mu^2} \le \frac{2\sqrt{\mu}}{\mu^2}.$$

For the second inequality consider the formula,

$$\Delta_\Gamma f = \Delta \bar{f} - \nabla^2 \bar{f}(\nu,\nu) - H\nu \cdot \nabla \bar{f},$$

where $\bar{f}\colon U \to \mathbb{R}$ is an extension of the sufficiently smooth function f to an open neighborhood $U \subset \mathbb{R}^{m+1}$ of $\Gamma(t)$, $\nabla^2 \bar{f}$ denotes the Hessian of $\bar{f}$ and H denotes the trace of the Weingarten map of $\Gamma(t)$.

(ii) In order to show these estimates we use the bounds (C.17) obtained above. ■

C.3.5. Interpolation and inverse estimates

Here we collect some results involving evolving surface finite element functions.

For a sufficiently regular function $u\colon \Gamma(t) \to \mathbb{R}$ we denote by $\widetilde{I}_h u \in S_h(t)$ its interpolation on $\Gamma_h(t)$. Then the finite element interpolation is given by $I_h u = (\widetilde{I}_h u)^l \in S_h^l(t)$, having the error estimate below, cf. [34].

Lemma C.3.10. *For $m \leq 3$ and $p \in \{2, \infty\}$, there exists a constant $c > 0$ independent of h and t such that for $u \in W^{2,p}(\Gamma(t))$:*

$$\begin{aligned} &\|u - I_h u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma(u - I_h u)\|_{L^p(\Gamma(t))} \\ &\qquad \leq ch^2 \Big(\|\nabla_\Gamma^2 u\|_{L^p(\Gamma(t))} + h\, \|\nabla_\Gamma u\|_{L^p(\Gamma(t))} \Big). \end{aligned}$$

The interpolation estimates hold also if weighted norms are considered.

Lemma C.3.11. *There exists a constant $c > 0$ such that for $u \in W^{2,\infty}(\Gamma(t))$ it holds*

$$\|u - I_h u\|_{L^2,2}^2 + \|u - I_h u\|_{H^1,1}^2 \leq ch^2\, |\log h|\, \|u\|_{W^{2,\infty}(\Gamma(t))}^2. \quad \text{(C.20)}$$

Proof. Use a Hölder inequality, lemma C.3.10 and lemma C.3.7 (C.12), (C.13) with the choice $u \equiv 1$. ■

Lemma C.3.12. *There exists $h_0 > 0$, $\gamma_0 > 0$ such that for all $\alpha \in \mathbb{R}$ there exists a constant $c = c(h_0, \gamma_0) > 0$ independent of t and h such that for all $\gamma > \gamma_0$ for the weight μ, cf.* (C.9), *and for all $h < h_0$ the following inequalities holds:*

(i) Let $u \in H^1(\Gamma(t))$ be curved element-wise H^2. The interpolation $I_h u \in S_h^l(t)$ satisfies

$$\begin{aligned} &\|u - I_h u\|_{L^2,\alpha} + h\|\nabla_\Gamma(u - I_h u)\|_{L^2,\alpha} \\ &\quad \leq ch^2(\|\nabla_\Gamma^2 u\|_{L^2,\alpha} + ch\|\nabla_\Gamma u\|_{L^2,\alpha}), \end{aligned} \tag{C.21}$$

where $\|\nabla_\Gamma^2 u\|_{L^2,\alpha}$ is understood curved element-wise.

(ii) For any $\varphi_h \in S_h^l(t)$ the following estimate holds:

$$\begin{aligned} &\|\mu^{-1}\varphi_h - I_h(\mu^{-1}\varphi_h)\|_{H^1,-1} \\ &\quad \leq c\left(\frac{h}{\rho} + h\right)(\|\varphi_h\|_{L^2,2} + \|\nabla_\Gamma\varphi_h\|_{L^2,1}). \end{aligned} \tag{C.22}$$

Proof. (i): To prove inequality (C.21) it suffices to show that there exists a constant $c = c(\alpha) > 0$ independent of t, h such that for each element $K \in \mathcal{T}_h(t)$ it holds

$$\begin{aligned} &\int_{K^l} \mu^\alpha((w - I_h w)^2 + h\,|\nabla_\Gamma(w - I_h w)|^2) \\ &\quad \leq ch^2 \int_{K^l} \mu^\alpha(|\nabla_\Gamma^2 w|^2 + ch\,|\nabla_\Gamma w|^2), \end{aligned}$$

where $K^l \subset \Gamma(t)$ denote the lifted curved element of K. It is easy to show that there exists $\gamma_0 = \gamma_0(h_0) > 0$ and $c = c(\gamma_0) > 0$ such that for all $\gamma > \gamma_0$ it holds

$$\max_{K \in \mathcal{T}_h} \left(\frac{\max_{x \in K^l} \mu(x,y)}{\min_{x \in K^l} \mu(x,y)} \right) \le c.$$

A straightforward calculation finishes the proof.

(ii): For an arbitrary function $f\colon \Gamma_h(t) \to \mathbb{R}$, which is element-wise H^2, a short calculation, similar to the one done in Dziuk [29, lemma 3], shows that

$$|(\nabla_\Gamma)_i(\nabla_\Gamma)_j(f^l)| \le c(|((\nabla_{\Gamma_h})_i(\nabla_{\Gamma_h})_j f)^l| + ch\,|\nabla_\Gamma(f^l)|),$$

for a sufficiently small $h_0 > h > 0$. A straightforward calculation combined with (i) and (C.17) shows the claim. ■

The following general version of inverse estimates for finite element functions plays a key role later on, cf. [75].

Lemma C.3.13 (Inverse estimate). *Let $1 \le q \le p \le \infty$ and $0 \le m \le k \le 1$. Then here exists $c(q,p,m,k) > 0$ such that for each triangle $E_h(t) \subset \Gamma_h(t)$ the following inequality holds for all $\varphi_h \in S_h(t)$*

$$\|\varphi_h(t)\|_{W^{k,p}(E_h(t))} \le ch^{m-k-2(1/q-1/p)}\,\|\varphi_h(t)\|_{W^{m,q}(E_h(t))}.$$

The lemma above does not require a separate proof, since it uses the referent element technique.

Lemma C.3.14. *There exists $c > 0$ with*

$$\|\varphi_h\|_{L^\infty(\Gamma(t))} \le \left| \frac{1}{V} \int_{\Gamma(t)} \varphi_h(y) \mathrm{d}V(y) \right| + c\,|\log h|^{1/2}\,\|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}.$$

Proof. Follow the steps in Schatz, Thomée, Wahlbin [75] using the Green's function from theorem C.8.1 and calculating with geodesic polar coordinates. ■

C.3.6. Estimates for an L^2-projection

This section shows some technical results for the L^2-projection, which is denoted by $P_{h,0}$ (in contrast with the Ritz map which will be denoted by P_1).

Definition C.3.15 (L^2-projection). We define

$$P_{h,0}(t)\colon L^2(\Gamma_h(t)) \to S_h(t)$$

as follows: Let $u_h \in L^2(\Gamma_h(t))$ be given. Then there exits a unique finite element function $P_{h,0}(t)u \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ it holds

$$m_h(P_{h,0}(t)u_h, \phi_h) = m_h(u_h, \phi_h). \tag{C.23}$$

Theorem C.3.16. *For $p \in [1, \infty]$ let $u_h \in L^p(\Gamma_h(t))$. Then there exists a constant $c > 0$ independent of h and t such that*

$$\|P_{h,0}(t)u_h\|_{L^p(\Gamma_h(t))} \leq c\,\|u_h\|_{L^p(\Gamma_h(t))}.$$

Further there exists $c_2, c_3 > 0$ independent of h and t such that for $A_h^1(t)$ and $A_h^2(t)$ disjoint subsets of $\Gamma_h(t)$ with $\operatorname{supp}(u_h) \subseteq A_h^1$ we have

$$\|P_{h,0}(t)u_h\|_{L^2(A_h^2(t))} \leq c_2 e^{-c_3 \operatorname{dist}_h(A_h^1, A_h^2) h^{-1}}\,\|u_h\|_{L^2(A_h^1(t))}, \tag{C.24}$$

where $\operatorname{dist}_h(x, y) = \operatorname{dist}_{\Gamma_h(t)}(x, y)$ is the intrinsic Riemannian distance of $\Gamma_h(t)$.

We do not need to reprove this result from Douglas, Dupont and Wahlbin [27, equation (6) and (7)], since their proof holds without any serious modification in our setting.

For the proof of our discrete weak maximum principle we are going to use a different weight function then (C.9). Let $[0,T] \to \mathbb{R}^{m+1}, t \mapsto y(t)$ be a curve with the property $y(t) \in \Gamma(t)$. In the following we write y instead of $y(t)$. We define

$$\sigma(x) := \sigma^y(x) := \sigma(x,y) := \left(|x-y|^2 + h^2\right)^{1/2}. \tag{C.25}$$

We gather some estimates concerning σ in the next lemma.

Lemma C.3.17. *There exists a constant $c > 0$ independent of t and h such that the following estimates hold*

$$\|\partial^\bullet \sigma\|_{L^\infty(\Gamma(t))} \le c, \quad \|\partial_h^\bullet \sigma\|_{L^\infty(\Gamma(t))} \le c, \tag{C.26}$$

$$\begin{aligned} \|\nabla_\Gamma \sigma\|_{L^\infty(\Gamma(t))} \le 1, \quad |\nabla_\Gamma^2 \sigma| \le c\left(\frac{1}{\sigma} + 1\right), \\ \|\nabla_\Gamma^2(\sigma^2)\|_{L^\infty(\Gamma(t))} \le c. \end{aligned} \tag{C.27}$$

The proof of this lemma is a straightforward calculation and is omitted here.

Lemma C.3.18. *There exists $c > 0$ such for fixed $t \in [0,T]$, $x_h \in \Gamma_h(t)$, $\sigma = \sigma^{x_h}$, $\phi_h \in S_h(t)$ and $\psi_h := P_{h,0}(\sigma^2 \varphi_h)$ the following inequality holds:*

$$\begin{aligned} \|\sigma^2\phi_h - \psi_h\|_{L^2(\Gamma_h(t))} + h\,\|\nabla_{\Gamma_h}(\sigma^2\phi_h - \psi_h)\|_{L^2(\Gamma_h(t))} \\ \le ch^2\big(\,\|\phi_h\|_{L^2(\Gamma_h(t))} + \|\sigma\nabla_{\Gamma_h}\phi_h\|_{L^2(\Gamma_h(t))}\big). \end{aligned}$$

Proof. Consider a triangle $E_h \subset \Gamma_h(t)$ and set $g_h := \widetilde{I}_h(\sigma^2\phi_h)$. Use lemma C.3.17 and (C.55) and follow the steps in Schatz, Thomée and Wahlbin [75, lemma 1.4]. ∎

C.4. A Ritz map and some error estimates

Just as in the usual L^2-theory the Ritz map plays a very important role for our L^∞-error estimates. This section is devoted to the careful L^∞- and weighted norm analysis of the errors in the Ritz map.

Definition C.4.1 (Ritz map, [62]). We define $P_{h,1}(t)\colon H^1(\Gamma(t)) \to S_h(t)$ as follows: Let $u \in H^1(\Gamma(t))$ be given. Then there exits a unique finite element function $P_{h,1}(t)u \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ with $\varphi_h = \phi_h^l$ it holds

$$a_h^*(P_{h,1}(t)u, \phi_h) = a^*(u, \varphi_h). \tag{C.28}$$

This naturally defines the Ritz map on the continuous surface:

$$P_1(t)u = \big(P_{h,1}(t)u\big)^l \in S_h^l(t).$$

Note that the Ritz map does not satisfy the Galerkin orthogonality, however it satisfies, using (C.15), the following estimate, cf. [62]. For all $\varphi_h \in S_h^l(t)$ we have

$$|a^*(u - P_1(t)u, \varphi_h)| \leq ch^2 \|P_1(t)u\|_{H^1,\alpha} \|\varphi_h\|_{H^1,-\alpha}. \tag{C.29}$$

In this section we aim to bound the following errors of the Ritz map:

$$u - P_1(t)u \quad \text{and} \quad \partial_h^\bullet(u - P_1(t)u),$$

in the L^∞- and $W^{1,\infty}$-norms. Previously, H^1- and L^2-error estimates have been shown in [31, 35].

C.4.1. Weighted a priori estimates

Before turning to the maximum norm error estimates, we state and prove some technical regularity results involving weighted norms.

Lemma C.4.2 (Weighted a priori estimates). *For $f \in L^2(\Gamma(t))$, the problem*

$$-\Delta_\Gamma w + w = f \qquad \text{on } \Gamma(t),$$

has a unique weak solution $w \in H^1(\Gamma(t))$. Furthermore, $w \in H^2(\Gamma(t))$ and we have the following weighted a priori estimates

$$\|w\|_{H^1,-1} \leq c(\|f\|_{L^2,-1} + \|w\|_{L^2}) \tag{C.30}$$
$$\|w\|_{H^2,-1} \leq c(\|f\|_{L^2,-1} + \|w\|_{H^1}), \tag{C.31}$$

where the constant $c > 0$ is independent of t, h and γ.

Proof. Existence and uniqueness of a weak solution follows from [5]. Using integration by parts, Young inequality and $|\nabla_\Gamma \mu| \leq \sqrt{\mu}$ a short calculation shows (C.30). For the details on elliptic regularity and a derivation of the a priori estimate

$$\|w\|_{H^2} \leq c\, \|-\Delta_\Gamma w + w\|_{L^2},$$

where $c > 0$ is independent of t, we refer to [56, appendix A].

Because of (C.30) it suffices to prove (C.31) for $\|\nabla_\Gamma^2 w\|_{L^2,-1}^2$ as the left-hand side instead of $\|w\|_{H^2,-1}^2$. Apply the usual elliptic a priori estimate on $(x^i - y^i)w$ for $i = 1, \ldots, m+1$ to get the desired estimate. ■

Lemma C.4.3. *For $g \in L^2(\Gamma(t))$ the problem*

$$-\Delta_\Gamma w + w = \mu^{-2} g.$$

has a unique weak solution $w \in H^1(\Gamma(t))$. Furthermore, $w \in H^2(\Gamma(t))$, and there exists a constant $c > 0$ independent of t and h

such that

$$\|w\|_{H^1}^2 \le c\rho^{-2} |\log \rho| \|g\|_{L^2,2}^2. \tag{C.32}$$

Proof. Lemma C.4.2 gives us existence, uniqueness and regularity of w. Consider the number

$$\lambda^{-1}(t) := \sup\{ \|f\|_{H^1}^2 \mid f \in H^2(\Gamma(t)),\ \|-\Delta_\Gamma f + f\|_{L^2,-2}^2 \le 1\}.$$

Inequality (C.32) is proven if we show

$$\lambda^{-1}(t) \le c\rho^{-2} |\log \rho|,$$

where c is t independent. A short calculation shows that the smallest eigenvalue $\widetilde{\lambda}_{\min}(t)$ of the elliptic eigenvalue problem

$$-\Delta_\Gamma f + f = \widetilde{\lambda}\mu^{-2} f \quad \text{on } \Gamma(t)$$

is equal to $\lambda(t)$. The weighted Rayleight quotient implies

$$\widetilde{\lambda}_{\min} = \inf_{f \in H^1} \frac{\|f\|_{H^1}^2}{\|f\|_{L^2,2}^2}.$$

Hence it suffices to prove

$$\|f\|_{L^2,2}^2 \le c\rho^{-2} |\log(\rho)| \, \|f\|_{H^1}^2, \tag{C.33}$$

for a $f \in H^1$. With a Hölder estimate we arrive at

$$\begin{aligned} \|f\|_{L^2,2}^2 &\le \left(\int_{\Gamma(t)} \mu^{-2p} \right)^{1/p} \left(\int_{\Gamma(t)} f^{2q} \right)^{1/q} \\ &= \left(\int_{\Gamma(t)} \mu^{-2p} \right)^{1/p} \|f\|_{L^{2q}(\Gamma(t))}^2, \end{aligned}$$

where $1 < p, q < \infty$ satisfies $p^{-1} + q^{-1} = 1$. We take the choice $q = \sqrt{|\log \rho|}$. It is easy to prove the following quantitative Sobolev-Nierenberg inequality for moving surfaces:

$$\|f\|_{L^q(\Gamma(t))} \le cq \, \|f\|_{H^1(\Gamma(t))},$$

where c is independent of t and q. A straightforward calculation with geodesic polar coordinates using lemma C.9.2 and lemma C.9.1 shows inequality (C.33). ∎

C.4.2. Maximum norm error estimates

Before showing L^∞- and $W^{1,\infty}$-norm error estimates for the Ritz map, we show similar estimates for weighted norms. Then, by connecting the norms, use these results to obtain our original goal.

Throughout this subsection, we write $P_1 u$ instead of $P_1(t)u$.

Lemma C.4.4. *There exists $h_0 > 0$ sufficiently small and $\gamma_0 > 0$ sufficiently large and a constant $c = c(h_0, \gamma_0) > 0$ such that for $u \in W^{2,\infty}(\Gamma(t))$ it holds*

$$\|u - P_1 u\|^2_{L^2,2} + \|u - P_1 u\|^2_{H^1,1} \leq ch^2 \,|\log h|\, \|u\|^2_{W^{2,\infty}(\Gamma(t))}. \quad \text{(C.34)}$$

Proof. **Step 1:** Our goal is to show

$$\|u - P_1 u\|^2_{H^1,1} \leq ch^2 \,|\log h|\, \|u\|^2_{W^{2,\infty}(\Gamma(t))} + \hat{c}\|u - P_1 u\|^2_{L^2,2}. \quad \text{(C.35)}$$

Similarly as in Nitsche [70, theorem 1], (C.17) and partial integration yields

$$\begin{aligned}\|u - P_1 u\|^2_{H^1,1} &\leq a^*(u - P_1 u, \mu^{-1}(u - P_1 u)) + c\int_{\Gamma(t)} (\Delta_\Gamma \mu^{-2})(u - P_1 u)^2 \\ &\leq a^*(u - P_1 u, \mu^{-1}(u - P_1 u)) + c\|u - P_1 u\|^2_{L^2,2}.\end{aligned}$$

For simplicity we set $e = u - P_1 u$, and use $I_h u = (\widetilde{I}_h u)^l$ to obtain

$$\begin{aligned}a^*(e, \mu^{-1}e) = a^*&(e, \mu^{-1}(u - I_h u)) \\ &+ a^*\Big(e, \mu^{-1}(I_h u - P_1 u) - I_h(\mu^{-1}(I_h u - P_1 u))\Big) \\ &+ a^*\Big(e, I_h(\mu^{-1}(I_h u - P_1 u))\Big) = I_1 + I_2 + I_3.\end{aligned}$$

Using lemma C.3.8 (C.14), lemma C.3.9 (C.18), lemma C.3.11 (C.20) and ε-Young inequality we estimate as

$$|I_1| \leq \varepsilon \|e\|^2_{H^1,1} + ch^2 \,|\log h|\, \|u\|^2_{W^{2,\infty}(\Gamma(t))}.$$

For the second term use in addition lemma C.3.12 (C.22) and a $0 < h < h_0$ sufficiently small to get

$$|I_2| \leq \varepsilon \|e\|^2_{H^1,1} + c(h^2 \,|\log h|\, \|u\|_{W^{2,\infty}(\Gamma(t))} + \|e\|_{L^2,2}).$$

For the last term use in addition lemma C.3.8 (C.29) to reach at

$$|I_3| \leq \varepsilon \|e\|^2_{H^1,1} + c(h^2 \,|\log h|\, \|u\|_{W^{2,\infty}(\Gamma(t))} + \|e\|_{L^2,2})$$

These estimates together, and absorbing $\|e\|^2_{H^1,1}$, imply (C.35).

Step 2: Using an Aubin-Nitsche argument we prove that there exists $\gamma > \gamma_0 > 0$ sufficiently large such that for all $\delta > 0$ the following estimate holds

$$\|u - P_1 u\|^2_{L^2,2} \leq ch^4 \,\|u\|^2_{W^{2,\infty}(\Gamma(t))} + \delta \|u - P_1 u\|^2_{H^1,1}. \tag{C.36}$$

Let $w \in H^2(\Gamma(t))$ be the weak solution of

$$-\Delta_\Gamma w + w = \mu^{-2} e.$$

Then by testing with e we obtain

$$\begin{aligned} \|e\|^2_{L^2,2} &= (a^*(e,w) - a^*(e, I_h w)) + a^*(e, I_h w) \\ &= a^*(e, w - I_h w) + a^*(e, I_h w) \end{aligned}$$

In addition to the already mentioned lemmata in Step 1 use lemma C.4.2 (C.31), lemma C.3.9 (C.19), lemma C.4.3 (C.32) and a sufficiently large $\gamma > \gamma_0 > 0$ to estimate

$$|a^*(e, w - I_h w)| \leq \frac{1}{4} \|e\|^2_{L^2,2} + \frac{\delta}{2} \|e\|^2_{H^1,1}.$$

For the other term we estimate

$$|a^*(e, I_h w)| \leq ch^2 \|e\|_{H^1} \|I_h w\|_{H^1} \leq ch^4 \|u\|^2_{W^{2,\infty}(\Gamma(t))} + \frac{1}{4}\|e\|^2_{L^2,2}.$$

By absorption, this implies (C.36).

The final estimate is shown by combining (C.35) and (C.36), and choosing $\delta > 0$ such that $c\delta < 1$. Then an absorbtion finishes the proof. ■

Theorem C.4.5. *There exist constants $c > 0$ independent of h and t such that for all $u \in W^{2,\infty}(\Gamma(t))$ it holds*

$$\begin{aligned} \|u - (P_{h,1}(t)u)^l\|_{L^\infty(\Gamma(t))} &\leq ch^2 |\log h|^{3/2} \|u\|_{W^{2,\infty}(\Gamma(t))}, \\ \|u - (P_{h,1}(t)u)^l\|_{W^{1,\infty}(\Gamma(t))} &\leq ch |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}. \end{aligned}$$

Proof. Using lemma C.3.10, lemma C.3.6 (C.11) and lemma C.3.7 (C.13) we get

$$\begin{aligned} \|u - P_1 u\|_{W^{1,\infty}(\Gamma(t))} &\leq \|u - I_h u\|_{W^{1,\infty}(\Gamma(t))} + c \|\widetilde{I}_h u - P_{h,1} u\|_{W^{1,\infty}(\Gamma_h(t))} \\ &\leq ch \|u\|_{W^{2,\infty}(\Gamma(t))} + c |\log h|^{1/2} \|\widetilde{I}_h u - P_{h,1} u\|_{H^1,1} \\ &\leq ch |\log h| \|u\|_{W^{2,\infty}(\Gamma_h(t))} + c\|u - (P_{h,1} u)^l\|_{H^1,1}. \end{aligned}$$

For the $W^{1,\infty}$-estimate use lemma C.4.4 to estimate the weighted norms. The L^∞-estimate is obtained in a similar way. ■

C.4.3. Maximum norm material derivative error estimates

Since in general time derivatives are not commuting with the Ritz map, i.e. $\partial_h^\bullet P_1 u \neq P_1 \partial_h^\bullet u$, we bound the error $\partial_h^\bullet(u - P_1 u)$. Again we first show our estimates in the weighted norms, and then use these results for the L^∞- and $W^{1,\infty}$-norm error estimates.

For this subsection we write $P_{h,1}u$ instead of $P_{h,1}(t)u$ and further $P_1 u$ instead of $P_1(t)u$.

We first state a substitute for our weighted pseudo Galerkin inequality (C.29).

Lemma C.4.6. *There exists a constant $c > 0$ independent of h and t such that for all $u \in W^{2,\infty}(\mathcal{G}_T)$ and $\varphi_h \in S_h^l(t)$ it holds*

$$|a^*(\partial_h^\bullet(u - P_1 u), \varphi_h)| \leq c\Big(h^2 \|\partial_h^\bullet(u - P_1 u)\|_{H^1,1} + h\,|\log h|^{1/2}\big(\,\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{1,\infty}(\Gamma(t))}\big)\Big)\|\varphi_h\|_{H^1,-1}. \tag{C.37}$$

Proof. The main idea is given by Dziuk and Elliott in [35]. Using (C.5) and lemma C.3.7 (C.13) it is easy to verify

$$\|\partial_h^\bullet P_1 u\|_{H^1,1} \leq \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{H^1,1} + c\,|\log h|^{1/2}\big(\,\|\partial^\bullet u\|_{W^{1,\infty}(\Gamma(t))} + h\,\|u\|_{W^{2,\infty}(\Gamma(t))}\big). \tag{C.38}$$

Let $\phi_h \in S_h(t)$ such that $\varphi_h = \phi_h^l$. Taking time derivative of the definition of the Ritz map (C.28), using the discrete transport properties (C.7) lemma C.3.2, and the definition of the Ritz map, we obtain

$$\begin{aligned} a^*(\partial_h^\bullet u - \partial_h^\bullet P_1 u, \varphi_h) = \; & a_h^*(\partial_h^\bullet P_{h,1} u, \phi_h) - a^*(\partial_h^\bullet P_1 u, \varphi_h) \\ & + (g_h + b_h)(V_h; u^{-l}, \phi_h) - (g + b)(v_h; u, \varphi_h) \\ & - (g_h + b_h)(V_h; u^{-l} - P_{h,1}u, \phi_h). \end{aligned} \tag{C.39}$$

Then estimate using lemma C.3.8 (C.15), (C.16), lemma C.4.4 (C.34) and the above inequality to finish the proof (cf. [66, Theorem 7.2]). ■

Lemma C.4.7. *For $k \in \{0,1\}$ there exists $c = c(k) > 0$ independent of t and h such that for $u \in W^{3,\infty}(\mathcal{G}_T)$ the following inequalities hold*

$$\begin{aligned} &\|\partial_h^\bullet u - I_h \partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))} \\ &\quad \le ch^{2-k}\big(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}\big), \end{aligned} \tag{C.40}$$

$$\begin{aligned} &\|\partial_h^\bullet u - I_h \partial^\bullet u\|_{L^2,2}^2 + \|\partial_h^\bullet u - I_h \partial^\bullet u\|_{H^1,1}^2 \\ &\quad \le ch^2 \,|\log h| \big(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2\big). \end{aligned} \tag{C.41}$$

Proof. Using (C.5) we get

$$\begin{aligned} &\|\partial_h^\bullet u - I_h \partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))} \\ &\qquad \le \|(v - v_h) \cdot \nabla_\Gamma u\|_{W^{k,\infty}(\Gamma(t))} + \|\partial^\bullet u - I_h \partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))}. \end{aligned}$$

Use lemma C.3.10 and (C.6) to show the first estimate.

For the second inequality use a Hölder estimate, and (C.40) with lemma C.3.7 (C.12) and (C.13). ∎

Lemma C.4.8. *There exists $h_0 > 0$ sufficiently small and $\gamma_0 > 0$ sufficiently large and a constant $c = c(h_0, \gamma_0) > 0$ such that for $u \in W^{3,\infty}(\mathcal{G}_T)$ the following holds*

$$\begin{aligned} &\|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{L^2,2}^2 + \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{H^1,1}^2 \\ &\quad \le ch^2 \,|\log h|^4 \big(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2\big). \end{aligned} \tag{C.42}$$

Proof. This proof has a similar structure as lemma C.4.4, and since it also uses similar arguments, we only give references if new lemmata are needed. For the ease of presentation we set $e = u - P_1 u$ and split the error as follows

$$\partial_h^\bullet e = (\partial_h^\bullet u - I_h \partial^\bullet u) + (I_h \partial^\bullet u - \partial_h^\bullet P_1 u) =: \sigma + \theta_h.$$

Step 1: Our goal is to prove

$$\|\partial_h^\bullet e\|_{H^1,1}^2 \le ch^2 \left|\log h\right|\left(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2\right) + \hat{c}\|\partial_h^\bullet e\|_{L^2,2}^2. \tag{C.43}$$

We start with

$$\|\partial_h^\bullet e\|_{H^1,1}^2 \le a^*\left(\partial_h^\bullet e, \mu^{-1}\partial_h^\bullet e\right) + c\|\partial_h^\bullet e\|_{L^2,2}^2$$

and continue with

$$\begin{aligned} a^*(\partial_h^\bullet e, \mu^{-1}\partial_h^\bullet e) &= a^*(\partial_h^\bullet e, \mu^{-1}\sigma) \\ &\quad + a^*(\partial_h^\bullet e, \mu^{-1}\theta_h - I(\mu^{-1}\theta_h)) \\ &\quad + a^*(\partial_h^\bullet e, I(\mu^{-1}\theta_h)) = I_1 + I_2 + I_3. \end{aligned}$$

We estimate the three terms separately. For the first ε-Young inequality and lemma C.4.7 (C.41) yields

$$|I_1| \le \varepsilon\|\partial_h^\bullet e\|_{H^1,1}^2 + ch^2 \left|\log h\right|(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2).$$

For a sufficiently small $0 < h < h_0$ we obtain

$$\begin{aligned} |I_2| \le \varepsilon\|\partial_h^\bullet e\|_{H^1,1}^2 &+ c(\|\partial_h^\bullet e\|_{L^2,2}^2 \\ &+ h^2 \left|\log h\right|(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2)). \end{aligned}$$

Using lemma C.4.6 (C.37) and a $0 < h < h_1$ sufficiently small we arrive at

$$\begin{aligned} |I_3| \le \varepsilon\|\partial_h^\bullet e\|_{H^1,1}^2 &+ c(\|\partial_h^\bullet e\|_{L^2,2}^2 \\ &+ h^2 \left|\log h\right|(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2)). \end{aligned}$$

These estimates together, and absorbing $\|\partial_h^\bullet e\|_{H^1,1}$, imply (C.43).

Step 2: Using again an Aubin-Nitsche like argument we show that, for any $\delta > 0$ sufficiently small, we have

$$\|\partial_h^\bullet e\|_{L^2,2}^2 \le \delta\|\partial_h^\bullet e\|_{H^1,1}^2 + ch^2 \left|\log h\right|^4 \left(\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2\right). \tag{C.44}$$

Let $w \in H^2(\Gamma(t))$ be the weak solution of

$$-\Delta_\Gamma w + w = \mu^{-2}\partial_h^\bullet e.$$

Then we have

$$\|\partial_h^\bullet e\|_{L^2,2} = a^*(\partial_h^\bullet e, w - I_h w) + a^*(\partial_h^\bullet e, I_h w).$$

Again let $\varepsilon > 0$ be a small number. For $\gamma > \gamma_0$ sufficiently big we get

$$|a^*(\partial_h^\bullet e, w - I_h w)| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + \delta \|\partial_h^\bullet e\|_{L^2,2}^2$$

Using equation (C.39) and proceeding similar like in Dziuk and Elliott [35, theorem 6.2], by adding and subtracting terms, we get

$$\begin{aligned}
&- a^*(\partial_h^\bullet e, I_h w) \\
&\quad = \left(a^*(\partial_h^\bullet P_1 u, I_h w) - a_h^*(\partial_h^\bullet P_{h,1} u, \widetilde{I}_h w)\right) \\
&\qquad + \left((g+b)(v_h; u, I_h w) - (g_h + b_h)(V_h; u^{-l}, \widetilde{I}_h w)\right) \\
&\qquad + \left((g_h + b_h)(V_h; u^{-l} - P_{h,1} u, \widetilde{I}_h w) - (g+b)(v_h; u - P_1 u, I_h w)\right) \\
&\qquad + \left((g+b)(v_h; u - P_1 u, I_h w) - (g+b)(v; u - P_1 u, I_h w)\right) \\
&\qquad + \left((g+b)(v; u - P_1 u, I_h w) - (g+b)(v; u - P_1 u, w)\right) \\
&\qquad + (g+b)(v; u - P_1 u, w) \\
&\quad = J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned}$$

Use lemma C.3.8 (C.16), (C.38), lemma C.4.4 (C.34) and the inequality

$$h\|I_h w\|_{H^1,1} \leq \varepsilon \|\partial_h^\bullet e\|_{L^2,2},$$

for $\gamma > \gamma_1$ sufficiently big, we reach at

$$|J_1| + \cdots + |J_4| \leq \delta \|\partial_h^\bullet e\|_{H^1,1}^2 + \varepsilon \|\partial_h^\bullet e\|_{L^2,2}^2 + ch^2(\|u\|_{W^{2,\infty}}^2 + \|\partial^\bullet u\|_{W^{1,\infty}}^2).$$

With the same arguments like for $a^*(\partial_h^\bullet e, w - I_h w)$ we estimate

$$|J_5| \leq \varepsilon \|\partial_h^\bullet e\|_{H^1,1}^2 + \delta \|\partial_h^\bullet e\|_{L^2,2}^2,$$

for $\gamma > \gamma_2$ sufficiently big. For $\gamma > \gamma_3$ sufficiently big we estimate the last term as follows

$$\begin{aligned}
|J_6| &\leq c\|e\|_{L^2,1}\|w\|_{H^2,-1} \\
&\leq c\,\|e\|_{L^\infty}\,|\log\rho|^{1/2}\|w\|_{H^2,-1} \\
&\leq ch^2\,|\log h|^{3/2}\,\|u\|_{W^{2,\infty}}\|w\|_{H^2,-1} \\
&\leq \varepsilon\|\partial_h^\bullet e\|_{L^2,2}^2 + ch^2\,|\log h|^4\,\|u\|_{W^{2,\infty}}^2.
\end{aligned}$$

By absorption, these estimates together imply (C.44).

The final estimate is shown by combining (C.43) and (C.44), and choosing $\delta > 0$ such that $c\delta < 1$. Then an absorbtion finishes the proof. ∎

From the weighted version of the error estimate in the material derivatives, the L^∞-norm estimate follows easily.

Theorem C.4.9 (Errors in the material derivative of the Ritz projection). *Let $z \in W^{3,\infty}(\mathcal{G}_T)$. For a sufficiently small $h < h_0$ and a sufficiently big $\gamma > \gamma_0$ there exists $c = c(h_0,\gamma_0) > 0$ independent of t and h such that*

$$\begin{aligned}
&\|\partial_h^\bullet(z - (P_{h,1}(t)z)^l)\|_{L^\infty(\Gamma(t))} \\
&\quad \leq ch^2\,|\log h|^3\big(\|z\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet z\|_{W^{2,\infty}(\Gamma(t))}\big), \\
&\|\partial_h^\bullet(z - (P_{h,1}(t)z)^l)\|_{W^{1,\infty}(\Gamma(t))} \\
&\quad \leq ch\,|\log h|^{5/2}\big(\|z\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet z\|_{W^{2,\infty}(\Gamma(t))}\big).
\end{aligned}$$

Proof. The above results are shown by exactly following the proof of theorem C.4.5, lemma C.4.8 (C.42) being the main tool. ∎

C.5. Maximum norm parabolic stability

The purpose of this section is to derive a ESFEM weak discrete maximum principle. The proof is modeled on the weak discrete maximum principle from Schatz, Thomée, Wahlbin [75]. For this we are going to need a well known matrix formulation of (C.3), which is due to Dziuk and Elliott [31]. It was first used in Dziuk, Lubich, Mansour [37] for theoretical reasons, namely a time discretization of (C.3). Using the matrix formulation we derive a discrete adjoint problem of (B.7), which does not arise in Schatz, Thomé, Wahlbin [75], but arises here, since the ESFEM evolution operator is not self adjoint. Then we deduce a corresponding a priori estimate and finally prove our weak discrete maximum principle.

C.5.1. A discrete adjoint problem

A matrix ODE version of (C.3) can be derived by setting

$$U_h(\,\cdot\,,t) = \sum_{j=1}^{N} \alpha_j(t)\chi_j(\,\cdot\,,t),$$

testing with the basis function $\phi_h = \chi_j$, where $S_h(t) = \operatorname{span}\{\chi_j \mid j = 1, \dots, N\}$, and using the transport property (C.2).

Proposition C.5.1 (ODE system). *The spatially semidiscrete problem* (C.3) *is equivalent to the following linear ODE system for the vector $\alpha(t) = (\alpha_j(t)) \in \mathbb{R}^N$, collecting the nodal values of $U_h(\,\cdot\,,t)$:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Big(M(t)\alpha(t)\Big) + A(t)\alpha(t) = 0 \\ \alpha(0) = \alpha_0 \end{cases} \tag{C.45}$$

where the evolving mass matrix $M(t)$ and stiffness matrix $A(t)$ are *defined as*

$$M(t)_{kj} = \int_{\Gamma_h(t)} \chi_j \chi_k, \qquad A(t)_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h} \chi_j \cdot \nabla_{\Gamma_h} \chi_k.$$

Definition C.5.2. Let $0 \leq s \leq t \leq T$. For given initial value $w_h \in S_h(s)$ at time s, there exists unique solution u_h (cf. [31]). This defines a linear evolution operator

$$E_h(t,s) \colon S_h(s) \to S_h(t), \quad w_h \mapsto u_h(t).$$

We define the adjoint of $E_h(t,s)$

$$E_h(t,s)^* \colon S_h(t) \to S_h(s)$$

via the equation

$$m_h(t; E_h(t,s)\varphi_h(s), w_h(t)) = m_h(s; \varphi_h(s), E_h(t,s)^* w_h(t)), \quad \text{(C.46)}$$

where $\varphi_h(s) \in S_h(s)$ and $w_h(t) \in S_h(t)$ are some arbitrary finite element functions.

Lemma C.5.3 (Adjoint problem). *Let $s \in [0,t]$ where $t \in [0,T]$ and $w_h(t) \in S_h(t)$. Then $s \mapsto E(t,s)^* w_h(t)$ is the unique solution of*

$$\begin{cases} m_h(s; \partial_h^{\bullet,s} u_h, \varphi_h) - a_h(s; u_h, \varphi_h) = 0, & \text{on } \Gamma(s) \\ \qquad\qquad\qquad\qquad u_h(t) = w_h(t), & \text{on } \Gamma(t). \end{cases} \quad \text{(C.47)}$$

where $\partial_h^{\bullet,s}$ is the discrete material derivative with respect to s.

Remark C.5.4. The problem (C.47) has the structure of a backward heat equation, where s is going backward in time. Hence we considered (C.47) as a PDE of parabolic type. We recall, that using lemma C.3.2 we may write equation (C.3) equivalently as

$$\begin{cases} m_h(t;\partial_h^\bullet u_h + (\nabla_{\Gamma_h}\cdot V_h)u_h,\varphi_h) + a_h(t;u_h,\varphi_h) = 0, & \text{on } \Gamma(t),\\ u_h(0) = w_h, & \text{on } \Gamma(0) \end{cases} \tag{C.48}$$

The problems (C.48) and (C.47) differ in the following way: If the initial data for (C.47) is constant then it remains so for all times. In general this does not hold for solutions of (C.48). On the other hand (C.48) preserves the mean value of its initial data, which is in general not true for a solution of (C.47).

Proof of lemma C.5.3. First we investigate the finite element matrix representation of $E_h(t,s)$ with respect to the standard finite element basis, which we denote by $\boldsymbol{E}_h(t,s)$. From (C.45) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\boldsymbol{E}_h(t,0)\boldsymbol{u}_h(0)\big) + A(t)\boldsymbol{E}_h(t,0)\boldsymbol{u}_h(0) = 0.$$

Let $\Lambda(t,s)$ the resolvent matrix of the ODE

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} + A(t)M(t)^{-1}\xi = 0.$$

Then obviously it holds

$$\boldsymbol{E}_h(t,s) = M(t)^{-1}\Lambda(t,s)M(s).$$

Denote by $\boldsymbol{E}_h(t,s)^*$ the matrix representation of $E_h(t,s)^*$. From equation (C.46) it follows

$$\boldsymbol{E}_h(t,s)^* = M(s)^{-1}\boldsymbol{E}_h(t,s)^T M(t) = \Lambda(t,s)^T.$$

Now we calculate $\frac{\mathrm{d}\Lambda(t,s)}{\mathrm{d}s}$. Note that $\Lambda(t,s) = \Lambda(s,t)^{-1}$ and it holds

$$\frac{\mathrm{d}\Lambda(s,t)^{-1}}{\mathrm{d}s} = -\Lambda(s,t)^{-1}\frac{\mathrm{d}\Lambda(s,t)}{\mathrm{d}s}\Lambda(s,t)^{-1}.$$

From that it easily follows

$$\frac{\mathrm{d}\Lambda(t,s)}{\mathrm{d}s} = \Lambda(t,s)A(s)M(s)^{-1},$$

which now implies

$$\frac{\mathrm{d}\boldsymbol{E}_h(t,s)^*}{\mathrm{d}s} = M(s)^{-1}A(s)\boldsymbol{E}_h(t,s)^*. \qquad \blacksquare$$

C.5.2. A discrete delta and Green's function

Let $\delta_h = \delta_h^{x_h} = \delta_h^{t,x_h} \in S_h(t)$ be a finite element discrete delta function defined as

$$m_h(t;\delta_h^{t,x_h},\varphi_h) = \varphi_h(x_h,t) \qquad (\varphi_h \in S_h(t)). \tag{C.49}$$

If $\delta^{x_h}\colon \Gamma_h(t) \to \mathbb{R}$ is a smooth function having support in the triangle E_h containing x_h, then since $\dim \Gamma_h(t) = 2$ one easily calculates $\|\delta^{x_h}\sigma^{x_h}\|_{L^2(\Gamma_h(t))} \leq c$ for some constant independent of h and t. For the discrete delta function δ_h a similar result holds.

Lemma C.5.5. *There exists $c > 0$ independent of t and h:*

$$\|\sigma^{x_h}\delta_h^{x_h}\|_{L^2(\Gamma_h(t))} \leq c \qquad (x_h \in \Gamma_h(t)).$$

The proof is a straight forward extension of the corresponding one in Schatz, Thomée, Wahlbin [75] and uses the exponential decay property of the L^2-projection, cf. theorem C.3.16 (C.24).

Next we define a finite element discrete Green's function as follows. Let $s \in [0,T]$. For given $u_h \in S_h(s)$ there exists a unique $\psi_h \in S_h(s)$ such that

$$a_h^*(s;\psi_h,\varphi_h) = m_h(s;u_h,\varphi_h) \quad \forall \varphi_h \in S_h(s).$$

This defines an operator

$$T_h^{*,s}\colon S_h(s) \to S_h(s), \quad T_h^{*,s} u_h := \psi_h.$$

We call $G_h^{s,x} := T_h^{*,s} \delta_h^{s,x}$ a discrete Green's function.

A short calculation shows that for all $0 \neq \varphi_h \in S_h(s)$ it holds

$$m_h(s; T_h^{*,s} \varphi_h, \varphi_h) > 0,$$

which implies that $G_h^{s,x}(x) > 0$. Actually we can bound the singularity x with $c\,|\log h|$.

Lemma C.5.6. *For the discrete Green's function $G_h^{s,x}$ we have the estimate*

$$G_h^{s,x}(x) \leq c\,|\log h|.$$

Proof. Using lemma C.3.14 with (C.4) we estimate as

$$\|G_h^{s,x}\|_{L^\infty(\Gamma_h(s))} \leq c\,|\log h|^{1/2}\,\|G_h^{s,x}\|_{H^1(\Gamma_h(s))} = c\,|\log h|^{1/2}\sqrt{G_h^{s,x}(x)}.$$

■

The next lemma needs a different treatment then the one presented in Schatz, Thomée and Wahlbin [75]. The reason for that is that the mass and stiffness matrix depend on time and further the stiffness matrix is singular.

Lemma C.5.7. *Let be u_h a solution of* (C.47)*. Then we have the estimate*

$$\int_0^t \|u_h\|^2_{L^2(\Gamma_h(s))} \mathrm{d}s \leq c \cdot m_h(t; T_h^{*,t} u_h, u_h).$$

Proof. Note that lemma C.3.2 (C.8) reads with the matrix notation as follows: If $\boldsymbol{Z}_h$ and $\boldsymbol{\phi}_h$ are the coefficient vectors of some finite element function, then we have the estimate

$$\begin{aligned} \boldsymbol{Z}_h^T \frac{\mathrm{d}M(s)}{\mathrm{d}s}\phi_h &\leq c\sqrt{\boldsymbol{Z}_h^T M(s)\boldsymbol{Z}_h}\sqrt{\phi_h^T M(s)\phi_h}, \\ \boldsymbol{Z}_h^T \frac{\mathrm{d}A(s)}{\mathrm{d}s}\phi_h &\leq c\sqrt{\boldsymbol{Z}_h^T A(s)\boldsymbol{Z}_h}\sqrt{\phi_h^T A(s)\phi_h}. \end{aligned} \tag{C.50}$$

In the following with drop the s dependency. Let $\boldsymbol{u}$ be the time dependent coefficient vector of u_h. Then we have

$$0 = -M\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s} + A\boldsymbol{u} = -M\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s} + (A+M)\boldsymbol{u} - M\boldsymbol{u}.$$

Equivalently we write this equation as

$$\begin{aligned} &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}[\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}] \\ &\quad = -\boldsymbol{u}^T M\boldsymbol{u} + \boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^T\frac{\mathrm{d}}{\mathrm{d}s}[M(A+M)^{-1}M]\boldsymbol{u}. \end{aligned}$$

The last term expanded reads

$$\begin{aligned} &\frac{1}{2}\boldsymbol{u}^T\frac{\mathrm{d}}{\mathrm{d}s}[M(A+M)^{-1}M]\boldsymbol{u} \\ &\quad = \boldsymbol{u}^T\frac{\mathrm{d}M}{\mathrm{d}s}(A+M)^{-1}M\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^T M\frac{\mathrm{d}(A+M)^{-1}}{\mathrm{d}s}M\boldsymbol{u} = I_1 + I_2. \end{aligned}$$

Using (C.50) and a Young inequality we estimate as

$$\begin{aligned} |I_1| &\leq c\cdot\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^T M\boldsymbol{u}. \\ |I_2| &= \frac{1}{2}\left|\boldsymbol{u}^T M(A+M)^{-1}\frac{\mathrm{d}(A+M)}{\mathrm{d}s}(A+M)^{-1}M\boldsymbol{u}\right| \\ &\leq c\cdot\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u} \end{aligned}$$

Putting everything together we reach at

$$-\frac{\mathrm{d}}{\mathrm{d}s}[\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}] \leq -\boldsymbol{u}^T M\boldsymbol{u} + c\cdot\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}.$$

The claim then follows from lemma C.10.1. ■

C.5.3. A weak discrete maximum principle

Proposition C.5.8. *Let $U_h(x,t) \in S_h(t)$ the ESFEM solution of our linear heat problem. Then there exists a constant $c = c(T,v) > 0$, which depends exponentially on T and v such that*

$$\|U_h(t)\|_{L^\infty(\Gamma_h(t))} \leq c\,|\log h|\,\|U_h(0)\|_{L^\infty(\Gamma_h(0))}.$$

Proof. There exists $x_h \in \Gamma_h(t)$ such that

$$\begin{aligned}\|U_h(t)\|_{L^\infty} &= |U_h(x_h,t)| = m_h(t;U_h(t),\delta_h^{t,x_h}) = m_h(t;E(t,0)U_h^0,\delta_h^{t,x_h})\\ &= m_h(0;U_h^0,E(t,0)^*\delta_h^{t,x_h}) \leq \|U_h^0\|_{L^\infty}\,\|E(t,0)^*\delta_h^{t,x_h}\|_{L^1}.\end{aligned}$$

The claim follows from lemma C.5.9. ■

Lemma C.5.9. *For $G_h^x(t,s) := E_h(t,s)^*\delta_h^{t,x}$, where $\delta_h^{t,x}$ is defined via* (C.49) *and $E_h(t,s)^*$ is defined via* (C.46), *it holds*

$$\|G^x(t,0)\|_{L^1(\Gamma_h(0))} \leq c\,|\log h|,$$

where the constant $c = c(T,v)$ depending exponentially on T and v such and is independent of x, h, t and s.

Proof. The proof presented here is a modification of the proof from Schatz, Thomée and Wahlbin [75, lemma 2.1]. We estimate

$$\|G_h^x(t,0)\|_{L^1(\Gamma_h(0))} \leq \|1/\sigma^x\|_{L^2(\Gamma_h(0))}\,\|\sigma^x G_h^x(t,0)\|_{L^2(\Gamma_h(0))}.$$

With subsection C.9.1 it follows

$$\|1/\sigma^x\|^2_{L^2(\Gamma_h(0))} \leq c\,|\log h|.$$

It remains to show

$$\|\sigma^x G_h^x(t,0)\|^2_{L^2(\Gamma_h(0))} \leq c\,|\log h|.$$

In the following we abbreviate $\sigma = \sigma^x$ and $G_h = G_h^x(t,s)$ With equation (C.47) and the discrete transport property we proceed as follows

$$
\begin{aligned}
-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} & \|\sigma G_h\|^2_{L^2(\Gamma_h(s))} + \|\sigma \nabla_{\Gamma_h} G_h\|^2_{L^2(\Gamma_h(s))} \\
&= -m_h(s; \partial_h^{\bullet,s} G_h, \sigma^2 G_h) + a_h(s; G_h, \sigma^2 G_h) \\
&\quad - 2 m_h(s; \sigma \nabla_{\Gamma_h} G_h, G_h \nabla_{\Gamma_h} \sigma) \\
&\quad - m_h(s; \partial_h^{\bullet,s} \sigma, \sigma G_h^2) - \frac{1}{2} m_h(s; \sigma^2 G_h^2, \nabla_{\Gamma_h} \cdot V_h) \\
&= -m_h(s; \partial_h^{\bullet,s} G_h, \sigma^2 G_h - \psi_h) + a_h(s; G_h, \sigma^2 G_h - \psi_h) \\
&\quad - 2 m_h(s; \sigma \nabla_{\Gamma_h} G_h, G_h \nabla_{\Gamma_h} \sigma) \\
&\quad - m_h(s; G_h \partial_h^{\bullet,s} \sigma, \sigma G_h) - \frac{1}{2} m_h(s; \sigma^2 G_h^2, \nabla_{\Gamma_h} \cdot V_h) \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

For the choice $\psi_h = P_{h,0}(\sigma^2 G_h)$ we have $I_1 = 0$. Using Cauchy-Schwarz inequality, lemma C.3.18 and an inverse estimate C.3.13 we get

$$
|I_2| \le c\big(\|G_h\|^2_{L^2(\Gamma_h(s))} + \|G_h\|_{L^2(\Gamma_h(s))} \cdot \|\sigma \nabla_{\Gamma_h} G_h\|_{L^2(\Gamma_h(s))} \big).
$$

Using lemma C.3.17 (C.27) we reach at

$$
|I_3| \le c \, \|G_h\|_{L^2(\Gamma_h(s))} \, \|\sigma \nabla_{\Gamma_h} G_h\|_{L^2(\Gamma_h(s))}.
$$

Using lemma C.3.17 (C.26) we have

$$
\begin{aligned}
|I_4| &\le c \, \|G_h\|_{L^2(\Gamma_h(s))} \, \|\sigma G_h\|_{L^2(\Gamma_h(s))}, \\
|I_5| &\le c \, \|\sigma G_h\|^2_{L^2(\Gamma_h(s))}.
\end{aligned}
$$

After a Young inequality we have

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}s} & \|\sigma G_h\|^2_{L^2(\Gamma_h(s))} + \|\sigma \nabla_{\Gamma_h} G_h\|^2_{L^2(\Gamma_h(s))} \\
&\le c \, \|G_h\|^2_{L^2(\Gamma_h(s))} + c \, \|\sigma G_h\|^2_{L^2(\Gamma_h(s))}.
\end{aligned}
$$

Lemma C.10.1 yields

$$\|\sigma G_h(t,0)\|^2_{L^2(\Gamma_h(0))} \leq c\left(\int_0^t \|G_h(t,s)\|^2_{L^2(\Gamma_h(s))}\mathrm{d}s + \|\sigma^x \delta_h^x\|_{L^2(\Gamma_h(0))}\right).$$

For the first term we get from lemma C.5.7 and lemma C.5.6 the bound

$$\int_0^t \|G_h(t,s)\|^2_{L^2(\Gamma_h(s))}\mathrm{d}s \leq c\,|\log h|.$$

The last term is bounded according to lemma C.5.5. ■

C.6. Convergence of the semidiscretization

Theorem C.6.1. *Let* $\Gamma(t)$ *be an evolving surface, let* $u\colon \Gamma(t) \to \mathbb{R}$ *be the solution of* (C.1) *and let* $u_h = U_h^l \in H^1(\Gamma(t))$ *be the solution of* (C.3)*. If it holds*

$$\|P_{h,1}(t)u - U_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2,$$

then there exists $h_0 > 0$ *sufficiently small and* $c = c(h_0) > 0$ *independent of* t*, such that for all* $0 < h < h_0$ *we have the estimate*

$$\begin{aligned}&\|u - u_h\|_{L^\infty(\Gamma(t))} + h\,\|u - u_h\|_{W^{1,\infty}(\Gamma(t))}\\ &\qquad \leq ch^2\,|\log h|^4(1+t)\big(\,\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}\big).\end{aligned}$$

Proof. It suffices to prove the L^∞-estimate, since an inverse inequality implies the $W^{1,\infty}$-estimate.

For this proof we denote by $P_{h,1}u = P_{h,1}(t)u$, $P_1u = (P_{h,1}u)^l$ and $u_h = U_h^l$. We split the error as follows

$$u - u_h = (u - P_1u) + (P_{h,1}u - U_h)^l = \sigma + \theta_h^l.$$

Because of theorem C.4.5 it remains to bound θ_h. Obviously there exists $R_h \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} \theta_h\phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h}\theta_h \cdot \nabla_{\Gamma_h}\phi_h - \int_{\Gamma_h(t)} \theta_h\partial_h^\bullet\phi_h = \int_{\Gamma_h(t)} R_h\phi_h.$$

By the variation of constant formula we deduce

$$\theta_h(t) = E_h(t,0)\theta_h(0) + \int_0^t E_h(t,s)R_h(s)\,\mathrm{d}s.$$

With Proposition C.5.8 we get

$$\|\theta_h\|_{L^\infty(\Gamma_h(t))} \leq c\,|\log h|\big(\|\theta_h(0)\|_{L^2(\Gamma_h(t))} + t\max_{s\in[0,t]}\|R_h(s)\|_{L^\infty(\Gamma_h(t))}\big).$$

Observe that if we denote by $\varphi_h \coloneqq \phi_h^l$, then a quick calculation reveals

$$\begin{aligned} m_h(R_h,\phi_h) &= m_h(\partial_h^\bullet P_{h,1}u,\phi_h) + g_h(V_h;P_{h,1}u,\phi_h) + a_h(P_{h,1}u,\phi_h) \\ &\quad - \big(m(\partial_h^\bullet u,\varphi_h) + g(v_h;u,\varphi_h) + a(u,\varphi_h)\big) \end{aligned} \tag{C.51}$$

lemma C.6.2 finishes the proof. ∎

Lemma C.6.2. *Assume that $R_h \in S_h(t)$ satisfies for all $\phi_h \in S_h(t)$ with $\varphi_h \coloneqq \phi_h^l$ equation* (C.51). *Then it holds*

$$\|R_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2\,|\log h|^3\big(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}\big).$$

Proof. Using Definition C.3.15 (C.23), (C.51) and since L^∞ is the dual of L^1 we deduce

$$\|R_h\|_{L^\infty(\Gamma_h(t))} = \sup_{\substack{f_h\in L^1(\Gamma_h(t))\\ \|f_h\|_{L^1(\Gamma_h(t))}=1}} m_h(R_h,f_h) = \sup_{\substack{f_h\in L^1(\Gamma_h(t))\\ \|f_h\|_{L^1(\Gamma_h(t))}=1}} m_h(R_h,P_{h,0}f_h).$$

Now consider

$$\begin{aligned} m_h(R_h, P_{h,0}f_h) &= m_h(\partial_h^\bullet P_{h,1}u, P_{h,0}f_h) - m(\partial_h^\bullet u, P_{h,0}f_h^l) \\ &\quad + g_h(V_h; P_{h,1}u, P_{h,0}f_h) - g(v_h; u, P_{h,0}f_h^l) \\ &\quad + a_h(P_{h,1}u, P_{h,0}f_h) - a(u, P_{h,0}f_h^l) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Using lemma C.3.3 and theorem C.3.16 it is easy to see

$$\begin{aligned} |I_1| &\leq c(\|\partial_h^\bullet u - \partial_h^\bullet (P_{h,1}u)^l\|_{L^\infty(\Gamma(t))} \\ &\qquad + h^2(\|\partial^\bullet u\|_{L^\infty(\Gamma(t))} + h^2 \|u\|_{W^{1,\infty}(\Gamma(t))}))\, \|f_h\|_{L^1(\Gamma_h(t))} \\ |I_2| &\leq c(\|u - P_{h,1}u^l\|_{L^\infty(\Gamma(t))} + h^2 \|u\|_{L^\infty(\Gamma(t))})\, \|f_h\|_{L^1(\Gamma_h(t))} \\ |I_3| &\leq c(h^2 \|u\|_{L^\infty(\Gamma(t))} + \|u - (P_{h,1}u)^l\|_{L^\infty(\Gamma(t))})\, \|f_h\|_{L^1(\Gamma_h(t))} \end{aligned}$$

Theorem C.4.5 and theorem C.4.9 imply the claim. ■

C.7. A numerical experiment

We present a numerical experiment for an evolving surface parabolic problem discretized in space by the evolving surface finite element method. As a time discretization method we choose backward difference formula 4 with a sufficiently small time step (in all the experiments we choose $\tau = 0.001$).

As initial surface Γ_0 we choose the unit sphere $S^2 \subset \mathbb{R}^3$. The dynamical system is given by

$$\Phi(x, y, z, t) = (\sqrt{1 + 0.25\sin(2\pi t)}x, y, z),$$

which implies the velocity

$$v(x, y, z, t) = (\pi\cos(2\pi t)/(4 + \sin(2\pi t))x, 0, 0),$$

over the time interval $[0,1]$. As the exact solution we choose $u(x,y,z,t) = xye^{-6t}$. The complicated right-hand side was calculated using the computer algebra system Sage [24].

We give the errors in the following norm and seminorm

$$L^\infty(L^\infty): \qquad \max_{1\le n\le N} \|u_h^n - u(\,\cdot\,,t_n)\|_{L^\infty(\Gamma(t_n))},$$

$$L^2(W^{1,\infty}): \qquad \Big(\tau \sum_{n=1}^{N} |u_h^n - u(\,\cdot\,,t_n)|^2_{W^{1,\infty}(\Gamma(t_n))}\Big)^{1/2}.$$

The experimental order of convergence (EOC) is given as

$$EOC_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \qquad (k = 2,3,\dots,n),$$

where e_k denotes the error of the k-th level.

level	dof	$L^\infty(L^\infty)$	EOCs	$L^2(W^{1,\infty})$	EOCs
1	126	0.00918195	-	0.01921707	-
2	516	0.00308305	1.57	0.01481673	0.37
3	2070	0.00100752	1.61	0.00851267	0.80
4	8208	0.00025326	1.99	0.00399371	1.09

Table C.1.: Errors and EOCs in the $L^\infty(L^\infty)$ and $L^2(W^{1,\infty})$ norms

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Appendix

C.8. A Green's function for evolving surfaces

Aubin [5, section 4.2] proves existence of a Green's function on a closed manifold M, that is a function which satisfies in $M \times M$

$$\Delta_{Q\text{ distr.}}\mathbf{G}(P,Q) = \delta_P(Q),$$

where Δ is the Laplace–Beltrami operator on M. The Green's function is unique up to an constant. For lemma C.3.14 we need that the first derivative of a Green's function can be bounded independent of t.

Theorem C.8.1 (Green's function). *Let $\Gamma(t)$ with $t \in [0,T]$ be an evolving surface. There exists a Green's function $\mathbf{G}(t;x,y)$ for $\Gamma(t)$. The value of $\mathbf{G}(x,y)$ depends only on the value of $d_\Gamma(x,y)$. $\mathbf{G}(x,y)$ satisfies the inequality*

$$|\nabla_\Gamma^x \mathbf{G}(t;x,y)| \le c\frac{1}{d_\Gamma(x,y)}.$$

for some $c > 0$ independent of t.

Furthermore for all functions $\varphi \in C^2(\mathcal{G}_T)$ it holds

$$\varphi(x,t) = \frac{1}{V}\int_{\Gamma(t)}\varphi(y,t)\mathrm{d}y - \int_{\Gamma(t)}\mathbf{G}(t;x,y)\Delta_\Gamma\varphi(y,t)\mathrm{d}y. \quad \text{(C.52)}$$

Proof. As noted in Aubin [5, 4.10] the distance $r = d_\Gamma(x,y)$ is only a Lipschitzian function on $\Gamma(t)$. To use his construction we therefore need to revise that the injectivity radius at any point $P \in \Gamma(t)$ can be bounded by below by a number independent of P and t. This follows if the Riemannian exponential map is continuous in t and from lemma C.10.2. To prove that the Riemannian exponential map is continuous one carefully revises the construction of exponential map as it is given in Chavel [18, Chapter 1]. Formula (C.52) follows from Aubin [5, theorem 4.13] and that the constant is independent of t is a straightforward calculation. ■

C.9. Calculations with some weight functions on evolving surfaces

C.9.1. Integration with geodesic polar coordinates on evolving surfaces

Assume we have sufficiently smooth function $f\colon \Gamma(t) \times \Gamma(t) \to \mathbb{R}$, where the value $f(x,y)$ depends on the distance $r = d_\Gamma(x,y)$ and we want to estimate the quantity $\int_{\Gamma(t)} f(x,y)\,\mathrm{d}y$ for a fix y.

Applying the well known coarea formulae to the distance function r, cf. Chavel [18, theorem 3.13] and Morgan [68, theorem 3.13], we reach at

$$\int_{\Gamma(t)} f(x,y)\,\mathrm{d}y = \int_0^\infty \int_{\{d_\Gamma(x,y)=r\}} f(r)\,\mathrm{d}\omega\,\mathrm{d}r$$
$$= \int_0^\infty \frac{\mathcal{H}^m(\{d_\Gamma(x,y)=r\})}{r^m} f(r) r^m\,\mathrm{d}r,$$

where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure. If $\Gamma(t)$ is not a closed surface but $\mathbb{R}^{m+1}$ then $\frac{\mathcal{H}^m(\{d_\Gamma(x,y)=r\})}{r^m}$ would be constant. For closed surfaces the situation is different. Obviously there exists a positive number $R > 0$ independent of t and $x, y \in \Gamma(t)$ such that for all $r \geq R$ it holds

$$\mathcal{H}^m(d_\Gamma(x,y) = r) = 0.$$

Lemma C.9.1. *There exists $c > 0$ depending on $t \in [0,T]$ and*

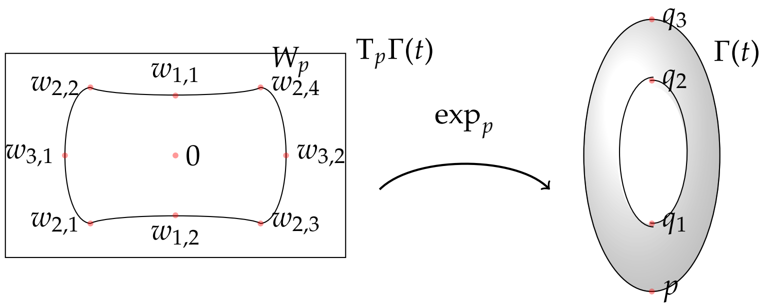

Figure C.1.: Illustration of a possible W_p for the Torus as a subset of $\mathbb{R}^3$ with induced metric. Note that the opposite boundary of W_p are identified. It holds $\exp_p(w_{i,*}) = q_i$ and $\exp_p(0) = p$.

$x \in \Gamma(t)$ such that

$$\sup_{r>0} \frac{\mathcal{H}^m(\{d_\Gamma(x,y) = r\})}{r^m} \leq c.$$

Proof. It holds

$$\lim_{r\to 0} \frac{\mathcal{H}^m(\{d_\Gamma(x,y) = r\})}{r^m} = \omega_m,$$

where ω_m is the volume of the m-dimensional sphere in $\mathbb{R}^{m+1}$, cf. Gray [47, theorem 3.1]. Thus the proof is finished if find a $c > 0$ such that

$$\mathcal{H}^m(\{d_\Gamma(x,y) = r\}) \leq c \quad \forall r \in [0,\infty).$$

This can be seen as follows. For a fix point $p \in \Gamma(t)$ it is possible to use the Riemannian exponential to flat out $\Gamma(t)$, cf. figure C.1 for an illustration on the torus. We make this argument precise.

For $r \in [0,\infty)$ let

$$S_p(r) := \{v \in \mathrm{T}_p\Gamma(t) \mid g_p(v,v) = r^2\}$$

be the sphere of radius r and for $v \in S_p(1)$ consider the geodesic

$$f_v \colon [0,\infty) \to \Gamma, \quad \lambda \mapsto \exp_p(\lambda v).$$

It is well known that a geodesic is just locally length minimizing. Hence there exists a unique $\lambda_*(v) > 0$, such that $f_v|[0,\lambda_*(v)]$ is a length minimizing geodesic and for every $\varepsilon > 0$ $f_v|[0,\lambda_*(v)+\varepsilon]$ is not anymore length minimizing. We define

$$W_p(t) \coloneqq \{w \in \mathrm{T}_p\Gamma(t) \mid w = \lambda \cdot v \text{ with } v \in S_p \text{ and } \lambda \in [0,\lambda_*(v)]\}.$$

Obviously it holds for every $w \in W_p(t)$ that $d_\Gamma(p,\exp_p(w)) \leq R$. Further there exists for every $q \in \Gamma$ a unique $w \in W_p$ with $\exp_p(w) = q$. Clearly it holds

$$\exp_p(W_p \cap S_p(r)) = \{d_\Gamma(x,y) = r\}.$$

Now apply a general Area-coarea Formula, cf. [68, theorem 3.13], to finish the proof. ■

Using this lemma we have the estimate

$$\int_{\Gamma(t)} f(x,y)\,\mathrm{d}y \leq c\int_0^R r^m f(r)\,\mathrm{d}r.$$

C.9.2. Comparison of extrinsic and intrinsic distance

Lemma C.9.2. *There exists a constant $c > 0$ independent of t such that for all $x,y \in \Gamma(t)$ the following inequality holds*

$$c \cdot d_\Gamma(x,y) \leq |x-y|. \tag{C.53}$$

Proof. For simplicity we assume that $\Gamma(t) = \Gamma_0$ for all $t \in [0,T]$. The basic idea is to find a radius $r > 0$ and two constant $c_1, c_2 > 0$ such that (C.53) holds with c_1 for $d_\Gamma(x,y) \leq r$ and with c_2 for $d_\Gamma(x,y) \geq r$.

Observe that from the compactness from Γ_0 it follows that there exists $r > 0$ such that for all $d_\Gamma(x, y) \le r$ it holds

$$\nu(x) \cdot \nu(y) \ge \cos(\pi/6).$$

After rotation we may assume $x = 0$, $\nu(x) = e_{n+1}$ and that Γ_0 may be written as graph of a smooth function, that means that there exits $f\colon U(x) \to \mathbb{R}$ smooth with $U(x) \subset \mathbb{R}^n$ an open subset, such that $z = (z', w) \in \Gamma_0 \subset \mathbb{R}^m \times \mathbb{R}$ with $d_\Gamma(z, x) \le r$ if and only if $z' \in U(x)$ and $w = f(z')$. For $x = (0, 0)$ and $y = (y', f(y'))$ consider the path $t \mapsto (ty', f(ty'))$. We calculate

$$\begin{aligned} d_\Gamma(x, y) \le \int_0^1 \sqrt{1 + \mathrm{d}f_{ty'} y'}\, \mathrm{d}t \le \sqrt{1 + \|f\|^2_{W^{1,\infty}}}\, |y'| \\ \le \sqrt{1 + \|f\|^2_{W^{1,\infty}}}\, |y - x|. \end{aligned}$$

Now the derivatives of f are bounded by $m \cdot \tan(\pi/6)$.

To get the existence of $c_2 > 0$ observe that d_Γ is continuous and hence the set $d_\Gamma^{-1}\{r > 0\}$ is compact. On this set the function $|x - y|$ does not vanish and takes it maximum and minimum. ■

C.9.3. Weight functions

Definition C.9.3. Let μ and $\tilde{\mu}$ be like (C.9) resp. (C.25). For given μ, $\tilde{\mu}$ with curve $y = y(t)$, we define a curve $y_h = y_h(t) :=$ $y(t)^{-l} \in \Gamma_h(t)$. Now we define a weight function on the discrete surface

$$\mu_h\colon \Gamma_h(t) \to \mathbb{R}, \quad \text{resp.} \quad \tilde{\mu}_h\colon \Gamma_h(t) \to \mathbb{R},$$

via the same formula like (C.9) resp. (C.25).

Lemma C.9.4. *There exists a constant $h_0 = h_0(\gamma) > 0$ sufficiently small and $c = c(h_0) > 0$ independent of t and h such that for all $0 < h < h_0$ it holds*

$$\frac{1}{c}\mu \leq \mu_h^l \leq c\mu, \tag{C.54}$$

$$\frac{1}{c}\tilde{\mu} \leq \tilde{\mu}_h^l \leq c\tilde{\mu}. \tag{C.55}$$

Proof. The main idea is to observe that we have the inequalities

$$|x^{-l} - y_h| \leq 2d + |x - y|,$$
$$|x - y| \leq 2d + |x^{-l} - y_h|,$$

where $d = d(t) := \max_{x \in \Gamma(t)} \operatorname{dist}_{\mathbb{R}^{n+1}}(x, \Gamma_h(t))$. ■

C.10. Modified analytic results for evolving surface problems

Lemma C.10.1 (modified Gronwall inequality). *Let $c > 0$ be a positive constant, let φ, ψ and ρ be some positive functions defined on $[t, T]$ and assume for all $s \in [t, T]$ we have the inequality*

$$-\frac{\mathrm{d}\varphi}{\mathrm{d}s}(s) + \psi(s) \leq c\varphi(s) + \rho(s).$$

Then it holds

$$\varphi(t) + \int_t^T \psi(s)\mathrm{d}s \leq e^{c(T-t)}\left(\varphi(T) + \int_t^T \rho(s)\mathrm{d}s\right).$$

Proof. Calculate $-\frac{\mathrm{d}}{\mathrm{d}s}[\varphi e^{-c(T-s)}]$ and integrate from t to T. ■

Lemma C.10.2 (modified inverse function theorem). *Let*

$$f\colon \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$$

be a smooth map, denote by $f(t)(x) \coloneqq f(x,t)$ *and assume that for all* $t \in [0,T]$ *the map* $\mathrm{d}f(t)_0 = \frac{\partial f}{\partial x}(0,t)$ *is invertible. Then there exists* $r > 0$ *independent of* t *such that*

$$f(t)\colon f(t)^{-1}\{B_r(0)\} \to \mathbb{R}^n, \quad x \mapsto f(x,t),$$

is a diffeomorphism onto its image and we have

$$B_{r/2}(0) \subset f(t)^{-1}\{B_r(0)\}$$

for all t*, where* $B_r(0) \coloneqq \{x \in \mathbb{R}^n \mid |x| \leq r\}$*. The map*

$$g\colon [0,T] \times B_r(0) \to \mathbb{R}^n, \quad (t,x) \mapsto f(t)^{-1}(x)$$

is smooth. In particular g *is smooth in* t*.*

Proof. The results follows from the compactness of $[0,T]$ and the smoothness of f.

∎

Appendix D.

Convergence of finite elements on an evolving surface driven by diffusion on the surface

The content of this chapter is based on [58].

Abstract

For a parabolic surface partial differential equation coupled to surface evolution, convergence of the spatial semidiscretization is studied in this paper. The velocity of the evolving surface is not given explicitly, but depends on the solution of the parabolic equation on the surface. Various velocity laws are considered: elliptic regularization of a direct pointwise coupling, a regularized mean curvature flow and a dynamic velocity law. A novel stability and convergence analysis for evolving surface finite elements for the coupled problem of surface diffusion and surface evolution

is developed. The stability analysis works with the matrix-vector formulation of the method and does not use geometric arguments. The geometry enters only into the consistency estimates. Numerical experiments complement the theoretical results.

D.1. Introduction

Starting from a paper by Dziuk and Elliott [31], much insight into the stability and convergence properties of finite elements on evolving surfaces has been obtained by studying a linear parabolic equation on a given moving closed surface $\Gamma(t)$. The strong formulation of this model problem is to find a solution $u(x,t)$ (for $x \in \Gamma(t)$ and $0 \leq t \leq T$) with given initial data $u(x,0) = u_0(x)$ to the linear partial differential equation

$$\partial^\bullet u(x,t) + u(x,t)\nabla_{\Gamma(t)} \cdot v(x,t) - \Delta_{\Gamma(t)} u(x,t) = 0,$$

where $\partial^\bullet$ denotes the material time derivative, $\Delta_{\Gamma(t)}$ is the Laplace-Beltrami operator on the surface, and $\nabla_{\Gamma(t)} \cdot v$ is the tangential divergence of the *given* velocity v of the surface. We refer to [34] for an excellent review article (up to 2012) on the numerical analysis of this and related problems. Optimal-order L^2 error bounds for piecewise linear finite elements are shown in [35] and maximum-norm error bounds in [58]. Stability and convergence of full discretizations obtained by combining the evolving surface finite element method (ESFEM) with various time discretizations are shown in [33, 37, 63]. Convergence of semi- and full discretizations using high-order evolving surface finite elements is studied in [54]. Arbitrary Euler-Lagrangian (ALE) variants of the ESFEM method for this equation are studied in [42, 43, 57]. Convergence properties of the ESFEM and of full discretizations for quasilinear parabolic equations on prescribed moving surfaces are studied in [56].

Beyond the above model problem, there is considerable interest in cases where the velocity of the evolving surface is *not given explicitly*, but depends on the solution u of the parabolic equation; see, e.g., [6, 17, 42, 44] for physical and biological models where such situations arise. Contrary to the case of surfaces with prescribed motion, there exists so far no numerical analysis for solution-driven surfaces in $\mathbb{R}^3$, to the best of our knowledge.

For the case of evolving *curves* in $\mathbb{R}^2$, there are recent preprints by Pozzi & Stinner [73] and Barrett, Deckelnick & Styles [7], who couple the curve-shortening flow with diffusion on the curve and study the convergence of finite element discretizations without and with a tangential part in the discrete velocity, respectively. The analogous problem for two- or higher-dimensional surfaces would be to couple mean curvature flow with diffusion on the surface. Studying the convergence of finite elements for these coupled problems, however, remains illusive as long as the convergence of ESFEM for mean curvature flow of closed surfaces is not understood. This has remained an open problem since Dziuk's formulation of such a numerical method for mean curvature flow in his 1990 paper [30].

In this paper we consider different velocity laws for coupling the surface motion with the diffusion on the surface. Conceivably the simplest velocity law would be to prescribe the normal velocity at any surface point as a function of the solution value and possibly its tangential gradient at this point:

$$v(x,t) = g\left(u(x,t), \nabla_{\Gamma(t)} u(x,t)\right) \nu_{\Gamma(t)}(x), \quad \text{for } x \in \Gamma(t),$$

where $\nu_{\Gamma(t)}(x)$ denotes the outer normal vector and g is a given smooth scalar-valued function. This does, however, not appear to lead to a well-posed problem, and in fact we found no mention of this seemingly obvious choice in the literature. Here we study instead a *regularized velocity law*:

$$v(x,t) - \alpha \Delta_{\Gamma(t)} v(x,t) = g\left(u(x,t), \nabla_{\Gamma(t)} u(x,t)\right) \nu_{\Gamma(t)}(x), \quad x \in \Gamma(t),$$

with a fixed regularization parameter $\alpha > 0$. This elliptic regularization will turn out to permit us to give a complete stability and convergence analysis of the ESFEM semidiscretization, for finite elements of polynomial degree at least two. The case of linear finite elements is left open in the theory of this paper, but will be considered in our numerical experiments. The stability and convergence results can be extended to full discretizations with linearly implicit backward difference time-stepping, as we plan to show in later work.

Our approach also applies to the ESFEM discretization of coupling a *regularized mean curvature flow* and diffusion on the surface:

$$v - \alpha\Delta_{\Gamma(t)}v = \Big(-H + g\left(u, \nabla_{\Gamma(t)}u\right)\Big)\nu_{\Gamma(t)},$$

where H denotes mean curvature on the surface $\Gamma(t)$.

The error analysis is further extended to a *dynamic velocity law*

$$\partial^{\bullet}v + v\nabla_{\Gamma(t)} \cdot v - \alpha\Delta_{\Gamma(t)}v = g\left(u, \nabla_{\Gamma(t)}u\right)\nu_{\Gamma(t)}.$$

A physically more relevant dynamic velocity law would be based on momentum and mass balance, such as incompressible Navier–Stokes motion of the surface coupled to diffusion on the surface. We expect that our analysis extends to such a system, but this is beyond the scope of this paper. Surface evolutions under Navier-Stokes equations and under Willmore flow have recently been considered in [11, 10, 9]. The paper is organized as follows.

In section D.2 we describe the considered problems and give the weak formulation. We recall the basics of the evolving surface finite element method and describe the semidiscrete problem. Its matrix-vector formulation is useful not only for the implementation, but will play a key role in the stability analysis of this paper.

In section D.3 we present the main result of the paper, which gives convergence estimates for the ESFEM semidiscretization with finite elements of polynomial degree at least 2. We further outline the main ideas and the organization of the proof.

In section D.4 we present auxiliary results that are used to relate different surfaces to one another. They are the key technical results used later on in the stability analysis. Section D.5 contains the stability analysis for the regularized velocity law with a prescribed driving term. In section D.6 this is extended to the stability analysis for coupling surface PDEs and surface motion. The stability analysis works with the matrix-vector formulation of the ESFEM semidiscretization and does not use geometric arguments.

In section D.7 we briefly recall some geometric estimates used for estimating the consistency errors, which are the defects obtained on inserting the interpolated exact solution into the scheme. Section D.8 deals with the defect estimates. Section D.9 proves the main result by combining the results of the previous sections.

In section D.10 we give extensions to other velocity laws: the regularized mean curvature flow and the dynamic velocity law addressed above.

Section D.11 presents numerical experiments that are complementary to our theoretical results in that they show the numerical behaviour of piecewise linear finite elements on some examples.

We use the notational convention to denote vectors in $\mathbb{R}^3$ by italic letters, but to denote finite element nodal vectors in $\mathbb{R}^N$ and $\mathbb{R}^{3N}$ by boldface lowercase letters and finite element mass and stiffness matrices by boldface capitals. All boldface symbols in this paper will thus be related to the matrix-vector formulation of the ESFEM.

D.2. Problem formulation and evolving surface finite element semidiscretization

D.2.1. Basic notions and notation

We consider the evolving two-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^3$ as the image

$$\Gamma(t) = \{X(p,t) \, : \, p \in \Gamma^0\}$$

of a sufficiently regular vector-valued function $X : \Gamma^0 \times [0,T] \to \mathbb{R}^3$, where Γ^0 is the smooth closed initial surface, and $X(p,0) = p$. In view of the subsequent numerical discretization, it is convenient to think of $X(p,t)$ as the position at time t of a moving particle with label p, and of $\Gamma(t)$ as a collection of such particles. To indicate the dependence of the surface on X, we will write

$$\Gamma(t) = \Gamma(X(\,\cdot\,,t)), \qquad \text{or briefly} \quad \Gamma(X)$$

when the time t is clear from the context. The *velocity* $v(x,t) \in \mathbb{R}^3$ at a point $x = X(p,t) \in \Gamma(t)$ equals

$$\partial_t X(p,t) = v(X(p,t),t). \tag{D.1}$$

Note that for a known velocity field $v : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$, the position $X(p,t)$ at time t of the particle with label p is obtained by solving the ordinary differential equation (D.1) from 0 to t for a fixed p.

For a function $u(x,t)$ ($x \in \Gamma(t)$, $0 \le t \le T$) we denote the *material derivative* as

$$\partial^\bullet u(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}\, u(X(p,t),t) \quad \text{ for } \; x = X(p,t).$$

At $x \in \Gamma(t)$ and $0 \le t \le T$, we denote by $\nu_{\Gamma(X)}(x,t)$ the outer normal, by $\nabla_{\Gamma(X)} u(x,t)$ the tangential gradient of u, by $\Delta_{\Gamma(X)} u(x,t)$ the Laplace-Beltrami operator applied to u, and by $\nabla_{\Gamma(X)} \cdot v(x,t)$ the tangential divergence of v; see, e.g., [34] for these notions.

D.2.2. Surface motion coupled to a surface PDE: strong and weak formulation

As outlined in the introduction, we consider a parabolic equation on an evolving surface that moves according to an elliptically regularized velocity law:

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u &= f(u, \nabla_{\Gamma(X)} u), \\ v - \alpha\Delta_{\Gamma(X)} v &= g(u, \nabla_{\Gamma(X)} u)\nu_{\Gamma(X)}. \end{aligned} \tag{D.2}$$

Here, $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are given continuously differentiable functions, and $\alpha > 0$ is a fixed parameter. This system is considered together with the collection of ordinary differential equations (D.1) for every label p. Initial values are specified for u and X.

On applying the Leibniz formula as in [31], the weak formulation reads as follows: Find $u(\cdot, t) \in W^{1,\infty}(\Gamma(X(\cdot, t)))$ and $v(\cdot, t) \in W^{1,\infty}(\Gamma(X(\cdot, t)))^3$ such that for all test functions $\varphi(\,\cdot\,, t) \in H^1(X(t))$ with $\partial^\bullet \varphi = 0$ and $\psi(\,\cdot\,, t) \in H^1(X(t); \mathbb{R}^3)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(X)} u\varphi + \int_{\Gamma(X)} \nabla_{\Gamma(X)} u \cdot \nabla_{\Gamma(X)} \varphi &= \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u)\varphi, \\ \int_{\Gamma(X)} v \cdot \psi + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)} \psi &= \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u)\, \nu_{\Gamma(X)} \cdot \psi, \end{aligned} \tag{D.3}$$

alongside with the ordinary differential equations (D.1) for the positions X determining the surface $\Gamma(X)$.

We assume throughout this paper that the problem (D.2) or (D.3) admits a unique solution with sufficiently high Sobolev regularity on the time interval $[0, T]$ for the given initial data $u(\cdot, 0)$ and $X(\cdot, 0)$. We assume further that the flow map $X(\cdot, t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ is non-degenerate for $0 \leq t \leq T$, so that $\Gamma(t)$ is a regular surface.

D.2.3. Evolving surface finite elements

We describe the surface finite element discretization of our problem, following [29] and [23]. We use simplicial elements and continuous piecewise polynomial basis functions of degree k, as defined in [23, Section 2.5].

We triangulate the given smooth surface Γ^0 by an admissible family of triangulations $\mathcal{T}_h$ of decreasing maximal element diameter h; see [31] for the notion of an admissible triangulation, which includes quasi-uniformity and shape regularity. For a momentarily fixed h, we denote by $\boldsymbol{x}^0 = (x_1^0, \dots, x_N^0)$ the vector in $\mathbb{R}^{3N}$ that collects all N nodes of the triangulation. By piecewise polynomial interpolation of degree k, the nodal vector defines an approximate surface Γ_h^0 that interpolates Γ^0 in the nodes x_j^0. We will evolve the jth node in time, denoted $x_j(t)$ with $x_j(0) = x_j^0$, and collect the nodes at time t in a vector

$$\boldsymbol{x}(t) = (x_1(t), \dots, x_N(t)) \in \mathbb{R}^{3N}.$$

Provided that $x_j(t)$ is sufficiently close to the exact position $x_j^*(t) := X(p_j, t)$ (with $p_j = x_j^0$) on the exact surface $\Gamma(t) = \Gamma(X(\,\cdot\,, t))$, the nodal vector $\boldsymbol{x}(t)$ still corresponds to an admissible triangulation. In the following discussion we omit the omnipresent argument t and just write $\boldsymbol{x}$ for $\boldsymbol{x}(t)$ when the dependence on t is not important.

By piecewise polynomial interpolation on the plane reference triangle that corresponds to every curved triangle of the triangulation, the nodal vector $\boldsymbol{x}$ defines a closed surface denoted by $\Gamma_h(\boldsymbol{x})$. We can then define finite element *basis functions*

$$\phi_j[\boldsymbol{x}] : \Gamma_h(\boldsymbol{x}) \to \mathbb{R}, \quad j = 1, \dots, N,$$

which have the property that on every triangle their pullback to the reference triangle is polynomial of degree k, and which satisfy

$$\phi_j[\boldsymbol{x}](x_k) = \delta_{jk} \qquad \text{for all } j, k = 1, \dots, N.$$

These functions span the finite element space on $\Gamma_h(\boldsymbol{x})$,

$$S_h(\boldsymbol{x}) = \operatorname{span}\{\phi_1[\boldsymbol{x}], \phi_2[\boldsymbol{x}], \ldots, \phi_N[\boldsymbol{x}]\}.$$

For a finite element function $u_h \in S_h(\boldsymbol{x})$ the tangential gradient $\nabla_{\Gamma_h(\boldsymbol{x})} u_h$ is defined piecewise.

We set

$$X_h(p_h, t) = \sum_{j=1}^{N} x_j(t)\, \phi_j[\boldsymbol{x}(0)](p_h), \qquad p_h \in \Gamma_h^0,$$

which has the properties that $X_h(p_j, t) = x_j(t)$ for $j = 1, \ldots, N$, that $X_h(p_h, 0) = p_h$ for all $p_h \in \Gamma_h^0$, and

$$\Gamma_h(\boldsymbol{x}(t)) = \Gamma(X_h(\,\cdot\,, t)).$$

The *discrete velocity* $v_h(x, t) \in \mathbb{R}^3$ at a point $x = X_h(p_h, t) \in \Gamma(X_h(\,\cdot\,, t))$ is given by

$$\partial_t X_h(p_h, t) = v_h(X_h(p_h, t), t).$$

A key property of the basis functions is the *transport property* [31]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_j[\boldsymbol{x}(t)](X_h(p_h, t))\Big) = 0,$$

which by integration from 0 to t yields

$$\phi_j[\boldsymbol{x}(t)](X_h(p_h, t)) = \phi_j[\boldsymbol{x}(0)](p_h).$$

This implies for $x \in \Gamma_h\,(\boldsymbol{x}(t))$ that the discrete velocity is simply

$$v_h(x, t) = \sum_{j=1}^{N} v_j(t)\phi_j[\boldsymbol{x}(t)](x), \quad \text{with } v_j(t) = \dot{x}_j(t),$$

where the dot denotes the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}$.

The *discrete material derivative* of a finite element function

$$u_h(x,t) = \sum_{j=1}^{N} u_j(t)\, \phi_j[\boldsymbol{x}(t)](x), \qquad x \in \Gamma_h\,(\boldsymbol{x}(t)),$$

is defined as

$$\partial_h^\bullet u_h(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} u_h(X_h(p_h,t),t) \qquad \text{for } \; x = X_h(p_h,t).$$

By the transport property of the basis functions, this is just

$$\partial_h^\bullet u_h(x,t) = \sum_{j=1}^{N} \dot u_j(t)\, \phi_j[\boldsymbol{x}(t)](x), \qquad x \in \Gamma_h\,(\boldsymbol{x}(t)).$$

D.2.4. Semidiscretization of the evolving surface problem

The finite element spatial semidiscretization of the problem (D.3) reads as follows: Find the unknown nodal vector $\boldsymbol{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element functions $u_h(\,\cdot\,,t) \in S_h(\boldsymbol{x}(t))$ and $v_h(\,\cdot\,,t) \in S_h(\boldsymbol{x}(t))^3$ such that, for all $\varphi_h(\,\cdot\,,t) \in S_h(\boldsymbol{x}(t))$ with $\partial_h^\bullet \varphi_h = 0$ and all $\psi_h(\,\cdot\,,t) \in S_h(\boldsymbol{x}(t))^3$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(\boldsymbol{x})} u_h \varphi_h + \int_{\Gamma_h(\boldsymbol{x})} \nabla_{\Gamma_h(\boldsymbol{x})} u_h \cdot \nabla_{\Gamma_h(\boldsymbol{x})} \varphi_h \\
&\qquad = \int_{\Gamma_h(\boldsymbol{x})} f(u_h, \nabla_{\Gamma_h(\boldsymbol{x})} u_h) \varphi_h, \\
&\int_{\Gamma_h(\boldsymbol{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\boldsymbol{x})} \nabla_{\Gamma_h(\boldsymbol{x})} v_h \cdot \nabla_{\Gamma_h(\boldsymbol{x})} \psi_h \\
&\qquad = \int_{\Gamma_h(\boldsymbol{x})} g(u_h, \nabla_{\Gamma_h(\boldsymbol{x})} u_h) \nu_{\Gamma_h(\boldsymbol{x})} \cdot \psi_h,
\end{aligned} \tag{D.4}$$

and

$$\partial_t X_h(p_h,t) = v_h(X_h(p_h,t),t), \quad p_h \in \Gamma_h^0. \tag{D.5}$$

The initial values for the nodal vector $\boldsymbol{u}$ corresponding to u_h and the nodal vector $\boldsymbol{x}$ of the initial positions are taken as the exact initial values at the nodes x_j^0 of the triangulation of the given initial surface Γ^0:

$$x_j(0) = x_j^0, \quad u_j(0) = u(x_j^0, 0) \qquad (j = 1, \dots, N).$$

D.2.5. Differential-algebraic equations of the matrix-vector formulation

We now show that the nodal vectors $\boldsymbol{u} \in \mathbb{R}^N$ and $\boldsymbol{v} \in \mathbb{R}^{3N}$ of the finite element functions u_h and v_h, respectively, together with the surface nodal vector $\boldsymbol{x} \in \mathbb{R}^{3N}$ satisfy a system of differential-algebraic equations (DAEs). Using the above finite element setting, we set (omitting the argument t)

$$\begin{aligned} u_h &= \sum_{j=1}^{N} u_j \phi_j[\boldsymbol{x}], \qquad u_h(x_j) = u_j \in \mathbb{R}, \\ v_h &= \sum_{j=1}^{N} v_j \phi_j[\boldsymbol{x}], \qquad v_h(x_j) = v_j \in \mathbb{R}^3, \end{aligned}$$

and collect the nodal values in column vectors $\boldsymbol{u} = (u_j) \in \mathbb{R}^N$ and $\boldsymbol{v} = (v_j) \in \mathbb{R}^{3N}$.

We define the surface-dependent mass matrix $M(\boldsymbol{x})$ and stiffness matrix $A(\boldsymbol{x})$ on the surface determined by the nodal vector $\boldsymbol{x}$:

$$\begin{aligned} M(\boldsymbol{x})|_{jk} &= \int_{\Gamma_h(\boldsymbol{x})} \phi_j[\boldsymbol{x}] \phi_k[\boldsymbol{x}], \\ A(\boldsymbol{x})|_{jk} &= \int_{\Gamma_h(\boldsymbol{x})} \nabla_{\Gamma_h} \phi_j[\boldsymbol{x}] \cdot \nabla_{\Gamma_h} \phi_k[\boldsymbol{x}], \end{aligned} \qquad (j, k = 1, \dots, N).$$

We further let (with the identity matrix $I_3 \in \mathbb{R}^{3\times 3}$)

$$M^*(\boldsymbol{x}) = I_3 \otimes \Big(M(\boldsymbol{x}) + \alpha A(\boldsymbol{x}) \Big). \tag{D.6}$$

The right-hand side vectors $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^N$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u}) \in \mathbb{R}^{3N}$ are given by

$$\begin{aligned}
\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})|_j &= \int_{\Gamma_h(\boldsymbol{x})} f(u_h, \nabla_{\Gamma_h} u_h)\, \phi_j[\boldsymbol{x}], \\
\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u})|_{3(j-1)+\ell} &= \int_{\Gamma_h(\boldsymbol{x})} g(u_h, \nabla_{\Gamma_h} u_h)\ \big(\nu_{\Gamma_h(\boldsymbol{x})}\big)_\ell\, \phi_j[\boldsymbol{x}],
\end{aligned}$$

for $j = 1,\dots,N$, and $\ell = 1,2,3$.

We then obtain from (D.4)–(D.5) the following coupled DAE system for the nodal values $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{x}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x})\boldsymbol{u}\Big) + A(\boldsymbol{x})\boldsymbol{u} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}), \\
M^*(\boldsymbol{x})\boldsymbol{v} &= \boldsymbol{g}(\boldsymbol{x},\boldsymbol{u}), \\
\dot{\boldsymbol{x}} &= \boldsymbol{v}.
\end{aligned} \tag{D.7}$$

With the auxiliary vector $\boldsymbol{w} = M(\boldsymbol{x})\boldsymbol{u}$, this system becomes

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{v}, \\
\dot{\boldsymbol{w}} &= -A(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}), \\
\mathbf{0} &= -M^*(\boldsymbol{x})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{x},\boldsymbol{u}), \\
\mathbf{0} &= -M(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{w}.
\end{aligned}$$

This is of a form to which standard DAE time discretization can be applied; see, e.g., [49, Chap. VI].

As will be seen in later sections, the matrix-vector formulation is very useful in the stability analysis of the ESFEM, beyond its obvious role for practical computations.

D.2.6. Lifts

In the error analysis we need to compare functions on three different surfaces: the *exact surface* $\Gamma(t) = \Gamma(X(\,\cdot\,,t))$, the *discrete surface* $\Gamma_h(t) = \Gamma_h(\boldsymbol{x}(t))$, and the *interpolated surface* $\Gamma_h^*(t) = \Gamma_h(\boldsymbol{x}^*(t))$, where $\boldsymbol{x}^*(t)$ is the nodal vector collecting the grid points $x_j^*(t) = X(p_j, t)$ on the exact surface. In the following definitions we omit the argument t in the notation.

A finite element function $w_h : \Gamma_h \to \mathbb{R}^m$ ($m = 1$ or 3) on the discrete surface, with nodal values w_j, is related to the finite element function $\widehat{w}_h$ on the interpolated surface that has the same nodal values:

$$\widehat{w}_h = \sum_{j=1}^{N} w_j \phi_j[\boldsymbol{x}^*].$$

The transition between the interpolated surface and the exact surface is done by the *lift operator*, which was introduced for linear surface approximations in [29]; see also [31, 35]. Higher-order generalizations have been studied in [23]. The lift operator l maps a function on the interpolated surface Γ_h^* to a function on the exact surface Γ, provided that Γ_h^* is sufficiently close to Γ.

The exact regular surface $\Gamma(X(\cdot, t))$ can be represented by a (sufficiently smooth) signed distance function $d : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$, cf. [31, Section 2.1], such that

$$\Gamma(X(\cdot, t)) = \{x \in \mathbb{R}^3 \mid d(x, t) = 0\} \subset \mathbb{R}^3. \tag{D.8}$$

Using this distance function, the lift of a continuous function $\eta_h \colon \Gamma_h^* \to \mathbb{R}$ is defined as

$$\eta_h^l(y) \coloneqq \eta_h(x), \qquad x \in \Gamma_h^*,$$

where for every $x \in \Gamma_h^*$ the point $y = y(x) \in \Gamma$ is uniquely defined via

$$y = x - \nu(y)d(x).$$

For functions taking values in $\mathbb{R}^3$ the lift is componentwise. By η^{-l} we denote the function on Γ_h^* whose lift is η.

We denote the composed lift L from finite element functions on Γ_h to functions on Γ via Γ_h^* by

$$w_h^L = (\widehat{w}_h)^l.$$

D.3. Statement of the main result: semidiscrete error bound

We are now in the position to formulate the main result of this paper, which yields optimal-order error bounds for the finite element semidiscretization of a surface PDE on a solution-driven surface as specified in (D.2), for finite elements of polynomial degree $k \geq 2$. We denote by $\Gamma(t) = \Gamma(X(\,\cdot\,,t))$ the exact surface and by $\Gamma_h(t) = \Gamma(X_h(\,\cdot\,,t)) = \Gamma_h(\boldsymbol{x}(t))$ the discrete surface at time t. We introduce the notation

$$x_h^L(x,t) = X_h^L(p,t) \in \Gamma_h(t) \qquad \text{for} \quad x = X(p,t) \in \Gamma(t).$$

Theorem D.3.1. *Consider the space discretization* (D.4)–(D.5) *of the coupled problem* (D.1)–(D.2)*, using evolving surface finite elements of polynomial degree $k \geq 2$. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for u. Suppose that the problem admits an exact solution (u, v, X) that is sufficiently smooth (say, in the Sobolev class H^{k+1}) on the time interval $0 \leq t \leq T$, and that the flow map $X(\cdot,t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ is non-degenerate for $0 \leq t \leq T$, so that $\Gamma(t)$ is a regular surface.*

Then, there exists $h_0 > 0$ such that for all mesh widths $h \leq h_0$ the following error bounds hold over the exact surface $\Gamma(t) = \Gamma(X(\,\cdot\,,t))$ for $0 \leq t \leq T$:

$$\|u_h^L(\,\cdot\,,t) - u(\,\cdot\,,t)\|^2_{L^2(\Gamma(t))} + \int_0^t \|u_h^L(\,\cdot\,,s) - u(\,\cdot\,,s)\|^2_{H^1(\Gamma(s))}\,\mathrm{d}s \leq Ch^{2k}$$

and

$$\left(\int_0^t \| v_h^L(\,\cdot\,,s) - v(\,\cdot\,,s) \|_{H^1(X(s);\mathbb{R}^3)} \, \mathrm{d}s \right)^{\frac{1}{2}} \leq Ch^k,$$

$$\| x_h^L(\,\cdot\,,t) - \mathrm{id}_{\Gamma(t)} \|_{H^1(X(t);\mathbb{R}^3)} \leq Ch^k.$$

The constant C is independent of t and h, but depends on bounds of the H^{k+1} norms of the solution (u, v, X), on local Lipschitz constants of f and g, on the regularization parameter $\alpha > 0$ and on the length T of the time interval.

We note that the last error bound is equivalent to

$$\| X_h^L(\,\cdot\,,t) - X(\,\cdot\,,t) \|_{H^1(X(0);\mathbb{R}^3)} \leq ch^k.$$

Moreover, in the case of a coupling function g in (D.2) that is independent of the solution gradient, so that $g = g(u)$, we obtain an error bound for the velocity that is pointwise in time: uniformly for $0 \leq t \leq T$,

$$\| v_h^L(\cdot,t) - v(\cdot,t) \|_{H^1(\Gamma(t))^3} \leq Ch^k.$$

A key issue in the proof is to ensure that the $W^{1,\infty}$ norm of the position error of the curves remains small. The H^1 error bound and an inverse estimate yield an $O(h^{k-1})$ error bound in the $W^{1,\infty}$ norm. This is small only for $k \geq 2$, which is why we impose the condition $k \geq 2$ in the above result.

Since the exact flow map $X(\cdot,t) : \Gamma_0 \to \Gamma(t)$ is assumed to be smooth and non-degenerate, it is locally close to an invertible linear transformation, and (using compactness) it therefore preserves the admissibility of grids with sufficiently small mesh width $h \leq h_0$. Our assumptions guarantee that the triangulations formed by the nodes $x_j^*(t) = X(p_j, t)$ remain admissible uniformly for $t \in [0, T]$ (though the admissibility bounds may degrade with growing t). Since $k \geq 2$, the position error estimate implies that for sufficiently small h also the triangulations formed by the numerical nodes

$x_j(t)$ remain admissible uniformly for $t \in [0, T]$. This cannot be concluded for $k = 1$.

The error bound will be proven by clearly separating the issues of consistency and stability. The consistency error is the defect on inserting a projection (interpolation or Ritz projection) of the exact solution into the discretized equation. The defect bounds involve *geometric estimates* that were obtained for the time dependent case and for higher order $k \geq 2$ in [54], by combining techniques of Dziuk & Elliott [31, 35] and Demlow [23]. This is done with the ESFEM formulation of section D.2.4.

The main issue in the proof of theorem D.3.1 is to prove *stability* in the form of an h-independent bound of the error in terms of the defect. The stability analysis is done in the matrix-vector formulation of section D.2.5. It uses energy estimates and transport formulae that relate the mass and stiffness matrices and the coupling terms for different nodal vectors $\boldsymbol{x}$. *No geometric estimates* enter in the proof of stability.

In section D.4 we prove important auxiliary results for the stability analysis. The stability is first analysed for the discretized velocity law without coupling to the surface PDE in section D.5 and is then extended to the coupled problem in section D.6. The necessary geometric estimates for the consistency analysis are collected in section D.7, and the defects are then bounded in section D.8. The proof of theorem D.3.1 is then completed in section D.9 by putting together the results on stability, defect bounds and interpolation error bounds.

D.4. Auxiliary results for the stability analysis: relating different surfaces

The finite element matrices of section D.2.5 induce discrete versions of Sobolev norms. For any $\boldsymbol{w} = (w_j) \in \mathbb{R}^N$ with corresponding finite element function $w_h = \sum_{j=1}^N w_j\phi_j[\boldsymbol{x}] \in S_h(\boldsymbol{x})$ we note

$$|\boldsymbol{w}|^2_{M(\boldsymbol{x})} \coloneqq \boldsymbol{w}^T M(\boldsymbol{x})\boldsymbol{w} = \|w_h\|^2_{L^2(\Gamma_h(\boldsymbol{x}))}, \tag{D.9}$$

$$|\boldsymbol{w}|^2_{A(\boldsymbol{x})} \coloneqq \boldsymbol{w}^T A(\boldsymbol{x})\boldsymbol{w} = \|\nabla_{\Gamma_h(\boldsymbol{x})} w_h\|^2_{L^2(\Gamma_h(\boldsymbol{x}))}. \tag{D.10}$$

In our stability analysis we need to relate finite element matrices corresponding to different nodal vectors. We use the following setting. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{3N}$ be two nodal vectors defining discrete surfaces $\Gamma_h(\boldsymbol{x})$ and $\Gamma_h(\boldsymbol{y})$, respectively. We let $\boldsymbol{e} = (e_j) = \boldsymbol{x} - \boldsymbol{y} \in \mathbb{R}^{3N}$. For the parameter $\theta \in [0,1]$, we consider the intermediate surface $\Gamma_h^\theta = \Gamma_h(\boldsymbol{y} + \theta\boldsymbol{e})$ and the corresponding finite element functions given as

$$e_h^\theta = \sum_{j=1}^N e_j\phi_j[\boldsymbol{y} + \theta\boldsymbol{e}]$$

and, for any vectors $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{R}^N$,

$$w_h^\theta = \sum_{j=1}^N w_j\phi_j[\boldsymbol{y} + \theta\boldsymbol{e}] \quad \text{and} \quad z_h^\theta = \sum_{j=1}^N z_j\phi_j[\boldsymbol{y} + \theta\boldsymbol{e}].$$

Lemma D.4.1. *In the above setting the following identities hold:*

$$\boldsymbol{w}^T(M(\boldsymbol{x}) - M(\boldsymbol{y}))\boldsymbol{z} = \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta(\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) z_h^\theta \,\mathrm{d}\theta,$$

$$\boldsymbol{w}^T(A(\boldsymbol{x}) - A(\boldsymbol{y}))\boldsymbol{z} = \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} w_h^\theta \cdot (D_{\Gamma_h^\theta} e_h^\theta)\nabla_{\Gamma_h^\theta} z_h^\theta \,\mathrm{d}\theta,$$

with $D_{\Gamma_h^\theta} e_h^\theta = \mathrm{trace}(E)I_3 - (E + E^T)$ *for* $E = \nabla_{\Gamma_h^\theta} e_h^\theta \in \mathbb{R}^{3\times 3}$.

Proof. Using the fundamental theorem of calculus and the Leibniz formula we write

$$\begin{aligned}\boldsymbol{w}^T\left(M(\boldsymbol{x})-M(\boldsymbol{y})\right)\boldsymbol{z} &= \int_{\Gamma_h(\boldsymbol{x})} w_h^1 z_h^1 - \int_{\Gamma_h(\boldsymbol{y})} w_h^0 z_h^0 \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\Gamma_h^\theta} w_h^\theta z_h^\theta \,\mathrm{d}\theta \\ &= \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta(\nabla_{\Gamma_h^\theta}\cdot e_h^\theta) z_h^\theta \,\mathrm{d}\theta.\end{aligned}$$

In the last formula we used that the material derivatives (with respect to θ) of w_h^θ and z_h^θ vanish, thanks to the transport property of the basis functions. The second identity is shown in the same way, using the formula for the derivative of the Dirichlet integral; see [31] and also [37, lemma 3.1]. ■

A direct consequence of lemma D.4.1 is the following conditional equivalence of norms:

Lemma D.4.2. *If* $\|\nabla_{\Gamma_h^\theta}\cdot e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \le \mu$ *for* $0\le\theta\le 1$, *then*

$$|\boldsymbol{w}|_{M(\boldsymbol{y}+\boldsymbol{e})} \le e^{\mu/2}\,|\boldsymbol{w}|_{M(\boldsymbol{y})}.$$

If $\|D_{\Gamma_h^\theta} e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \le \eta$ *for* $0\le\theta\le 1$, *then*

$$|\boldsymbol{w}|_{A(\boldsymbol{y}+\boldsymbol{e})} \le e^{\eta/2}\,|\boldsymbol{w}|_{A(\boldsymbol{y})}.$$

Proof. By lemma D.4.1 we have for $0\le\tau\le 1$

$$\begin{aligned}&|\boldsymbol{w}|^2_{M(\boldsymbol{y}+\tau\boldsymbol{e})} - |\boldsymbol{w}|^2_{M(\boldsymbol{y})} = \boldsymbol{w}^T(M(\boldsymbol{y}+\tau\boldsymbol{e})-M(\boldsymbol{y}))\boldsymbol{w} \\ &= \int_0^\tau\int_{\Gamma_h^\theta} w_h^\theta\cdot(\nabla_{\Gamma_h^\theta}\cdot e_h^\theta) w_h^\theta\,\mathrm{d}\theta \le \mu\int_0^\tau \|w_h^\theta\|^2_{L^2(\Gamma_h^\theta)}\,\mathrm{d}\theta \\ &= \mu\int_0^\tau |\boldsymbol{w}|^2_{M(\boldsymbol{y}+\theta\boldsymbol{e})}\,\mathrm{d}\theta,\end{aligned}$$

and the first result follows from Gronwall's inequality. The second result is proved in the same way. ■

The following result, when used with w_h^θ equal to components of e_h^θ, reduces the problem of checking the conditions of the previous lemma for $0 \le \theta \le 1$ to checking the condition just for the case $\theta = 0$.

Lemma D.4.3. *In the above setting, assume that*

$$\|\nabla_{\Gamma_h[\boldsymbol{y}]} e_h^0\|_{L^\infty(\Gamma_h[\boldsymbol{y}])} \le \frac{1}{2}. \tag{D.11}$$

Then, for $0 \le \theta \le 1$ *the function* $w_h^\theta = \sum_{j=1}^N w_j \phi_j[\boldsymbol{y} + \theta \boldsymbol{e}]$ *on* $\Gamma_h^\theta = \Gamma[\boldsymbol{y} + \theta \boldsymbol{e}]$ *is bounded by*

$$\|\nabla_{\Gamma_h^\theta} w_h^\theta\|_{L^p(\Gamma_h^\theta)} \le c_p \ \|\nabla_{\Gamma_h^0} w_h^0\|_{L^p(\Gamma_h^0)}, \quad 1 \le p \le \infty,$$

where c_p *depends only on* p *(we have* $c_\infty = 2$*).*

Proof. We describe the finite element parametrization of the discrete surfaces Γ_h^θ in the same way as in section D.2.3, with θ instead of t in the role of the time variable. We set

$$Y_h^\theta(q_h) = Y_h(q_h, \theta) = \sum_{j=1}^N (y_j + \theta e_j)\phi_j[\boldsymbol{y}](q_h), \qquad q_h \in \Gamma_h[\boldsymbol{y}] \tag{D.12}$$

so that

$$\Gamma(Y_h^\theta) = \Gamma_h[\boldsymbol{y} + \theta \boldsymbol{e}] = \Gamma_h^\theta.$$

Since $Y_h^0(q_h) = q_h$ for all $q_h \in \Gamma_h^0 = \Gamma_h[\boldsymbol{y}]$, the above formula can be rewritten as

$$Y_h^\theta(q_h) = q_h + \theta e_h^0(q_h).$$

Tangent vectors to Γ_h^θ at $y_h^\theta = Y_h^\theta(q_h)$ are therefore of the form

$$\delta y_h^\theta = DY_h^\theta(q_h)\, \delta q_h = \delta q_h + \theta \left(\nabla_{\Gamma_h^0} e_h^0(q_h)\right)^T \delta q_h,$$

where δq_h is a tangent vector to Γ_h^0 at q_h, or written more concisely, $\delta q_h \in T_{q_h}\Gamma_h^0$.

Letting $|\cdot|$ denote the Euclidean norm of a vector in $\mathbb{R}^3$, we have at $y_h^\theta = Y_h^\theta(q_h)$

$$|\nabla_{\Gamma_h^\theta} w_h^\theta(y_h^\theta)| = \sup_{\delta y_h^\theta \in T_{y_h^\theta}\Gamma_h^\theta} \frac{\left(\nabla_{\Gamma_h^\theta} w_h^\theta(y_h^\theta)\right)^T \delta y_h^\theta}{|\delta y_h^\theta|} = \sup_{\delta y_h^\theta \in T_{y_h^\theta}\Gamma_h^\theta} \frac{D w_h^\theta(y_h^\theta)\delta y_h^\theta}{|\delta y_h^\theta|}$$
$$= \sup_{\delta q_h \in T_{q_h}\Gamma_h^0} \frac{D w_h^\theta(y_h^\theta) D Y_h^\theta(q_h)\, \delta q_h}{|D Y_h^\theta(q_h)\, \delta q_h|}.$$

By construction of w_h^θ and the transport property of the basis functions, we have

$$w_h^\theta(Y_h^\theta(q_h)) = \sum_{j=1}^N w_j \phi_j[\boldsymbol{y} + \theta \boldsymbol{e}](Y_h^\theta(q_h)) = \sum_{j=1}^N w_j \phi_j[\boldsymbol{y}](q_h) = w_h^0(q_h).$$

By the chain rule, this yields

$$D w_h^\theta(y_h^\theta) D Y_h^\theta(q_h) = D w_h^0(q_h)$$

Under the imposed condition $\|\nabla_{\Gamma_h^0} e_h^0\|_{L^\infty(\Gamma_h[\boldsymbol{y}])} \leq \frac{1}{2}$ we have for $0 \leq \theta \leq 1$

$$|D Y_h^\theta(q_h)\, \delta q_h| \geq |\delta q_h| - \theta \left| \left(\nabla_{\Gamma_h^0} \delta e_h^0(q_h)\right)^T \delta q_h \right| \geq \frac{1}{2} |\delta q_h|.$$

Hence we obtain

$$|\nabla_{\Gamma_h^\theta} w_h^\theta(y_h^\theta)| = \sup_{\delta q_h \in T_{q_h}\Gamma_h^0} \frac{D w_h^0(q_h)\, \delta q_h}{|D Y_h^\theta(q_h)\, \delta q_h|}$$
$$\leq \sup_{\delta q_h \in T_{q_h}\Gamma_h^0} \frac{D w_h^0(q_h)\, \delta q_h}{\frac{1}{2}|\delta q_h|} = 2\, |\nabla_{\Gamma_h^0} w_h^0(q_h)|.$$

This yields the stated result for $p = \infty$. For $1 \le p < \infty$ we note in addition that in using the integral transformation formula we have a uniform bound between the surface elements, since DY_h^θ is close to the identity matrix by our smallness assumption on $\nabla_{\Gamma_h^0} e_h^0$. ■

The arguments of the previous proof are also used in estimating the changes of the normal vectors on the various surfaces $\Gamma_h^\theta = \Gamma_h[\boldsymbol{y} + \theta \boldsymbol{e}]$.

Lemma D.4.4. *Suppose that condition* (D.11) *is satisfied. Let* $y_h^\theta = Y_h^\theta(q_h) \in \Gamma_h^\theta$ *be related by the parametrization* (D.12) *of* Γ_h^θ *over* Γ_h^0, *for* $0 \le \theta \le 1$. *Then, the corresponding unit normal vectors differ by no more than*

$$|\nu_{\Gamma_h^\theta}(y_h^\theta) - \nu_{\Gamma_h^0}(y_h^0)| \le C\theta |\nabla_{\Gamma_h^0} e_h^0(y_h^0)|$$

with some constant C.

Proof. Let δq_h^1 and δq_h^2 be two linearly independent tangent vectors of Γ_h^0 at $q_h \in \Gamma_h^0$ (which may be chosen orthogonal to each other and of unit length with respect to the Euclidean norm). With $\delta y_h^{\theta,i} = DY_h^\theta(q_h)\, \delta q_h^i = \delta q_h^i + \theta \left(\nabla_{\Gamma_h^0} e_h(q_h)\right)^T \delta q_h^i$ for $i = 1, 2$ we then have, for $0 \le \theta \le 1$,

$$\nu_{\Gamma_h^\theta}(y_h^\theta) = \frac{\delta y_h^{\theta,1} \times \delta y_h^{\theta,2}}{|\delta y_h^{\theta,1} \times \delta y_h^{\theta,2}|}.$$

Since this expression is a locally Lipschitz continuous function of the two vectors, the result follows. (The imposed bound (D.11) is sufficient to ensure the linear independence of the vectors $\delta y_h^{\theta,i}$.) ■

We denote by $\partial_\theta^\bullet f$ the material derivative of a function $f = f(y_h^\theta, \theta)$ depending on $\theta \in [0, 1]$ and $y_h^\theta \in \Gamma_h^\theta$:

$$\partial_\theta^\bullet f = \frac{\mathrm{d}}{\mathrm{d}\theta} f(y_h^\theta, \theta).$$

From lemma D.4.4 together with lemma D.4.3 we obtain the following bound:

Lemma D.4.5. *If condition* (D.11) *is satisfied, then*

$$\|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}\|_{L^p(\Gamma_h^\theta)} \le C\, \|\nabla_{\Gamma_h^0} e_h^0\|_{L^p(\Gamma_h^0)}$$

where C is independent of $0 \le \theta \le 1$ *and* $1 \le p \le \infty$.

Proof. By lemma D.4.4 with Γ_h^θ in the role of Γ_h^0, we obtain

$$|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}(y_h^\theta)| = |\lim_{\tau\to 0} (\nu_{\Gamma_h^{\theta+\tau}}(y_h^{\theta+\tau}) - \nu_{\Gamma_h^\theta}(y_h^\theta))/\tau| \le C|\nabla_{\Gamma_h^\theta} e_h^\theta(y_h^\theta)|,$$

which implies

$$\|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}\|_{L^p(\Gamma_h^\theta)} \le C\, \|\nabla_{\Gamma_h^\theta} e_h^\theta\|_{L^p(\Gamma_h^\theta)},$$

and lemma D.4.3 completes the proof. ∎

A bound for the time derivatives of the mass and stiffness matrices corresponding to nodes on the $\Gamma(t)$ is a direct consequence of [37, lemma 4.1].

Lemma D.4.6. *Let* $\Gamma(t) = \Gamma(X(\,\cdot\,, t))$, $t \in [0, T]$, *be a smoothly evolving family of smooth closed surfaces, and let the vector* $\boldsymbol{x}^*(t) \in \mathbb{R}^{3N}$ *collect the nodes* $x_j^*(t) = X(p_j, t)$. *Then,*

$$\boldsymbol{w}^T \frac{\mathrm{d}}{\mathrm{d}t} M(\boldsymbol{x}^*(t))\boldsymbol{z} \le C\, |\boldsymbol{w}|_{M(\boldsymbol{x}^*(t))}\, |\boldsymbol{z}|_{M(\boldsymbol{x}^*(t))},$$
$$\boldsymbol{w}^T \frac{\mathrm{d}}{\mathrm{d}t} A(\boldsymbol{x}^*(t))\boldsymbol{z} \le C\, |\boldsymbol{w}|_{A(\boldsymbol{x}^*(t))}\, |\boldsymbol{z}|_{A(\boldsymbol{x}^*(t))},$$

for all $\boldsymbol{w}, \boldsymbol{z} \in \mathbb{R}^N$ *and* $s, t \in [0, T]$. *The constant C depends only on a bound of the* $W^{1,\infty}$ *norm of the surface velocity.*

D.5. Stability of discretized surface motion under a prescribed driving-term

In this section we begin the stability analysis by first studying the stability of the spatially discretized velocity law with a given inhomogeneity instead of a coupling to the surface PDE. This allows us to present, in a technically simpler setting, some of the basic arguments that are used in our approach to stability estimates, which works with the matrix-vector formulation. The stability of the spatially discretized problem including coupling with the surface PDE is then studied in Section D.6 by similar, but more elaborate arguments.

D.5.1. Uncoupled velocity law and its semidiscretization

In this section we consider the velocity law without coupling to a surface PDE:

$$v - \alpha \Delta_{\Gamma(X)} v = g\, \nu_{\Gamma(X)},$$

where $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given continuous function of (x, t), and $\alpha > 0$ is a fixed parameter. This problem is considered together with the ordinary differential equations (D.1) for the positions X determining the surface $\Gamma(X)$. Initial values are specified for X.

The weak formulation is given by the second formula of (D.3) with the function g considered here. This is considered together with the ordinary differential equations (D.1) for the positions X. Then the finite element spatial semidiscretization of this problem reads as: Find the unknown nodal vector $\boldsymbol{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element function $v_h(\,\cdot\,, t) \in S_h(\boldsymbol{x}(t))^3$ such that the following semidiscrete equation holds for every $\psi_h \in S_h(\boldsymbol{x}(t))^3$:

$$\int_{\Gamma_h(\boldsymbol{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\boldsymbol{x})} \nabla_{\Gamma_h(\boldsymbol{x})} v_h \cdot \nabla_{\Gamma_h(\boldsymbol{x})} \psi_h = \int_{\Gamma_h(\boldsymbol{x})} g\, \nu_{\Gamma_h(\boldsymbol{x})} \cdot \psi_h, \tag{D.13}$$

together with the ordinary differential equations (D.5). As before, the nodal vector of the initial positions $\boldsymbol{x}(0)$ is taken from the exact initial values at the nodes x_j^0 of the triangulation of the given initial surface Γ^0: $x_j(0) = x_j^0$ for $j = 1, \dots, N$.

As in section D.2.5, the nodal vectors $\boldsymbol{v} \in \mathbb{R}^{3N}$ of the finite element function v_h together with the surface nodal vector $\boldsymbol{x} \in \mathbb{R}^{3N}$ satisfy a system of differential-algebraic equations (DAEs). We obtain from (D.13) and (D.5) the following coupled DAE system for the nodal values $\boldsymbol{v}$ and $\boldsymbol{x}$:

$$\begin{aligned} M^*(\boldsymbol{x})\boldsymbol{v} &= \boldsymbol{g}(\boldsymbol{x}, t) \\ \dot{\boldsymbol{x}} &= \boldsymbol{v} \end{aligned} \tag{D.14}$$

Here the matrix $M^*(\boldsymbol{x}) = I_3 \otimes (M(\boldsymbol{x}) + \alpha A(\boldsymbol{x}))$ is from (D.6), and the driving term $\boldsymbol{g}(\boldsymbol{x}, t)$ is given by

$$\boldsymbol{g}(\boldsymbol{x}, t))|_{3(j-1)+\ell} = \int_{\Gamma_h(\boldsymbol{x})} g(\,\cdot\,, t)\ \big(\nu_{\Gamma_h(\boldsymbol{x})}\big)_\ell\, \phi_j[\boldsymbol{x}],$$

where $j = 1, \dots, N$, and $\ell = 1, 2, 3$.

D.5.2. Error equations

We denote by

$$\boldsymbol{x}^*(t) = \big(x_j^*(t)\big) \in \mathbb{R}^{3N} \quad \text{with} \quad x_j^*(t) = X(p_j, t) \qquad (j = 1, \dots, N)$$

the nodal vector of the *exact* positions on the surface $\Gamma(X(\,\cdot\,, t))$. This defines a discrete surface $\Gamma_h(\boldsymbol{x}^*(t))$ that interpolates the exact surface $\Gamma(X(\,\cdot\,, t))$.

We consider the interpolated exact velocity

$$v_h^*(\cdot, t) = \sum_{j=1}^N v_j^*(t)\phi_j[\boldsymbol{x}^*(t)] \qquad \text{with} \qquad v_j^*(t) = \dot{x}_j^*(t),$$

with the corresponding nodal vector

$$\boldsymbol{v}^*(t) = \big(v_j^*(t)\big) = \dot{\boldsymbol{x}}^*(t) \in \mathbb{R}^{3N}.$$

Inserting v_h^* and $\boldsymbol{x}^*$ in place of the numerical solution v_h and $\boldsymbol{x}$ into (D.13) yields a defect $d_h(\,\cdot\,,t) \in S_h(\boldsymbol{x}^*(t))^3$: for every $\psi_h \in S_h(\boldsymbol{x}^*(t))^3$,

$$\begin{aligned}\int_{\Gamma_h(\boldsymbol{x}^*)} v_h^* \cdot \psi_h + \alpha \int_{\Gamma_h(\boldsymbol{x}^*)} \nabla_{\Gamma_h(\boldsymbol{x}^*)} v_h^* \cdot \nabla_{\Gamma_h(\boldsymbol{x}^*)} \psi_h \\ = \int_{\Gamma_h(\boldsymbol{x}^*)} g\, \nu_{\Gamma_h(\boldsymbol{x}^*)} \cdot \psi_h + \int_{\Gamma_h(\boldsymbol{x}^*)} d_h \cdot \psi_h.\end{aligned}$$

With $d_h(\,\cdot\,,t) = \sum_{j=1}^N d_j(t)\phi_j[\boldsymbol{x}^*(t)]$ and the corresponding nodal vector $\boldsymbol{d_v}(t) = \big(d_j(t)\big) \in \mathbb{R}^{3N}$ we then have $(I_3 \otimes M(\boldsymbol{x}^*(t)))\boldsymbol{d_v}(t)$ as the defect on inserting $\boldsymbol{x}^*$ and $\boldsymbol{v}^*$ in the first equation of (D.14), and the defect in the second equation is denoted $\boldsymbol{d_x}$. With $M^{[3]}(\boldsymbol{x}^*) = I_3 \otimes M(\boldsymbol{x}^*)$, we thus have

$$\begin{aligned} M^*(\boldsymbol{x}^*)\boldsymbol{v}^* &= \boldsymbol{g}(\boldsymbol{x}^*) + M^{[3]}(\boldsymbol{x}^*)\boldsymbol{d_v}, \\ \dot{\boldsymbol{x}}^* &= \boldsymbol{v}^*. \end{aligned} \tag{D.15}$$

We denote the errors in the surface nodes and in the velocity by $\boldsymbol{e_x} = \boldsymbol{x} - \boldsymbol{x}^*$ and $\boldsymbol{e_v} = \boldsymbol{v} - \boldsymbol{v}^*$, respectively. We rewrite the velocity law in (D.14) as

$$M^*(\boldsymbol{x}^*)\boldsymbol{v} = -\big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{v}^* - \big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{e_v} + \boldsymbol{g}(\boldsymbol{x}).$$

Then, by subtracting (D.15) from the above version of (D.14), we obtain the following error equations for the uncoupled problem:

$$\begin{aligned} M^*(\boldsymbol{x}^*)\boldsymbol{e_v} =&\; -\big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{v}^* - \big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{e_v} \\ &+ \big(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)\big) - M^{[3]}(\boldsymbol{x}^*)\boldsymbol{d_v}, \\ \dot{\boldsymbol{e}}_{\boldsymbol{x}} =&\; \boldsymbol{e_v}. \end{aligned} \tag{D.16}$$

When no confusion can arise, we write in the following $M(\boldsymbol{x}^*)$ for $M^{[3]}(\boldsymbol{x}^*)$ and $\|\cdot\|_{H^1(\Gamma)}$ for $\|\cdot\|_{H^1(X(t);\mathbb{R}^3)}$, etc.

D.5.3. Norms

We recall that $M^*(\boldsymbol{x}) = I_3 \otimes (M(\boldsymbol{x}) + \alpha A(\boldsymbol{x}))$ and, for $\boldsymbol{w} \in \mathbb{R}^{3N}$ and the corresponding finite element function $w_h = \sum_{j=1}^N w_j \phi_j[\boldsymbol{x}] \in S_h(\boldsymbol{x})^3$, we consider the norm

$$
\begin{aligned}
|\boldsymbol{w}|^2_{M^*(\boldsymbol{x}^*)} :=& \boldsymbol{w}^T M^*(\boldsymbol{x}^*)\boldsymbol{w} \\
=& \; \|w_h\|^2_{L^2(\Gamma_h(\boldsymbol{x}^*))} + \alpha \, \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} w_h\|^2_{L^2(\Gamma_h(\boldsymbol{x}^*))} \sim \|w_h\|^2_{H^1(\Gamma_h(\boldsymbol{x}^*))}.
\end{aligned}
$$

For convenience, we will take $\alpha = 1$ in the remainder of this section, so that the last norm equivalence becomes an equality. For the defect $d_h \in S_h(\boldsymbol{x}^*)^3$ we use the dual norm (cf. [63, Proof of theorem 5.1])

$$
\|d_h\|_{H_h^{-1}(\Gamma_h(\boldsymbol{x}^*))} := \sup_{0 \neq \psi_h \in S_h(\boldsymbol{x}^*)^3} \frac{\int_{\Gamma_h(\boldsymbol{x}^*)} d_h \cdot \psi_h}{\|\psi_h\|_{H^1(\boldsymbol{x}^*;\mathbb{R}^3)}}. \tag{D.17}
$$

Further, a quick calculation shows

$$
\|d_h\|_{H_h^{-1}(\Gamma_h(\boldsymbol{x}^*))} = (\boldsymbol{d}_{\boldsymbol{v}}^T M(\boldsymbol{x}^*) M^*(\boldsymbol{x}^*)^{-1} M(\boldsymbol{x}^*) \boldsymbol{d}_{\boldsymbol{v}})^{\frac{1}{2}}. \tag{D.18}
$$

We denote

$$
\|\boldsymbol{d}_{\boldsymbol{v}}\|^2_{*,\boldsymbol{x}^*} := \boldsymbol{d}_{\boldsymbol{v}}^T M(\boldsymbol{x}^*) M^*(\boldsymbol{x}^*)^{-1} M(\boldsymbol{x}^*) \boldsymbol{d}_{\boldsymbol{v}}
$$

so that

$$
\|\boldsymbol{d}_{\boldsymbol{v}}\|_{*,\boldsymbol{x}^*} = \|d_h\|_{H_h^{-1}(\boldsymbol{x}^*)}.
$$

D.5.4. Stability estimate

The following stability result holds for the errors $\boldsymbol{e}_{\boldsymbol{v}}$ and $\boldsymbol{e}_{\boldsymbol{x}}$, under an assumption of small defects. It will be shown in section D.8 that this assumption is satisfied if the exact solution is sufficiently smooth.

Proposition D.5.1. *Suppose that the defect is bounded as follows, with $\kappa > 1$:*

$$\|\boldsymbol{d_v}(t)\|_{*,\boldsymbol{x}^*(t)} \le ch^\kappa, \quad t \in [0, T].$$

Then there exists $h_0 > 0$ such that the following error bounds hold for $h \le h_0$ and $0 \le t \le T$:

$$\|\boldsymbol{e_x}(t)\|^2_{M^*(\boldsymbol{x}^*(t))} \le C \int_0^t \|\boldsymbol{d_v}(s)\|^2_{\star,\boldsymbol{x}^*} \,\mathrm{d}s, \tag{D.19}$$

$$\|\boldsymbol{e_v}(t)\|^2_{M^*(\boldsymbol{x}^*(t))} \le C \,\|\boldsymbol{d_v}(t)\|^2_{\star,\boldsymbol{x}^*} + C \int_0^t \|\boldsymbol{d_v}(s)\|^2_{\star,\boldsymbol{x}^*} \,\mathrm{d}s \tag{D.20}$$

The constant C is independent of t and h, but depends on the final time T and on the regularization parameter α.

We note that the error functions $e_v(\,\cdot\,, t), e_x(\,\cdot\,, t) \in S_h(\boldsymbol{x}^*(t))^3$ with nodal vectors $\boldsymbol{e_v}(t)$ and $\boldsymbol{e_x}(t)$, respectively, are then bounded for $t \in [0, T]$ by

$$\|e_v(\,\cdot\,, t)\|_{H^1(\Gamma_h(\boldsymbol{x}^*(t)))} \le Ch^k$$

and

$$\|e_x(\,\cdot\,, t)\|_{H^1(\Gamma_h(\boldsymbol{x}^*(t)))} \le Ch^k,$$

Proof. The proof uses energy estimates for the error equations (D.16) in the matrix-vector formulation, and it relies on the results of Section D.4. In the course of this proof c and C will be generic constants that take on different values on different occurrences.

In view of condition (D.11) for $\boldsymbol{y} = \boldsymbol{x}^*(t)$, we will need to control the $W^{1,\infty}$ norm of the position error $e_x(\,\cdot\,, t)$. Let $0 < t^* \le T$ be the maximal time such that

$$\|\nabla_{\Gamma_h(\boldsymbol{x}^*(t))} e_x(\,\cdot\,, t)\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*(t)))} \le h^{(\kappa-1)/2} \qquad \text{for} \quad t \in [0, t^*]. \tag{D.21}$$

At $t = t^*$ either this inequality becomes an equality, or else we have $t^* = T$.

We will first prove the stated error bounds for $0 \le t \le t^*$. Then the proof will be finished by showing that in fact t^* coincides with T.

By testing the first equation in (D.16) with $\boldsymbol{e_v}$, and dropping the omnipresent argument $t \in [0, t^*]$, we obtain:

$$\begin{aligned} |\boldsymbol{e_v}|^2_{M^*(\boldsymbol{x}^*)} = \boldsymbol{e_v}^T M^*(\boldsymbol{x}^*)\boldsymbol{e_v} = & - \boldsymbol{e_v}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{v}^* \\ & - \boldsymbol{e_v}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{e_v} \\ & + \boldsymbol{e_v}^T \left(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)\right) - \boldsymbol{e_v}^T M(\boldsymbol{x}^*)\boldsymbol{d_v}. \end{aligned}$$

We separately estimate the four terms on the right-hand side in an appropriate way, with lemmas D.4.1 – D.4.4 as our main tools.

(i) We denote, for $0 \le \theta \le 1$, by e_v^θ and v_h^θ the finite element functions in $S_h(\Gamma_h^\theta)^3$ for $\Gamma_h^\theta = \Gamma_h(\boldsymbol{x}^* + \theta\boldsymbol{e_x})$ with nodal vectors $\boldsymbol{e_v}$ and $\boldsymbol{v}^*$, respectively. Lemma D.4.1 then gives us

$$\begin{aligned} & \boldsymbol{e_v}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{v}^* \\ & = \int_0^1 \int_{\Gamma_h^\theta} e_v^\theta \cdot (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) v_h^\theta \, \mathrm{d}\theta + \alpha \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_v^\theta \cdot (D_{\Gamma_h^\theta} e_x^\theta) \nabla_{\Gamma_h^\theta} v_h^\theta \, \mathrm{d}\theta. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we estimate the integral with the product of the $L^2 - L^2 - L^\infty$ norms of the three factors. We thus have

$$\begin{aligned} & \boldsymbol{e_v}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{v}^* \\ & \le \int_0^1 \|e_v^\theta\|_{L^2(\Gamma_h^\theta)} \, \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)} \, \|v_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\ & \quad + \alpha \int_0^1 |e_v^\theta|_{H^1(\Gamma_h^\theta)} \, \|D_{\Gamma_h^\theta} e_x^\theta\|_{L^2(\Gamma_h^\theta)} \, \|\nabla_{\Gamma_h^\theta} v_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\ & \le c \int_0^1 \|e_v^\theta\|_{H^1(\Gamma_h^\theta)} \, \|e_x^\theta\|_{H^1(\Gamma_h^\theta)} \, \|v_h^\theta\|_{W^{1,\infty}(\Gamma_h^\theta)} \, \mathrm{d}\theta. \end{aligned}$$

By (D.21) and lemma D.4.3, this is bounded by

$$\begin{aligned} & \boldsymbol{e_v}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{v}^* \\ & \qquad \le c \, \|e_v\|_{H^1(\Gamma_h(\boldsymbol{x}^*))} \, \|e_x\|_{H^1(\Gamma_h(\boldsymbol{x}^*))} \, \|v_h^*\|_{W^{1,\infty}(\Gamma_h(\boldsymbol{x}^*))}, \end{aligned}$$

where the last factor is bounded independently of h. By the Young inequality, we thus obtain

$$\begin{aligned} \boldsymbol{e}_{\boldsymbol{v}}^T\,(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*))\boldsymbol{v}^* &\leq \frac{1}{6}\,\|e_v\|^2_{H^1(\Gamma_h(\boldsymbol{x}^*))} + C\,\|e_x\|^2_{H^1(\Gamma_h(\boldsymbol{x}^*))} \\ &= \frac{1}{6}\,|\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} + C\,|\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}. \end{aligned}$$

(ii) Similarly, estimating the three factors in the integrals by $L^2 - L^\infty - L^2$, we obtain

$$\begin{aligned} \boldsymbol{e}_{\boldsymbol{v}}^T\,(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*))\boldsymbol{e}_{\boldsymbol{v}} &\leq c\,\|e_{\boldsymbol{v}}\|^2_{L^2(\Gamma_h(\boldsymbol{x}^*))}\,\|\nabla_{\Gamma_h}\cdot e_{\boldsymbol{x}}\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\ &\quad + c\alpha\,\|\nabla_{\Gamma_h} e_{\boldsymbol{v}}\|^2_{L^2(\Gamma_h(\boldsymbol{x}^*))}\,\|D_{\Gamma_h} e_{\boldsymbol{x}}\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\ &\leq c h^{(\kappa-1)/2}\,|\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)}, \end{aligned}$$

where in the last inequality we used the bound (D.21).

(iii) In the following estimate we use lemma D.4.5. With the finite element function $e_v^\theta = \sum_{j=1}^N (\boldsymbol{e}_{\boldsymbol{v}})_j \phi_j[\boldsymbol{x}^* + \theta \boldsymbol{e}_{\boldsymbol{x}}]$ on the surface $\Gamma_h^\theta = \Gamma_h(\boldsymbol{x}^* + \theta \boldsymbol{e}_{\boldsymbol{x}})$, for $0 \leq \theta \leq 1$, we write

$$\begin{aligned} \boldsymbol{e}_{\boldsymbol{v}}^T\,(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)) &= \int_{\Gamma_h^1} g\nu_{\Gamma_h^1}\cdot e_v^1 - \int_{\Gamma_h^0} g\nu_{\Gamma_h^0}\cdot e_v^0 \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\Gamma_h^\theta} g\nu_{\Gamma_h^\theta}\cdot e_v^\theta\,\mathrm{d}\theta. \end{aligned}$$

Using the Leibniz formula, this becomes

$$\boldsymbol{e}_{\boldsymbol{v}}^T\,(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)) = \int_0^1 \int_{\Gamma_h^\theta} \Big(\partial_\theta^\bullet\,(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta) + (g\nu_{\Gamma_h^\theta}\cdot e_v^\theta)(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta)\Big)\,\mathrm{d}\theta.$$

Here we have, noting that $\partial_\theta^\bullet e_v^\theta = 0$,

$$\partial_\theta^\bullet\,(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta) = g' e_x^\theta\,\nu_{\Gamma_h^\theta}\cdot e_v^\theta + g\,\partial_\theta^\bullet \nu_{\Gamma_h^\theta}\cdot e_v^\theta.$$

With lemmas D.4.3 and D.4.5 we therefore obtain via the Cauchy-Schwarz inequality

$$\int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g\nu_{\Gamma_h^\theta} \cdot e_v^\theta\right) \leq c_2^2 \|g'\|_{L^\infty} \, \|e_x\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \, \|e_v\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} + c_2^2 \|g\|_{L^\infty} \, \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \, \|e_v\|_{L^2(\Gamma_h(\boldsymbol{x}^*))},$$

and again with lemma D.4.3,

$$\int_{\Gamma_h^\theta} (g\nu_{\Gamma_h^\theta} \cdot e_v^\theta)(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) \leq c_2^2 \|g\|_{L^\infty} \, \|e_v\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \, \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} \cdot e_x\|_{L^2(\Gamma_h(\boldsymbol{x}^*))}.$$

In total, we obtain a bound of the same type as for the terms in (i) and (ii):

$$\begin{aligned} \boldsymbol{e}_{\boldsymbol{v}}^T \left(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)\right) &\leq c \, \|e_x\|_{H^1(\Gamma_h(\boldsymbol{x}^*))} \, \|e_v\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \\ &= c \, |\boldsymbol{e}_{\boldsymbol{x}}|_{M^*(\boldsymbol{x}^*)} \, |\boldsymbol{e}_{\boldsymbol{v}}|_{M(\boldsymbol{x}^*)} \\ &\leq \frac{1}{6} \, |\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} + C \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}. \end{aligned}$$

The combination of the estimates of the three terms (i)–(iii) with absorptions (for sufficiently small $h \leq h_0$), and a simple dual norm estimate, based on (D.18), for the defect term, yield the bound

$$|\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} \leq c \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} + c \, \|\boldsymbol{d}_{\boldsymbol{v}}\|^2_{*,\boldsymbol{x}^*}. \tag{D.22}$$

Using this estimate, together with taking the $|\cdot|_{M^*(\boldsymbol{x}^*)}$ norm of both sides of the second equation in (D.16), we obtain

$$\frac{1}{2} \, |\dot{\boldsymbol{e}}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} = |\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} \leq c \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} + c \, \|\boldsymbol{d}_{\boldsymbol{v}}\|^2_{*,\boldsymbol{x}^*}. \tag{D.23}$$

In order to apply Gronwall's inequality, we connect $\frac{\mathrm{d}}{\mathrm{d}t} \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}$ and $|\dot{\boldsymbol{e}}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}$ as follows:

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} &= \boldsymbol{e}_{\boldsymbol{x}}^T M^*(\boldsymbol{x}^*) \dot{\boldsymbol{e}}_{\boldsymbol{x}} + \frac{1}{2} \boldsymbol{e}_{\boldsymbol{x}}^T \left(\frac{\mathrm{d}}{\mathrm{d}t} M^*(\boldsymbol{x}^*)\right) \boldsymbol{e}_{\boldsymbol{x}} \\ &\leq \, |\dot{\boldsymbol{e}}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} + c \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}, \end{aligned}$$

where we use the Cauchy-Schwarz inequality and lemma D.4.6 in the estimate. Inserting (D.23), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} \leq c\, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)} + c\, \|\boldsymbol{d}_{\boldsymbol{v}}\|^2_{*,\boldsymbol{x}^*}.$$

A Gronwall inequality then yields (D.19), using $e_j(0) = x_j(0) - x_j^0 = 0$ for $j = 1, \ldots, N$. Inserting this estimate in (D.22), we can bound $\boldsymbol{e}_{\boldsymbol{v}}(t)$ for $0 \leq t \leq t^*$ by (D.20).

Now it only remains to show that $t^* = T$ for h sufficiently small. For $0 \leq t \leq t^*$ we use an inverse inequality and (D.19) to bound the left-hand side in (D.21):

$$\begin{aligned}
|e_x(\,\cdot\,,t)|_{W^{1,\infty}(\Gamma_h(\boldsymbol{x}^*(t)))} &\leq ch^{-1}\, |e_x(\,\cdot\,,t)|_{H^1(\Gamma_h(\boldsymbol{x}^*(t)))} \\
&\leq ch^{-1} \|\boldsymbol{e}_{\boldsymbol{x}}(t)\|_{M^*(\boldsymbol{x}^*(t))} \\
&\leq cCh^{\kappa-1} \leq \frac{1}{2} h^{(\kappa-1)/2}
\end{aligned}$$

for sufficiently small h. Hence, we can extend the bound (D.21) beyond t^*, which contradicts the maximality of t^* unless we have already $t^* = T$. ■

D.6. Stability of coupling surface PDEs to surface motion

Now we turn to the stability bounds of the original problem (D.4)–(D.5), or in DAE form (D.7), which is the formulation we will actually use for the stability analysis.

D.6.1. Error equations

Similarly as before, in order to derive stability estimates we consider the DAE system when we insert the nodal values $\boldsymbol{u}^*(t) \in \mathbb{R}^N$ of the exact solution $u(\,\cdot\,,t)$, the nodal values $\boldsymbol{x}^*(t) \in \mathbb{R}^{3N}$ of the exact positions $X(\cdot\,,t)$, and the nodal values $\boldsymbol{v}^*(t) \in \mathbb{R}^{3N}$ of the exact velocity $v(\cdot\,,t)$. Inserting them into (D.7) yields the defects $\boldsymbol{d_u}(t) \in \mathbb{R}^N$ and $\boldsymbol{d_v} \in \mathbb{R}^{3N}$: omitting the argument t in the notation, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{u}^*\Big) + A(\boldsymbol{x}^*)\boldsymbol{u}^* &= \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*) + M(\boldsymbol{x}^*)\boldsymbol{d_u}, \\ M^*(\boldsymbol{x}^*)\dot{\boldsymbol{v}}^* &= \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{u}^*) + M(\boldsymbol{x}^*)\boldsymbol{d_v}, \\ \dot{\boldsymbol{x}}^* &= \boldsymbol{v}^*, \end{aligned} \tag{D.24}$$

where again $M^{[3]}(\boldsymbol{x}^*) = I_3 \otimes M(\boldsymbol{x}^*)$. As no confusion can arise, we write again $M(\boldsymbol{x}^*)$ for $M^{[3]}(\boldsymbol{x}^*)$.

We denote the PDE error by $\boldsymbol{e_u} = \boldsymbol{u} - \boldsymbol{u}^*$, and as in the previous section, $\boldsymbol{e_v} = \boldsymbol{v} - \boldsymbol{v}^*$ and $\boldsymbol{e_x} = \boldsymbol{x} - \boldsymbol{x}^*$ denote the velocity error and surface error, respectively. Subtracting (D.24) from (D.7), we obtain the following error equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{e_u}\Big) + A(\boldsymbol{x}^*)\boldsymbol{e_u} = &-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(M(\boldsymbol{x}) - M(\boldsymbol{x}^*)\big)\boldsymbol{u}^*\Big) \\ &-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(M(\boldsymbol{x}) - M(\boldsymbol{x}^*)\big)\boldsymbol{e_u}\Big) \\ &-\big(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\big)\boldsymbol{u}^* \\ &-\big(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\big)\boldsymbol{e_u} \\ &+\big(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*)\big) - M(\boldsymbol{x}^*)\boldsymbol{d_u} \\ M^*(\boldsymbol{x}^*)\boldsymbol{e_v} = &-\big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{v}^* \\ &-\big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{e_v} \\ &+\big(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{u}^*)\big) - M(\boldsymbol{x}^*)\boldsymbol{d_v}, \\ \dot{\boldsymbol{e}}_{\boldsymbol{x}} = &\ \boldsymbol{e_v}. \end{aligned} \tag{D.25}$$

D.6.2. Stability estimate

We now formulate the stability result for the errors $\boldsymbol{e_u}$, $\boldsymbol{e_v}$ and $\boldsymbol{e_x}$ of the surface motion coupled to the surface PDE. Here, we use the norms (D.9)-(D.10) and those of Section D.5.3.

Proposition D.6.1. *Assume that the following bounds hold for the defects, for some $\kappa > 1$ and $0 \le t \le T$:*

$$\|\boldsymbol{d_u}(t)\|_{*,\boldsymbol{x}^*(t)} \le ch^\kappa, \ \|\boldsymbol{d_v}(t)\|_{*,\boldsymbol{x}^*(t)} \le ch^\kappa, \text{ for } t \in [0,T].$$

Then there exists $h_0 > 0$ such that the following stability estimate holds for all $h \le h_0$ and $0 \le t \le T$:

$$\begin{aligned}|\boldsymbol{e_u}(t)|^2_{M(\boldsymbol{x}^*)} + \int_0^t |\boldsymbol{e_u}(s)|^2_{A(\boldsymbol{x}^*)}\,\mathrm{d}s + |\boldsymbol{e_x}(t)|^2_{M^*(\boldsymbol{x}^*)} \\ + \int_0^t |\boldsymbol{e_v}(s)|^2_{M^*(\boldsymbol{x}^*)}\,\mathrm{d}s \le C \int_0^t \Big(\|\boldsymbol{d_u}(s)\|^2_{*,\boldsymbol{x}^*} + \|\boldsymbol{d_v}(s)\|^2_{*,\boldsymbol{x}^*} \Big)\,\mathrm{d}s.\end{aligned} \tag{D.26}$$

The constant C is independent of t and h, but depends on the final time T and on the regularization parameter α.

We note that the error functions $e_u(\cdot,t) \in S_h(\boldsymbol{x}^*(t))$ and the error functions $e_v(\cdot,t), e_x(\cdot,t) \in S_h(\boldsymbol{x}^*(t))^3$ with nodal vectors $\boldsymbol{e_u}(t)$ and $\boldsymbol{e_v}(t), \boldsymbol{e_x}(t)$, respectively, are then bounded by

$$\begin{aligned}&\|e_u(\cdot,t)\|_{L^2(\Gamma_h(\boldsymbol{x}^*(t)))} + \left(\int_0^t \|e_u(\cdot,t)\|^2_{H^1(\Gamma_h(\boldsymbol{x}^*(t)))}\,\mathrm{d}s \right)^{1/2} \le Ch^\kappa, \\ &\left(\int_0^t \|e_v(\cdot,t)\|^2_{H^1(\Gamma_h(\boldsymbol{x}^*(t)))^3}\mathrm{d}s \right)^{1/2} \le Ch^\kappa, \\ &\|e_x(\cdot,t)\|_{H^1(\Gamma_h(\boldsymbol{x}^*(t)))^3} \le Ch^\kappa, \qquad t \in [0,T].\end{aligned} \tag{D.27}$$

Proof. The proof is an extension of the proof of proposition D.5.1, again based on the matrix-vector formulation and the auxiliary results of Section D.4. We handle the surface PDE and the surface equations separately: we first estimate the errors of the PDE, while those for the surface equation are based on section D.5. Finally we will combine the results to obtain the stability estimates for the coupled problem. In the course of this proof c and C will be generic constants that take on different values on different occurrences.

Let $0 < t^* \leq T$ be the maximal time such that the following inequalities hold:

$$\begin{aligned} \|\nabla_{\Gamma_h(\boldsymbol{x}^*(t))} e_x(\,\cdot\,,t)\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*(t)))} &\leq h^{(\kappa-1)/2}, \\ \|e_u(\,\cdot\,,t)\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*(t)))} &\leq 1, \end{aligned} \qquad \text{for} \quad t \in [0,t^*], \tag{D.28}$$

Note that $t^* > 0$ since initially both $e_{\boldsymbol{x}}(\,\cdot\,,0) = 0$ and $e_{\boldsymbol{u}}(\,\cdot\,,0) = 0$.

We first prove the stated error bounds for $0 \leq t \leq t^*$. At the end, the proof will be finished by showing that in fact t^* coincides with T.

Testing the first two equations of (D.25) with $\boldsymbol{e_u}$ and $\boldsymbol{e_v}$, and dropping the omnipresent argument $t \in [0,t^*]$, we obtain:

$$\begin{aligned}
\boldsymbol{e}_{\boldsymbol{u}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{e_u}\Big) + \boldsymbol{e}_{\boldsymbol{u}}^T A(\boldsymbol{x}^*)\boldsymbol{e_u} = {} & -\boldsymbol{e}_{\boldsymbol{u}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{u}^*\Big) \\
& - \boldsymbol{e}_{\boldsymbol{u}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{e_u}\Big) \\
& - \boldsymbol{e}_{\boldsymbol{u}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{u}^* \\
& - \boldsymbol{e}_{\boldsymbol{u}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{e_u} \\
& + \boldsymbol{e}_{\boldsymbol{u}}^T \left(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*)\right) \\
& - \boldsymbol{e}_{\boldsymbol{u}}^T M(\boldsymbol{x}^*)\boldsymbol{d_u}, \\
|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)}^2 = {} & -\boldsymbol{e}_{\boldsymbol{v}}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{v}^* \\
& - \boldsymbol{e}_{\boldsymbol{v}}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{e_v} \\
& + \boldsymbol{e}_{\boldsymbol{v}}^T \left(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{g}(\boldsymbol{x}^*,\boldsymbol{u}^*)\right) \\
& - \boldsymbol{e}_{\boldsymbol{v}}^T M(\boldsymbol{x}^*)\boldsymbol{d_v}, \\
\dot{\boldsymbol{e}}_{\boldsymbol{x}} = {} & \boldsymbol{e_v} - \boldsymbol{d_x}.
\end{aligned}$$

(A) *Estimates for the surface PDE:* We estimate the terms separately, with Lemmas D.4.1 – D.4.3 as our main tools.

(i) The symmetry of $M(\boldsymbol{x}^*)$ and a simple calculation yields

$$\boldsymbol{e}_{\boldsymbol{u}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{e_u}\Big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\boldsymbol{e}_{\boldsymbol{u}}^T M(\boldsymbol{x}^*)\boldsymbol{e_u}\Big) + \frac{1}{2}\boldsymbol{e}_{\boldsymbol{u}}^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}M(\boldsymbol{x}^*)\Big)\boldsymbol{e_u}$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \, |\boldsymbol{e_u}|^2_{M(\boldsymbol{x}^*)} + \frac{1}{2}\boldsymbol{e}_{\boldsymbol{u}}^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}M(\boldsymbol{x}^*)\Big)\boldsymbol{e_u},$$

where the last term is bounded by lemma D.4.6 as

$$\Big|\boldsymbol{e}_{\boldsymbol{u}}^T\frac{\mathrm{d}}{\mathrm{d}t}M(\boldsymbol{x}^*)\boldsymbol{e_u}\Big| \leq c\, |\boldsymbol{e_u}|^2_{M(\boldsymbol{x}^*)}.$$

(ii) By the definition of the A-norm we have

$$\boldsymbol{e}_{\boldsymbol{u}}^T A(\boldsymbol{x}^*)\boldsymbol{e_u} = |\boldsymbol{e_u}|^2_{A(\boldsymbol{x}^*)}.$$

(iii) With the product rule we write

$$\begin{aligned}&\boldsymbol{e}_{\boldsymbol{u}}^T\frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{u}^*\Big) \\ &\quad = \boldsymbol{e}_{\boldsymbol{u}}^T\,(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\dot{\boldsymbol{u}}^* + \boldsymbol{e}_{\boldsymbol{u}}^T\Big(\frac{\mathrm{d}}{\mathrm{d}t}\,(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\Big)\boldsymbol{u}^*.\end{aligned} \tag{D.29}$$

With $\Gamma_h^\theta(t) = \Gamma_h[\boldsymbol{x}^*(t) + \theta\boldsymbol{e_x}(t)]$ and with the finite element functions $e_u^\theta(\cdot,t)$, $u_h^{*,\theta}(\cdot,t) \in S_h(\boldsymbol{x}^*(t) + \theta\boldsymbol{e_x}(t))$ with nodal vectors $\boldsymbol{e_u}(t)$, $\boldsymbol{u}^*(t)$, resp., Lemma D.4.1 (with $\boldsymbol{x}^*(t)$ in the role of $\boldsymbol{y}$) yields for the first term, omitting again the argument t,

$$\boldsymbol{e}_{\boldsymbol{u}}^T\,(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\dot{\boldsymbol{u}}^* = \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta\,(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta)\,\partial_h^\bullet u_h^\theta \;\mathrm{d}\theta.$$

Using the Cauchy-Schwarz inequality we obtain for the first term

$$\begin{aligned}&|\boldsymbol{e}_{\boldsymbol{u}}^T\,(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\dot{\boldsymbol{u}}^*| \\ &\qquad\qquad \leq \int_0^1 \|e_u^\theta\|_{L^2(\Gamma_h^\theta)}\,\|\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)}\,\|\partial_h^\bullet u_h^\theta\|_{L^\infty(\Gamma_h^\theta)}\,\mathrm{d}\theta.\end{aligned}$$

Under condition (D.28) we obtain from lemmas D.4.2 and D.4.3 that for $0 \leq t \leq t^*$,

$$|\boldsymbol{e}_{\boldsymbol{u}}^T\,(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\dot{\boldsymbol{u}}^*| \leq c\;\|e_u^0\|_{L^2(\Gamma_h^0)}\;\|e_x^0\|_{H^1(\Gamma_h^0)}\;\|\partial_h^\bullet u_h^0\|_{L^\infty(\Gamma_h^0)}$$

Now, the last factor is bounded by

$$\|\partial_h^\bullet u_h^{*,0}\|_{L^\infty(\Gamma_h^0)} \leq c\|\dot{\boldsymbol{u}}^*\|_\infty \leq C$$

because of the assumed smoothness of the exact solution u and hence of its material derivative $\partial^\bullet u(\cdot, t)$, whose values at the nodes are the entries of the vector $\dot{\boldsymbol{u}}^*(t)$. Hence we obtain, on recalling the definitions of the discrete norms,

$$-\boldsymbol{e}_{\boldsymbol{u}}^T(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\dot{\boldsymbol{u}}^* \leq C\,|\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)}\,|\boldsymbol{e}_{\boldsymbol{x}}|_{A(\boldsymbol{x}^*)}.$$

Using lemma D.4.1 together with the Leibniz formula, the last term in (D.29) becomes

$$\begin{aligned}\boldsymbol{e}_{\boldsymbol{u}}^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}\,(M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\Big)\boldsymbol{u}^* &= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta\, \partial_h^\bullet(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta)\, u_h^\theta\, d\theta \\ &\quad + \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta\, (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta)\, u_h^\theta\, (\nabla_{\Gamma_h^\theta} \cdot v_h^\theta)\, d\theta.\end{aligned}$$

where v_h^θ is the velocity of Γ_h^θ (as a function of t), which is the finite element function in $S_h(\boldsymbol{x}^* + \theta\boldsymbol{e}_{\boldsymbol{x}})$ with nodal vector $\dot{\boldsymbol{x}}^* + \theta\dot{\boldsymbol{e}}_{\boldsymbol{x}} = \boldsymbol{v}^* + \theta\boldsymbol{e}_{\boldsymbol{v}}$, so that

$$v_h^\theta = v_h^{*,\theta} + \theta e_v^\theta, \tag{D.30}$$

where $v_h^{*,\theta}$ and e_v^θ are the finite element functions on Γ_h^θ with nodal vectors $\boldsymbol{v}^*$ and $\boldsymbol{e}_{\boldsymbol{v}}$, respectively. In the first integral we further use, cf. [36, lemma 2.6],

$$\partial_h^\bullet(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) = \nabla_{\Gamma_h^\theta} \cdot \partial_h^\bullet e_x^\theta - ((I_3 - \nu_h^\theta(\nu_h^\theta)^T)\nabla_{\Gamma_h^\theta} v_h^\theta) : \nabla_{\Gamma_h^\theta} e_x^\theta$$

where : symbolizes the Euclidean inner product of the vectorization of two matrices. Here we note that $\partial_h^\bullet e_x^\theta$ is the finite element function on Γ_h^θ with nodal vector $\dot{\boldsymbol{e}}_{\boldsymbol{x}} = \boldsymbol{e}_{\boldsymbol{v}}$, so that $\partial_h^\bullet e_x^\theta = e_v^\theta$.

We then estimate, using the Cauchy-Schwarz inequality in the first step, Lemmas D.4.2 and D.4.3 in the second step (using (D.28) to ensure the smallness condition in these lemmas), the definition of the discrete norms in the third step, and using the first bound of (D.28) and the boundedness of the discrete gradient of the interpolated exact velocity $\nabla_{\Gamma_h(\boldsymbol{x}^*)} v_h^*$ and of the interpolated exact solution u_h^* in the fourth step,

$$
\begin{aligned}
&\left| \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \, \partial_h^\bullet (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) \, u_h^{*,\theta} \, \mathrm{d}\theta \right| \\
&\leq \int_0^1 \int_{\Gamma_h^\theta} \| e_u^\theta \|_{L^2(\Gamma_h^\theta)} \Big(\| \nabla_{\Gamma_h^\theta} \cdot e_v^\theta \|_{L^2(\Gamma_h^\theta)} \\
&\quad + \| \nabla_{\Gamma_h^\theta} v_h^{*,\theta} \|_{L^\infty(\Gamma_h^\theta)} \cdot \| \nabla_{\Gamma_h^\theta} e_x^\theta \|_{L^2(\Gamma_h^\theta)} \\
&\quad + \| \nabla_{\Gamma_h^\theta} e_v^\theta \|_{L^2(\Gamma_h^\theta)} \cdot \| \nabla_{\Gamma_h^\theta} e_x^\theta \|_{L^\infty(\Gamma_h^\theta)} \Big) \| u_h^{*,\theta} \|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\
&\leq c \, \| e_u \|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \Big(\| \nabla_{\Gamma_h(\boldsymbol{x}^*)} e_v \|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \\
&\quad + \| \nabla_{\Gamma_h(\boldsymbol{x}^*)} v_h^* \|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \cdot \| \nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x \|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \\
&\quad + \| \nabla_{\Gamma_h(\boldsymbol{x}^*)} e_v \|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \cdot \| \nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x \|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \Big) \| u_h^* \|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\
&\leq c \; |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e_v}|_{A(\boldsymbol{x}^*)} + \| \nabla_{\Gamma_h(\boldsymbol{x}^*)} v_h^* \|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \, |\boldsymbol{e_x}|_{A(\boldsymbol{x}^*)} \\
&\quad + |\boldsymbol{e_v}|_{A(\boldsymbol{x}^*)} \| \nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x \|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \Big) \| \boldsymbol{u}^* \|_\infty \\
&\leq c \, |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} + C \, |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} h^{(\kappa-1)/2} \Big) C \\
&\leq C' \, |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} \Big).
\end{aligned}
$$

With the same arguments we estimate, on inserting (D.30),

$$
\begin{aligned}
&\left| \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \, (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) \, u_h^{*,\theta} \, (\nabla_{\Gamma_h^\theta} \cdot v_h^\theta) \, \mathrm{d}\theta \right| \\
&\leq \int_0^1 \int_{\Gamma_h^\theta} \| e_u^\theta \|_{L^2(\Gamma_h^\theta)} \, \| \nabla_{\Gamma_h^\theta} \cdot e_x^\theta \|_{L^2(\Gamma_h^\theta)} \, \| u_h^{*,\theta} \|_{L^\infty(\Gamma_h^\theta)} \, \| \nabla_{\Gamma_h^\theta} \cdot v_h^{*,\theta} \|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\
&+ \int_0^1 \int_{\Gamma_h^\theta} \| e_u^\theta \|_{L^2(\Gamma_h^\theta)} \, \| \nabla_{\Gamma_h^\theta} \cdot e_x^\theta \|_{L^\infty(\Gamma_h^\theta)} \, \| u_h^{*,\theta} \|_{L^\infty(\Gamma_h^\theta)} \, \| \nabla_{\Gamma_h^\theta} \cdot e_v^\theta \|_{L^2(\Gamma_h^\theta)} \, \mathrm{d}\theta
\end{aligned}
$$

$$
\begin{aligned}
&\leq c \ |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \ |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} \|\boldsymbol{u}^*\|_\infty \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} \cdot v_h^*\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\
&+ c \ |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \|\boldsymbol{u}^*\|_\infty |\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} \\
&\leq C \ |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} \Big).
\end{aligned}
$$

Altogether we obtain the bound

$$
\begin{aligned}
&- \boldsymbol{e}_{\boldsymbol{u}}^T \frac{\mathrm{d}}{\mathrm{d}t} \Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \boldsymbol{u}^* \Big) \\
&\quad \leq C \ |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} \Big).
\end{aligned}
$$

(iv) We obtain similarly

$$
\begin{aligned}
&- \boldsymbol{e}_{\boldsymbol{u}}^T \frac{\mathrm{d}}{\mathrm{d}t} \Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \boldsymbol{e_u} \Big) \\
&= -\frac{1}{2} \boldsymbol{e}_{\boldsymbol{u}}^T \Big(\frac{\mathrm{d}}{\mathrm{d}t} (M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \Big) \boldsymbol{e_u} \\
&\quad - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\boldsymbol{e}_{\boldsymbol{u}}^T (M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \boldsymbol{e_u} \Big) \\
&\leq c \ |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)} \Big) \|e_u\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\
&\quad - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\boldsymbol{e}_{\boldsymbol{u}}^T (M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \boldsymbol{e_u} \Big) \\
&\leq \ C \, |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)} (|\boldsymbol{e_v}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)}) \\
&\quad - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \Big(\boldsymbol{e}_{\boldsymbol{u}}^T (M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \boldsymbol{e_u} \Big),
\end{aligned}
$$

where we used the second bound of (D.28) in the last inequality.

(v) Lemma D.4.1 and the Cauchy-Schwarz inequality yield

$$
\begin{aligned}
&- \boldsymbol{e}_{\boldsymbol{u}}^T (A(\boldsymbol{x}) - A(\boldsymbol{x}^*)) \boldsymbol{u}^* \\
&= - \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_u^\theta \cdot (D_{\Gamma_h^\theta} e_x^\theta) \nabla_{\Gamma_h^\theta} u_h^{*,\theta} \ \mathrm{d}\theta \\
&\leq c \, |\boldsymbol{e_u}|_{A(\boldsymbol{x}^*)} \ |\boldsymbol{e_x}|_{A(\boldsymbol{x}^*)} \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} u_h^*\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\
&\leq C \, |\boldsymbol{e_u}|_{A(\boldsymbol{x}^*)} \ |\boldsymbol{e_x}|_{M^*(\boldsymbol{x}^*)}.
\end{aligned}
$$

(vi) Similarly we estimate

$$\boldsymbol{e}_{\boldsymbol{u}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{e}_{\boldsymbol{u}} \leq c\, |\boldsymbol{e}_{\boldsymbol{u}}|^2_{A(\boldsymbol{x}^*)} \, \|D_{\Gamma_h} e_x\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \leq c h^{(\kappa-1)/2}\, |\boldsymbol{e}_{\boldsymbol{u}}|^2_{A(\boldsymbol{x}^*)},$$

where we used the first bound of (D.28).

(vii) The coupling term is estimated similarly to (iii) in the proof of proposition D.5.1:

$$\boldsymbol{e}_{\boldsymbol{u}}^T \left(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*)\right) = \int_{\Gamma_h^1} f(u_h, \nabla_{\Gamma_h^1} u_h) e_u^1 - \int_{\Gamma_h^0} f(u_h^*, \nabla_{\Gamma_h^0} u_h^*) e_u^0$$

With

$$u_h^\theta = \sum_{j=1}^N (u_j^* + \theta(\boldsymbol{e}_{\boldsymbol{u}})_j)\, \phi_j[\boldsymbol{x}^* + \theta \boldsymbol{e}_{\boldsymbol{x}}] = u_h^{*,\theta} + \theta e_u^\theta \tag{D.31}$$

we therefore have

$$\boldsymbol{e}_{\boldsymbol{u}}^T \left(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*)\right) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\Gamma_h^\theta} f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\, e_u^\theta \,\mathrm{d}\theta$$

and with the Leibniz formula (noting that e_x^θ is the velocity of the surface Γ_h^θ considered as a function of θ), we rewrite this as

$$\begin{aligned} &\boldsymbol{e}_{\boldsymbol{u}}^T \left(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*)\right) \\ &= \int_0^1 \int_{\Gamma_h^\theta} \left(\partial_\theta^\bullet f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\, e_u^\theta \,\mathrm{d}\theta + f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\, e_u^\theta (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) \right) \mathrm{d}\theta. \end{aligned}$$

Here we use the chain rule

$$\partial_\theta^\bullet f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) = \partial_1 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\, \partial_\theta^\bullet u_h^\theta + \partial_2 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\, \partial_\theta^\bullet \nabla_{\Gamma_h^\theta} u_h^\theta$$

and observe the following: by the assumed smoothness of f and the exact solution u, and by the bound (D.28) for e_u (and hence for e_u^θ by Lemmas D.4.2 and D.4.3), we have on recalling (D.31)

$$\|\partial_i f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)\|_{L^\infty(\Gamma_h^\theta)} \leq C, \qquad i = 1, 2.$$

We note

$$\partial_\theta^\bullet u_h^\theta = e_u^\theta$$

and the relation, see [36, Lemma 2.6],

$$\partial_\theta^\bullet \nabla_{\Gamma_h^\theta} u_h^\theta = \nabla_{\Gamma_h^\theta} \partial_\theta^\bullet u_h^\theta - \nabla_{\Gamma_h^\theta} e_x^\theta \, \nabla_{\Gamma_h^\theta} u_h^\theta + \nu_h^\theta (\nu_h^\theta)^T (\nabla_{\Gamma_h^\theta} e_x^\theta)^T \nabla_{\Gamma_h^\theta} u_h^\theta.$$

We then have, on inserting (D.31) and using once again Lemmas D.4.2 and D.4.3 and the bound (D.28),

$$\begin{aligned}
&\boldsymbol{e}_{\boldsymbol{u}}^T(\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{x}^*,\boldsymbol{u}^*)) \\
&= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \Big(\partial_1 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta) e_u^\theta + \partial_2 f(u_h^\theta, \nabla_{\Gamma_h^\theta} u_h^\theta)(\nabla_{\Gamma_h^\theta} e_u^\theta \\
&\quad - \nabla_{\Gamma_h^\theta} e_x^\theta \, \nabla_{\Gamma_h^\theta} u_h^\theta + \nu_h^\theta (\nu_h^\theta)^T (\nabla_{\Gamma_h^\theta} e_x^\theta)^T \nabla_{\Gamma_h^\theta} u_h^\theta) \Big) \, \mathrm{d}\theta \\
&\leq c \|e_u\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \Big(\|e_u\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \\
&\quad + \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} e_u\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} + \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \, \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} u_h^*\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\
&\quad + \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} e_x\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \, \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} e_u\|_{L^2(\Gamma_h(\boldsymbol{x}^*))} \Big) \\
&\leq C \, |\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{u}}|_{A(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{x}}|_{A(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{u}}|_{A(\boldsymbol{x}^*)} \Big) \\
&\leq C \, |\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{u}}|_{A(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{x}}|_{M^*(\boldsymbol{x}^*)} \Big).
\end{aligned}$$

The above estimates combined, yield the following inequality: Combined, the above estimates yield the following inequality:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\boldsymbol{e}_{\boldsymbol{u}}|^2_{M(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{u}}|^2_{A(\boldsymbol{x}^*)} \leq \; & C \, |\boldsymbol{e}_{\boldsymbol{u}}|^2_{M(\boldsymbol{x}^*)} \\
& + C \, |\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} \Big(|\boldsymbol{e}_{\boldsymbol{v}}|_{M^*(\boldsymbol{x}^*)} + |\boldsymbol{e}_{\boldsymbol{x}}|_{M^*(\boldsymbol{x}^*)} \Big) \\
& + C \, |\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} \, |\boldsymbol{e}_{\boldsymbol{v}}|_{M^*(\boldsymbol{x}^*)} + c \, |\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} \, |\boldsymbol{e}_{\boldsymbol{x}}|_{M^*(\boldsymbol{x}^*)} \\
& + C \, |\boldsymbol{e}_{\boldsymbol{u}}|_{M(\boldsymbol{x}^*)} \, |\boldsymbol{e}_{\boldsymbol{v}}|_{M^*(\boldsymbol{x}^*)} \\
& - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \Big(\boldsymbol{e}_{\boldsymbol{u}}^T (M(\boldsymbol{x}) - M(\boldsymbol{x}^*)) \boldsymbol{e}_{\boldsymbol{u}} \Big)
\end{aligned}$$

$$
\begin{aligned}
&+ C\, |e_u|_{A(x^*)}\, |e_x|_{M^*(x^*)} \\
&+ C h^{(\kappa-1)/2}\, |e_u|^2_{A(x^*)} \\
&+ C\, |e_u|_{M(x^*)} \Big(|e_u|_{M(x^*)} + |e_u|_{A(x^*)} + |e_x|_{M^*(x^*)} \Big) \\
&+ C\, |e_u|_{A(x^*)} \|d_u\|_{\star,x^*}.
\end{aligned}
$$

Estimating further, using Young's inequality and absorptions into $|e_u|^2_{A(x^*)}$ (using $h \leq h_0$ for a sufficiently small h_0), we obtain the following estimate, where we can choose $\rho > 0$ small at the expense of enlarging the constant in front of $|e_u|^2_{M(x^*)}$:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, |e_u|^2_{M(x^*)} + \frac{1}{2}\, |e_u|^2_{A(x^*)} \leq c\, |e_u|^2_{M(x^*)} + c\, |e_x|^2_{M^*(x^*)} \\
&+ \rho\, |e_v|^2_{M^*(x^*)} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(e_u^T (M(x) - M(x^*)) e_u \Big) + c\|d_u\|^2_{\star,x^*}.
\end{aligned} \tag{D.32}
$$

(B) *Estimates in the surface equation:* Based on section D.5, we obtain

$$
|e_v|^2_{M^*(x^*)} \leq c\, |e_x|^2_{M^*(x^*)} + |e_v^T\, (g(x,u) - g(x^*,u^*))| + c\, \|d_v\|^2_{*,x^*},
$$

where the coupling term can be estimated based on (iii) in the proof of proposition D.5.1 and (vii) above:

$$
\begin{aligned}
&|e_v^T (g(x,u) - g(x^*,u^*))| \\
&\quad \leq |e_v|_{M(x^*)} (|e_u|_{M(x^*)} + |e_u|_{A(x^*)} + |e_x|_{M^*(x^*)}).
\end{aligned}
$$

We then obtain

$$
|\dot{e}_x|^2_{M^*(x^*)} \leq C(|e_x|^2_{M^*(x^*)} + |e_u|^2_{M(x^*)} + |e_u|^2_{A(x^*)} + \|d_v\|^2_{\star,x^*}). \tag{D.33}
$$

As in Section D.5, this provides the estimate

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\, |e_x|^2_{M^*(x^*)} \leq C(|e_x|^2_{M^*(x^*)} + |e_u|^2_{M(x^*)} + |e_u|^2_{A(x^*)} + \|d_v\|^2_{\star,x^*}). \tag{D.34}
$$

(C) *Combination:* We first insert (D.33) into (D.32), where we can choose $\rho > 0$ so small that $C\rho \leq 1/2$ for the constant C in (D.33). Then we take a linear combination of (D.32) and (D.34) to obtain, for a sufficiently small $\sigma > 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \, |\boldsymbol{e_u}|^2_{M(\boldsymbol{x}^*)} + \frac{1}{2} \, |\boldsymbol{e_u}|^2_{A(\boldsymbol{x}^*)} + \sigma \frac{\mathrm{d}}{\mathrm{d}t} \, |\boldsymbol{e_x}|^2_{M^*(\boldsymbol{x}^*)} \\ \leq c \, |\boldsymbol{e_u}|^2_{M(\boldsymbol{x}^*)} + c \, |\boldsymbol{e_x}|^2_{M^*(\boldsymbol{x}^*)} + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\boldsymbol{e}_{\boldsymbol{u}}^T (M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{e_u}\Big) \\ + c \|\boldsymbol{d_u}\|^2_{\star,\boldsymbol{x}^*} + c \|\boldsymbol{d_v}\|^2_{\star,\boldsymbol{x}^*}.\end{aligned}$$

We integrate both sides over $[0, t]$, for $0 \leq t \leq t^*$, to get

$$\begin{aligned}&|\boldsymbol{e_u}(t)|^2_{M(\boldsymbol{x}^*)} + \frac{1}{2} \int_0^t |\boldsymbol{e_u}(s)|^2_{A(\boldsymbol{x}^*)} \, \mathrm{d}s + \sigma \, |\boldsymbol{e_x}(t)|^2_{M^*(\boldsymbol{x}^*)} \\ &\leq |\boldsymbol{e_u}(0)|^2_{M(\boldsymbol{x}^*)} + |\boldsymbol{e_x}(0)|^2_{M^*(\boldsymbol{x}^*)} + c \int_0^t |\boldsymbol{e_u}(s)|^2_{M(\boldsymbol{x}^*)} + |\boldsymbol{e_x}(s)|^2_{M^*(\boldsymbol{x}^*)} \, \mathrm{d}s \\ &\quad - \boldsymbol{e_u}(t)^T \, (M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{e_u}(t) \\ &\quad + c \int_0^t \|\boldsymbol{d_u}(s)\|^2_{*,\boldsymbol{x}^*} + \|\boldsymbol{d_v}(s)\|^2_{*,\boldsymbol{x}^*} \, \mathrm{d}s.\end{aligned}$$

The middle term can be further bounded using lemmas D.4.1–D.4.3 and an $L^2 - L^\infty - L^2$ estimate, as

$$\begin{aligned}\boldsymbol{e_u}(t)^T \, (M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{e_u}(t) &= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \cdot (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) e_u^\theta \; \mathrm{d}\theta \\ &\leq c \, |\boldsymbol{e_u}(t)|^2_{M(\boldsymbol{x}^*)} \, \|\nabla_{\Gamma_h(\boldsymbol{x}^*)} \cdot e_x\|_{L^\infty(\Gamma_h(\boldsymbol{x}^*))} \\ &\leq c h^{(\kappa-1)/2} \, |\boldsymbol{e_u}(t)|^2_{M(\boldsymbol{x}^*)},\end{aligned}$$

where we used the first bound from (D.28) in the last inequality.

Absorbing this to the left-hand side and using Gronwall's inequality yields the stability estimates:

$$\begin{aligned}|\boldsymbol{e_u}(t)|^2_{M(\boldsymbol{x}^*)} + \int_0^t |\boldsymbol{e_u}(s)|^2_{A(\boldsymbol{x}^*)} \, \mathrm{d}s + |\boldsymbol{e_x}(t)|^2_{M^*(\boldsymbol{x}^*)} \\ \leq c \int_0^t \|\boldsymbol{d_u}(s)\|^2_{*,\boldsymbol{x}^*} + \|\boldsymbol{d_v}(s)\|^2_{*,\boldsymbol{x}^*} \, \mathrm{d}s,\end{aligned}$$

Inserting this bound in (D.33), squaring and integrating from 0 to t yields

$$\int_0^t |\boldsymbol{e}_{\boldsymbol{v}}(s)|^2_{M^*(\boldsymbol{x}^*)}\,\mathrm{d}s \le c\int_0^t \Big(\|\boldsymbol{d}_{\boldsymbol{u}}(s)\|^2_{\star,\boldsymbol{x}^*} + \|\boldsymbol{d}_{\boldsymbol{v}}(s)\|^2_{\star,\boldsymbol{x}^*} \Big)\,\mathrm{d}s.$$

With the assumed bounds of the defects, we obtain $O(h^k)$ error estimates for $0 \le t \le t^*$. Finally, to show that $t^* = T$, we use the same argument as at the end of the proof of Proposition D.5.1. ■

D.7. Geometric estimates

In this section we give further notations and some technical lemmas from [54] that will be used later on. Most of the results are high-order and time-dependent extensions of geometric approximation estimates shown in [29, 31, 35] and [23].

D.7.1. The interpolating surface

We return to the setting of section D.2, where $X(\,\cdot\,,t)$ defines a smooth surface $\Gamma(t) = \Gamma(X(\,\cdot\,,t))$. For an admissible triangulation of $\Gamma(t)$ with nodes $x_j^*(t) = X(p_j,t)$ and the corresponding nodal vector $\boldsymbol{x}^*(t) = (x_j^*(t))$, we define the interpolating surface by

$$X_h^*(p_h,t) = \sum_{j=1}^N x_j^*(t)\,\phi_j[\boldsymbol{x}(0)](p_h), \qquad p_h \in \Gamma_h^0,$$

which has the properties that $X_h^*(p_j,t) = x_j^*(t) = X(p_j,t)$ for $j = 1,\dots,N$, and

$$\Gamma_h^*(t) := \Gamma_h(\boldsymbol{x}^*(t)) = \Gamma(X_h^*(\,\cdot\,,t)).$$

In the following we drop the argument t when it is not essential. The velocity of the interpolating surface Γ_h^*, defined as in section D.2.3, is denoted by v_h^*.

D.7.2. Approximation results

The lift of a function $\eta_h : \Gamma_h^*(t) \to \mathbb{R}$ is again denoted by $\eta_h^l : \Gamma(t) \to \mathbb{R}$, defined via the oriented distance function d between $\Gamma_h^*(t)$ and $\Gamma(t)$ provided that the surfaces are sufficiently close (which is the case if h is sufficiently small).

Lemma D.7.1 (Equivalence of norms [29, lemma 3], [23]). *Let $\eta_h : \Gamma_h^*(t) \to \mathbb{R}$ with lift $\eta_h^l : \Gamma(t) \to \mathbb{R}$. Then the L^p and $W^{1,p}$ norms on the discrete and continuous surfaces are equivalent for $1 \le p \le \infty$, uniformly in the mesh size $h \le h_0$ (with sufficiently small $h_0 > 0$) and in $t \in [0,T]$.*

In particular, there is a constant c such that for $h \le h_0$ and $0 \le t \le T$,

$$c^{-1} \|\eta_h\|_{L^2(\Gamma_h^*(t))} \le \|\eta_h^l\|_{L^2(\Gamma(t))} \le c \|\eta_h\|_{L^2(\Gamma_h^*(t))},$$
$$c^{-1} \|\eta_h\|_{H^1(\Gamma_h^*(t))} \le \|\eta_h^l\|_{H^1(\Gamma(t))} \le c \|\eta_h\|_{H^1(\Gamma_h^*(t))}.$$

Later on the following estimates will be used. They have been shown in [54], based on [23] and [35].

Lemma D.7.2. *Let $\Gamma(t)$ and $\Gamma_h^*(t)$ be as above in Section D.7.1. Then, for $h \le h_0$ with a sufficiently small $h_0 > 0$, we have the following estimates for the distance function d from* (D.8)*, and for the error in the normal vector:*

$$\|d\|_{L^\infty(\Gamma_h^*(t))} \le ch^{k+1}, \qquad \|\nu_{\Gamma(t)} - \nu_{\Gamma_h^*(t)}^l\|_{L^\infty(\Gamma(t))} \le ch^k,$$

with constants independent of $h \le h_0$ and $t \in [0,T]$.

D.7.3. Bilinear forms and their estimates

We use surface-dependent bilinear forms defined similarly as in [35]: Let X be a given surface with velocity v, with interpolation surface X_h^* with velocity v_h^*. For arbitrary $z, \varphi \in H^1(\Gamma(X))$ we set

$$\begin{aligned} m(X; z, \varphi) &= \int_{\Gamma(X)} z\varphi, \\ a(X; z, \varphi) &= \int_{\Gamma(X)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, \\ q(X; v; z, \varphi) &= \int_{\Gamma(X)} (\nabla_\Gamma \cdot v) z\varphi, \end{aligned}$$

and for $Z_h, \phi_h \in S_h(\boldsymbol{x}^*)$ we set

$$\begin{aligned} m(X_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} Z_h \phi_h, \\ a(X_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\ q(X_h^*; v_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} (\nabla_{\Gamma_h} \cdot v_h^*) Z_h \phi_h, \end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense. For more details see [35, lemma 2.1] (and the references in the proof), or [34, lemma 5.2].

We start by defining a discrete velocity on the smooth surface, denoted by $\hat{v}_h$. We follow sections 5.3 of [54], where the high-order ESFEM generalization of the discrete velocity on $\Gamma(X)$ from Sections 4.3 and 5.3 of [35] is discussed. Using the lifted elements, $\Gamma(X)$ is decomposed into curved elements whose Lagrange points move with the velocity $\hat{v}_h$ defined by

$$\hat{v}_h \left((X_h^*)^l(\,\cdot\,, t), t \right) = \frac{\mathrm{d}}{\mathrm{d}t} (X_h^*)^l(\,\cdot\,, t).$$

Discrete material derivatives on $\Gamma(X_h^*)$ and $\Gamma(X)$ are given by for $\phi_h \in S_h(\boldsymbol{x}^*)$ by

$$\begin{aligned}\partial^\bullet_{v_h^*}\varphi_h &:= \partial_t\varphi_h + v_h^* \cdot \nabla\varphi_h,\\ \partial^\bullet_{\hat{v}_h}\varphi_h^l &:= \partial_t\varphi_h^l + \hat{v}_h \cdot \nabla\varphi_h^l.\end{aligned}$$

In [35, lemma 4.1] it was shown that the transport property of the basis functions carries over to the lifted basis functions $\phi_j[\boldsymbol{x}^*]$:

$$\partial^\bullet_{\hat{v}_h}\phi_j[\boldsymbol{x}^*]^l = (\partial^\bullet_{v_h^*}\phi_j[\boldsymbol{x}^*])^l = 0 \qquad (j = 1, \dots, N).$$

Therefore, the above discrete material derivatives and the lift operator satisfy,

$$\partial^\bullet_{\hat{v}_h}\varphi_h^l = (\partial^\bullet_{v_h^*}\varphi_h)^l, \tag{D.35}$$

for $\varphi_h \in S_h(X_h^*)$.

Lemma D.7.3 (Transport properties [35, lemma 4.2]). *For any $z(t), \varphi(t) \in H^1(\Gamma(X(\,\cdot\,, t)))$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} m(X; z, \varphi) = m(X; \partial^\bullet z, \varphi) + m(X; z, \partial^\bullet\varphi) + q(X; v; z, \varphi).$$

The same formulas hold when $\Gamma(X)$ is considered as the lift of the discrete surface $\Gamma(X_h^*)$ (i.e. $\Gamma(X)$ can be decomposed into curved elements which are lifts of the elements of $\Gamma(X_h^*)$), moving with the velocity $\hat{v}_h$:

$$\frac{\mathrm{d}}{\mathrm{d}t} m(X; z, \varphi) = m(X; \partial^\bullet_{\hat{v}_h} z, \varphi) + m(X; z, \partial^\bullet_{\hat{v}_h}\varphi) + q(X; \hat{v}_h; z, \varphi).$$

Similarly, in the discrete case, for arbitrary $z_h(t), \varphi_h(t) \in S_h(\boldsymbol{x}^*(t))$ we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m(X_h^*; z_h, \varphi_h) = m(X_h^*; \partial^\bullet_{v_h^*} z_h, \varphi_h) &+ m(X_h^*; z_h, \partial^\bullet_{v_h^*}\varphi_h)\\ &+ q(X_h^*; v_h^*; z_h, \varphi_h),\end{aligned}$$

where v_h^* is the velocity of the surface $\Gamma(X_h^*)$.

The following estimates, proved in lemma 5.6 of [54], will play a crucial role in the defect bounds later on.

Lemma D.7.4 (Geometric perturbation errors). *For any $Z_h, \psi_h \in S_h(\boldsymbol{x}^*)$ where $\Gamma(X_h^*)$ is the interpolation surface of piecewise polynomial degree k, we have the following bounds, for $h \leq h_0$ with a sufficiently small $h_0 > 0$,*

$$\begin{aligned} |m(X; Z_h^l, \varphi_h^l) - m(X_h^*; Z_h, \varphi_h)| & \\ &\leq ch^{k+1} \, \|Z_h^l\|_{L^2(\Gamma(X))} \, \|\varphi_h^l\|_{L^2(\Gamma(X))}, \\ |a(X; Z_h^l, \varphi_h^l) - a(X_h^*; Z_h, \varphi_h)| & \\ &\leq ch^{k+1} \, |Z_h^l|_{H^1(\Gamma(X))} \, |\varphi_h^l|_{H^1(\Gamma(X))}, \\ |q(X; \hat{v}_h; Z_h^l, \varphi_h^l) - q(X_h^*; v_h^*; Z_h, \varphi_h)| & \\ &\leq ch^{k+1} \, \|Z_h^l\|_{L^2(\Gamma(X))} \, \|\varphi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

The constant c is independent of h and $t \in [0, T]$.

D.7.4. Interpolation error estimates for evolving surface finite element functions

For any $u \in H^{k+1}(\Gamma(X))$, there is a unique piecewise polynomial surface finite element interpolation of degree k in the nodes x_j^*, denoted by $\widetilde{I}_h u \in S_h(\boldsymbol{x}^*)$. We set $I_h u \coloneqq (\widetilde{I}_h u)^l : \Gamma(X) \to \mathbb{R}$. Error estimates for this interpolation are obtained from [23, proposition 2.7] by carefully studying the time dependence of the constants, cf. [54].

Lemma D.7.5. *There exists a constant $c > 0$ independent of $h \leq h_0$,* *with a sufficiently small* $h_0 > 0$*, and* t *such that for* $u(\,\cdot\,, t) \in H^{k+1}(\Gamma(t))$*, for* $0 \le t \le T$*,*

$$\|u - I_h u\|_{L^2(\Gamma(X))} + h\,|u - I_h u|_{H^1(\Gamma(X))} \le ch^{k+1}\,\|u\|_{H^{k+1}(\Gamma(X))}.$$

The same result holds for vector valued functions. As it will always be clear from the context we do not distinguish between interpolations for scalar and vector valued functions.

D.8. Defect bounds

In this section we show that the assumed defect estimates of proposition D.5.1 and D.6.1 are indeed fulfilled when the projection Π_h is chosen to be the piecewise kth-degree polynomial interpolation operator I_h for $k \ge 2$.

The interpolations satisfy the discrete problem (D.4)–(D.5) only up to some defects. These defects are denoted by $d_u \in S_h(\boldsymbol{x}^*)$, $d_v \in S_h(\boldsymbol{x}^*)^3$, with $\boldsymbol{x}^*(t)$ the vector of exact nodal values $x_j^*(t) = X(p_j, t) \in \Gamma(t)$, and are given as follows: for all $\varphi_h \in S_h(\boldsymbol{x}^*)$ with $\partial^\bullet_{v_h^*}\varphi_h = 0$ and $\psi_h \in S_h(\boldsymbol{x}^*; \mathbb{R}^3)$,

$$\begin{aligned}
\int_{\Gamma_h(\boldsymbol{x}^*)} d_u \varphi_h &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(\boldsymbol{x}^*)} \widetilde{I}_h u\, \varphi_h + \int_{\Gamma_h(\boldsymbol{x}^*)} \nabla_{\Gamma_h(\boldsymbol{x}^*)} \widetilde{I}_h u \cdot \nabla_{\Gamma_h(\boldsymbol{x}^*)}\varphi_h \\
&\quad - \int_{\Gamma_h(\boldsymbol{x}^*)} f(\widetilde{I}_h u, \nabla_{\Gamma_h(\boldsymbol{x}^*)}\widetilde{I}_h u)\, \varphi_h, \\
\int_{\Gamma_h(\boldsymbol{x}^*)} d_v \cdot \psi_h &= \int_{\Gamma_h(\boldsymbol{x}^*)} \widetilde{I}_h v \cdot \psi_h + \alpha \int_{\Gamma_h(\boldsymbol{x}^*)} \nabla_{\Gamma_h(\boldsymbol{x}^*)} \widetilde{I}_h v \cdot \nabla_{\Gamma_h(\boldsymbol{x}^*)}\psi_h \\
&\quad - \int_{\Gamma_h(\boldsymbol{x}^*)} g(\widetilde{I}_h u, \nabla_{\Gamma_h(\boldsymbol{x}^*)}\widetilde{I}_h u)\nu_{\Gamma_h(\boldsymbol{x}^*)} \cdot \psi_h.
\end{aligned}$$

Later on the vectors of nodal values of the defects d_u and d_v are denoted by $\boldsymbol{d_u} \in \mathbb{R}^N$ and $\boldsymbol{d_v} \in \mathbb{R}^{3N}$, respectively. These vectors, together with $\boldsymbol{d_x} = \boldsymbol{0}$, satisfy (D.24).

Lemma D.8.1. *Let the solution u, the surface X and its velocity v be all sufficiently smooth. Then there exists a constant $c > 0$ such that for all $h \leq h_0$, with a sufficiently small $h_0 > 0$, and for all $t \in [0, T]$, the defects d_u and d_v of the kth-degree finite element interpolation are bounded as*

$$\|\boldsymbol{d_u}\|_{*,\boldsymbol{x}^*} = \|d_u\|_{H_h^{-1}(\Gamma(X_h^*))} \leq ch^k,$$
$$\|\boldsymbol{d_v}\|_{*,\boldsymbol{x}^*} = \|d_v\|_{H_h^{-1}(\Gamma(X_h^*))} \leq ch^k,$$

where the H_h^{-1}-norm is defined in (D.17). *The constant c is independent of h and $t \in [0, T]$.*

Proof. (i) We start from an identity for the dual norm as in (D.18), (omitting the argument $\boldsymbol{x}^*$ of the matrices):

$$\|\boldsymbol{d_u}\|_{*,\boldsymbol{x}^*} = (\boldsymbol{d_u}^T M(M + A)^{-1} M\boldsymbol{d_u})^{\frac{1}{2}} = \|d_u\|_{H_h^{-1}(\Gamma(X_h^*))}.$$

In order to estimate the defect in u, we subtract (D.3) from the above equation, and perform almost the same proof as in [35, section 7]. We use the bilinear forms and the discrete versions of the transport properties from lemma D.7.3. We obtain, for any $\varphi_h \in S_h(\boldsymbol{x}^*)$ with $\partial^\bullet_{v_h^*}\varphi_h = 0$,

$$\begin{aligned} m(X_h^*; d_u, \varphi_h) &= \frac{\mathrm{d}}{\mathrm{d}t} m(X_h^*; \widetilde{I}_h u, \varphi_h) + a(X_h^*; \widetilde{I}_h u, \varphi_h) \\ &\quad - m(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) \\ &= m(X_h^*; \partial^\bullet_{v_h^*} \widetilde{I}_h u, \varphi_h) + q(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h) + a(X_h^*; \widetilde{I}_h u, \varphi_h) \\ &\quad - m(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h), \end{aligned}$$

and

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} m(X; u, \varphi_h^l) + a(X; u, \varphi_h^l) - m(X; f(u, \nabla_{\Gamma(X)} u), \varphi_h^l)$$

$$= m(X; \partial^\bullet_{\hat{v}_h} u, \varphi_h^l) + q(X; \hat{v}_h; u, \varphi_h^l)$$
$$+ a(X; u, \varphi_h^l) - m(X; f(u, \nabla_{\Gamma(X)} u), \varphi_h^l).$$

Subtracting the two equation yields

$$\begin{aligned}
&m(X_h^*; d_u, \varphi_h) \\
&\quad = m(X_h^*; \partial^\bullet_{v_h^*} \widetilde{I}_h u, \varphi_h) - m(X; \partial^\bullet_{\hat{v}_h} u, \varphi_h^l) + q(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h) \\
&\qquad - q(X; \hat{v}_h; u, \varphi_h^l) + a(X_h^*; \widetilde{I}_h u, \varphi_h) - a(X; u, \varphi_h^l) \\
&\qquad - \Big(m(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X; f(u, \nabla_\Gamma u), \varphi_h^l) \Big).
\end{aligned}$$

We bound all the terms pairwise, by using the interpolation estimates of lemma D.7.5 and the estimates for the geometric perturbation errors of the bilinear forms of lemma D.7.4. For the first pair, using that $(\partial^\bullet_{v_h^*} \widetilde{I}_h u)^l = \partial^\bullet_{\hat{v}_h} I_h u$, we obtain

$$\begin{aligned}
&|m(X_h^*; \partial^\bullet_{v_h^*} \widetilde{I}_h u, \varphi_h) - m(X; \partial^\bullet_{\hat{v}_h} u, \varphi_h^l)| \\
&\quad \leq |m(X_h^*; \partial^\bullet_{v_h^*} \widetilde{I}_h u, \varphi_h) - m(X; \partial^\bullet_{\hat{v}_h} I_h u, \varphi_h^l)| + |m(X; I_h \partial^\bullet_{\hat{v}_h} u - \partial^\bullet_{\hat{v}_h} u, \varphi_h^l)| \\
&\quad \leq c h^{k+1} \, \|\varphi_h^l\|_{L^2(\Gamma(X))}.
\end{aligned}$$

For the second pair we obtain

$$\begin{aligned}
&|q(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h) - q(X; \hat{v}_h; u, \varphi_h^l)| \\
&\quad \leq |q(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h) - q(X; \hat{v}_h; I_h u, \varphi_h^l)| + |q(X; v_h^*; I_h u - u, \varphi_h)| \\
&\quad \leq c h^{k+1} \, \|\varphi_h^l\|_{L^2(\Gamma(X))}.
\end{aligned}$$

The third pair is estimated by

$$\begin{aligned}
|a(X_h^*; \widetilde{I}_h u, \varphi_h) - a(X; u, \varphi_h^l)| &\leq |a(X_h^*; \widetilde{I}_h u, \varphi_h) - a(X; I_h u, \varphi_h^l)| \\
&\qquad + |a(X; I_h u - u, \varphi_h^l)| \\
&\leq c h^k \, \|\nabla_\Gamma \varphi_h^l\|_{L^2(\Gamma(X))}.
\end{aligned}$$

For the last pair we use the fact that $(f(u, \nabla_\Gamma u))^{-l} = f(u^{-l}, (\nabla_\Gamma u)^{-l})$ and the local Lipschitz continuity of the function f, to obtain

$$\begin{aligned}
&|m(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X; f(u, \nabla_\Gamma u), \varphi_h^l)| \\
&\leq |m(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) - f(u^{-l}, (\nabla_\Gamma u)^{-l}), \varphi_h)| \\
&\quad + |m(X_h^*; f(u, \nabla_\Gamma u)^{-l}, \varphi_h) - m(X; f(u, \nabla_\Gamma u), \varphi_h^l)| \\
&\leq c \, \|f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) - f(u^{-l}, (\nabla_\Gamma u)^{-l})\|_{L^2(\Gamma(X_h^*))} \, \|\varphi_h^l\|_{L^2(\Gamma(X))} \\
&\quad + c h^{k+1} \, \|\varphi_h^l\|_{L^2(\Gamma(X))}.
\end{aligned}$$

The first term is estimated, using the local Lipschitz continuity of f and equivalence of norms, by

$$\begin{aligned}
&\|f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) - f(u^{-l}, (\nabla_\Gamma u)^{-l})\|_{L^2(\Gamma(X_h^*))} \\
&\leq \|f\|_{W^{1,\infty}} \Big(c \, \|I_h u - u\|_{L^2(\Gamma(X))} + c \, \|\nabla_\Gamma (I_h u - u)\|_{L^2(\Gamma(X))} \\
&\quad + c \, \|(\nabla_{\Gamma_h} u^{-l})^l - \nabla_\Gamma u\|_{L^2(\Gamma(X))} \Big),
\end{aligned}$$

where the first two terms are bounded by $O(h^k)$ using interpolation estimates, while the third term is bounded, using remark 4.1 in [35] and lemma D.7.2, as

$$\|(\nabla_{\Gamma_h} u^{-l})^l - \nabla_\Gamma u\|_{L^2(\Gamma(X))} \leq c h^k.$$

Thus for the fourth pair we obtained

$$|m(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X; f(u, \nabla_\Gamma u), \varphi_h^l)| \leq c h^k \, \|\varphi_h^l\|_{L^2(\Gamma(X))}$$

Altogether, we have

$$m(X_h^*; d_u, \varphi_h) \leq c h^k \, \|\varphi_h^l\|_{H^1(\Gamma(X))},$$

which, by the equivalence of norms given by lemma D.7.1, shows the first bound of the stated lemma.

(ii) In order to estimate the defect in v, similarly as previously we subtract (D.3) from the above equation and use the bilinear forms to obtain

$$\begin{aligned} &m(X_h^*; d_v, \psi_h) \\ &= m(X_h^*; \widetilde{I}_h v, \psi_h) - m(X; v, \psi_h^l) \\ &\quad + \alpha \left(a(X_h^*; \widetilde{I}_h v, \psi_h) - a(X; v, \psi_h^l) \right) \\ &\quad + m(X_h^*; g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) \nu_{\Gamma(X_h^*)}, \psi_h) - m(X; g(u, \nabla_\Gamma u) \nu_{\Gamma(X)}, \psi_h^l). \end{aligned}$$

Similarly as in the previous part, these three pairs are bounded pairwise. For the first pair we have

$$\begin{aligned} |m(X_h^*; \widetilde{I}_h v, \psi_h) - m(X; v, \psi_h^l)| &\leq |m(X_h^*; \widetilde{I}_h v, \psi_h) - m(X; I_h v, \psi_h^l)| \\ &\quad + |m(X; I_h v - v, \psi_h^l)| \\ &\leq c h^{k+1} \|\psi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

For the second pair we use the interpolation estimate to bound

$$\begin{aligned} |a(X_h^*; \widetilde{I}_h v, \psi_h) - a(X; v, \psi_h^l)| &\leq |a(X_h^*; \widetilde{I}_h v, \psi_h) - a(X; I_h v, \psi_h^l)| \\ &\quad + |a(X; I_h v - v, \psi_h^l)| \\ &\leq c h^k \|\nabla_\Gamma \psi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

The third pair we estimate, similarly to the nonlinear pair above, by

$$\begin{aligned} &|m(X_h^*; g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) \nu_{\Gamma(X_h^*)}, \psi_h) - m(X; g(u, \nabla_\Gamma u) \nu_{\Gamma(X)}, \psi_h^l)| \\ &\leq |m(X_h^*; (g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) - g(u, \nabla_\Gamma u)^{-l}) \nu_{\Gamma(X_h^*)}, \psi_h)| \\ &\quad + |m(X_h^*; g(u, \nabla_\Gamma u)^{-l} (\nu_{\Gamma(X_h^*)} - \nu_{\Gamma(X)}^{-l}), \psi_h)| \\ &\quad + |m(X_h^*; g(u, \nabla_\Gamma u)^{-l} \nu_{\Gamma(X)}^{-l}, \psi_h) - m(X; g(u, \nabla_\Gamma u) \nu_{\Gamma(X)}, \psi_h^l)| \\ &\leq c h^k \|g\|_{W^{1,\infty}} \|\psi_h^l\|_{L^2(\Gamma(X))} + c \|\nabla_\Gamma (X - X_h^*)\|_{L^2(\Gamma(X))} \|\psi_h^l\|_{L^2(\Gamma(X))} \\ &\quad + c h^{k+1} \|g\|_{L^2} \|\psi_h^l\|_{L^2(\Gamma(X))} \end{aligned}$$

$$\leq ch^k \, \|g\|_{W^{1,\infty}} \, \|\psi_h^l\|_{L^2(\Gamma(X))} + ch^k \, \|\psi_h^l\|_{L^2(\Gamma(X))}$$
$$\leq ch^k \, \|\psi_h^l\|_{L^2(\Gamma(X))},$$

where we have used the local Lipschitz boundedness of the function g, the interpolation estimate, lemma D.7.2, and lemma D.7.4, through a similar argument as above for the semilinear term with f.

Finally, the combination of these bounds yields

$$m(X_h^*; d_v, \psi_h) \leq ch^k \, \|\psi_h^l\|_{H^1(\Gamma(X))},$$

providing the asserted bound on $\boldsymbol{d_v}$. ∎

D.9. Proof of theorem D.3.1

The errors are decomposed using interpolations and the definition of lifts from section D.2.6: omitting the argument t,

$$\begin{aligned} u_h^L - u &= \left(\widehat{u}_h - \widetilde{I}_h u\right)^l + (I_h u - u), \\ v_h^L - v &= \left(\widehat{v}_h - \widetilde{I}_h v\right)^l + (I_h v - v), \\ X_h^L - X &= \left(\widehat{X}_h - \widetilde{I}_h X\right)^l + (I_h X - X). \end{aligned}$$

The last terms in these formulas can be bounded in the $H^1(\Gamma)$ norm by Ch^k, using the interpolation bounds of lemma D.7.5.

To bound the first terms on the right-hand sides, we first use the defect bounds of lemma D.8.1, which then together with the stability estimate of proposition D.6.1 proves the result, since by the norm equivalences of lemma D.7.1 and equations (D.9)–(D.10) we have (again omitting the argument t)

$$\begin{aligned} \|\left(\widehat{u}_h - \widetilde{I}_h u\right)^l\|_{L^2(\Gamma)} &\leq c \, \|\widehat{u}_h - \widetilde{I}_h u\|_{L^2(\Gamma_h^*)} = c \, |\boldsymbol{e_u}|_{M(\boldsymbol{x}^*)}, \\ \|\nabla_\Gamma \left(\widehat{u}_h - \widetilde{I}_h u\right)^l\|_{L^2(\Gamma)} &\leq c \, \|\nabla_{\Gamma_h^*} \left(\widehat{u}_h - \widetilde{I}_h u\right)\|_{L^2(\Gamma_h^*)} = c \, |\boldsymbol{e_u}|_{A(\boldsymbol{x}^*)}, \end{aligned}$$

and similarly for $\widehat{v}_h - \widetilde{I}_h v$ and $\widehat{X}_h - \widetilde{I}_h X$.

D.10. Extension to other velocity laws

In this section we consider the extension of our results to different velocity laws: adding a mean curvature term to the regularized velocity law considered so far, and a dynamic velocity law. We concentrate on the velocity laws without coupling to the surface PDE, since the coupling can be dealt with in the same way as previously. We only consider the stability of the evolving surface finite element discretization, since bounds for the consistency error are obtained by the same arguments as before.

D.10.1. Regularized mean curvature flow

We next extend our results to the case where the velocity law contains a mean curvature term:

$$v - \alpha \Delta_{\Gamma(X)} v - \beta \Delta_{\Gamma(X)} X = g(\,\cdot\,, t)\nu_{\Gamma(X)} \tag{D.36}$$

where $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given Lipschitz continuous function of (x, t), and $\alpha > 0$ and $\beta > 0$ are fixed parameters. Here $\Delta_{\Gamma(X)} X$ is a suggestive notation for $-H\mathrm{n}$, where H denotes the mean curvature of the surface $\Gamma(X)$. (More precisely, $\Delta_{\Gamma(X)} \mathbb{1} = -H\nu_{\Gamma(X)}$.)

The corresponding differential-algebraic system has the following form:

$$M^*(\boldsymbol{x})\boldsymbol{v} + A(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{x}). \tag{D.37}$$

Similarly as before the corresponding error equation is given as

$$\begin{aligned} M^*(\boldsymbol{x}^*)\boldsymbol{e_v} + A(\boldsymbol{x}^*)\boldsymbol{e_x} = &- \big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{v}^* \\ &- \big(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\big)\boldsymbol{e_v} \\ &- \big(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\big)\boldsymbol{x}^* - \big(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\big)\boldsymbol{e_x} \\ &+ \big(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)\big) - M(\boldsymbol{x}^*)\boldsymbol{d}_v, \end{aligned}$$

together with $\dot{\boldsymbol{e}}_{\boldsymbol{x}} = \boldsymbol{e_v}$.

Proposition D.10.1. *Under the assumptions of proposition D.5.1, there exists $h_0 > 0$ such that the following stability estimate holds for all $h \leq h_0$, for $0 \leq t \leq T$:*

$$\begin{aligned}|\boldsymbol{e}_{\boldsymbol{x}}(t)|^2_{M^*(\boldsymbol{x}^*(t))} &\leq C \int_0^t \|\boldsymbol{d}_{\boldsymbol{v}}(s)\|^2_{*,\boldsymbol{x}^*} + |\boldsymbol{d}_{\boldsymbol{x}}(s)|^2_{M^*(\boldsymbol{x}^*(s))} \, \mathrm{d}s, \\ |\boldsymbol{e}_{\boldsymbol{v}}(t)|^2_{M^*(\boldsymbol{x}^*(t))} &\leq C \, \|\boldsymbol{d}_{\boldsymbol{v}}(t)\|^2_{*,\boldsymbol{x}^*} + C \int_0^t \|\boldsymbol{d}_{\boldsymbol{v}}(s)\|^2_{*,\boldsymbol{x}^*} \\ &\qquad + |\boldsymbol{d}_{\boldsymbol{x}}(s)|^2_{M^*(\boldsymbol{x}^*(s))} \, \mathrm{d}s.\end{aligned}$$

The constant C is independent of t and h, but depends on the final time T, and on the parameters α and β.

Proof. We detail only those parts of the proof of proposition D.5.1 where the mean curvature term introduces differences, otherwise exactly the same proof applies.

In order to prove the stability estimate we again test with $\boldsymbol{e}_{\boldsymbol{v}}$, and obtain

$$\begin{aligned}|\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} = &-\boldsymbol{e}_{\boldsymbol{v}}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{v}^* - \boldsymbol{e}_{\boldsymbol{v}}^T \left(M^*(\boldsymbol{x}) - M^*(\boldsymbol{x}^*)\right)\boldsymbol{e}_{\boldsymbol{v}} \\ &- \boldsymbol{e}_{\boldsymbol{v}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{x}^* - \boldsymbol{e}_{\boldsymbol{v}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{e}_{\boldsymbol{x}} \\ &- \boldsymbol{e}_{\boldsymbol{v}}^T A(\boldsymbol{x}^*)\boldsymbol{e}_{\boldsymbol{x}} + \boldsymbol{e}_{\boldsymbol{v}}^T \left(\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)\right) - \boldsymbol{e}_{\boldsymbol{v}}^T M(\boldsymbol{x}^*)\boldsymbol{d}_{\boldsymbol{v}}.\end{aligned}$$

Every term is estimated exactly as previously in the proof of proposition D.5.1, except the terms corresponding to the mean curvature term, involving the stiffness matrix A. They are estimated by the same techniques as previously:

$$\begin{aligned}&\boldsymbol{e}_{\boldsymbol{v}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{x}^* + \boldsymbol{e}_{\boldsymbol{v}}^T \left(A(\boldsymbol{x}) - A(\boldsymbol{x}^*)\right)\boldsymbol{e}_{\boldsymbol{x}} \\ &\quad \leq \frac{1}{6} \, |\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} + c \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}, \\ &\boldsymbol{e}_{\boldsymbol{v}}^T A(\boldsymbol{x}^*)\boldsymbol{e}_{\boldsymbol{x}} \leq \frac{1}{6} \, |\boldsymbol{e}_{\boldsymbol{v}}|^2_{M^*(\boldsymbol{x}^*)} + c \, |\boldsymbol{e}_{\boldsymbol{x}}|^2_{M^*(\boldsymbol{x}^*)}.\end{aligned}$$

Altogether we obtain the error bound

$$|\boldsymbol{e_v}|^2_{M^*(\boldsymbol{x}^*)} \leq c\, |\boldsymbol{e_x}|^2_{M^*(\boldsymbol{x}^*)} + c\, \|\boldsymbol{d_v}\|^2_{*,\boldsymbol{x}^*},$$

which is exactly (D.22). The proof is then finished as before. ■

With proposition D.10.1 and the appropriate defect bounds, theorem D.3.1 extends directly to the system with mean curvature term in the regularized velocity law.

D.10.2. A dynamic velocity law

Let us consider the dynamic velocity law, again without coupling to a surface PDE:

$$\partial^\bullet v + v\nabla_{\Gamma(X)} \cdot v - \alpha\Delta_{\Gamma(X)} v = g(\,\cdot\,, t)\, \nu_{\Gamma(X)},$$

where again $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given Lipschitz continuous function of (x, t), and $\alpha > 0$ is a fixed parameter. This problem is considered together with the ordinary differential equations (D.1) for the positions X determining the surface $\Gamma(X)$. Initial values are specified for X and v.

The weak formulation and the semidiscrete problem can be obtained by a similar argument as for the PDE on the surface in section D.6. Therefore we immediately present the ODE formulation of the semidiscretization. As in section D.2.5, the nodal vectors $\boldsymbol{v} \in \mathbb{R}^{3N}$ of the finite element function v_h, together with the surface nodal vector $\boldsymbol{x} \in \mathbb{R}^{3N}$ satisfy a system of ODEs with matrices and driving term as in section D.5:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x})\boldsymbol{v}\Big) + A(\boldsymbol{x})\boldsymbol{v} &= \boldsymbol{g}(\boldsymbol{x}, t) \\ \dot{\boldsymbol{x}} &= \boldsymbol{v}, \end{aligned} \tag{D.38}$$

By using the same notations for the exact positions $\boldsymbol{x}^*(t) \in \mathbb{R}^{3N}$, the interpolated exact velocity $\boldsymbol{v}^*(t) \in \mathbb{R}^{3N}$, and for the defects $\boldsymbol{d}_v(t)$

and $\boldsymbol{d_x}(t)$, we obtain that they fulfill the following equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{v}^*\Big) + A(\boldsymbol{x}^*)\boldsymbol{v}^* &= \boldsymbol{g}(\boldsymbol{x}^*, t) + M(\boldsymbol{x}^*)\boldsymbol{d_v} \\ \dot{\boldsymbol{x}}^* &= \boldsymbol{v}^*. \end{aligned}$$

By subtracting this from (D.38), and using similar arguments as before, we obtain the error equations for the surface nodes and velocity:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{e_v}\Big) + A(\boldsymbol{x}^*)\boldsymbol{e_v} &= -\frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{v}^*\Big) \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{e_v}\Big) \\ &\quad - (A(\boldsymbol{x}) - A(\boldsymbol{x}^*))\boldsymbol{v}^* \\ &\quad - (A(\boldsymbol{x}) - A(\boldsymbol{x}^*))\boldsymbol{e_v} \\ &\quad + (\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)) - M(\boldsymbol{x}^*)\boldsymbol{d_v} \\ \dot{\boldsymbol{e}}_{\boldsymbol{x}} &= \boldsymbol{e_v} - \boldsymbol{d_x}. \end{aligned}$$

We then have the following stability result.

Proposition D.10.2. *Under the assumptions of Proposition D.5.1, there exists $h_0 > 0$ such that the following error estimate holds for all $h \leq h_0$, uniformly for $0 \leq t \leq T$:*

$$\begin{aligned} &\|\boldsymbol{e_x}(t)\|^2_{M^*(\boldsymbol{x}^*(t))} + \|\boldsymbol{e_v}(t)\|^2_{M(\boldsymbol{x}^*(t))} \\ &\quad + \int_0^t \|\boldsymbol{e_v}(s)\|^2_{A(\boldsymbol{x}^*(s))}\,\mathrm{d}s \leq C\int_0^t \|\boldsymbol{d_v}(s)\|^2_{\star,\boldsymbol{x}^*}\,\mathrm{d}s. \end{aligned}$$

The constant $C > 0$ is independent of t and h, but depends on the final time T and the parameter α.

Proof. By testing the error equation with $\boldsymbol{e}_{\boldsymbol{v}}$ we obtain

$$
\begin{aligned}
\boldsymbol{e}_{\boldsymbol{v}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(M(\boldsymbol{x}^*)\boldsymbol{e}_{\boldsymbol{v}}\Big) + \boldsymbol{e}_{\boldsymbol{v}}^T A(\boldsymbol{x}^*)\boldsymbol{e}_{\boldsymbol{v}} = & -\boldsymbol{e}_{\boldsymbol{v}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{v}^*\Big) \\
& -\boldsymbol{e}_{\boldsymbol{v}}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big((M(\boldsymbol{x}) - M(\boldsymbol{x}^*))\boldsymbol{e}_{\boldsymbol{v}}\Big) \\
& -\boldsymbol{e}_{\boldsymbol{v}}^T (A(\boldsymbol{x}) - A(\boldsymbol{x}^*))\boldsymbol{v}^* \\
& -\boldsymbol{e}_{\boldsymbol{v}}^T (A(\boldsymbol{x}) - A(\boldsymbol{x}^*))\boldsymbol{e}_{\boldsymbol{v}} \\
& +\boldsymbol{e}_{\boldsymbol{v}}^T (\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x}^*)) - \boldsymbol{e}_{\boldsymbol{v}}^T M(\boldsymbol{x}^*)\boldsymbol{d}_{\boldsymbol{v}}.
\end{aligned}
$$

The terms are bounded in the same way as the corresponding terms in the proofs of Propositions D.5.1 and D.6.1. With these estimates, a Gronwall inequality yields the result. ■

With proposition D.10.2 and the appropriate defect bounds, theorem D.3.1 extends directly to the parabolic surface PDE coupled with the dynamic velocity law.

D.11. Numerical results

In this section we complement theorem D.3.1 by showing the numerical behaviour of piecewise linear finite elements, which are not covered by theorem D.3.1, but nevertheless perform remarkably well. Moreover, we compare our regularized velocity law with regularization by mean curvature flow.

D.11.1. A coupled problem

Our test problem is a combination of (D.2) with a mean curvature term as in (D.36):

$$
\begin{aligned}
\partial^\bullet u + u\nabla_\Gamma \cdot v - \Delta_\Gamma u &= f(t,x), \\
v - \alpha\Delta_\Gamma v - \beta\Delta_\Gamma X &= \delta u \mathrm{n}_\Gamma + g(t,x)\nu_\Gamma,
\end{aligned}
\tag{D.39}
$$

for non-negative parameters α, β, δ. The velocity law here is a special case of (D.2) for $\beta = 0$, and reduces to (D.36) for $\delta = 0$. The matrix-vector form reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(M\,(\boldsymbol{x}(t))\boldsymbol{u}(t)\Big) + A\,(\boldsymbol{x}(t))\boldsymbol{u}(t) &= \boldsymbol{f}\,(t, \boldsymbol{x}(t)) \\ M^*\,(\boldsymbol{x}(t))\dot{\boldsymbol{x}}(t) + \beta A\,(\boldsymbol{x}(t))\boldsymbol{x}(t) &= \delta \mathbf{N}\,(\boldsymbol{x}(t))\boldsymbol{u}(t) + \boldsymbol{g}\,(t, \boldsymbol{x}(t)) \end{aligned}$$

for $0 \leq t \leq T$ and given $\boldsymbol{x}(0)$ and $\boldsymbol{u}(0)$, where

$$\mathbf{N}\,(\boldsymbol{x})\boldsymbol{u}|_{3(j-1)+\ell} = \int_{\Gamma_h(\boldsymbol{x})} (\mathrm{n}_{\Gamma_h})_\ell \boldsymbol{u}_j \phi_j[\boldsymbol{x}],$$

for $j = 1, \dots, N$ and $\ell = 1, 2, 3$.

In our numerical experiments we used a linearly implicit Euler discretization of this system with step sizes chosen so small that the error is dominated by the spatial discretization error.

Example D.11.1. We consider (D.39) and choose f and g such that $X(p,t) = r(t)p$ with

$$r(t) = \frac{r_0 r_K}{r_K e^{-kt} + r_0(1 - e^{-kt})}$$

and $u(X,t) = X_1 X_2 e^{-6t}$ are the exact solution of the problem. The parameters are set to be $T = 1$, $\alpha = 1$, $\beta = 0$, $\delta = 0.4$, $r_0 = 1$, $r_K = 2$ and $k = 0.5$.

We choose $(\mathcal{T}_k)$ as a series of meshes such that $2h_k \approx h_{k-1}$. In table D.1 we report on the errors and the corresponding experimental orders of convergence (EOC). Using the notation of section D.2.6, the following norms are used:

$$\begin{aligned} \|\mathrm{err}_u\|_{L^\infty(L^2)} &= \sup_{[0,T]} \|\widehat{u}_h(\,\cdot\,, t) - \widetilde{I}_h u(\,\cdot\,, t)\|_{L^2(\Gamma_h^*(t))}, \\ \|\mathrm{err}_u\|_{L^2(H^1)} &= \left(\int_0^T \Big\| \widehat{u}_h(\,\cdot\,, s) - \widetilde{I}_h u(\,\cdot\,, s) \Big\|^2_{H^1(\Gamma_h^*(s))} \mathrm{d}s \right)^{\frac{1}{2}}, \end{aligned}$$

$$\|\mathrm{err}_v\|_{L^\infty(H^1)} = \sup_{[0,T]} \|\widehat{v}_h(\,\cdot\,,t) - \widetilde{I}_h v(\,\cdot\,,t)\|_{H^1(\Gamma_h^*(t))},$$
$$\|\mathrm{err}_x\|_{L^\infty(H^1)} = \sup_{[0,T]} \|\widehat{x}_h(\,\cdot\,,t) - \mathbb{1}_{\Gamma_h^*(t)}\|_{H^1(\Gamma_h^*(t))}.$$

The EOCs for the errors $E(h_{k-1})$ and $E(h_k)$ with mesh sizes h_{k-1}, h_k are given via

$$EOC(h_{k-1}, h_k) = \frac{\log\left(\frac{E(h_{k-1})}{E(h_k)}\right)}{\log\left(\frac{h_{k-1}}{h_k}\right)} \qquad (k = 2, \ldots, n).$$

The degree of freedoms (DOF) and maximum mesh size at time T are also reported in the tables.

In table D.1 we report on the errors and EOCs observed using example D.11.1. The EOCs in the PDE are expected to be 2 for the $L^\infty(L^2)$ norm and 1 for the $L^2(H^1)$ norm, while the errors in the surface and in the surface velocity are expected to be 1 in the $L^\infty(H^1)$ norm.

Example D.11.2. Again we consider (D.39), but this time we quantitatively compare the two different regularized velocity laws. Hence, we let δ vanish. We use a g like in example D.11.1 and run two tests with the common parameters $T = 2$, $r_0 = 1$, $r_K = 2$ and $k = 0.5$, and use the same mesh and time step levels as before. The first test uses $\alpha = 0$ and $\beta = 1$ and the second test uses $\alpha = 1$ and $\beta = 0$. The results are captured in table D.2.

Our regularized velocity law provides smaller errors as regularizing with mean curvature flow. The EOCs in the errors in the surface and in the errors for the surface velocity are expected to be 1 in $L^\infty(H^1)_v$ and $L^\infty(H^1)_x$ norm, see table D.2.b. While it can be observed that for this particular example the convergence rates for $\alpha \neq 0$ are higher then for $\beta \neq 0$.

DOF	$h(T)$	$\|\mathrm{err}_u\|_{L^\infty(L^2)}$	EOC	$\|\mathrm{err}_u\|_{L^2(H^1)}$	EOC
126	0.6664	0.1519165	-	0.2727214	-
516	0.4088	0.0896624	1.08	0.1498895	1.22
2070	0.1799	0.0222349	1.70	0.0344362	1.79
8208	0.0988	0.0070552	1.91	0.0109074	1.92
32682	0.0499	0.0018319	1.98	0.0029375	1.92

(a) Errors for u

DOF	$h(T)$	$\|\mathrm{err}_v\|_{L^\infty(H^1)}$	EOC	$\|\mathrm{err}_x\|_{L^\infty(H^1)}$	EOC
126	0.6664	0.2260428	-	0.1473157	-
516	0.4088	0.0595755	2.73	0.0298673	3.27
2070	0.1799	0.0158342	1.61	0.0106836	1.25
8208	0.0988	0.0053584	1.81	0.0042312	1.54
32682	0.0499	0.0019341	1.50	0.0017838	1.27

(b) Surface and velocity errors

Table D.1.: Errors and EOCs for example D.11.1

D.11.2. A model for tumor growth

Our next test problem is the coupled system of equations

$$\begin{aligned} \partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u &= f_1(u, w), \\ \partial^\bullet w + w \nabla_\Gamma \cdot v - D_c \Delta_\Gamma w &= f_2(u, w), \\ v - \alpha \Delta_\Gamma v - \beta \Delta_\Gamma X &= \delta u \mathrm{n}_\Gamma, \end{aligned} \tag{D.40}$$

where

$$f_1(u, w) = \gamma(a - u + u^2 w), \quad f_2(u, w) = \gamma(b - u^2 w),$$

with non-negative parameters $D_c, \gamma, a, b, \alpha, \beta$.

For $\alpha = 0$ this system has been used as a simplified model for tumor growth; see Barreira, Elliott and Madzvamuse [6] and [42, 17].

$h(T)$	$L^\infty(L^2)_v$	EOC	$L^\infty(H^1)_v$	EOC	$L^\infty(H^1)_x$	EOC
0.6664	0.756045	-	1.31532	-	1.601255	-
0.4088	0.393067	1.34	0.78538	1.06	0.522342	2.29
0.1799	0.095914	1.72	0.96206	-0.25	0.137396	1.63
0.0988	0.035166	1.67	1.48784	-0.73	0.044666	1.87
0.0499	0.019755	0.85	2.73584	-0.89	0.013507	1.75

(a) Surface and velocity errors with parameters $\alpha = 0$ and $\beta = 1$.

$h(T)$	$L^\infty(L^2)_v$	EOC	$L^\infty(H^1)_v$	EOC	$L^\infty(H^1)_x$	EOC
0.6664	0.149836	-	0.225114	-	0.143419	-
0.4088	0.036118	2.91	0.058147	2.77	0.024087	3.65
0.1799	0.009286	1.65	0.015843	1.58	0.009702	1.11
0.0988	0.002705	2.06	0.005361	1.81	0.003990	1.48
0.0499	0.000686	2.01	0.001935	1.49	0.001746	1.21

(b) Surface and velocity errors with parameters $\alpha = 1$ and $\beta = 0$.

Table D.2.: Errors and EOCs for example D.11.2.

These authors used the mean curvature term with a small parameter $\beta > 0$ to regularize their velocity law.

We used piecewise linear finite elements and the same time discretization scheme as in [6, 42].

Example D.11.3. We consider (D.40) and want to compare qualitatively the two different regularized velocity laws $\alpha \neq 0$ and $\beta \neq 0$. As common parameters we use $D_c = 10$, $\gamma = 100$, $a = 0.1$, $b = 0.9$ and $T = 5$. The initial surface is a sphere and the initial values u_0 and w_0 are calculated by solving an auxiliary surface PDE as follows. We take small perturbations around the steady state

$$\begin{pmatrix} \widetilde{u}_0 \\ \widetilde{w}_0 \end{pmatrix} = \begin{pmatrix} a + b + \varepsilon_1(x) \\ \frac{b}{(a+b)^2} + \varepsilon_2(x) \end{pmatrix},$$

where $\varepsilon_1(x), \varepsilon_2(x) \in [0, 0.01]$ take random values. We solve the auxiliary coupled diffusion equations with the stationary initial surface until time $\widetilde{T} = 5$. We set $u_0 = \widetilde{u}(\widetilde{T})$ and $w_0 = \widetilde{w}(\widetilde{T})$, which we used as initial values for (D.40).

We perform two experiments with $(\alpha, \beta) = (0, 0.01)$ and $(\alpha, \beta) = (0.01, 0)$. We present snapshots in figure D.1. We observe that both velocity laws display the same qualitative behavior, also agreeing with [42].

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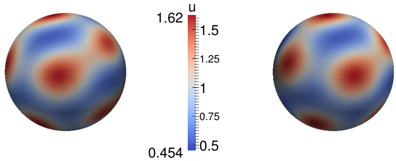
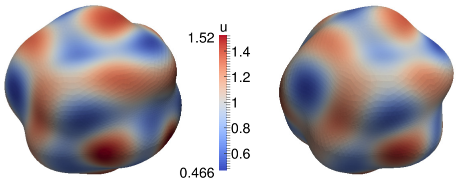
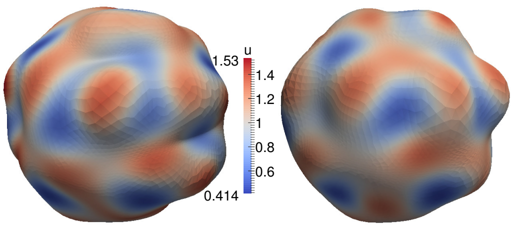


(a) time $t = 0$



(b) time $t = 1$



(c) time $t = 2$

Figure D.1.: Simulation for example D.11.3. The first column corresponds to $(\alpha, \beta) = (0, 0.01)$ and the second column to $(\alpha, \beta) = (0.01, 0)$.

Bibliography

[1] R. A. Adams and J. J. F. Fournier. Sobolev spaces. 2 edition. *Pure and Applied Mathematics*, **140**. Elsevier/ Academic press, 2003. ISBN:0-12-044143-8.

[2] G. Akrivis and Ch. Lubich. Fully implicit, linearly implicit and implicit–explicit backward difference formulae for quasilinear parabolic equations. *Numer. Math.* (2015), p. 1–23.

[3] A. Alphonse, C. M. Elliott and B. Stinner. An abstract framework for parabolic PDEs on evolving spaces. Preprint, 2014. arXiv:1403.4500v1.

[4] K. E. Atkinson and W. Han. Theoretical numerical analysis. *Texts in Applied Mathematics*, **39**. Springer Verlag, New York, 2001. ISBN:0-387-95142-3.

[5] T. Aubin. Some Nonlinear Problems in Riemannian Geometry. *Springer Monographs in Mathematics*. Springer Verlag, 1998. ISBN:3-540-60752-8.

[6] R. Barreira, C. M. Elliott and A. Madzvamuse. The surface finite element method for pattern formation on evolving biological surfaces. *J. Math. Biol.* **63** (2011), no. 6, p. 1095–1119. DOI:10.1007/s00285-011-0401-0.

[7] J. W. Barrett, K. Deckelnick and V. Styles. Numerical analysis for a system coupling curve evolution to reaction-diffusion on the curve. Preprint, 2016. arXiv:1607.01726.

[8] J. W. Barrett, H. Garcke and R. Nürnberg. On the parametric finite element approximation of evolving hypersurfaces in $\mathbb{R}^3$. *J. Comput. Phys.* **227** (2008), no. 9, p. 4281–4307.

[9] J. W. Barrett, H. Garcke and R. Nürnberg. Numerical computations of the dynamics of fluidic membranes and vesicles. *Phys. Rev. E* **92** (2015), no. 5.

[10] J. W. Barrett, H. Garcke and R. Nürnberg. A stable parametric finite element discretization of two-phase navier–stokes flow. *J. Sci. Comput.* **63** (2015), no. 1, p. 78–117.

[11] J. W. Barrett, H. Garcke and R. Nürnberg. A stable numerical method for the dynamics of fluidic membranes. *Numer. Math.* (2016), p. 1–41. DOI:10.1007/s00211-015-0787-5.

[12] A. Bonito, I. Kiza and R. H. Nochetto. Time–discrete higher–order ALE formulations: A priori error analysis. *Numer. Math.* **125** (2013), no. 2, p. 225–257.

[13] A. Bonito, I. Kiza and R. H. Nochetto. Time–discrete higher–order ALE formulations: Stability. *SIAM J. Numer. Anal.* **51** (2013), no. 1, p. 577–604.

[14] S. C. Brenner and L. R. Scott. The Mathematical Theory of Finite Element Methods. *Texts in Applied Mathematics*, **15**. Springer Verlag, New York, 1994. ISBN:0-387-94193-2.

[15] J. W. Cahn, C. M. Elliott and A. Novick-Cohen. The Cahn–Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature. *European J. Appl. Math.* **7** (1996), no. 3, p. 287–301.

[16] J. W. Cahn, P. C. Fife and O. Penrose. A phase-field model for diffusion-induced grain-boundary motion. *Acta Mat.* **45** (1997), no. 10, p. 4397–4413.

[17] M. A. J. Chaplain, M. Ganesh and I. G. Graham. Spatio-temporal pattern formation on spherical surfaces: numerical simulation and application to solid tumour growth. *J. Math. Biol.* **42** (2001), no. 5, p. 387–423.

[18] I. Chavel. Riemannian Geometry: A Modern Introduction. Cambridge university press, 2006. ISBN:0-521-48578-9.

[19] L.-Q. Chen. Phase-field models for microstructure evolution. *Annual Review of Materials Research* **32** (2002), no. 1, p. 113–140.

[20] G. Dahlquist. G–stability is equivalent to A–stability. *BIT* **18** (1978), no. 4, p. 384–401.

[21] K. Deckelnick, C. M. Elliott and V. Styles. Numerical diffusion induced grain boundary motion. *Interfaces Free Bound.* **3** (2001), no. 4, p. 393–414.

[22] A. Dedner, R. Klöfkorn, M. Nolte and M. Ohlberger. A generic interface for parallel and adaptive scientific computing: abstraction principles and the DUNE-FEM module. *Computing* **90** (2010), no. 3–4, p. 165–196.

[23] A. Demlow. Higher–order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.* **47** (2009), no. 2, p. 805–807. DOI:10.1137/070708135.

[24] T. S. Developers. Sage Mathematics Software (Version 6.1.1), 2014. http://www.sagemath.org.

[25] G. Di Fazio. L^p estimates for divergence form elliptic equations with discontinuous coefficients. *Boll. Un. Mat. Ital. A (7)* **10** (1996), no. 2, p. 409–420.

[26] J. Douglas, Jr and T. Dupont. Galerkin methods for parabolic equations. *SIAM J. Numer. Anal.* **7** (1970), no. 4, p. 575–626.

[27] J. Douglas, Jr, T. Dupont and L. B. Wahlbin. The Stability in L^q of the L^2-projection into finite element function spaces. *Numer. Math.* **23** (1974), no. 3, p. 193–197.

[28] Q. Du and L. Ju. Approximations of a Ginzburg–Landau model for superconducting hollow spheres based on spherical centroidal voronoi tessellations. *Math. Comp.* **74** (2005), no. 251, p. 1257–1280.

[29] G. Dziuk. Finite elements for the Beltrami operator on arbitrary surfaces. *Cal. Var. Partial Differ. Equ.* (1988), p. 142–155.

[30] G. Dziuk. An algorithm for evolutionary surfaces. *Numer. Math.* **58** (1990), no. 1, p. 603–611.

[31] G. Dziuk and C. M. Elliott. Finite elements on evolving surfaces. *IMA J. Numer. Anal.* **27** (2007), no. 2, p. 262–292. DOI:10.1093/imanum/drl023.

[32] G. Dziuk and C. M. Elliott. Surface finite elements for parabolic equations. *J. Comput. Math.* **25** (2007), no. 4, p. 385–407.

[33] G. Dziuk and C. M. Elliott. A fully discrete evolving surface finite element method. *SIAM J. Numer. Anal.* **50** (2012), no. 5, p. 2677–2694.

[34] G. Dziuk and C. M. Elliott. Finite element methods for surface PDEs. *Acta Numer.* **22** (2013), p. 289–396. DOI:10.1017/S0962492913000056.

[35] G. Dziuk and C. M. Elliott. L^2–estimates for the evolving surface finite element method. *Math. Comp.* **82** (2013), no. 281, p. 1–24.

[36] G. Dziuk, D. Kröner and T. Müller. Scalar conservation laws on moving hypersurfaces. *Interfaces Free Bound.* **15** (2013), no. 2, p. 203–236. arXiv:1307.1056v1.

[37] G. Dziuk, Ch. Lubich and D. E. Mansour. Runge–Kutta time discretization of parabolic differential equations on evolving surfaces. *IMA J. Numer. Anal.* **32** (2012), no. 2, p. 394–416. DOI:10.1093/imanum/drr017.

[38] K. Ecker. Regularity theory for mean curvature flow. *Progress in Nonlinear Differential Equations and their Applications*, **57**. Birkhäuser, Boston, MA., 2004. ISBN:0-8176-3243-3.

[39] C. M. Elliott and H. Garcke. On the Cahn–Hilliard equation with degenerate mobility. *SIAM J. Math. Anal.* **27** (1996), no. 2, p. 404–423.

[40] C. M. Elliott and T. Ranner. Evolving surface finite element method for the Cahn–Hilliard equation. *Numer. Math.* **129** (2015), no. 3, p. 483–534.

[41] C. M. Elliott and B. Stinner. Modeling and computation of two phase geometric biomembranes using surface finite elements. *J. Comput. Phys.* **229** (2010), no. 18, p. 6585–6612. DOI:10.1016/j.jcp.2010.05.014.

[42] C. M. Elliott and V. Styles. An ALE ESFEM for solving PDEs on evolving surfaces. *Milan J. Math.* **80** (2012), no. 2, p. 469–501. DOI:10.1007/s00032-012-0195-6.

[43] C. M. Elliott and C. Venkataraman. Error analysis for an ALE evolving surface finite element method. *Numer. Methods Partial Differential Equations* **31** (2015), no. 2, p. 459–499. arXiv:1403.1402v2.

[44] P. C. Fife, J. W. Cahn and C. M. Elliott. A free boundary model for diffusion induced grain boundary motion. *Interfaces Free Bound.* **3** (2001), no. 3, p. 291–336.

[45] L. Formaggia and F. Nobile. A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements. *East–West J. Numer. Math.* **7** (1999), no. 2, p. 105–131.

[46] D. Gilbarg and N. S. Trudinger. Elliptic Partial Differential Equations of Second Order. 2 edition. Springer Verlag, 1998. ISBN:3-540-13025-X; 0-387-13025-X.

[47] A. Gray. The volume of a small geodesic ball of a Riemannian manifold. *Michigan Math. J.* **20** (1974), no. 4, p. 329–344.

[48] E. Hairer, S. P. Nørsett and G. Wanner. Solving Ordinary Differential Equations I: Nonstiff problems. 2 edition. Springer Verlag, 1993.

[49] E. Hairer and G. Wanner. Solving Ordinary Differential Equations II.: Stiff and differetial–algebraic problems. 2 edition. Springer Verlag, 1996.

[50] C. Handwerker. Diffusion-induced grain boundary migration in thin films. *Diffusion Phenomena in Thin Films and Microelectronic Materials* (1989), p. 245–322.

[51] A. J. James and J. Lowengrub. A surfactant-conserving volume-of-fluid method for interfacial flows with insoluble surfactant. *J. Comput. Phys.* **201** (2004), no. 2, p. 685–722. DOI:10.1016/j.jcp.2004.06.013.

[52] H. L. Jin, A. J. Yezzi and S. Soatto. Region–based segmentation on evolving surfaces with application to 3D reconstruction of shape and piecewise constant radiance. *UCLA preprint* (2004).

[53] B. Kovács. Computing arbitrary lagrangian eulerian maps for evolving surfaces. Preprint, 2016. arXiv:1612.01701.

[54] B. Kovács. High-order evolving surface finite element method for parabolic problems on evolving surfaces. Preprint, 2016. arXiv:1606.07234.

[55] B. Kovács, B. Li, Ch. Lubich and Ch. A. Power Guerra. Convergence of finite elements on an evolving surface driven by diffusion on the surface. *Numer. Math.* **137** (2017), no. 3, p. 643–689. arXiv:1607.07170.

[56] B. Kovács and Ch. A. Power Guerra. Error analysis for full discretizations of quasilinear parabolic problems on evolving surfaces. *Numer. Methods Partial Differential Equations* **32** (2016), no. 4, p. 1200–1231. arXiv:1503.08990.

[57] B. Kovács and Ch. A. Power Guerra. Higher-order time discretizations with ALE finite elements for parabolic problems on evolving surfaces. *IMA J. Numer. Anal.* (2017). DOI:10.1093/imanum/drw074, arXiv:1410.0486.

[58] B. Kovács and Ch. A. Power Guerra. Maximum norm stability and error estimates for the evolving surface finite element method. *Numer. Methods Partial Differential Equations* (2017). DOI:10.1002/num.22212, arXiv:1510.00605.

[59] H. Kröner. Error estimate for a finite element approximation of the solution of a linear parabolic equation on a two-dimensional surface. Preprint, 2016. arXiv:1604.04665.

[60] C. H. Leung and M. Berzins. A computational model for organism growth based on surface mesh generation. *J. Comput. Phys.* (2002).

[61] D. Leykekhman and B. Li. Maximum-norm stability of the finite element ritz projection with mixed boundary conditions. Preprint, 2017.

[62] Ch. Lubich and D. E. Mansour. Variational discretization of wave equations on evolving surfaces. *Math. Comp.* **84** (2015), no. 292, p. 513–542.

[63] Ch. Lubich, D. E. Mansour and C. Venkataraman. Backward difference time discretization of parabolic differential equations on evolving surfaces. *IMA J. Numer. Anal.* **33** (2013), no. 4, p. 1365–1385. DOI:10.1093/imanum/drs044.

[64] Ch. Lubich and A. Ostermann. Runge–Kutta approximation of quasilinear parabolic equations. *Math. Comp.* **64** (1995), p. 601–627. DOI:10.1090/S0025-5718-1995-1284670-0.

[65] A. Madzvamuse and R. Barreira. Exhibiting cross-diffusion-induced patterns for reaction-diffusion systems on evolving domains and surfaces. *Physical Review E* **90** (2014), no. 4.

[66] D. E. Mansour. Numerical Analysis of Partial Differential Equations on Evolving Surfaces. PhD thesis, Universität Tübingen, 2013. http://hdl.handle.net/10900/49925.

[67] D. E. Mansour. Gauss–Runge–Kutta time discretization of wave equations on evolving surfaces. *Numer. Math.* **129** (2015), p. 21–54.

[68] F. Morgan. Geometric Measure Theory. Elsevier/ Academic press, Boston, MA., 1988. ISBN:0-12-506855-7.

[69] O. Nevanlinna and F. Odeh. Multiplier techniques for linear multistep methods. *Numer. Funct. Anal. Optim.* **3** (1981), no. 4, p. 377–423.

[70] J. Nitsche. L^∞-convergence of finite element approximations. *Math. Asp. Finite Element Metho.* (1977), p. 261–274.

[71] P.-O. Persson and G. Strang. A simple mesh generator in MATLAB. *SIAM Rev.* **46** (2004), no. 2, p. 329–345.

[72] Ch. A. Power Guerra. Numerical analysis of the evolving surface finite element method for some parabolic problems. Phd. Thesis (in preparation).

[73] P. Pozzi and B. Stinner. Curve shortening flow coupled to lateral diffusion. Preprint, 2015. arXiv:1510.06173.

[74] R. Rannacher and L. R. Scott. Some optimal error estimates for piecewise linear finite element approximation. *Math. Comp.* **38** (1982), no. 158, p. 437–445.

[75] A. H. Schatz, V. Thomée and L. B. Wahlbin. Maximum Norm Stability and Error Estimates in Parabolic Finite Element Equations. *Commun. Pure Appl. Anal.* **33** (1980), no. 3, p. 265–304.

[76] G. Strang and G. J. Fix. An Analysis of the Finite Element Method. *Series in Automatic Computation*, **212**. Prentice-Hall, Englewood Cliffs, NJ., 1973.

[77] V. Thomée. Galerkin Finite Element Methods for Parabolic Problems. 2 edition. *Springer Series in Computational Mathematics*, **25**. Springer Verlag, Berlin, 2006. ISBN:978-3-540-33121-6.

[78] M. F. Wheeler. A priori L^2-error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal.* **10** (1973), p. 723–759.