Cox sheaves on graded schemes, algebraic actions and $\mathbb{F}_1$-schemes

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Introduction

This thesis deals with the theory of *Cox sheaves*, i.e. $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebras $\mathcal{R}$ with homogeneous parts $\mathcal{R}_{[D]} = \mathcal{O}_X(D)$. Their global sections $\mathcal{R}(X)$ are called *Cox rings* and the structure homomorphisms $\mathcal{O}(X) \to \mathcal{R}(X)$ are *Cox algebras*. For the present X denotes a normal prevariety over an algebraically closed field $\mathbb{K}$, but later, more general spaces will take its place. Of particular interest will be the role of finiteness conditions and the generalization of existing results on algebraic and geometric properties of Cox sheaves and rings. Generalizing [**17**], we show that normal prevarieties X have a Cox ring $\mathcal{R}(X)$ of finite type over the base field $\mathbb{K}$ if and only if they admit a *very neat* embedding into a normal toric prevariety Z, see Theorem VII.3.1.1. Such embeddings are described entirely by toric morphisms which are relative spectra of toric Cox sheaves, termed *toric characteristic spaces*, and $\mathbb{K}$-algebras R with the characterizing algebraic properties of Cox rings and certain finite systems of generators, see Theorem VII.2.2.8.

A minimal set of those characterizing properties is formed by factoriality of the set $R^{\mathrm{hom}} \setminus 0$ of non-zero homogeneous elements, all homogeneous units having degree zero and the grading group $gr(R)$ being generated by the degrees of the units of the localization $R_{\mathfrak{p}}$ at each homogeneously prime divisor $\mathfrak{p}$ of R, see Theorem VII.3.3.1. Of the three characterizing conditions, the latter is easily checked in terms of the generators of R. For factoriality of $R^{\mathrm{hom}} \setminus 0$ we provide a reduction to factoriality of $(S^{-1}R)_0$ in Theorem II.2.2.4, where S is generated as a monoid by the homogeneously prime elements among the generators of R, which in turn may be verified in terms of conditions on the exponents occuring in defining relations of R, see Proposition II.2.2.6. This new characterization of Cox rings of finite type over $\mathbb{K}$ differs from the original one from [**17**] in that integrality and normality are no longer necessary conditions for a ring to be a Cox ring, which means we also have obtained a sufficient criterion for those two in terms of graded algebra, see Corollary VII.3.3.4.

A key object of study for a Cox sheaf $\mathcal{R}$ which is locally of finite type (e.g. has a finitely generated Cox ring) is its relative spectrum $\widehat{X} = \mathrm{Spec}_X(\mathcal{R}) \to X$, termed the *characteristic space* of $\mathcal{R}$, which is a good quotient by the canonical action by $\widehat{H} := \mathrm{Spec}(\mathbb{K}[\mathrm{Cl}(X)])$. We establish a criterion for a good quotient $q\colon \widehat{X} \to X$ by a quasi-torus $\widehat{H}$ to be a characteristic space in terms of $\widehat{H}$-factoriality of $\widehat{X}$, coincidence of homogeneous units of $\mathcal{O}(\widehat{X})$ and units of $\mathcal{O}(X)$, and existence of a big saturated open subset of $\widehat{X}$ on which $\widehat{H}$ acts freely, see Theorem VI.4.2.7 for the case $H = \{1\}$. This criterion differs from its model from [**4**] in that irreducibility and normality of $\widehat{X}$ are not among the sufficient conditions.

A strong motivation for this work has been the question which properties still hold for Cox sheaves and Cox algebras which are not (locally) of finite type and indeed, for Cox sheaves and algebras of spaces X which are not (locally) noetherian. Moreover, it would still be desirable to have geometric realizations, i.e. characteristic spaces, for such Cox sheaves and also a characterization in geometric terms similar to the one given above. In order to work in full generality and within a setting that allows geometric realizations of graded sheaves without loss of information we consider the category of *graded schemes* (over $A = \mathbb{Z}$ or the mulitiplicative monoid $A = \mathbb{F}_1 = \{0,1\}$). These are topological spaces X with graded structure sheaves $\mathcal{O}_X$, which are locally sets of *homogeneously prime* ideals of a graded A-algebra, the latter being graded ideals whose homogeneous elements form a prime ideal of the set of all homogeneous elements.

In order to have a suitable notion of Weil divisors and class groups, we require X to be *of Krull type*, meaning quasi-compact such that each $\mathcal{O}_X(U)^{\mathrm{hom}} \setminus 0$ is a Krull monoid. In particular, classical noetherian normal schemes over $\mathbb{Z}$ are of Krull type. Here, a *Krull monoid* is a cancellative monoid M for which the partially ordered monoid $\mathrm{Div}(M)$ of non-empty proper M-submodules of $Q(M)$ which are intersections of principal submodules Mf is a group whose minimal positive elements form a $\mathbb{Z}$-basis. Equivalently, there exist a group L containing M and a family $\nu_i, i \in I$ of homomorphisms from L to $\mathbb{Z}$, called *valuations*, such that M is the intersection of all $\nu_i^{-1}(\mathbb{N}_0)$ and for each $f \in M$ there exist only finitely many $i \in I$ with $\nu_i(f) \neq 0$. A monoid M is *factorial*, i.e. M/M^* is a direct sum over copies of $\mathbb{N}_0$, if and only if M is a Krull monoid for which the *class group* $\mathrm{Cl}(M)$, i.e. the quotient of $\mathrm{Div}(M)$ by the group $\mathrm{PDiv}(M)$ of principal submodules Mf, is zero.

In this general setting, we provide the first algebraic criterion for an $\mathcal{O}_X$-algebra $\mathcal{R}$ to be a Cox sheaf. Using the model of Krull monoids resp. rings, we define what it means for a (pre-)sheaf of (graded) monoids, $\mathbb{F}_1$-algebras or rings to be *of Krull type*, see Section III.4, a condition which structure sheaves of normal prevarieties, actions of Krull type and graded schemes X of Krull type (over $\mathbb{F}_1$ or $\mathbb{Z}$) all satisfy. Specifically, each prime divisor Y on X defines a *graded valuation* ν_Y on the constant sheaf $\mathcal{K}$ of *graded fraction rings* as well as a *graded valuation sheaf* $\mathcal{K}_{\nu_Y} \subset \mathcal{K}$, and $\mathcal{O}_X$ is the intersection over all $\mathcal{K}_{\nu_Y}$. Adjoining to $\mathcal{K}$ the group $\mathrm{WDiv}(X)$ gives a constant sheaf $\mathcal{K}[\mathrm{WDiv}(X)]$ in which one takes the direct sum over all $\mathcal{O}_X(D)\chi^D$ to obtain the *divisorial* $\mathcal{O}_X$-algebra $\mathcal{O}_X(\mathrm{WDiv}(X))$. A *Cox sheaf* on a graded scheme X of Krull type is then an $\mathcal{O}_X$-algebra $\mathcal{R}$ allowing a homomorphism of (graded) $\mathcal{O}_X$-algebras

$$\pi\colon \mathcal{O}_X(\mathrm{WDiv}(X)) \longrightarrow \mathcal{R}, \quad \psi\colon gr(\mathcal{O}_X) \oplus \mathrm{WDiv}(X) \longrightarrow gr(\mathcal{R})$$

such that π is an epimorphism of presheaves which restricts to isomorphisms of homogeneous components and $\psi^{-1}(gr(\mathcal{O}_X)) \cap \mathrm{WDiv}(X)$ is the group $\mathrm{PDiv}(X)$ of principal divisors.

We show that an $\mathcal{O}_X$-algebra $\mathcal{R}$ is a Cox sheaf if and only if it is Veronesean and of Krull type such that its defining graded valuations restrict bijectively to $\{\nu_Y\}_Y$ and, modulo its intersection with $gr(\mathcal{O}_X)$, the degree support set of $\mathcal{R}$ is isomorphic to $\mathrm{Cl}(X)$ via a map which is induced by the defining valuations of $\mathcal{R}$, see Theorem V.2.2.4. The defining graded valuations of $\mathcal{R}$ are determined by its stalks at the prime divisors of X so that we have obtained an intrinsic characterization of Cox sheaves. For a Cox algebra $R_G = \mathcal{O}(X) \to R = \mathcal{R}(X)$ the monoid $R^{\mathrm{hom}} \setminus 0$ is then factorial with the same units as $R_G^{\mathrm{hom}} \setminus 0$ and each homogeneously prime divisor $\mathfrak{p}$ of R satisfies $\deg((R_{\mathfrak{p}}^{\mathrm{hom}})^*) + G = gr(R)$, and conversely, a Veronesean inclusion $R_G \subseteq R$ satisfying these conditions is a Cox algebra of some graded scheme provided that $gr(R)/G$ is finitely generated, see Theorem V.2.2.5.

With view towards geometric properties of Cox sheaves we show that *Veronesean good quotients*, i.e affine morphisms whose accompanying sheaf homomorphisms are Veronesean, are closed and compatible with intersections of closed subsets and have special points in each fibre, see Proposition IV.2.2.3. The classical **Proj**-construction of a $\mathbb{N}_0$-graded ring offers a well-known example of a bijective Veronesean good quotient. The characterizing properties of *graded relative spectra* $q\colon \mathrm{Spec}_{gr,X}(\mathcal{R}) \to X$ of Cox sheaves, called *graded characteristic spaces*, are then q

being a Veronesean good quotient, $\widehat{X}$ being of Krull type with $\mathrm{Cl}(\widehat{X}) = 0$, order-preserving invertibility of the pullback $q_X^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ of the sheaves of Weil divisors, coincidence of homogeneous units of $\mathcal{O}(\widehat{X})$ and $\mathcal{O}(X)$, and canonical isomorphy of grading group modulo degree support set of $\mathcal{O}_{\widehat{X}}$ and $\mathcal{O}_X$, respectively, see Theorem V.3.1.4.

We translate these results in to the realm of actions on prevarieties. For the case that $\mathbb{K}$ is an algebraically closed field we establish an equivalence of reduced graded schemes of finite type over $\mathbb{K}$ and morphical quasi-torus actions on prevarietes over $\mathbb{K}$, see Theorem VI.2.4.1. As a first step, each graded scheme X over $\mathbb{K}$ is assigned its canonical action $\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[gr(\mathcal{O}_X)]) \circlearrowright X \to X$ which is induced on affines by the coaction $f_w \mapsto \chi^w \otimes f_w$. Secondly, we apply the functor from graded schemes to (0-graded) schemes which sends W to the (graded) relative spectrum of the composition of $\mathcal{O}_W$ with the forgetful functor from graded rings to rings. Thirdly, we apply the equivalence of schemes over $\mathbb{K}$ to prevarieties over $\mathbb{K}$. In the converse direction, we send $H \circlearrowright Z$ to Z, equipped with the topology $\Omega_{Z,H}$ of H-invariant Zariski open sets and the (canonically $\mathbb{X}(H)$-graded) structure sheaf $\mathcal{O}_{Z,H} := (\mathcal{O}_Z)_{|\Omega_{Z,H}}$, after which we apply the soberification functor.

For an action $H \circlearrowright Z$ *of Krull type*, meaning that $\mathcal{O}(U)^{\mathrm{hom}} \setminus 0$ is a Krull monoid for each $U \in \Omega_{Z,H}$ we define H-prime divisors as invariant closed $\Omega_{Z,H}$-irreducible subsets $Y \subset Z$ of $\Omega_{Z,H}$-codimension one. On the constant sheaf $\mathcal{K}_H$ of *graded fraction rings* which assigns the stalk of $\mathcal{O}_{Z,H}$ at the $\Omega_{Z,H}$-irreducible closed subset Z each Y then defines a *graded valuation* ν_Y whose image is the skyscraper sheaf $\mathbb{Z}^{(Y)}$ on $\Omega_{Z,H}$ which sends $\Omega_{Z,H}$-neighbourhoods of Y to $\mathbb{Z}$ and all other invariant opens to 0. $\mathcal{O}_{Z,H}$ is then the intersection over the *graded valuation sheaves* $(\mathcal{K}_H)_{\nu_Y}$ defined by $\{\nu_Y\}_Y$. In other words, $\mathcal{O}_{Z,H}$ is of Krull type. The direct sum over all $\mathbb{Z}^{(Y)}$ gives the sheaf WDiv_H of *H-Weil divisors* on $\Omega_{Z,H}$ and the sum over all ν_Y gives a homomorphism $\mathrm{div}_H\colon (\mathcal{K}_H^{\mathrm{hom}})^* \to \mathrm{WDiv}_H$. Its image and cokernel presheaves are PDiv_H and Cl_H. Each $D \in \mathrm{WDiv}_H(Z)$ defines an $\mathcal{O}_{Z,H}$-module $\mathcal{O}_{Z,H}(D)$ in the usual way and taking the direct sum over all these gives the $\mathcal{O}_{Z,H}$-algebra $\mathcal{O}_{Z,H}(\mathrm{WDiv}_H(Z))$. A *Cox sheaf on* $\Omega_{Z,H}$ is now an $\mathcal{O}_{Z,H}$-algebra $\mathcal{R}$ which allows a homomorphism $\pi\colon \mathcal{O}_{Z,H}(\mathrm{WDiv}_H(Z)) \to \mathcal{R}$ with accompanying group homomorphism $\psi\colon \mathbb{X}(H) \oplus \mathrm{WDiv}_H(Z) \to gr(\mathcal{R})$ such that π restricts to isomorphisms of homogeneous components and $\mathrm{WDiv}_H(Z) \cap \psi^{-1}(\mathbb{X}(H)) = \mathrm{PDiv}_H(X)$ holds. An affine morphism of actions $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{Z} \to H \circlearrowright Z$ for which $q_* \mathcal{O}_{\widehat{Z},\widehat{H}}$ is a Cox sheaf is a *characteristic space over* $H \circlearrowright Z$. An already well-known example of such a characteristic space is the toric morphism associated to the *homogeneous coordinate ring* of a toric variety as introduced by Cox in [**10**].

The obtained equivalences preserve good quotients and characteristic spaces, and commute with formation of (invariant) Weil divisors and class groups. Moreover, the singleton closure map *sc* from Z to the soberification of $(Z, \Omega_{Z,H}, \mathcal{O}_{Z,H})$ defines a natural correspondence between algebras and modules over $\mathcal{O}_{Z,H}$ and those over $\mathcal{O}_X$, and this correspondence sends Cox sheaves to Cox sheaves, see Section VI.4. We also show that the pullback property of characteristic spaces may be translated to the existence of a big q-saturated $\widehat{H}$-invariant open set on which $\ker(\theta)$ acts with constant isotropy, see Proposition VI.4.2.6. The condition on degree support sets corresponds to θ restricting to an isomorphism $\widehat{H}_{\widehat{Z}} \to H_Z$ of the generic isotropy groups. The resulting criterion is the general version of Theorem VI.4.2.7.

The connection between the respective Cox sheaf theories of (quasi-)toric prevarieties and $\mathbb{F}_1$-schemes is also investigated. Similarily to quasi-tori over $\mathbb{K}$ we define *graded quasi-tori* over $A = \mathbb{F}_1$ or $A = \mathbb{K}$ as group objects G in the category of affine graded schemes over A for which the degree map is bijective on the set

$G(\mathcal{O}(H))$ of homogeneous group-like elements of $\mathcal{O}(H)$ and the canonical morphism $H \to \operatorname{Spec}_{\mathrm{gr}}(A[G(\mathcal{O}(H))])$ is an isomorphism. A graded torus additionally has a free set of group-like elements, see Section IV.2.3. *Quasi-toric graded schemes over* A are then degree-preserving open embeddings of quasi-tori of finite type over A into graded schemes of finite type over A, morphisms being compatible pairs of morphisms (of group objects) in the category of graded schemes over A. Applying $\mathbb{K}[-]$ resp. $(-)/\mathbb{K}^*$ on the level of structure sheaves then defines mutually inverse equivalences between these two categories of quasi-toric graded schemes. For $A = \mathbb{F}_1$ the forgetful functor is an equivalence onto the category of *integral* $\mathbb{F}_1$-schemes (with dominant morphisms), the essential inverse sending X to the canonical action of $\operatorname{Spec}(\mathbb{F}_1[\mathcal{K}(X)^*])$ on X where both are endowed with the natural $\mathcal{K}(X)^*$-grading. Here, X is integral if each $\mathcal{O}_X(U) \setminus 0$ is cancellative. For $A = \mathbb{K}$ the equivalence of graded schemes and quasi-torus actions induces an equivalence of quasi-toric graded schemes with *quasi-toric prevarieties* over $\mathbb{K}$, which are invariant open embeddings of form $H \circlearrowright H \to H \circlearrowright Z$. Theorem VII.1.1.12 formulates the resulting equivalence of integral $\mathbb{F}_1$-schemes of finite type and quasi-toric prevarieties over $\mathbb{K}$, thus extending a result from [**14**]. Intuitively, the $\mathbb{F}_1$-scheme corresponding to a quasi-toric prevariety may be viewed as its orbit space.

A scheme X of finite type over $\mathbb{F}_1$ is then of Krull type if and only if the corresponding quasi-toric prevariety $H \circlearrowright H \to H \circlearrowright Z$ is of Krull type, meaning $H \circlearrowright Z$ is of Krull type. Normal Toric prevarieties then correspond to Krull schemes X of finite type over $\mathbb{F}_1$ for which $\mathcal{K}(X)^*$ is free. The canonical map $sc\colon Z \to X$ establishes correspondences for sheaves on Ω_X and $\Omega_{Z,H}$. In particular, Weil divisors and H-Weil divisors are in correspondence, and divisorial $\mathcal{O}_{Z,H}$-modules are naturally of type $\mathbb{K}[sc^{-1}\mathcal{O}_X(D)]$ with grading group $\mathcal{K}(X)^*$, while divisorial $\mathcal{O}_X$-modules are of type $sc_*\mathcal{O}_{Z,H}(D)^{\mathrm{hom}}/\mathbb{K}^*$ with the grading being forgotten. Moreover, each Cox sheaf on $H \circlearrowright Z$ is of the form $\mathbb{K}[sc^{-1}\mathcal{R}]$ with grading group $\mathcal{R}_X^*$ for some Cox sheaf $\mathcal{R}$ on Ω_X and conversely, each Cox sheaf on X is of type $sc_*\mathcal{S}^{\mathrm{hom}}/\mathbb{K}^*$ with grading group $\mathrm{Cl}_H(Z)$ for some Cox sheaf $\mathcal{S}$ on $\Omega_{Z,H}$, see Proposition VII.1.3.3. Geometrically, we have induced equivalences between graded characteristic spaces over X and toric characteristic spaces over $H \circlearrowright H \to H \circlearrowright Z$, see Proposition VII.1.3.7.

A Krull scheme X of finite type over $\mathbb{F}_1$ with $\mathcal{K}(X)^*$ free may be viewed as a combinatorial object because the specialisation preorder and the sets of non-trivial valuations at each point determine X, see Section V.3.4. More generally, for a graded scheme X over $\mathbb{F}_1$ or $\mathbb{Z}$ the restriction of $\mathcal{O}_X$ to the set $\mathcal{B}_X$ of non-empty affine open sets forms a *schematic cofunctor*. The latter form a category $\mathfrak{J}$ of which graded algebras over $\mathbb{F}_1$ or $\mathbb{Z}$ are a subcategory, and $\operatorname{Spec}_{\mathrm{gr}}$ extends naturally to a contravariant equivalence of $\mathfrak{J}$ with the category of graded schemes over $\mathbb{F}_1$ or $\mathbb{Z}$, see Section IV.2.3.

This thesis is organized as follows: In Chapter I we develop divisibility theory for monoids and $\mathbb{F}_1$-algebras, i.e. multiplicative monoids with zero. Some results on Krull monoids have been stated without proof in [**9**]. In the interest of completeness we provide the proofs omitted there as far as needed for our purposes, although they are just analoga of the ring-theoretic proofs. Our (non-standard) results on homomorphisms of divisor monoids and essential valuations will be a crucial step for the characterization of characteristic spaces, see Proposition I.2.6.9. With view towards $\mathbb{F}_1$-schemes we also provide basics on prime ideals and their complements, faces. Furthermore, we introduce regular noetherian $\mathbb{F}_1$-algebras and prove the Auslander-Buchsbaum-Theorem and its converse for integral $\mathbb{F}_1$-algebras, i.e. we show that a noetherian integral $\mathbb{F}_1$-algebra is regular if and only if it is factorial, see Proposition I.2.7.6.

Chapter II firstly provides basic constructions in the setting of graded algebra like localizations, (co)limits, monoid algebras as well as the basic theory on graded noetherianity - including a graded version of Hilbert's basis theorem, and on homogeneously prime ideals. Our results on graded ideals under Veronesean inclusions ascertain the good behaviour of good quotients of graded schemes, see Proposition II.1.8.14. Secondly, we develop the divisibility theory of graded rings, from *graded integrality*, the localization behaviour of *graded factoriality*, *graded valuations* on *simply graded rings* to the characterization of *graded rings of Krull type* in terms of their monoid of *graded divisors*, finally treating *graded normality*. As a class of examples we treat *natural divisorial algebras* in Section II.2.6 which are motivated by divisorial $\mathcal{O}_X$-algebras. We also show that for a $K \oplus F$-graded ring R, where F is free, the discussed properties are well-behaved under coarsening to the induced K-grading. In particular, R is of Krull type with respect to the coarsened K-grading if and only if it is so with respect to the $K \oplus F$-grading, and the class groups then coincide, see Theorem II.2.5.15. This gives a graded version of Gauß's Theorem. Parts of these results, in addition to the result on graded factoriality and localization, were published by the author in [**5**]. All concepts are studied with respect to their behaviour under graded homomorphisms which restrict to isomorphisms on homogeneous components (CBEs).

Chapter III gives generalities on the continuity behaviour of sheaves to an arbitrary category as well as basics like the sheafification construction for presheaves to the various categories of graded objects. Moreover, we give first results on sheaves of Krull type and the behaviour of graded (pre)sheaves under homomorphisms which restrict to isomorphisms of homogeneous components (CBEs). In Chapter IV we develop the theory of graded schemes over $A = \mathbb{Z}$ and $A = \mathbb{F}_1$, their quasi-coherent modules and algebras and graded relative spectra. Moreover, we introduce (Veronesean) good quotients as well as canonical actions by graded quasi-tori. While graded schemes over A behave analogously to schemes over A in many ways, it is noteworthy that the structure sheaf $\mathcal{O}_X$ of a graded scheme X will not generally be a sheaf with respect to the category of sets but only with respect to the category of $gr(\mathcal{O}_X)$-graded rings, see Example IV.1.1.8.

Chapter V introduces graded schemes of Krull type, their Weil divisors and class groups, as well as divisorial algebras, Cox sheaves and graded characteristic spaces, and gives proofs of our stated results surrounding the two latter's characterization. We also differentiate between finite Weil divisors and locally finite Weil divisors on graded schemes which are locally of Krull type, and consequently consider quasi-Cox sheaves as well as proper Cox sheaves. Parts of the results of this Chapter were published by the author in [**6**]. These results are then translated into the realm of quasi-torus actions in Chapter VI. After the proof of equivalence of graded schemes and quasi-torus actions in Section VI.2 we treat the connection between stalks of the invariant structure sheaf $\mathcal{O}_{Z,H}$ and geneneric H-isotropy groups in Section VI.3.

In Chapter VII we treat the connection between $\mathbb{F}_1$-schemes and quasi-toric prevarieties, as well as very neat embeddings into toric characteristic spaces. Working in full generality, we show that an *action* of Krull type allows a very neat embedding into a toric characteristic space if and only if it has a finitely generated Cox ring. Moreover, a Veronesean algebra $R_G \subseteq R$ with R finitely generated over $\mathbb{K}$ is shown to be Cox algebra $\mathcal{O}(Z) \subseteq \mathcal{R}(Z)$ of some action $H \circlearrowright Z$ if and only if $R^{\mathrm{hom}} \setminus 0$ is factorial, we have $(R^{\mathrm{hom}})^* = (R_G^{\mathrm{hom}})^*$ and each homogeneously prime divisor $\mathfrak{p}$ of R satisfies $\deg((R_{\mathfrak{p}}^{\mathrm{hom}})^*) + G = gr(R)$. Furthermore, we generalize statements from [**17**] and show that in the setting of a very neat embedding, ambient and embedded space have the same formulae for (semi-)ample and moving cones of classes of invariant divisors.

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Conventions

By $\mathbb{N}_0$ and $\mathbb{N}$ we denote the natural numbers with and without zero, respectively. $\mathbb{F}_1$ denotes the multiplicative monoid $\{0, 1\}$. $\mathbb{K}$ denotes an algebraically closed field of characteristic 0, e.g. the field $\mathbb{C}$ of complex numbers. All semigroups, rings, modules etc. are assumed to be *commutative*. Regarding concepts from graded algebra which are analoga of classical algebraic concepts we distinguish the former from the latter either by denoting the grading group as a prefix in terminology or an index in notation, leading to the set $\operatorname{Spec}_K(R)$ of *K-prime* ideals of R, or by using modifiers like “graded” or “homogeneous(ly)” in terminology and an index “gr” in notation, as in *homogeneously prime ideal*, *graded spectrum*, $\operatorname{Spec}_{\mathrm{gr}}(R)$. Similarily, in the context of quasi-torus actions $H \circlearrowright Z$ we use modifiers like “(H-)invariant(ly)” or an index H to distinguish a concept that is formulated with respect to the topology $\Omega_{Z,H}$ of H-invariant open subsets or $\mathcal{O}_{Z|\Omega_{Z,H}}$ from its classical model. We also employ some conventions resp. general concepts from category theory as featured below.

With respect to an arbitrary category $\mathfrak{C}$ we speak of *limits* and *colimits* in the following sense. Given a (contravariant) functor $D\colon I \to \mathfrak{C}$ (also called a (co-)diagram) we may consider the categories of natural transformations (i.e. morphisms of (contravariant) functors) from constant I-shaped (co-)diagrams to D and vice versa. A limit of D is a terminal object in the former category, while a colimit is an initial object in the latter category. Products are limits, while coproducts, stalks and gluing along open subschemes are examples of colimits. The $\mathfrak{C}$-sheaf property may be defined in terms of $\mathfrak{C}$-limits. This way is neccessary e.g. for categories of graded rings. Existence of (co-)limits often depends on I being small, which means that the class of I-objects form a set and for two fixed I-objects the class of I-morphisms from one to the other also form a set. In cases where we provide an explicit construction we call the result *the* limit resp. colimit and use the notation $\lim_D$ or $\lim_{i \in I} D(i)$ resp. colim_D or $\operatorname{colim}_{i \in I} D(i)$.

Given a tensor category $\mathfrak{C}$ with a binary functor $\otimes_{\mathfrak{C}}\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$ and 1-object $1_{\mathfrak{C}}$, e.g. a category with binary products and terminal object or the category of graded $\mathbb{Z}$-modules with tensor product $\otimes_{\mathbb{Z}}$ and 1-object $\mathbb{Z}$, one may define the category of *$\mathfrak{C}$-monoid/-group objects* and that of *$\mathfrak{C}$-comonoid/-cogroup objects*. A monoid (resp. group) object is a $\mathfrak{C}$-object G together with a multiplication morphism $G \otimes G \to G$, a unit morphism $1 \to G$ (and an inversion morphism $G \to G$) such that these *structure morphisms* satisfy the diagram conditions that define monoids and groups in terms of multiplication map, inclusion of the unit element (and inversion map). Comonoid (resp. cogroup) objects are defined analogously with all arrows reversed. Morphisms are $\mathfrak{C}$-morphisms which respect structure morphisms. With respect to $\otimes_{\mathbb{Z}}$, graded rings are monoid objects in the category of graded $\mathbb{Z}$-modules and graded Hopf algebras over $\mathbb{Z}$ are cogroup objects in the category of graded rings. Affine algebraic groups resp. graded group schemes are group objects in the categories of affine varieties resp. graded schemes.

Likewise, one may consider *(co-)actions* of monoid/group objects on $\mathfrak{C}$-objects. These are given by a morphism $G \otimes X \to X$ (resp. $X \to G \otimes X$) which satisfies the diagram-theoretic conditions in which the definition of group actions may be phrased (resp. the dual conditions). It is then usual to call X a G-(co-)module, or to write $G \circlearrowright X$ for the action. Morphisms of (co)actions are pair consisting of a morphism of (co-)monoid/group objects and a $\mathfrak{C}$-morphism such that the canonical diagram conditions are satisfied. Examples include homomorphisms induced by scalar multiplications of graded rings R on graded $\mathbb{Z}$-modules M turning the latter into graded R-modules, morphical actions of an algebraic torus $\mathbb{T}$ on a prevariety Z, coactions of graded Hopf algebras on graded algebras.

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CHAPTER I

$\mathbb{F}_1$-algebras, monoids and their divisibility theory

This chapter provides the algebraic preparation for the theory of $\mathbb{F}_1$-schemes, in particular those of finite and Krull type. To this end, the first part features a version of Hilbert's basis theorem, a discussion of faces of monoids (and $\mathbb{F}_1$-algebras) and of their complements viz. prime ideals. In the second part we develop divisibility for monoids (and $\mathbb{F}_1$-algebras). We characterize Krull monoids in terms of their divisor monoid and define homomorphisms of divisor monoids induced by homomorphisms of Krull monoids. This lays the foundation for the definition of Weil divisors of $\mathbb{F}_1$-schemes and their pullbacks under dominant morphisms. The properties of divisor monoid homomorphisms featured in Proposition I.2.6.9 will be crucial for our results on Cox sheaves and characteristic spaces. We close the chapter with the observation that under presence of integrality the Auslander-Buchsbaum-Theorem and its converse hold, i.e. factoriality is equivalent to regularity. This is reflected geometrically in the fact that all smooth affine toric varieties are of type $\mathbb{K}^n \times (\mathbb{K}^*)^m$. All monoids and $\mathbb{F}_1$-algebras will be assumed to be *abelian* without further notice.

I.1. Modules and faces of monoids and $\mathbb{F}_1$-algebras

After the introduction of the basic concepts in Section I.1.1 we prove Hilbert's basis theorem for monoids in Section I.1.2. As a preparation for the discussion of $\mathbb{F}_1$-schemes of finite type we treat faces of monoids in Section I.1.3 and their complements, i.e. prime ideals, in Section I.1.4.

I.1.1. Monoids, $\mathbb{F}_1$-algebras and their modules. By a monoid we mean an associative semigroup with neutral element. If no other specification is given monoids will be written multiplicatively and the neutral element of a monoid M is denoted 1_M. Homomorphisms of monoids are by definition required to preserve neutral elements.

Definition I.1.1.1. An *absorbing* or *zero element* of a monoid A is an element 0_A such that $0_A a = 0_A$ holds for every $a \in A$. In the category of monoids with absorbing element a *morphism* is a monoid homomorphism that preserves absorbing elements.

Remark I.1.1.2. $\mathbb{F}_1 := \{0, 1\}$ is the initial object of the category of monoids with zero, which therefore in the interest of brevity is termed the category of $\mathbb{F}_1$-algebras, denoted $\mathbf{Alg}_{\mathbb{F}_1}$. Its terminal object is $\{0\}$.

Remark I.1.1.3. The category of $\mathbb{F}_1$-algebras is a full subcategory of the category of *sesquiads*, whose objects are pairs (M, R) consisting of a commutative ring R such that the underlying $\mathbb{F}_1$-algebra $(R, 0_R, 1_R)$ contains M as a $\mathbb{F}_1$-subalgebra and R is generated as a ring by the subset M. Morphisms of sesquiads are ring homomorphisms that restrict to $\mathbb{F}_1$-algebra homomorphisms. Other subcategories of sesquiads include rings.

Remark I.1.1.4. The forgetful functor $\mathfrak{i}\colon \mathbf{PtSet} \to \mathbf{Set}$ from pointed sets to sets is right adjoint to the functor sending S to $S \sqcup \{0\}$ due to Lemma A.0.0.2.

Remark I.1.1.5. The colimit of a small diagram $D\colon I \to \mathbf{PtSet}, i \mapsto (X_i, 0_i)$ of pointed sets is the colimit of the corresponding diagram of sets modulo the equivalence relation generated by $0_i \sim 0_j$ for $i, j \in I$. Another way to think of the coproduct in **PtSet** is as the set of all $x \in \prod_{i\in I} X_i$ whose coordinates are zero for all, or all but one $i \in I$.

The limit is the **Set**-limit, i.e. the subset of those $(x_i)_i \in \prod_i X_i$ such that $D(\alpha)(x_i) = x_j$ holds for each I-morphism $\alpha\colon i \to j$, together with the distinguished point $(0_{X_i})_i$.

For a fixed object A of **Mon** or $\mathbf{Alg}_{\mathbb{F}_1}$ the category of objects under A is denoted $\mathbf{Alg}_A$, its objects are called *A-algebras* or *algebras over A*. For a given A-algebra B the category $\mathbf{SubAlg}_A(B)$ of *A-subalgebras of B* has as its objects A-algebras C which are subsets of B such that the inclusion $C \subseteq B$ is an morphism of A-algebras, and as its morphisms inclusion maps.

The notion of modules over monoids was introduced in [**15**]. It is obtained from the theory of modules over rings by eliminating every occurrence of addition, i.e. one replaces rings with monoids and abelian groups with sets. Thus, modules over monoids are actions of monoids on sets. From now on, A denotes a fixed monoid/$\mathbb{F}_1$-algebra.

Definition I.1.1.6. A module over A is a set resp. pointed set M (with *stationary element* 0_M) together with an associative action $\mu_M\colon A \times M \to M$ such that 1_A acts identically on each element (and the product of 0_A with each element is the A-invariant element 0_M).

A morphism of M-modules is an equivariant map (which preserves stationary elements). A *submodule* is a module supported on a subset such that the inclusion map is a morphism.

Example I.1.1.7. The operation of a monoid/$\mathbb{F}_1$-algebra M turns M into an M-module. M-submodules of M are called *ideals*.

Remark I.1.1.8. Each pointed set carries a canonical (and unique) $\mathbb{F}_1$-action. Thus, pointed sets are equivalent to $\mathbb{F}_1$-modules.

Proposition I.1.1.9. *In the category of modules over a monoid A products and coproducts are given by Cartesian products resp. disjoint unions. In the category of A-submodules of an A-module M products are intersections and coproducts are unions.*

Construction I.1.1.10. Let M be an A-module. The A-submodule *generated by* a subset $N \subseteq M$ is the intersection over all A-submodules of M which contain N, i.e. the union over all Mf for $f \in N$.

Remark I.1.1.11. Under morphisms of A-algebras/-modules preimages of A-subalgebras/-submodules are again A-subalgebras/-submodules. Moreover, images of A-subalgebras are A-subalgebras. Under surjective morphisms, images of A-submodules are A-submodules, too.

Remark I.1.1.12. Under a monoid homomorphism $\phi\colon A \to B$ the preimage of a subgroup G is a subgroup if and only if $\phi^{-1}(G) \subseteq A^*$.

Definition I.1.1.13. Let B be an A-module (resp. an A-algebra). A *congruence* on B is an equivalence relation $\sim$ on B such that we have $ab \sim ab'$ (and $bc \sim b'c'$) for all $a \in A$ and $b, b', c, c' \in B$ with $b \sim b'$ (and $c \sim c'$).

Remark I.1.1.14. For a congruence $\sim$ on an A-algebra/-module B the set $B/\sim$ of congruence classes is again an A-algebra/-module and the canonical map $B \to B/\sim$ is a morphism. Conversely, for a morphism $\phi\colon B \to C$ the relation

$\sim$ defined by all pairs (b, b') with $\phi(b) = \phi(b')$ is a congruence, called the *kernel relation* of ϕ.

EXAMPLE I.1.1.15. Let B be an A-module. For an A-submodule C of B the equivalence relation $\sim_C := C \times C \cup \bigcup_{b \in B \setminus C} (b, b)$ is a congruence and the quotient $B/\sim_C$ is also denoted B/C, see citation[**12**]. Note that the class $C \times C$ is a stationary element of B/C even if A is only a monoid.

The following is known as the homomorphism theorem.

PROPOSITION I.1.1.16. *Let $\sim$ be a congruence on an A-algebra/-module B. Then $B \to B/\sim$ is the initial object in the category of B-algebras whose kernel relation contains $\sim$.*

EXAMPLE I.1.1.17. Let B be an A-algebra. Then each subalgebra $N \subseteq B$ defines a congruence $\sim_N$ where $b \sim_N c$ if and only if $bN = cN$. Note that $bN = Nb$ holds due to commutativity of B. Congruence classes are N-orbits (minus the zero point) if and only if N is simple.

CONSTRUCTION I.1.1.18. Let M be a module over the monoid/$\mathbb{F}_1$-algebra A and let $N \subseteq A$ be a submonoid. The *localization* $N^{-1}M$ of M by N is the set $M \times N$ modulo the equivalence relation given by all pairs $((m, n), (m', n'))$ for which there exists $k \in N$ with $kmn' = km'n$. Its elements are denoted m/n. If A is an $\mathbb{F}_1$-algebra then $0_{N^{-1}M} := 0_M/1_A$ is a stationary element. For $M = A$ one obtains the structure of a monoid/$\mathbb{F}_1$-algebra via $(m/n)(m'/n') := (mm'/nn')$ and $1_{N^{-1}M} := 1_A/1_A$. For arbitrary M one thus has an A-module structure defined by $a(m/n) = (am)/n$, and a $N^{-1}A$-module structure via $(a/k) \cdot (m/n) = (am)/(kn)$. The canonical map $\imath_M : M \to N^{-1}M$ is a map of M-modules.

CONSTRUCTION I.1.1.19. Let M be a module over the monoid/$\mathbb{F}_1$-algebra A and let $f \in A$ be an element. The *localization by f* is then the localization $M_f := N^{-1}M$ by the submonoid $N \subseteq A$ generated by f.

PROPOSITION I.1.1.20. *Let $\phi : M \to M'$ be a homomorphism of A-modules and let $N \subseteq A$ be a submonoid. Then the map*

$$N^{-1}\phi : N^{-1}M \longrightarrow N^{-1}M', \quad m/n \longmapsto \phi(m)/n$$

is the unique homomorphism of $N^{-1}A$-modules such that $N^{-1}\phi \circ \imath_M = \imath_{M'} \circ \phi$.

PROPOSITION I.1.1.21. *Let $\phi : M \to M'$ be a homomorphism of monoids and let $N \subseteq M$ be a submonoid with $\phi(N) \subseteq M'^*$. Then there is an induced homomorphism of monoids*

$$N^{-1}\phi : N^{-1}M \longrightarrow M', \quad m/n \longmapsto \phi(n)^{-1}\phi(m).$$

REMARK I.1.1.22. If $\sim$ is a congruence on an A-module/-algebra B and $N \subseteq A$ is a submonoid then there is an induced congruence $N^{-1} \sim$ on $N^{-1}B$ generated by all pairs $(b/n, b'/n')$ for which there exists $s \in N$ with $sb'n \sim sbn'$. Moreover, $N^{-1} \sim$ is the kernel relation of the canonical epimorphism of $N^{-1}A$-modules/-algebras $N^{-1}B \to N^{-1}(B/\sim)$.

If $\sim$ was induced by a submodule $\mathfrak{b}$ or a subalgebra C then $N^{-1} \sim$ is induced by $N^{-1}\mathfrak{b}$ resp. $N^{-1}C$.

REMARK I.1.1.23. Let B be a monoid/$\mathbb{F}_1$-algebra and let A be a submonoid/$\mathbb{F}_1$-subalgebra. For an A-submodule $\mathfrak{b}$ of B the intersection $N \cap \mathfrak{b}$ is empty if and only if $N^{-1}\mathfrak{b}$ is proper.

I.1.2. Noetherianity and Hilbert's basis theorem. This section serves to prove a version of Hilbert's basis theorem which in turn is used to show noetherianity of finitely generated abelian monoids (and $\mathbb{F}_1$-algebras). Again, A denotes a fixed monoid/$\mathbb{F}_1$-algebra.

Definition I.1.2.1. Let M be an A-module.

(i) An A-submodule $\mathfrak{m}$ of an A-module M is *finitely generated* if $\mathfrak{m} = AS$ holds with a finite subset $S \subseteq M$. If S is a singleton, then $\mathfrak{m}$ is called *principal*.
(ii) M is *noetherian* if all its submodules are finitely generated. M is *1-noetherian* if each ascending chain of principal submodules becomes stationary.
(iii) A is called *(1-)noetherian* if it is (1-)noetherian as a module over itself.

Remark I.1.2.2. For an A-module M the following are equivalent:

(i) M is noetherian,
(ii) every non-empty set of ideals of M has maximal elements with respect to inclusion,
(iii) every ascending chain of ideals in M becomes stationary.

Remark I.1.2.3. An A-module M is 1-noetherian if and only if each non-empty set of principal submodules has maximal elements with respect to inclusion.

The next statement is Hilbert's basis theorem for monoids.

Proposition I.1.2.4. *Let M be a noetherian (additive) monoid. Then $M \times \mathbb{N}_0$ is noetherian.*

Proof. Let $\mathfrak{a}$ be an ideal of $M \times \mathbb{N}_0$. For every $m \in \mathbb{N}_0$ let $\mathfrak{b}_m$ be the set of all elements $b \in M$ such that $(b, m) \in \mathfrak{a}$. $\mathfrak{b}_m$ is an ideal since whenever $r \in M$ and $b \in \mathfrak{b}_m$ we have $(r+b, m) = (r, 0) + (b, m) \in \mathfrak{b}_m$, i.e. $r + b \in \mathfrak{b}_m$. Moreover, $\mathfrak{b}_m \subseteq \mathfrak{b}_{m+1}$ because $(b, m) \in \mathfrak{a}$ implies $(b, m+1) = (0, 1)+(b, m) \in \mathfrak{a}$. By noetherianity the chain defined by all $\mathfrak{b}_m$ eventually terminates, i.e. there exists k such that $\mathfrak{b}_k$ contains all other $\mathfrak{b}_m$. Moreover, for $i \leq k$ we have $\mathfrak{b}_i = M + c_{i,1} \cup \ldots \cup M + c_{i,d_i}$ with some elements $c_{i,j} \in \mathfrak{b}_i$. We claim that $\mathfrak{a}$ is the (finite) union over all $M \times \mathbb{N}_0 + (c_{i,j}, i)$ where $i \leq k$ and $j \leq d_i$. Let $(a, m) \in \mathfrak{a}$. Then $a \in \mathfrak{b}_m$. If $m \leq k$ then there exists $j \leq d_m$ with $a = w + c_{m,j} \in M + c_{m,j}$ for some $w \in M$. Thus, $(a, m) = (w, 0) + (c_{m,j}, m)$ has the desired form. If otherwise $m > k$ then in particular $a \in \mathfrak{b}_k$, hence there exist $j \leq d_k$ and $w \in M$ with $a = w + c_{k,j}$. Therefore, $(a, m) = (w, m-k) + (c_{k,j}, k)$ has the desired form. □

Corollary I.1.2.5. *If M is noetherian, then so is $M \times \mathbb{N}_0^n$ for every $n \geq 1$.*

Remark I.1.2.6. Let $\phi\colon M \to N$ be a surjective homomorphism of monoids. If M is noetherian then so is N. Indeed, for an ideal $\mathfrak{a}$ of N the ideal $\pi^{-1}(\mathfrak{a})$ is finitely generated and hence the images of the generators generate $\mathfrak{a} = \pi(\pi^{-1}(\mathfrak{a}))$.

Corollary I.1.2.7. *Every finitely generated algebra M over a noetherian monoid A is noetherian.*

Proof. If $w_1, \ldots, w_d$ generate M as an A-algebra then we obtain an epimorphism $\pi\colon A \times \mathbb{N}_0^d \to M$ of A-algebras by sending (a, k) to $a + \sum_{i=1}^{d} k_i w_i$. □

I.1.3. Faces of modules over semirings. Here, we treat faces of modules over semirings, in particular, of monoids with focus on the finitely generated case. To keep notation simple we write monoids additively throughout this section. Depending on the context $\mathbb{N}_0$ denotes both the additive monoid $(\mathbb{N}_0, +, 0_{\mathbb{N}})$ and the semiring $(\mathbb{N}_0, +, \cdot, 0_{\mathbb{N}}, 1_{\mathbb{N}})$. Our discussion of the duality operation will be essential for the interpretation of $\mathbb{F}_1$-schemes of finite type in terms of combinatorics in Section V.3.4.

Definition I.1.3.1. A *module* M over a semiring S consists of a commutative (additive) monoid M on whose underlying pointed set the multiplicative $\mathbb{F}_1$-algebra underlying S defines a module structure which satisfies $a(v+w) = av + aw$ and $(a+b)v = av+bv$ for all $a, b \in S$ and $v, w \in M$. A *homomorphism* of modules over S is a homomorphism of monoids which is compatible with the scalar multiplications.

Example I.1.3.2. Modules over $\mathbb{Q}_{\geq 0}$ are rational *cones.*

Remark I.1.3.3. The category of commutative monoids is equivalent to the category of modules over the semiring $\mathbb{N}_0$ of non-negative integers in the same way that abelian groups are equivalent to $\mathbb{Z}$-modules.

Construction I.1.3.4. Let S be a semiring and let $T \subseteq S$ be a multiplicative submonoid. Then the $\mathbb{F}_1$-algebra $T^{-1}S$ constructed in Construction I.1.1.18 becomes a semiring via $a/t + b/u := (ua + tb)/tu$ for $a, b \in S$ and $t, u \in T$. For an S-module M the localization $T^{-1}M$ as constructed in Construction I.1.1.18 becomes an $T^{-1}S$-module via $v/t + w/u := (uv + tw)/tu$ for $v, w \in M$ and $t, u \in T$.

Remark I.1.3.5. In the above setting, the localization map $S \to T^{-1}S$ is a homomorphism of semirings, i.e. it respects both the additive and the multiplicative monoid structures. The localization map $\imath_T \colon M \to T^{-1}M$ then becomes a homomorphism of S-modules.

Definition I.1.3.6. A *face* of an S-module M is an S-submodule τ such that $u + v \in \tau$ implies $u, v \in \tau$ for all $u, v \in M$. To indicate that τ is a face of M we write $\tau \preceq M$. The set of all faces of M is denoted $\mathrm{faces}(M)$.

Remark I.1.3.7. Intersections of faces of a monoid M are again faces of M. Also, faces of faces of M are faces of M. Moreover, preimages of faces under homomorphisms are faces. Furthermore, if M is generated as an S-submodule by $\{v_i\}_{i \in I}$ then each face is generated as an S-submodule by $\{v_i\}_{i \in J}$ for some $J \subseteq I$.

Definition I.1.3.8. The *face generated by* a subset $A \subseteq M$ of an S-module M is the intersection $\mathrm{face}(A)$ over all faces of M which contain A. If $A = \{a\}$ is a singleton then $\mathrm{face}(a) := \mathrm{face}(A)$ is called a *principal face* of M.

Definition I.1.3.9. For a face τ of a monoid/$\mathbb{F}_1$-algebra M the *relative interior* is the set τ° of all elements of τ which do not belong to a proper face of τ.

Remark I.1.3.10. Let V be a finite-dimensional $\mathbb{Q}$-vector space. Let τ be a cone in V generated by $v_1, \dots, v_d \in V$. Then with respect to the standard metric topology $\tau^\circ = \sum_{i=1}^d \mathbb{Q}_{>0} v_i$ is the interior of τ in the vector space $\tau - \tau$ generated by τ. This standard fact can be found e.g. in [**11**].

Remark I.1.3.11. The relative interior of an S-module τ is an ideal of the underlying monoid τ. Moreover, a face η of τ is principal if and only if its relative interior η° is non-empty, in which case each element of η° generates η.

Definition I.1.3.12. A face of a monoid/$\mathbb{F}_1$-algebra M is a face of the associated module over the semiring $\mathbb{N}_0$. M is called *pointed* if $\{0_M\}$ is a face of M.

Remark I.1.3.13. Let M be a monoid, i.e. a module over $\mathbb{N}_0$. Then $\mathbb{N}^{-1}M$ is a cone and the maps $\alpha \colon \tau \mapsto \mathbb{N}^{-1}\tau$ and $\beta \colon \eta \mapsto \imath_{\mathbb{N}}^{-1}(\eta)$ form mutually inverse inclusion preserving bijections between $\mathrm{faces}(M)$ and $\mathrm{faces}(\mathbb{N}^{-1}M)$. In particular, we have $M^\circ = \imath_{\mathbb{N}}^{-1}((\mathbb{N}^{-1}M)^\circ)$ and $(\mathbb{N}^{-1}M)^\circ = \mathbb{N}^{-1}(M^\circ)$.

Indeed, for $v, v' \in M, n, n' \in \mathbb{N}$ with $(v/n)(v'/n') \in \mathbb{N}^{-1}\tau$, i.e. $nn'vv' \in \tau$ we have $v, v' \in \tau$, meaning that $\mathbb{N}^{-1}\tau$ is a face. For $\beta(\alpha(\tau)) = \tau$ let $v \in \phi^{-1}(\alpha(\tau))$.

Then there exist $w \in \tau$ and $n \in \mathbb{N}$ with $v/1 = w/n$, i.e. there exists $k \in \mathbb{N}$ with $knv = kw \in \tau$ and hence $v \in \tau$. Moreover, since η is invariant under $\mathbb{Q}_{>0}$ we have $\alpha(\beta(\eta)) = \eta$.

Definition I.1.3.14. Let $M \subseteq N$ be an inclusion of semigroups. The *saturation of M in N* is the set $\mathrm{sat}(M, N)$ of all $v \in N$ for which there exists $k \in \mathbb{N}$ such that $kv \in M$. M is *saturated in* N if $M = \mathrm{sat}(M, N)$. If in the last case it is clear which ambient semigroup N is meant we will also say that M is *saturated*.

Remark I.1.3.15. The saturation of M in N is a saturated subsemigroup of N. Moreover, if N is a monoid then $\mathrm{sat}(M, N)$ is a submonoid if and only if M is a submonoid. Moreover, for a subsemigroup $L \subseteq M$ we have $\mathrm{sat}(L, N) \subseteq \mathrm{sat}(M, N)$, $\mathrm{sat}(L, M) \subseteq \mathrm{sat}(L, N)$ and $\mathrm{sat}(L, \mathrm{sat}(M, N)) = \mathrm{sat}(L, N) = \mathrm{sat}(\mathrm{sat}(L, M), N)$. Indeed, if for $v \in N$ there exists $k \in \mathbb{N}$ such that $kv \in \mathrm{sat}(L, M)$ then there exists $l \in \mathbb{N}$ with $lkv \in L \subseteq M$ and in particular, we have $v \in \mathrm{sat}(M, N)$. Consequently, if L is saturated in M and M is saturated in N then L is saturated in N.

Example I.1.3.16. Faces of monoids are in particular saturated submonoids.

Remark I.1.3.17. Let $\psi\colon K \to L$ be a homomorphism of monoids. Then the preimage of the saturation of a submonoid $N \subseteq L$ is the saturation of $\psi^{-1}(N)$.

Lemma I.1.3.18. *Let M be an abelian monoid, i.e. a module over the semiring $\mathbb{N}_0$. Then the following hold:*

(i) *The saturation of a submonoid τ in M is $\imath_{\mathbb{N}}^{-1}(\mathbb{N}^{-1}\tau)$.*
(ii) *$\tau \mapsto \mathbb{N}^{-1}\tau$ and $\sigma \mapsto \imath_{\mathbb{N}}^{-1}(\sigma)$ define mutually inverse bijections between saturated $\mathbb{N}_0$-submodules of M and $\mathbb{Q}_{\geq 0}$-submodules of the $\mathbb{Q}_{\geq 0}$-module $\mathbb{N}^{-1}M$. Both assignments commute with arbitrary intersections.*
(iii) *If M is a finitely generated abelian group then a saturated submonoid τ of M is finitely generated over $\mathbb{N}_0$ if and only if $\mathbb{N}^{-1}\tau$ is finitely generated over $\mathbb{Q}_{\geq 0}$.*

Proof. In (i) let $w \in M$ and $n \in \mathbb{N}$ with $nw \in \tau$. Then $nw/1 \in \mathbb{N}^{-1}\tau$ and hence $w/1 \in \mathbb{N}^{-1}\tau$, i.e. $w \in \imath_{\mathbb{N}}^{-1}(\mathbb{N}^{-1}\tau)$. Conversely, if $w/1 = v/n$ holds with $v \in \tau$ and $n \in \mathbb{N}$ then there exists $k \in \mathbb{N}$ with $knw = kv \in \tau$, i.e. w belongs to the saturation of τ.

In (ii) note that if $nw/1 \in \sigma$ holds with $n \in \mathbb{N}$ then $w/1 \in \sigma$, which shows $\imath_{\mathbb{N}}^{-1}(\sigma)$ is saturated. Lastly, if for a family $\tau_i, i \in I$ of saturated $\mathbb{N}_0$-submodules $v \in M$ satisfies $v/k = w_i/k_i$ with $w_i \in \tau_i$ and $k, k_i \in \mathbb{N}$ for each i then there exist $l_i \in \mathbb{N}$ with $l_i k_i v = l_i k w_i \in \tau_i$ and saturatedness gives $v \in \tau_i$, i.e. $v/k \in \mathbb{N}^{-1} \bigcap_i \tau_i$.

For (iii) let $\mathbb{N}^{-1}\tau$ be finitely generated by $w_1/1, \ldots, w_d/1$ where $w_i \in \tau$. Let P denote the parallelepiped spanned by all $w_i/1$, i.e. the set of all linear combinations $\sum_{i=1}^d \lambda_i(w_i/1)$ where $0 \leq \lambda_i \leq 1$. Then $\imath_{\mathbb{N}}^{-1}(P)$ has the form $\bigcup_{j=1}^n v_j + t(M)$ with respect to certain $v_j \in M$ and the torsion module $t(M)$ of M. τ is now generated by $w_1, \ldots, w_d, v_1, \ldots, v_n$ together with finitely many generators of $t(M)$. Indeed, let $w \in \tau$ with $w/1 = \sum_{i=1}^d \alpha_i(w_i/1)$ where all $\alpha_i \in \mathbb{Q}_{\geq 0}$. Then upto $t(M)$ the element $w - \sum_{i=1}^d \lfloor \alpha_i \rfloor w_i \in \tau$ is one of the v_j. □

Proposition I.1.3.19. *Let K be a finitely generated abelian group, let M be a finitely generated submonoid and let $v_1, \ldots, v_n \in M$ be elements such that M is the saturation of $\sum_{i=1}^n \mathbb{N}_0 v_i$ in M. Then the following hold:*

(i) *M° is the saturation of $\sum_{i=1}^n \mathbb{N} v_i$ in M.*
(ii) *For a homomorphism $\phi\colon K \to L$ of finitely generated abelian groups $\phi(M)$ is the saturation of $\phi(\sum_i \mathbb{N}_0 v_i)$ in $\phi(M)$ and $\phi(M)^\circ$ is the saturation of $\phi(M^\circ)$ in $\phi(M)$.*

Proof. For (i) we apply Remark I.1.3.10 to $\mathbb{N}^{-1}M = \sum_i \mathbb{Q}_{\geq 0} v_i/1$ and use Remark I.1.3.13 and Lemma I.1.3.18 to obtain the assertion. In (ii) first note that for $\phi(v)$ where $kv = \sum_i k_i v_i$ with $k \in \mathbb{N}$ and $k_i \in \mathbb{N}_0$ we have $k\phi(v) = \sum_i k_i \phi(v_i)$. Secondly, if $w \in \phi(M)^\circ$ then $kw = \sum_{i=1}^n k_i \phi(v_i) = \phi(\sum_{i=1}^n k_i v_i)$ holds with $k, k_i \in \mathbb{N}$ and hence $kw \in \phi(M^\circ)$. Conversely, if for $w \in \phi(M)$ there exist $v \in M$ and $k, k_i \in \mathbb{N}$ with $kv = \sum_{i=1}^n k_i v_i$ as well as $l \in \mathbb{N}$ such that $lw = \phi(v)$ then $klw = \phi(kv) = \sum_{i=1}^n kk_i \phi(v_i)$, meaning that $w \in \phi(M)^\circ$. □

Proposition I.1.3.20. *Let $\phi\colon M \to N$ be a homomorphism of monoids. Then the inclusion-preserving maps $\alpha\colon \tau \mapsto \mathrm{face}(\phi(\tau))$ and $\beta\colon \eta \mapsto \phi^{-1}(\eta)$ are mutually inverse bijections in the following cases:*

(i) *ϕ is the inclusion of a submodule whose saturation is N. $\alpha(\tau)$ is then the saturation of τ.*
(ii) *ϕ is an epimorphism such that $\phi(v) = \phi(v')$ implies that there exists $a \in \mathbb{N}$ with $av = av'$. In this case, $\alpha(\tau) = \phi(\tau)$ holds for every face of M.*

Proof. In (i) note that the saturation τ' of $\tau \in \mathrm{faces}(M)$ is contained in $\mathrm{face}(\tau) \in \mathrm{faces}(N)$. To see that τ' is a face first observe that for $v \in \tau'$ and $a \in \mathbb{N}$ with $av \in \tau$ and $s \in S$ we have $a(sv) = s(av) \in \tau$, i.e. $sv \in \tau'$. If we have $v, v' \in N$ with $vv' \in \tau'$ then there exists $a \in \mathbb{N}$ with $(av)(av') = a(vv') \in \tau$. Now let $b, b' \in \mathbb{N}$ with $bv, b'v' \in M$. Then $(abb'v)(abb'v') \in \tau$ gives $abb'v, abb'v' \in \tau$ and hence $v, v' \in \tau'$.

For $\beta(\alpha(\tau)) = \tau$, let $v \in \beta(\alpha(\tau))$. Then there exists $k \in \mathbb{N}$ with $kv \in \tau$ and thus $v \in \tau$. The converse follows from $\tau \subseteq \alpha(\tau)$. For a face η of N let $v \in \alpha(\eta \cap M)$. Then there exists $k \in \mathbb{N}$ with $kv \in \eta \cap M$ and we obtain $v \in \eta$. Conversely, if $v \in \eta$ then each $k \in \mathbb{N}$ with $kv \in M$ satisfies $kv \in \eta \cap M$.

In (ii) let τ be a face of M. Let $v, v' \in M$ with $\phi(v)\phi(v') = \phi(w)$ for some $w \in \tau$. Then there exists $a \in \mathbb{N}$ with $a(vv') = aw \in \tau$ which implies $\phi(v), \phi(v') \in \phi(\tau)$. Consequently, we have $\beta(\alpha(\tau)) = \phi^{-1}(\phi(\tau)) = \tau$. □

Consider a finite-dimensional $\mathbb{Q}$-vector space V and set $W := V^* = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$. The canonical isomorphism $V \to W^*$ sends $v \in V$ to the map $\phi_v\colon W \to \mathbb{Q}$ which evaluates $\psi \in W$ at v. For $A \subseteq V$ and $B \subseteq W$ set

$$A^\vee := \{\psi \in W | \psi(A) \subseteq \mathbb{Q}_{\geq 0}\}, \quad B^\vee := \{v \in V | \phi_v(B) \subseteq \mathbb{Q}_{\geq 0}\}$$

as well as $A^\perp := A^\vee \cap (-A)^\vee$ and $B^\perp := B^\vee \cap (-B)^\vee$. For $x \in V$ and $\psi \in W$ we use the notations $x^\vee := \{x\}^\vee$, $x^\perp := \{x\}^\perp$, $\psi^\vee := \{\psi\}^\vee$ and $\psi^\perp := \{\psi\}^\perp$. The following facts on finitely generated cones and dualization are found e.g. in [**11**].

Remark I.1.3.21. In the above setting consider a cones $\tau, \sigma \subseteq V$. Then the following hold:

(i) τ is finitely generated over the semiring $\mathbb{Q}_{\geq 0}$ if and only if it is of the form $\tau = \{\psi_1, \ldots, \psi_d\}^\vee$, and in that case $\tau^\vee$ is also finitely generated.
(ii) If τ and σ are finitely generated over the semiring $\mathbb{Q}_{\geq 0}$ then we have $\tau = (\tau^\vee)^\vee$ as well as $(\tau + \sigma)^\vee = \tau^\vee \cap \sigma^\vee$ and $(\tau \cap \sigma)^\vee = \tau^\vee + \sigma^\vee$.
(iii) If τ is finitely generated over the semiring $\mathbb{Q}_{\geq 0}$ then $\eta \subseteq \tau$ is a face if and only if it is of the form $\eta = \tau \cap \{\psi\}^\perp$. Moreover, sending a face η of τ to $\tau^\vee \cap \eta^\perp$ and a face σ of $\tau^\vee$ to $\tau \cap \sigma^\perp$ defines mutually inverse inclusion reversing bijections between $\mathrm{faces}(\tau)$ and $\mathrm{faces}(\tau^\vee)$. Moreover, we have $\dim(V) = \dim(\mathrm{lin}_{\mathbb{Q}}(\eta)) + \dim(\mathrm{lin}_{\mathbb{Q}}(\tau^\vee \cap \eta^\perp))$.
(iv) We have $\dim(V) = \dim(U) + \dim(U^\perp)$ for each $\mathbb{Q}$-linear subspace $U \subseteq V$.

Consider a finitely generated abelian group K and set $L := \mathrm{Hom}_{\mathbb{Z}}(K, \mathbb{Z})$. Then the kernel of the canonical epimorphism $K \to \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}), v \mapsto \phi_v$ is the torsion part $t(K)$ of K. For $A \subseteq K$ and $B \subseteq L$ set

$$A^\vee := \{\psi \in L | \psi(A) \subseteq \mathbb{N}_0\}, \quad B^\vee := \{v \in K | \phi_v(B) \subseteq \mathbb{N}_0\}.$$

as well as $A^\perp := A^\vee \cap (-A)^\vee$ and $B^\perp := B^\vee \cap (-B)^\vee$. For $w \in K$ and $\psi \in L$ we use the notations $w^\vee := \{w\}^\vee$, $w^\perp := \{w\}^\perp$, $\psi^\vee := \{\psi\}^\vee$ and $\psi^\perp := \{\psi\}^\perp$.

Remark I.1.3.22. In the above notation the following hold:

(i) If A resp. B is a submonoid then $\mathrm{Hom}_{\mathbb{N}_0}(A, \mathbb{N}_0)$ resp. $\mathrm{Hom}_{\mathbb{N}_0}(B, \mathbb{N}_0)$ is canonically isomorphic to $A^\vee$ resp. $B^\vee/t(K)$.

(ii) A and B define evaluation homomorphisms

$$\varepsilon_A \colon L \to \prod_{a \in A} \mathbb{Z}, \psi \mapsto (\psi(a))_{a \in A}, \quad \delta_B \colon K \to \prod_{\psi \in B} \mathbb{Z}, w \mapsto (\psi(w))_{\psi \in B}$$

and we have $A^\vee = \varepsilon_A^{-1}(\prod_{a \in A} \mathbb{N}_0)$ and $B^\vee = \delta_B^{-1}(\prod_{\psi \in B} \mathbb{N}_0)$, in particular, $A^\vee$ and $B^\vee$ are saturated in L resp K.

(iii) $(-)^\vee$ translates unions to intersections, and gives the same result for a generating subset of a submonoid A, for A itself and for its saturation. In particular, $(-)^\vee$ translates sums of submonoids into intersections.

(iv) If A resp. B are submonoids then we have $\mathbb{N}^{-1}(A^\vee) = (\mathbb{N}^{-1}A)^\vee$ and $\mathbb{N}^{-1}(B^\vee) = (\mathbb{N}^{-1}B)^\vee$ in $\mathbb{N}^{-1}L = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{N}^{-1}K, \mathbb{Q})$ resp. $\mathbb{N}^{-1}K$.

Proposition I.1.3.23. *In the above setting the following hold for a submonoid M of K:*

(i) *If M is saturated then for any free abelian subgroup $F \subseteq K$ such that $K = t(K) + F$ and $t(K) \cap F = \{0_K\}$ we have $M = t(K) + (M \cap F)$.*

(ii) *If M is finitely generated over $\mathbb{N}_0$ then $M^\vee$ is finitely generated and $M^{\vee\vee}$ is the (also finitely generated) saturation of M.*

(iii) *If M is finitely generated over $\mathbb{N}_0$ then sending $\tau \preceq M$ to $M^\vee \cap \tau^\perp$ and $\eta \preceq M^\vee$ to $M \cap \eta^\perp$ defines mutually inverse bijections between the faces of M and $M^\vee$. Moreover, $\mathrm{rank}_{\mathbb{Z}}(K)$ is the sum of the ranks of the $\mathbb{Z}$-modules generated by η and $M^\vee \cap \eta^\perp$, respectively.*

Proof. In (i) consider $v \in t(K)$ and $w \in F$ with $v + w \in M$. For $n \in \mathbb{N}$ with $nv = 0$ we then have $nw = n(v + w) \in M$ and obtain $w \in M$ from saturatedness.

For (ii) note that $\mathbb{N}^{-1}M$ is finitely generated over $\mathbb{N}_0$ by Lemma I.1.3.18(iii). By Remark I.1.3.21 $\mathbb{N}^{-1}(M^\vee) = (\mathbb{N}^{-1}M)^\vee$ is finitely generated over $\mathbb{Q}_{\geq 0}$ and another application of Lemma I.1.3.18(iii) gives finite generation of $M^\vee$ over $\mathbb{N}_0$. Similarly, $M^{\vee\vee}$ is finitely generated over $\mathbb{N}_0$ because $\mathbb{N}^{-1}(M^{\vee\vee}) = (\mathbb{N}^{-1}M)^{\vee\vee} = \mathbb{N}^{-1}M$ is finitely generated over $\mathbb{Q}_{\geq 0}$. The inclusion $M^{\vee\vee} \subseteq \mathrm{sat}(M, K)$ follows from Lemma I.1.3.18(i) because $\mathbb{N}^{-1}(M^{\vee\vee}) = \mathbb{N}^{-1}M$.

In (iii) the map $\mathrm{faces}(M) \to \mathrm{faces}(M^\vee)$ factorizes into three steps the first being the bijection $\mathrm{faces}(M) \to \mathrm{faces}(\mathbb{N}^{-1}M)$ from Remark I.1.3.13, the second being the bijection $\mathrm{faces}(\mathbb{N}^{-1}M) \to \mathrm{faces}((\mathbb{N}^{-1}M)^\vee)$ from Remark I.1.3.21 and the third being the bijection $\mathrm{faces}((\mathbb{N}^{-1}M)^\vee) \to \mathrm{faces}(M^\vee)$. Indeed, with respect to the localization map $\imath_{\mathbb{N}} \colon M^\vee \to \mathbb{N}^{-1}M^\vee$ we have

$$\imath_{\mathbb{N}}^{-1}((\mathbb{N}^{-1}M)^\vee \cap (\mathbb{N}^{-1}\tau)^\perp) = \imath_{\mathbb{N}}^{-1}(\mathbb{N}^{-1}(M^\vee \cap \tau^\perp)) = M^\vee \cap \tau^\perp.$$

Likewise, the map $\mathrm{faces}(M^\vee)$ is the composition of the three corresponding inverse maps. Indeed, with respect to the localization map $\imath_{\mathbb{N}} \colon M \to \mathbb{N}^{-1}M$ we have

$$\imath_{\mathbb{N}}^{-1}(\mathbb{N}^{-1}M \cap \mathbb{N}^{-1}\eta^\perp) = \imath_{\mathbb{N}}^{-1}(\mathbb{N}^{-1}(M \cap \eta^\perp) = M \cap \eta^\perp.$$

The rank formula now follows from the dimension formula of Remark I.1.3.21 because $\mathrm{lin}_{\mathbb{Q}}(\mathbb{N}^{-1}(M^\vee \cap \tau^\perp)) = \mathbb{N}^{-1}\mathrm{lin}_{\mathbb{Z}}(M^\vee \cap \tau^\perp)$. □

Remark I.1.3.24. Let $\phi\colon K \to K'$ be a homomorphism and let $M \subseteq K$ and $M' \subseteq K'$ be submonoids. Then the following hold:

(i) We have $(\phi^*)^{-1}(M^\vee) = \phi(M)^\vee$ because $\phi^*(\psi)(M) = \psi(\phi(M))$.

(ii) If $\phi(M) \subseteq M'$ then $\phi^*(M'^\vee) \subseteq M^\vee$. The converse holds if M' is saturated in K' and finitely generated, because then $M'^\vee \subseteq (\phi^*)^{-1}(M^\vee) = \phi(M)^\vee$ implies $\phi(M) \subseteq M'^{\vee\vee} = M'$ by Proposition I.1.3.23.

LEMMA I.1.3.25. *For finitely generated submonoids M, N of K (resp. L) each $u \in (M-N)^\vee$ satisfies the following:*

(i) *If $M \cap N = M \cap u^\perp = N \cap u^\perp$ holds then we have*

$$(M \cap N)^\vee = M^\vee - \mathbb{N}_0 u = N^\vee + \mathbb{N}_0 u = M^\vee + N^\vee,$$

and the converse holds if M and N are saturated in K (resp. L).

(ii) *If $(M-N) \cap (N-M) = (M-N) \cap u^\perp$ then the condition of (i) is satisfied.*

(iii) *If $M \cap N$ is a face of M and N then each element of $((M-N)^\vee)^\circ$ satisfies the condition of (ii).*

PROOF. In (i) suppose that the first condition holds. Clearly, $M^\vee$ and $N^\vee$ are contained in $(M\cap N)^\vee$. Let $A \subseteq M$ be a finite generating set. For $w \in (M \cap N)^\vee$ let k be the maximum over all $|w(a)|$ where $a \in A \setminus u^\perp$. Then $(w+ku)(a) \geq 0$ holds for each $a \in A$ and hence $w \in M^\vee - \mathbb{N}_0 u$. Together, we obtain

$$M^\vee + N^\vee \subseteq (M\cap N)^\vee \subseteq M^\vee - \mathbb{N}_0 u \subseteq M^\vee + N^\vee.$$

In the same manner, one shows the above inclusions with $N^\vee + \mathbb{N}_0 u$ in the place of $M^\vee - \mathbb{N}_0 u$. In the converse direction we use stability of M and N under $(-)^{\vee\vee}$ to obtain $M \cap (-u)^\vee = N \cap u^\vee = M \cap N$. Now, $u \in M^\vee$ gives $M \cap (-u)^\vee = M \cap u^\perp$ and $u \in -N^\vee$ gives $N \cap u^\vee = N \cap u^\perp$.

In (ii) first note that $M \cap N \subseteq u^\perp$. For $w \in M \cap u^\perp$ we have $w = b - a$ with $a \in M$ and $b \in N$. Thus, $w + a = b \in M \cap N$ gives $w \in M \cap N$. Analogously, we obtain $M \cap N = N \cap u^\perp$.

For (iii) note that if $u \in ((M-N)^\vee)^\circ$ then according to Proposition I.1.3.23 $(M-N)\cap u^\perp = (M-N) \cap \mathrm{face}(u)^\perp$ is the minimal face of $M-N$, which is equal to $(M\cap N) - (M \cap N)$ because $M \cap N$ is a face of M and N. $\square$

The above is known as the separation lemma and an element u satisfying the condition of (i) is called a *separating linear form* for M and N.

REMARK I.1.3.26. Let $M, N \subseteq K$ be submonoids of an abelian group. Then $M + N = M - \mathbb{N}_0 u = N - \mathbb{N}_0 v$ holds with certain $u \in M$ and $v \in N$ if and only if there exists $w \in M \cap -N$ such that $M + N = M - \mathbb{N}_0 w = N + \mathbb{N}_0 w$.

Indeed, if the first condition holds then there exist $a, c \in M$ and $b, d \in N$ with $a + b = -u$ and $c + d = -v$. The element $w := c - b = c + a + u = -d - v - b$ now lies in $M \cap -N$ and we have

$$M + N \subseteq M - \mathbb{N}_0 u \subseteq M - \mathbb{N}_0 w \subseteq M + N \subseteq N - \mathbb{N}_0 v \subseteq N + \mathbb{N}_0 w \subseteq M + N$$

because $-u = a + c - w \in M - \mathbb{N}_0 w$ and $-v = b + d + w \in N + \mathbb{N}_0 w$.

EXAMPLE I.1.3.27. Consider a submonoid $L = \mathbb{N}_0^n \oplus G$ of the group $K = \mathbb{Z}^n \oplus G$ where G is a finitely generated additive abelian group. For elements $u, v \in L$ we set $M := L - \mathbb{N}_0 u$ and $N := L - \mathbb{N}_0 v$. Then we have $M + N = L - \mathbb{N}_0(u+v)$ and

$$\{w \in M \cap -N | M + N = M - \mathbb{N}_0 w = N + \mathbb{N}_0 w\} = \mathrm{face}(v)^\circ - \mathrm{face}(u)^\circ.$$

Indeed, let $w = \sum_{i=1}^n \gamma_i v_i + w'j \in S$ belong to the left-hand side where $\gamma_i \in \mathbb{Z}$ and $w' \in G$. Here, $v_i \in \mathbb{N}_0^n$ denote the element whose i-th coordinate is 1 and whose other coordinates are 0. We have $u = \sum_{i \in I} \alpha_i v_i + u'$ and $v = \sum_{i\in J} \beta_i v_i + v'$ for unique $u', v' \in G$, $I, J \subseteq \{1, \ldots, n\}$ and $\alpha_i, \beta_i \in \mathbb{N}$. In this notation, we obtain $M = L - \mathbb{N}_0 \sum_{i\in I} v_i$ and $N = L - \mathbb{N}_0 \sum_{i \in J} v_j$ as well as

$$L - \mathbb{N}_0 \sum_{i\in I} v_i - \mathbb{N}_0 w = L - \mathbb{N}_0 \sum_{i \in I \cup I} v_i = L - \mathbb{N}_0 \sum_{i \in J} v_i + \mathbb{N}_0 w.$$

In particular, both w and $-w$ lie in $L-\mathbb{N}_0\sum_{i\in I\cup J} v_i$ which means that for each i in the complement of $I\cup J$ we have $\gamma_i, -\gamma_i \geq 0$ and hence $\gamma_i = 0$. Since $-\sum_{i\in I\cup J} v_i$ lies in $L-\mathbb{N}_0\sum_{i\in J} v_i+\mathbb{N}_0 w$ we have $\gamma_i < 0$ for $i \in I\setminus J$. Likewise, since $-\sum_{i\in I\cup J} v_i$ belongs to $L-\mathbb{N}_0\sum_{i\in I} v_i - \mathbb{N}_0 w$ we have $\gamma_i > 0$ for $i \in J\setminus I$. In other words, we have

$$w \in \sum_{i\in J\setminus I}\mathbb{N}v_i - \sum_{i\in I\setminus J}\mathbb{N}v_i + \sum_{i\in I\cap J}\mathbb{Z}v_i + G = \mathrm{face}(v)^\circ - \mathrm{face}(u)^\circ.$$

I.1.4. Prime and radical ideals and faces of monoids and $\mathbb{F}_1$-algebras. In this section we study prime ideals of monoids and $\mathbb{F}_1$-algebras as preparation for the theory of $\mathbb{F}_1$-schemes as well as that of prime divisors of monoids and $\mathbb{F}_1$-algebras of Krull type. We start with the connection between prime ideals and faces.

Definition I.1.4.1. An ideal $\mathfrak{p}$ of a monoid/$\mathbb{F}_1$-algebra is *prime* if it is proper, and whenever it contains a product ab of elements of M it already contains one of the factors.

Proposition I.1.4.2. *For a monoid/$\mathbb{F}_1$-algebra M there is an inclusion-reversing bijection*

$$\begin{aligned}\{\textit{faces of } M\} &\longrightarrow \{\textit{prime ideals of } M\}\\ \tau &\longmapsto M\setminus\tau\\ M\setminus\mathfrak{p} &\longleftarrow\!\shortmid \mathfrak{p}\end{aligned}$$

Here, the group of units M^* is the unique minimal face of M. Its complement, the unique maximal proper ideal of M, is consequently prime. Since the intersection of proper faces is a proper face, every union of prime ideals is again a prime ideal.

Proof. Let τ be a face of M. First we ascertain that $\mathfrak{p} := M\setminus\tau$ is an ideal. Let $a\in M$ and $b\in\mathfrak{p}$. If we had $ab\notin\mathfrak{p}$, i.e. $ab\in\tau$ then in particular $b\in\tau$ - in contradiction to the choice of b. Thus, $ab\in\mathfrak{p}$. Now, let $a,b\in M$ with $ab\in\mathfrak{p}$ and suppose that $a\notin\mathfrak{p}$, i.e. $a\in\tau$. If we had $b\notin\mathfrak{p}$, i.e. $b\in\tau$ then $ab\in\tau = M\setminus\mathfrak{p}$ - a contradiction. Therefore, $b\in\mathfrak{p}$.

Conversely, let $\mathfrak{p}$ be a prime ideal and let $a,b\in M$. Then $1\in\tau := M\setminus\mathfrak{p}$ and whenever $a,b\in\tau$ we have $ab\in\tau$ because otherwise we had $a\notin\tau$ or $b\notin\tau$. If $ab\in\tau$ and we had $a\notin\tau$, i.e. $a\in\mathfrak{p}$ then the ideal property gives $ab\in\mathfrak{p}$ - a contradiction. Therefore, a belongs to τ and analogously we deduce $b\in\tau$.

We next show that M^* is indeed a face. Let $a,b\in M$ with $ab\in M^*$. Then there is a $c\in M$ with $a(bc) = 1_M\in M^*$. Thus, $a\in K^*$ and analogously $b\in K^*$. For minimality, let $\tau\preceq M$ be any face and let $a\in M^*$. Take any $b\in\tau$. Then $a(a^{-1}b) = b\in\tau$ and thus $a\in\tau$. □

Remark I.1.4.3. Unions of prime ideals are prime ideals.

Remark I.1.4.4. If M is an $\mathbb{F}_1$-algebra and τ a face containing 0 then $\tau = M$, in particular, M is a principal face. Indeed, for any $a\in M$ we have $a0 = 0\in\tau$ and consequently $a\in\tau$.

Construction I.1.4.5. Let M be a monoid/$\mathbb{F}_1$-algebra and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of M. Then the *product* $\mathfrak{ab}$ is the ideal generated by all elements ab where $a\in\mathfrak{a}$ and $b\in\mathfrak{b}$. Explicitely, this is the union over all Mab where $a\in\mathfrak{a}$ and $b\in\mathfrak{b}$.

Proposition I.1.4.6. *An ideal $\mathfrak{p}$ of a monoid/$\mathbb{F}_1$-algebra M is prime if and only if whenever a product $\mathfrak{ab}$ of ideals of M lies in $\mathfrak{p}$ then $\mathfrak{a}$ or $\mathfrak{b}$ does, too.*

PROOF. Suppose that $\mathfrak{p}$ is prime and let $\mathfrak{a}, \mathfrak{b}$ be ideals of M with $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. If $\mathfrak{a}$ is not contained in $\mathfrak{p}$ then there exists an element $a \in \mathfrak{a} \setminus \mathfrak{p}$. For any $b \in \mathfrak{b}$ we have $ab \in \mathfrak{p}$ and deduce $b \in \mathfrak{p}$ from primality of $\mathfrak{p}$. Thus, $\mathfrak{b}$ is contained in $\mathfrak{p}$.

For the converse, let $ab \in \mathfrak{p}$. Then $MaMb = Mab \subseteq \mathfrak{p}$ and we deduce $Ma \subseteq \mathfrak{p}$ or $Mb \subseteq \mathfrak{p}$, i.e. $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. □

PROPOSITION I.1.4.7. *Preimages of faces resp. prime ideals under homomorphisms of monoids/$\mathbb{F}_1$-algebras are faces resp. prime ideals.*

PROOF. Let $\phi\colon M \to N$ be a homomorphism and let η be a face of N. Since $\phi(1) = 1$ we know that $\tau := \phi^{-1}(\eta)$ is non-empty. Now, let $ab \in \tau$ for some $a, b \in M$. Then $\phi(a)\phi(b) = \phi(ab) \in \eta$ which implies $\phi(a), \phi(b) \in \eta$, i.e. $a, b \in \tau$. □

DEFINITION I.1.4.8. Let M be a monoid/$\mathbb{F}_1$-algebra.

(i) An ascending chain $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ of prime ideals $\mathfrak{p}_0, \ldots, \mathfrak{p}_n$ is said to have *length* n.
(ii) The *height* $\mathrm{ht}(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ of M is the supremum of all lengths n of chains of prime ideals which end in $\mathfrak{p}_n = \mathfrak{p}$.
(iii) The *Krull dimension* $\dim(M)$ of M is the supremum over all lengths of chains of prime ideals of M.

PROPOSITION I.1.4.9. *Let N be a subgroup of the unit group M^* of a monoid M and let $\phi\colon M \to M/N$ be the canonical map. Then there is an inclusion preserving bijection*

$$\begin{aligned} \mathrm{faces}(M) &\longleftrightarrow \mathrm{faces}(M/N) \\ \tau &\longmapsto \phi(\tau) \\ \phi^{-1}(\eta) &\longleftarrow\!\shortmid\ \eta \end{aligned}$$

and a bijection respecting products and inclusions

$$\begin{aligned} \{\textit{ideals of } M\} &\longleftrightarrow \{\textit{ideals of } M/N\} \\ \mathfrak{a} &\longmapsto \phi(\mathfrak{a}) \\ \phi^{-1}(\mathfrak{b}) &\longleftarrow\!\shortmid\ \mathfrak{b} \end{aligned}$$

which restricts to a bijection of the sets of prime ideals.

PROOF. Let $\mathfrak{b}$ be an ideal of M/N. Then $\mathfrak{b}$ contains an element $\overline{a}$ where $a \in M$, in particular, $\mathfrak{a} := \phi^{-1}(\mathfrak{b})$ is non-empty. If $m \in M$ and $b \in \mathfrak{a}$ then $\phi(mb) = \phi(m)\phi(b) \in \mathfrak{b}$, i.e. $mb \in \mathfrak{a}$. Hence, $\mathfrak{a}$ is an ideal.

For the equation $\phi^{-1}(\phi(\mathfrak{a})) = \mathfrak{a}$ let $m \in M$ with $\phi(m) = \phi(a)$ for some $a \in \mathfrak{a}$. Then there exists $u \in N$ with $m = au \in \mathfrak{a}$. Then converse is always true. The equation $\phi(\phi^{-1}(\mathfrak{b})) = \mathfrak{b}$ is due to surjectivity.

Now, Proposition I.1.4.6 gives the statement on the sets of prime ideals. The statement concerning faces follows from the one about prime ideals since we know that $\phi(M) = M/N$ and $\phi^{-1}(M/N) = M$. □

PROPOSITION I.1.4.10. *Let N be a submonoid of a monoid/$\mathbb{F}_1$-algebra M and consider the localization map $\imath_N\colon M \to N^{-1}M$. Then the following hold:*

(i) *For each ideal $\mathfrak{a}$ of M the ideal generated by $\imath_N(\mathfrak{a})$ is $N^{-1}\mathfrak{a}$. Moreover, $\mathfrak{a}$ has empty intersection with N if and only if $N^{-1}\mathfrak{a}$ is proper.*
(ii) *For each $(N^{-1}M)^*$-invariant subset $S \subseteq N^{-1}M$, in particular for ideals and faces, we have $N^{-1}(\imath_N^{-1}(S)) = S$. Consequently, if $\mathfrak{b}$ is a proper ideal of $N^{-1}M$, then $\imath_N^{-1}(\mathfrak{b})$ does not intersect N.*
(iii) *The assignments $\mathfrak{a} \mapsto N^{-1}\mathfrak{a}$ and $\mathfrak{b} \mapsto \imath_N^{-1}(\mathfrak{b})$ define inclusion preserving mutually inverse bijections between the set of prime ideals of M which do not intersect N and the set of prime ideals of $N^{-1}M$.*

(iv) *The assignments $\tau \mapsto N^{-1}\tau$ and $\eta \mapsto \imath_N^{-1}(\eta)$ define inclusion preserving mutually inverse bijections between the set of faces of M which contain N and the set of faces of $N^{-1}M$. In particular, $\imath_N^{-1}((N^{-1}M)^*)$ is the smallest face containing N. Moreover, for each face τ containing N we have $N^{-1}\tau \cap N^{-1}(M \setminus \tau) = \emptyset$. Furthermore, $m/n \in N^{-1}M$ is a unit if and only if $m \in \imath_N^{-1}((N^{-1}M)^*)$.*

Proof. For (i) note that if $a \in N \cap \mathfrak{a}$ then $a/1$ is a unit. Conversely, if $1/1 = a/n \in N^{-1}\mathfrak{a}$ then there exists $n' \in N$ with $nn' = an' \in N \cap \mathfrak{a}$.

In (ii) let $a/n \in S$. Then $\imath_N(a) = na/n \in S$ and thus $a \in \imath_N^{-1}(S)$ and $a/n \in N^{-1}\imath_N^{-1}(S)$. Conversely, if $a \in \imath_N^{-1}(S)$ then $a/n = (1/n)\imath_N(a) \in S$.

In (iii) note that if $\mathfrak{b}$ is prime, then so is $\imath_N^{-1}(\mathfrak{b})$ by Proposition I.1.4.7.Now, let $\mathfrak{p}$ be a prime ideal of M which does not intersect N. Then, whenever $(a/k)(b/n)$ lies in $N^{-1}\mathfrak{p}$ there exist $c \in \mathfrak{p}$ and $l \in N$ with $(a/k)(b/n) = c/l$ and hence there exists $h \in N$ with $(hl)(ab) = hknc \in \mathfrak{p}$ which implies $ab \in \mathfrak{p}$. Now, we may assume that $a \in \mathfrak{p}$ and deduce $a/k \in N^{-1}\mathfrak{p}$. Thus, $N^{-1}\mathfrak{p}$ is a prime ideal.

Next, we show that $\mathfrak{p} = \imath_N^{-1}(N^{-1}\mathfrak{p})$. Let $a \in M$ such that there exist $b \in \mathfrak{b}$ and $n \in N$ with $a/1 = b/n$. Then there exists $k \in N$ with $(kn)a = kb \in \mathfrak{p}$ and we deduce $a \in \mathfrak{p}$.

In (iv) for well-definedness we consider a face τ with $N \subseteq \tau$. For $m/n, m'/n' \in N^{-1}M$ with $mm'/nn' = v/k \in N^{-1}\tau$ for $v \in \tau$ and $k \in N$ there exists $l \in N$ with $lkmm' = lnn'v \in \tau$ and we deduce $m, m' \in \tau$. $\square$

Definition I.1.4.11. For a prime ideal $\mathfrak{p}$ of a monoid/$\mathbb{F}_1$-algebra M the localization $M_{\mathfrak{p}} := (M \setminus \mathfrak{p})^{-1}M$ is called the *localization at* $\mathfrak{p}$.

Remark I.1.4.12. By Proposition I.1.4.10 the height of a prime ideal $\mathfrak{p}$ is equal to the Krull dimension of the localization at $\mathfrak{p}$.

Remark I.1.4.13. Consider an element f of a monoid/$\mathbb{F}_1$-algebra M. The principal face τ generated by f is the preimage of $(M_f)^*$ under the localization map $\imath_f\colon M \to M_f$. Here, M_f is the localization by f from Construction I.1.1.19. In other words, τ is the set of $g \in M$ for which there exist $m, n \geq 0$ and $h \in M$ such that $f^m gh = f^{n+m}$.

Definition I.1.4.14. Following [**15**] we call a morphism $\phi : M \to M'$ of monoids/$\mathbb{F}_1$-algebras *local* if $\phi^{-1}(M'^*) = M^*$ holds.

Remark I.1.4.15. If $\phi : M \to M'$ is a morphism of monoids/$\mathbb{F}_1$-algebras, then for $N := \phi^{-1}(M'^*)$ the induced morphism $N^{-1}M \to M'$ is local.

Definition I.1.4.16. The *radical* of an ideal $\mathfrak{a}$ of a monoid/$\mathbb{F}_1$-algebra M is its saturation $\sqrt{\mathfrak{a}}$ in M. If $\mathfrak{a} = \sqrt{\mathfrak{a}}$ then $\mathfrak{a}$ is called *radical*.

Proposition I.1.4.17. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a monoid/$\mathbb{F}_1$-algebra M. Then the following hold:*

(i) *Every intersection of prime ideals is radical.*
(ii) *$\sqrt{\mathfrak{a}}$ is the intersection over all prime ideals containing $\mathfrak{a}$.*
(iii) *We have $\sqrt{\mathfrak{ab}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$*
(iv) *For a morphism $\phi\colon N \to M$ we have $\phi^{-1}(\sqrt{\mathfrak{a}}) = \sqrt{\phi^{-1}(\mathfrak{a})}$.*

Proof. In (ii) let $r \in M$ and set $S := \{r^n\}_{n \geq 0}$. If S intersects $\mathfrak{a}$ non-trivially than every prime ideal which contains $\mathfrak{a}$ also contains r. If $S \cap \mathfrak{a}$ is empty then $S^{-1}\mathfrak{a}$ is proper. Then $\imath_S^{-1}(S^{-1}M \setminus (S^{-1}M)^*)$ contains $\mathfrak{a}$ but not r because it intersects S trivially.

For (iii) we calculate $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subseteq \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{ab}} \subseteq \sqrt{\mathfrak{a} \cap \mathfrak{b}}$ using Proposition I.1.4.6. Lastly, $f \in N$ satisfies $\phi(f) \in \sqrt{\mathfrak{a}}$ if and only if there exists $n \in \mathbb{N}$ with $\phi(f^n) = \phi(f)^n \in \mathfrak{a}$ which holds if and only if $f^n \in \phi^{-1}(\mathfrak{a})$. $\square$

I.2. divisibility theory of monoids and $\mathbb{F}_1$-algebras

In this section, we develop key concepts from divisibility theory of monoids (and $\mathbb{F}_1$-algebras), including divisor (class) monoids and the Krull property. As remarked in [**9**], this works analogously to the case of rings. Standard references for the latter include [**16, 20**].

I.2.1. Cancellation and integrality.

Definition I.2.1.1. An $\mathbb{F}_1$-algebra M is said to have *no zero divisors* or to be *zero divisor free* (ZDF) if $M \setminus \{0_M\}$ is a submonoid. M is *integral* if $M \setminus \{0_M\}$ is a cancellative submonoid.

Definition I.2.1.2. A monoid/$\mathbb{F}_1$-algebra M is *simple* if it has only one proper ideal.

Remark I.2.1.3. A monoid/$\mathbb{F}_1$-algebra M is simple if and only if each (non-zero) element is invertible.

Definition I.2.1.4. For a cancellative monoid/integral $\mathbb{F}_1$-algebra M the *quotient group/$\mathbb{F}_1$-algebra* is the localization $Q(M)$ by all (non-zero) elements of M.

Definition I.2.1.5. A cancellative monoid resp. integral $\mathbb{F}_1$-algebra M is called *normal*, if it is saturated in $Q(M)$.

Remark I.2.1.6. Let $\phi\colon M \to N$ be a homomorphism of $\mathbb{F}_1$-algebras.

(i) Assume that $\phi^{-1}(0) = 0$. An element is a zero divisor of M if and only if its image is a zero divisor of N. Consequently, if N has no zero divisors, then neither has M. The converse holds if $N = N^*\phi(M)$.
(ii) If M is simple then $\phi^{-1}(0) = 0$, because images of units are units. Moreover, if $N = N^*\phi(M)$ then N is simple.

I.2.2. Factoriality. In addition to the canonical characterizations of factoriality in terms of freeness resp. generation by prime elements we discuss its behaviour under *pseudo*-faces and localization. Throughout, all monoids (and $\mathbb{F}_1$-algebras) are abelian. Recall that two elements a, b of a monoid M are *associated* if $aM^* = bM^*$.

Definition I.2.2.1. An (non-zero) non-unit p in a monoid/$\mathbb{F}_1$-algebra M is

(i) *irreducible*, if $p = ab$ with $a, b \in M$ implies $a \in M^*$ or $b \in M^*$,
(ii) *prime*, if $p|ab$ with $a, b \in M$ implies $p|a$ or $p|b$.

Remark I.2.2.2. An irreducible element that is divided by a prime element is associated to that element.

Remark I.2.2.3. In cancellative monoids resp. integral $\mathbb{F}_1$-algebras, prime elements are irreducible. Indeed, if $p \in M$ is prime and $p = ab$ then we may assume $a = pc$. Thus, we have $p = pcb$ and hence $cb = 1$ by the cancellation property.

The coproduct of a family M_i of $\mathbb{F}_1$-algebras is the coproduct (i.e. the direct sum) of the underlying monoids modulo the equivalence relation generated by all $0_{M_i} \sim 0_{M_j}$. An element $m = [(m_i)_{i\in I}]$ is non-zero if and only if all m_i are non-zero, in which case m_i is called the *i-th component* of m.

Proposition I.2.2.4. *Let $M = \coprod_{i\in I} M_i$ be the coproduct of a family of commutative monoids resp. $\mathbb{F}_1$-algebras. Consider an (non-zero) element m with components $m_i, i \in I$. Then the following hold:*

(i) *$m \in M$ is a unit if and only if all $m_i \in M_i$ are units,*
(ii) *m is a zero divisor of the $\mathbb{F}_1$-algebra M if and only if one m_{i_0} is a zero divisor of M_{i_0},*
(iii) *M is cancellative/integral if and only if each M_i is cancellative/integral,*

(iv) *$m \in M$ is irreducible/prime if and only if one m_{i_0} is irreducible/prime and all m_i for $i \neq i_0$ are units.*

PROOF. In (i) if m is a unit then there exists $w \in M$ with $1 = mw$ which means its components w_i satisfy $1_{M_i} = m_i w_i$ for every i. Conversely, if all m_i are units then let J be the finite set where $m_i \neq 1_{M_i}$ and define $w_i := m_i^{-1}$ for $i \in J$ and $w_i := 1_{M_i}$ for all other i. The element w with components w_i then satisfies $mw = 1$.

In (ii) suppose that m is a zero divisor, i.e. there exists a non-zero w with components w_i such that $mw = 0_M$. Then $m_{i_0} w_{i_0} = 0_{i_0}$ holds for some i_0. Conversely, if there exist $i_0 \in I$ and a non-zero $w_{i_0} \in M_{i_0}$ then the element w whose i_0-component is w_{i_0} and whose other components are 1 satisfies $mw = 0_M$.

For (iii) let M be cancellative/integral. For every i, if $a_i b_i = a_i c_i$ holds for (non-zero) elements of M_i then the elements $a, b, c \in M$ with i-th component a_i, b_i and c_i respectively and other components 1_{M_j} satisfy $ab = ac$ which implies $b = c$, i.e. $b_i = c_i$. Conversely, let every M_i be cancellative/integral and let $ab = ac$ hold with (non-zero) elements of M. Then the respective components satisfy $a_i b_i = a_i c_i$ and hence $b_i = c_i$ holds for all i meaning $b = c$.

In (iv) let $m \in M$ be irreducible. Then m is no unit, hence there exists i_0 such that m_{i_0} is no unit. We claim that all other m_i are units. Indeed, consider $i \neq i_0$ and let m' be the element which differs from m only in having i-th component 1, and let m'' be the element whose i-th component is m_i and whose other components are 1. Then by (i) $m = m'm''$ implies that m'' is a unit because m' is no unit. Thus, m_i is a unit. If $m_{i_0} = a_{i_0} b_{i_0}$ then the elements $a, b \in M$ with i_0-th component a_{i_0} resp. b_{i_0} and 1_{M_i} in all other components satisfy $m = ab$. Thus, a or b is a unit which means that a_{i_0} or b_{i_0} is a unit. Conversely, suppose the i_0-th component m_{i_0} of m is irreducible and all other components are units. If $m = ab$ then $a_i b_i$ is a unit for all $i \neq i_0$, and irreducibility of $m_{i_0} = a_{i_0} b_{i_0}$ implies that a_{i_0} or b_{i_0} is a unit, i.e. a or b is a unit.

Now, suppose that $m \in M$ is prime. Since m is no unit there exists $i_0 \in I$ such that the i_0-component m_{i_0} is no unit. We claim that all other m_i are units. Indeed, consider $i \neq i_0$ and let m' be the element which differs from m only in having i-th component 1, and let m'' be the element whose i-th component is m_i and whose other components are 1. m does not divide m'' because m_{i_0} does not divide 1. Hence m divides m', i.e. there exists an element $w \in M$ with $mw = m'$ and for the i-th components we have $m_i w_i = 1$. If $m_{i_0} c_{i_0} = a_{i_0} b_{i_0}$ in M_{i_0} then let a be the element whose i_0-th component is a_{i_0} and whose other components a_i are m_i, and let b, c be the elements whose i_0-th components are b_{i_0} resp. c_{i_0} and whose other components are 1. These elements then satisfy $mc = ab$. We may assume $m|a$ and deduce $m_{i_0}|a_{i_0}$. Conversely, suppose that m has i_0-th component m_{i_0} with a prime of M_{i_0} and all other components are units $m|ab$ implies $m_{i_0}|a_{i_0}b_{i_0}$ and we may assume $m_{i_0}|a_{i_0}$. By invertibility we have $m_i|a_i$ for all other i and conclude that m divides a. $\square$

PROPOSITION I.2.2.5. *Let M be a monoid/$\mathbb{F}_1$-algebra and $N \subseteq M^*$ a subgroup. Then*

(i) *$a|b$ in M if and only if $\overline{a}|\overline{b} \in M/N$,*
(ii) *$(M/M^*)^* = \{\overline{1}\}$,*
(iii) *M is cancellative/integral if and only if M/N is cancellative/integral,*
(iv) *p is prime in M if and only if $\overline{p}$ is prime in M/N,*
(v) *p is irreducible in M if and only if $\overline{p}$ is irreducible in M/N.*

PROOF. For (i) note that $ac = b$ implies $\overline{ac} = \overline{b}$ and thus $\overline{a}|\overline{b}$ whenever $a|b$. For the converse, suppose that $\overline{a}|\overline{b}$ which means there exist $c \in M$ with $\overline{ac} = \overline{b}$.

Then there exists a $d \in N$ with $acd = b$ and hence $a|b$. Assertions (ii) and (iv) are consequences of (i).

For (v) let p be irreducible in M. Then $\overline{p}$ is no unit and for $a, b \in M$ with $\overline{p} = \overline{a}\overline{b}$ there exists $u \in N$ with $p = (ua)b$. Then either (ua) or b is a unit and hence $\overline{ua} = \overline{a}$ or $\overline{b}$ is a unit. Conversely, let $\overline{p}$ be irreducible. Again, p does not divide 1 because $\overline{p}$ does not divide $\overline{1}$. For $a, b \in M$ with $p = ab$ we have $\overline{p} = \overline{a}\overline{b}$ and which implies $\overline{a}|\overline{1}$ or $\overline{b}|\overline{1}$, i.e. $a|1$ or $b|1$. □

Definition I.2.2.6. A cancellative monoid resp. integral $\mathbb{F}_1$-algebra is *factorial* if every (non-zero) non-unit is a product of primes.

Example I.2.2.7. Let I any index set and let $M := \bigoplus_{i \in I} \mathbb{N}_0$. Then $M^* = \{0\}$ and the primes of M are the precisely the standard basis elements. By construction, M is factorial.

Proposition I.2.2.8. *For a cancellative monoid M the following are equivalent:*

(i) *M is factorial,*
(ii) *the $(\mathbb{N}_0, +, \cdot)$-module M has a decomposition $M = M^* \oplus \bigoplus_{i \in I} \mathbb{N}_0 p_i$ with a family of pairwise non-associated primes $p_i \in M$,*
(iii) *M/M^* is a free $(\mathbb{N}_0, +, \cdot)$-module.*

Proof. Let M be factorial. Then M/M^* is generated by its primes, i.e. by the classes of the primes of M. We show that every element $\overline{a} \in M/M^*$ is a unique product of primes. Let $\overline{p_1}^{k_1} \cdot \overline{p_d}^{k_d} = \overline{q_1}^{l_1} \cdots \overline{q_s}^{l_s}$ with primes $\overline{p_i}$ and $\overline{q_j}$ and $k_i, l_j \in \mathbb{N}_0$. We may assume that all prime factors occuring on both sides have already been cancelled. Then both sides must be $\overline{1}$. Otherwise, we had $k_i \geq 1$ for some i and consequently p_i must divide some q_j with $l_j \geq 1$ which means that $\overline{p_i} = \overline{q_j}$ - a contradiction to our assumption. Thus, M/M^* is in particular free.

Now, suppose that (iii) holds and let P be a system of representatives of the primes of M/M^*. Since M/M^* is free, the primes of M/M^* form a basis. Indeed, let $\overline{q_i}, i \in I$ be a $\mathbb{N}_0$-basis of M/M^*. If $\overline{q_i}|\overline{a}\overline{b}$ then there exist $k_j, l_j, t_j \in \mathbb{N}_0$ for $j \in I$ such that $\overline{a} = \prod_{j \in J} \overline{q_j}^{l_j}, \overline{b} = \prod_{j \in J} \overline{q_j}^{t_j}$ and with $\overline{c} = \prod_{j \in J} \overline{q_j}^{t_j}$ we have

$$\overline{q_i} \prod_{j \in I} \overline{q_j}^{k_j} = \overline{q_i c} = \overline{a}\overline{b} = \prod_{j \in J} \overline{q_j}^{l_j + t_j}$$

which implies $k_i + 1 = l_i + t_i$. Hence l_i or t_i must be at least 1 which means that $\overline{q_i}$ divides $\overline{a}$ or $\overline{b}$. Conversely, any prime is a member of the basis, because if $\overline{q} = \prod_{j \in J} \overline{q_j}^{n_j}$ is prime then it divides one $\overline{q_i}$ which implies $\overline{q} = \overline{q_i}$.

We now claim that every $a \in M$ is a unique product $u \prod_{p \in P} p^{n_p}$ with only finitely many of the $n_p \geq 0$ being non-zero. Indeed, if $\overline{a} = \prod_{p \in P} \overline{p}^{n_p}$ then there exists $u \in M^*$ with $a = u \prod_{p \in P} p^{n_p}$. For uniqueness, note that if

$$v \prod_{p \in P} p^{k_p} = a = u \prod_{p \in P} p^{n_p}$$

with $v \in M^*$ then $\prod_{p \in P} \overline{p}^{k_p} = \prod_{p \in P} \overline{p}^{n_p}$ which implies $k_p = n_p$ for all $p \in P$. Then cancellation gives $v = u$. This establishes (ii). The implication from (ii) to (i) is obvious. □

Definition I.2.2.9. A submonoid N of a monoid M is a *pseudo-face* if whenever $a, b \in M$ satisfy $ab \in N$ and $a \in N$ then also $b \in N$.

Proposition I.2.2.10. *Let $N \subseteq M$ be a submonoid. If $N$is a pseudo-face then $\imath_N(N) = Q(N) \cap \imath_N(M)$ holds in $N^{-1}M$. If the localization map $\imath_N \colon M \to N^{-1}M$ is injective then the converse holds.*

PROOF. If N is a pseudo-face and $a, b \in N$ and $c \in M$ satisfy $a/b = c/1$ then there exists $s \in N$ with $as = (bs)c \in N$ and thus $c \in N$. If for the converse we consider $ab \in N$ and $a \in N$ then $b/1 = ab/a \in Q(N) \cap \imath_N(M) = \imath_N(N)$ and hence $b \in N$. □

REMARK I.2.2.11. Preimages of pseudo-faces under homomorphisms are pseudo-faces.

EXAMPLE I.2.2.12. Subgroups of groups are pseudo-faces.

Primality, irreducibility and thus also factoriality are not well-behaved under inclusions of submonoids but behave better under inclusions of (pseudo-)faces:

PROPOSITION I.2.2.13. *Let N be a submonoid of a monoid/$\mathbb{F}_1$-algebra M such that $N \subseteq M$ is a pseudo-face of M and let $s \in N$. Then the following hold:*

(i) $N^* = M^* \cap N$.
(ii) *If s is irreducible in M, then it is also irreducible in N. If NM^* is a face of M, then the converse statement also holds.*
(iii) *If s is prime in M, then it is also prime in N. If $NM^* = M$ then the converse also holds.*
(iv) *If NM^* is a face and M is factorial, then N is factorial, too, and $s \in N$ is prime in N if and only if it is prime in M. If $NM^* = M$ and N is factorial then M is factorial, too.*
(v) *If N maps onto M/M^*, then N is factorial if and only if M is factorial.*

PROOF. For assertion (i), consider $s \in M^* \cap N$. Then there exists $t \in M$ with $1 = st$. Since $1 \in N$ we obtain $t \in N$ from the pseudo-face property and $s \in N^*$.

For (ii), let $s = ab$ with $a, b \in N$. Then we may assume that $a \in M^*$. Thus $a, aa^{-1} = 1 \in M^*$, so $a^{-1} \in N$ by the pseudo-face property. If N is a face, consider $a, b \in M$ with $s = ab$. By the face property, we have $a, b \in N$ and may assume $a \in N^*$, in particular $a \in M^*$.

For (iii), let $a, b \in N$ with $s|ab$ in N. By primality we may assume that $s|a$ in M, i.e. there exists $t \in M$ with $st = a$. By the pseudo-face property, $t \in N$, so $s|a$ in N. Now suppose that $NM^* = M$ and s is prime in N. If $s|ab$ with $a, b \in M$ then there exist $u, v \in M^*$ with $ua, vb \in N$ and we have $s|(ua)(vb)$ in M, i.e $st = uavb$ with some $t \in M$. Since $s, st \in N$ the pseudo-face property of N implies $t \in N$. Thus, primality in N implies $s|ua$ or $s|vb$ in N and hence $s|a$ or $s|b$ in M.

In (iv) consider a (non-zero) $s' \in N \setminus N^*$. Then $s' = p_1 \cdots p_m$ with p_i prime in M. By the face property, the $p_i = p_i' u_i$ holds with $p_i' \in N$ and $u_i \in M^*$, and by (ii) each p_i' is prime in N. Thus, $s' = (u_1^{-1} \cdots u_m^{-1}) p_1' \cdots p_m'$ holds and the pseudo-face property yields $u_1^{-1} \cdots u_m^{-1} \in N$. If s' is prime in N then it is in particular irreducible in N and hence $s' = p_i$ for some i.

Now suppose that NM^*M and N is factorial and let $s' \in M$ be a (non-zero) non-unit. Let $t \in M^*$ with $ts' \in N$. Then we have $ts' = p_1 \cdots p_d$ with primes p_i of N which are also prime in M by (iii). Thus, $s' = (tp_1)p_2 \cdots p_d$ is a product of primes.

For (v), note that (i) implies $N/N^* \cong M/M^*$, so Proposition I.2.2.5 gives the assertion. □

We have seen that faces of factorial monoids are factorial. The following is a partial converse.

PROPOSITION I.2.2.14. *Let M be a cancellative monoid and let $N \subseteq M$ be a submonoid that is generated by M^* and a set of primes of M. Then N is a face.*

PROOF. Let $a, b \in M$ with $ab \in N$. Then there exist elements $p_1, \ldots, p_d \in N$ which are pairwise non-associated primes of M, natural numbers $k_1, \ldots, k_d$ and a unit $u \in M^*$ such that $ab = up_1^{k_1} \cdots p_d^{k_d}$. By primality, each p_i divides at least one of the elements a and b. Since the p_i are non-associated each p_i can occur at most k_i times as a factor of a resp. b. Thus, there exist $l_i, t_i \in \mathbb{N}_0$ and elements $v, w \in M$ with $a = vp_1^{l_1} \cdots p_d^{l_d}$ and $b = wp_1^{t_1} \cdots p_d^{t_d}$ such that the p_i divide neither v nor w. Then

$$up_1^{k_1} \cdots p_d^{k_d} = vwp_1^{l_1+t_1} \cdots p_d^{l_d+tt_d}$$

implies $k_i \geq l_i + t_i$ and thus

$$up_1^{k_1-l_1-t_1} \cdots p_d^{k_d-l_d-t_d} = vw$$

and since none of the p_i divide vw we deduce $k_i = l_i + t_i$ and $u = vw$. Hence v and w are units and we obtain $a, b \in N$. □

LEMMA I.2.2.15. *Let A be a 1-noetherian monoid/$\mathbb{F}_1$-algebra. Then every non-zero non-unit is a product of irreducible elements.*

PROOF. Suppose that the set M of all (non-zero) principal proper ideals $\langle a \rangle$ generated by elements $a \in A$ which are no products of irreducible elements is non-empty. Then, M by 1-noetherianity has a maximal element $\langle a' \rangle$ whose generator a' is in particular not irreducible. So there are $s, t \in A \setminus A^*$ with $a' = bc$ and $\langle a' \rangle \subsetneq \langle b \rangle, \langle b \rangle$ are proper inclusions. Thus, by maximality of $\langle a' \rangle$ the elements b and c are products of irreducible elements. But then, so is a' - a contradiction. □

By a *principal ideal* monoid/$\mathbb{F}_1$-algebra we mean a monoid resp. $\mathbb{F}_1$-algebra in which each ideal is principal.

PROPOSITION I.2.2.16. *Integral principal ideal monoids/$\mathbb{F}_1$-algebras are factorial.*

PROOF. For a principal ideal monoid/$\mathbb{F}_1$-algebra A Lemma I.2.2.15 reduces the problem to showing that every irreducible $p \in A$ is prime. By definition, $\langle p \rangle$ is maximal among all the principal ideals - so in our case among all ideals. Thus, $A/\langle p \rangle$ is simple and hence integral. □

I.2.3. The divisor monoid. In this section, we consider divisors of a multiplicative cancellative abelian monoid resp. integral $\mathbb{F}_1$-algebra M, i.e. intersections of submodules $Mf \leq Q(M)$. These form a monoid $\mathrm{Div}(M)$ in terms of which we later characterize complete integral closedness (Section I.2.4), the Krull property (Section I.2.5) and factoriality (Section I.2.6).

DEFINITION I.2.3.1. Principal M-submodules of $Q(M)$ are called *principal divisors*. An M-submodule of $Q(M)$ is called *fractional* or a *fractional ideal* if it is non-empty and is contained in a principal divisor.

REMARK I.2.3.2. For non-empty/-zero submodules $\mathfrak{a}, \mathfrak{b} \leq_M Q(M)$ the following hold:

(i) the localization of $\mathfrak{a}$ by (the non-zero elements of) M is $Q(M)$, because for $a, b \in M$ and $f/g \in \mathfrak{a}$ we have $af = (ag)(f/g) \in \mathfrak{a}$ and hence $a/b = (1/bf)(af) \in (M)^{-1}\mathfrak{a}$.
(ii) $\mathfrak{a} \cap \mathfrak{b}$ is non-empty/-zero, because it contains $(\mathfrak{a} \cap M)(\mathfrak{b} \cap M)$.
(iii) If $\mathfrak{a} \subseteq M(u/v)$ and $\mathfrak{b} \subseteq M(x/y)$ are fractional then $\mathfrak{ab} \subseteq M(ux/vy)$ and $\mathfrak{a} \cup \mathfrak{b} \subseteq M(1/vy)$ are also fractional.

CONSTRUCTION I.2.3.3. A *divisor* is a non-zero/-empty intersection over a (non-empty) family of principal divisors. For each fractional ideal $\mathfrak{a}$ the intersection $\mathrm{div}(\mathfrak{a})$ over all principal divisors containing $\mathfrak{a}$ is a divisor. The set $\mathrm{Div}(M)$ of divisors endowed with the operation sending $D, D' \in \mathrm{Div}(M)$ to $D + D' := \mathrm{div}(DD')$ is a monoid with neutral element $0_{Div(M)} := M$. Setting $D \leq D'$ if and only if $D \supseteq D'$

turns $\mathrm{Div}(M)$ into a partially ordered monoid, called the *divisor monoid* of M. It comes with the *divisor homomorphism*

$$\mathrm{div}\colon Q(M)^* \longrightarrow \mathrm{Div}(M), \quad f \longmapsto \mathrm{div}(f) := \mathrm{div}(Mf) = Mf$$

whose image $\mathrm{PDiv}(M)$ is the subgroup of principal divisors. The factor monoid $\mathrm{Cl}(M) = \mathrm{Div}(M)/\,\mathrm{PDiv}(M)$ is the *divisor class monoid* of M.

Remark I.2.3.4. For an integral $\mathbb{F}_1$-algebra M set $N := M \setminus \{0_M\}$. Then we have $Q(M)^* = Q(N)$ and there is a bijection

$$\begin{aligned} \{M\text{-submodules of } Q(M)\} &\longleftrightarrow \{N\text{-submodules of } Q(N)\} \\ \mathfrak{a} &\longmapsto \mathfrak{a} \setminus \{0\} \\ \mathfrak{b} \cup \{0\} &\longleftarrow\!\shortmid \mathfrak{b} \end{aligned}$$

which respects intersections, products, principal divisors and divisors, in particular it restricts to an isomorphism $\mathrm{Div}(M) \to \mathrm{Div}(N)$.

Example I.2.3.5. Every ideal of $\mathbb{N}_0$ is of the form $a+\mathbb{N}_0$ with $a \in \mathbb{N}_0$ and thus every fractional ideal, in particular every divisor, is of the form $a + \mathbb{N}_0$ with $a \in \mathbb{Z}$. Therefore the map

$$\mathrm{div}\colon \mathbb{Z} = Q(\mathbb{N}_0) \longrightarrow \mathrm{Div}(\mathbb{N}_0), \quad a \longmapsto a + \mathbb{N}_0$$

is an isomorphism. In particular, $\mathrm{Cl}(\mathbb{N}_0) = 0$.

Proposition I.2.3.6. *Let $N \subseteq M^*$ be a subgroup of a cancellative monoid M. Then the canonical homomorphism $\pi\colon Q(M) \to Q(M/N) = Q(M)/N$ satisfies $\pi^{-1}(M/N) = M$ and induces an isomorphism of $\mathbb{F}_1$-algebras*

$$\{M\text{-submodules of } Q(M)\} \longrightarrow \{(M/N)\text{-submodules of } Q(M/N)\}, \quad X \longmapsto \pi(X)$$

which respects intersections, unions, products and quotients. It restricts to an isomorphism of the monoids of (principal) divisors and hence induces an isomorphism of the divisors class monoids.

Proof. It suffices to observe that every M-submodule X of $Q(M)$ is π-saturated because whenever $\pi(x) = \pi(x')$ holds with $x' \in X$ we have $x = ux' \in X$ with some $u \in N$. In particular, $\pi^{-1}((M/N)fN) = \pi^{-1}(\pi(Mf)) = Mf$ holds for $f \in Q(M)$. □

Remark I.2.3.7. A divisor $D = \bigcap_{i\in I} Mf_i$ where $f_i \in Q(M)$ is equal to the intersection D' of all principal divisors Mf containing D. Indeed, D' is an intersection of supersets of D, and D is a subintersection D'.

Remark I.2.3.8. For a cancellative monoid M the following hold:

(i) For each $D \in \mathrm{Div}(M)$ we have
$$\begin{aligned} D &= \{f \in Q(M) \mid Mf \subseteq D\} = \{f \in Q(M) \mid \mathrm{div}(f) \geq D\} \\ &= \mathrm{div}^{-1}(\mathrm{Div}(M)_{\geq D}), \end{aligned}$$
in particular $M = \mathrm{div}^{-1}(\mathrm{Div}(M)_{\geq 0})$,

(ii) We have $\ker(\mathrm{div}) = M^*$ because $f \in M^*$ if and only if $Mf = M$,

(iii) We have $\mathrm{Div}(M) = \mathrm{Div}(M)_{\geq 0} + \mathrm{PDiv}(M)$ because for $D \in \mathrm{Div}(M)$ there exists $f \in Q(M)$ with $D \subseteq Mf$, i.e. $\mathrm{div}(f^{-1}) + D \geq 0$.

Proposition I.2.3.9. *For a cancellative monoid M, a fractional ideal $\mathfrak{a}$ and a set $S := \{D_i | i \in I\} \subseteq \mathrm{Div}(M)$ the following hold:*

(i) *S has a supremum if and only if it has an upper bound, which holds if and only if it has an upper bound which is a principal divisor. This holds if and only if $\bigcap_i D_i$ is non-empty, e.g. if I is finite, and then $\sup_i D_i = \bigcap_i D_i$. In particular, $\mathrm{div}(\mathfrak{a})$ is the supremum of all divisors (resp. all principal divisors) containing $\mathfrak{a}$.*

(ii) *S has an infimum if and only if it has a lower bound, which holds if and only if it has a lower bound which is a principal divisor, i.e. if $\bigcup_i D_i$ is fractional, e.g. if I is finite. In that case we have* $\inf_i D_i = \operatorname{div}(\bigcup_i D_i)$. *In particular, the infimum of all divisors (resp. all principal divisors) contained in* $\mathfrak{a}$ *is* $\operatorname{div}(\mathfrak{a})$.

Proof. In (i) denote by D' the intersection over all D_i. If D' is non-empty then it is a divisor and since it is contained in every D_i it is an upper bound of $\{D_i | i \in I\}$. Each further upper bound E is also contained in every D_i and hence is a subset of D', i.e. $E \geq D'$.

For the supplement let D' is the intersection over all divisors containing $\mathfrak{a}$. Then $D' \subseteq \operatorname{div}(\mathfrak{a})$ holds. Conversely, as an intersection of principal divisors containing $\mathfrak{a}$, D' is a subintersection of $\operatorname{div}(\mathfrak{a})$ and thus contains $\operatorname{div}(\mathfrak{a})$.

For (ii) let $\mathfrak{a}$ be the union over $\{D_i; i \in I\}$ and set $E := \operatorname{div}(\mathfrak{a})$. By definition, E is a lower bound of the set $\{D_i; i \in I\}$. If $E' \in \operatorname{Div}(M)$ is greater or equal to all D_i then it is contained in all of them set-theoretically. E' equals the intersection over all principal divisors containing E' and is thus a sub-intersection of E and hence greater or equal to E. □

All claims from the construction of $\operatorname{Div}(M)$ are proven in the remainder of the section. In particular, $\operatorname{Div}(M)$ is a partially ordered monoid due to Proposition I.2.3.14. For brevity denote $Q(M)$ by K.

Definition I.2.3.10. If $\mathfrak{a}, \mathfrak{b}$ are M-submodules of K, then their quotient is

$$[\mathfrak{a} : \mathfrak{b}] := \{w \in K \mid w\mathfrak{b} \subseteq \mathfrak{a}\}.$$

Remark I.2.3.11. $[\mathfrak{a} : \mathfrak{b}]$ is again a M-submodule, since $w\mathfrak{b} \subseteq \mathfrak{a}$ implies $mw\mathfrak{b} \subseteq w\mathfrak{b} \subseteq \mathfrak{a}$ for all $m \in M$.

Construction I.2.3.12. For the cancellative monoid/integral $\mathbb{F}_1$-algebra M the set $F(M)$ of non-empty/-zero fractional ideals of M forms a commutative partially ordered monoid, the operation being the product of submodules, the neutral element being M, and the order being the reverse inclusion order.

Remark I.2.3.13. For any submodule $\mathfrak{a} \subseteq K$ over the monoid/$\mathbb{F}_1$-algebra M the following hold:

(i) $\mathfrak{a}$ is fractional if and only if $[M : \mathfrak{a}]$ is non-empty/-zero.
(ii) $[M : [M : \mathfrak{a}]]$ is the intersection of all principal divisors containing $\mathfrak{a}$. Indeed, for $\mathfrak{a} \subseteq Mw$ we have $[M : [M : \mathfrak{a}]] \subseteq [M : [M : Mw] = Mw$. Conversely, if v lies in all principal divisors containing $\mathfrak{a}$, then for every $u \in [M : \mathfrak{a}]$ we have $v \in Mu^{-1}$ which shows $v \in [M : [M : \mathfrak{a}]]$.

Proposition I.2.3.14. *The following hold:*

(i) *If* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *are M-submodules of K such that* $\mathfrak{b} \subseteq \mathfrak{c}$ *then* $[\mathfrak{a} : \mathfrak{b}] \supseteq [\mathfrak{a} : \mathfrak{c}]$.
(ii) *If* $\mathfrak{a}, \mathfrak{b}$ *are M-submodules of K, then* $[\mathfrak{a} : [\mathfrak{a} : [\mathfrak{a} : \mathfrak{b}]]] = [\mathfrak{a} : \mathfrak{b}]$.
(iii) *If* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ *are M-submodules of K, then* $[\mathfrak{a} : \mathfrak{b}\mathfrak{c}] = [[\mathfrak{a} : \mathfrak{b}] : \mathfrak{c}]$.
(iv) *The relation* $\mathfrak{a} \sim \mathfrak{b}$ *if* $[M : \mathfrak{a}] = [M : \mathfrak{b}]$ *defines a congruence on* $F(M)$.
(v) *The map* $\phi \colon \mathfrak{a} \mapsto \operatorname{div}(\mathfrak{a})$ *induces an isomorphism* $F(M)/\sim \to \operatorname{Div}(M)$ *of partially ordered monoids.*

Proof. For (i) note that any $w \in [\mathfrak{a} : \mathfrak{c}]$ satisfies $w\mathfrak{b} \subseteq w\mathfrak{c} \subseteq \mathfrak{a}$, i.e. $w \in [\mathfrak{a} : \mathfrak{b}]$.

For (ii), we first observe that $\mathfrak{b} \subseteq [\mathfrak{a} : [\mathfrak{a} : \mathfrak{b}]]$. Indeed, for every $b \in \mathfrak{b}$ we obtain $b[\mathfrak{a} : \mathfrak{b}] \subseteq \mathfrak{a}$ because $bc \in \mathfrak{a}$ for all $c \in [\mathfrak{a} : \mathfrak{b}]$. Now, consider $w \in K$ with $w[\mathfrak{a} : [\mathfrak{a} : \mathfrak{b}]] \subseteq [\mathfrak{a}]$. Then

$$w\mathfrak{b} \subseteq w[\mathfrak{a} : [\mathfrak{a} : \mathfrak{b}]] \subseteq [\mathfrak{a}].$$

Conversely, if $w \in [\mathfrak{a} : \mathfrak{b}]$, then $w[\mathfrak{a} : [\mathfrak{a} : \mathfrak{b}]] \subseteq \mathfrak{a}$, because $wc \in \mathfrak{a}$ for all $c \in [\mathfrak{a} : [\mathfrak{a} : \mathfrak{b}]]$.

For (iii), observe that if $w\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a}$, then $wc\mathfrak{b} \subseteq \mathfrak{a}$, i.e. $wc \in [\mathfrak{a} : \mathfrak{b}]$ for all $c \in \mathfrak{c}$ and hence $w\mathfrak{c} \subseteq [\mathfrak{a} : \mathfrak{b}]$, i.e. $w \in [[\mathfrak{a} : \mathfrak{b}] : \mathfrak{c}]$. Conversely, if $w \in [[\mathfrak{a} : \mathfrak{b}] : \mathfrak{c}]$, then $wc \in [\mathfrak{a} : \mathfrak{b}]$, i.e. $wcb \in \mathfrak{a}$ for all $c \in \mathfrak{c}, b \in \mathfrak{b}$ and hence $w\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a}$, i.e. $w \in [\mathfrak{a} : \mathfrak{b}\mathfrak{c}]$.

In (iv) first note that the relation is an equivalence relation because it is defined in terms of an equality which is symmetric and transitive in its arguments. Secondly, if $\mathfrak{a} \sim \mathfrak{b}$ and $\mathfrak{c} \sim \mathfrak{d}$ then we have

$$[M : \mathfrak{a}\mathfrak{c}] = [[M : \mathfrak{a}] : \mathfrak{c}] = [[M : \mathfrak{b}] : \mathfrak{c}] = [[M : \mathfrak{c}] : \mathfrak{b}] = [[M : \mathfrak{d}] : \mathfrak{b}] = [M : \mathfrak{b}\mathfrak{d}].$$

For (v) first note that due to (iv) we always have $\mathfrak{a}\mathfrak{b} \sim \mathrm{div}(\mathfrak{a})\mathrm{div}(\mathfrak{b})$ and hence $\mathrm{div}(\mathfrak{a}\mathfrak{b}) = \mathrm{div}(\mathrm{div}(\mathfrak{a})\mathrm{div}(\mathfrak{b})) = \mathrm{div}(\mathfrak{a}) + \mathrm{div}(\mathfrak{b})$. Thus, ϕ is an isomorphism of magmas, i.e. of sets with binary operation. Consequently, the operation of $\mathrm{Div}(M)$ inherits associativity and commutativity. Clearly, M is neutral in both $F(M)$ and $\mathrm{Div}(M)$. By (i) ϕ also preserves the partial order. The inverse of ϕ is the composition of the inclusion $\mathrm{Div}(M) \subseteq F(M)$ with the quotient map $F(M) \to F(M)/\sim$, both of which preserve the respective partial orders. Thus, ϕ is an isomorphism of partially ordered monoids. □

I.2.4. Complete integral closure and valuations. Here, we treat complete integral closures and valuations, both of which will be relevant to the discussion of the Krull property in the next section.

Definition I.2.4.1. Let $M \subseteq N$ be an inclusion of integral $\mathbb{F}_1$-algebras resp. cancellative monoids. An element $f \in N$ is *almost integral* over M, if there exists $g \in M$ such that $gf^k \in M$ for all $k \geq 0$. The set of almost integral elements of N is denoted $\mathrm{CInt}(M, N)$.

Remark I.2.4.2. $\mathrm{CInt}(M, N)$ is a submonoid resp. $\mathbb{F}_1$-subalgebra of N. If M is a pseudo-face of N then $\mathrm{CInt}(M, N) = M$.

Definition I.2.4.3. A cancellative monoid resp. integral $\mathbb{F}_1$-algebra M is *completely integrally closed* (CIC) if $M = \mathrm{CInt}(M, Q(M))$.

Remark I.2.4.4. If M is CIC and N is a pseudo-face then N is CIC due to Proposition I.2.2.10.

Proposition I.2.4.5. *Let M be an integral $\mathbb{F}_1$-algebra resp. cancellative monoid. Then the saturation of M in $Q(M)$ is contained in* $\mathrm{CInt}(M, Q(M))$*. In particular, complete integral closedness implies normality. If M is noetherian then the converse inclusion also holds.*

Proof. Consider $f = g/h$ with (non-zero) $g, h \in M$. If $f^k = a \in M$ holds for some $k \in \mathbb{N}$ then we have $h^{k-1}f^m \in M$ for $m = 1, \dots, k-1$. For $m > k$ we have $h^{k-1}f^m = h^{k-1}f^{m-k}a \in M$ by induction.

For the converse, suppose that $bf^m \in M$ holds for some $b \in M$ and all $m \in \mathbb{N}$. The chain of ideals $\mathfrak{a}_m := \langle bf^k | 1 \leq k \leq m \rangle$ becomes stationary and hence there exists $n \in \mathbb{N}$ with $bf^{n+1} = cbf^k$ for some $c \in M$ and $1 \leq k < n$ and we conclude $f^{n+1-k} = c \in M$. □

Proposition I.2.4.6. *A cancellative monoid resp. integral $\mathbb{F}_1$-algebra M is CIC if and only if* $\mathrm{Div}(M)$ *is a group, and in that case the inverse of a divisor D is* $[M : D]$.

Proof. If M is CIC consider $D \in \mathrm{Div}(M)$ and set $D' := [M : D]$. Since $DD' \subseteq M$ every principal divisor containing M also contains DD'. Conversely, let $DD' \subseteq Mh$ and set $f := h^{-1}$. Then $fDD' \subseteq M$ and thus $fD \subseteq [M : D'] = D$ which gives $f^nD \subseteq D$ for each $n > 0$ by induction. Now let $d \in D$ and $w \in Q(M)$ with $D \subseteq Mw$. Then $g := w^{-1}d \in M$ and $gf^n \in w^{-1}f^nD \subseteq w^{-1}D \subseteq M$ holds for each $n > 0$, hence $f \in M$ by assumption and thus $M \subseteq Mh$.

For the converse suppose that $\mathrm{Div}(M)$ is a group and let $f \in Q(M)$ and $g \in M$ such that $gf^n \in M$ for all $n \geq 0$. Then $\mathfrak{a} := M\{f^n; n \geq 0\} \subseteq Mg^{-1}$ is fractional and since $(Mf)\mathfrak{a} \subseteq \mathfrak{a}$ and we obtain $\mathrm{div}(f) + \mathrm{div}(\mathfrak{a}) \geq \mathrm{div}(\mathfrak{a})$ and thus $f \in M$. □

REMARK I.2.4.7. Let M be CIC. Then the following hold:

(i) For every $D \in \mathrm{Div}(M)$ we have $\mathrm{Div}(M)_{\geq D} = \mathrm{Div}(M)_{\geq 0} + D$ and hence $D = \mathrm{div}^{-1}(\mathrm{Div}(M)_{\geq 0} + D)$.

(ii) If $D \in \mathrm{Div}(M)_{\geq 0}$ is a prime element then $D \leq_M M$ is a prime ideal.

PROPOSITION I.2.4.8. *Let $M_i, i \in I$ be CIC monoids resp. $\mathbb{F}_1$-algebras with $Q(M_i) = K$. Then $M := \bigcap_i M_i$ is CIC.*

PROOF. Let $f \in K$ be almost integral over M. Then there exists $g \in K$ such that $gf^n \in M$ for all $n \geq 0$. For every i we conclude that $f \in M_i$ and thus $f \in M$. □

DEFINITION I.2.4.9. A cancellative monoid resp. integral $\mathbb{F}_1$-algebra M is a *valuation monoid/$\mathbb{F}_1$-algebra* if for every pair of (non-zero) elements $v, w \in M$ we have $v/w \in M$ or $w/v \in M$ in $Q(M)$.

A *(normed) valuation* on a simple monoid/$\mathbb{F}_1$-algebra K is a (surjective) homomorphism $\nu\colon K^* \to G$ to a totally ordered group G whose group operation by convention is written additively. The associated valuation monoid/$\mathbb{F}_1$-algebra K_ν is $\nu^{-1}(G_{\geq 0})$ resp. $\nu^{-1}(G_{\geq 0}) \cup \{0\}$.

REMARK I.2.4.10. If M is a valuation monoid resp. $\mathbb{F}_1$-algebra, then the canonical map $\nu\colon Q(M)^* \to Q(M)^*/M^*$ is a normed valuation and we have $M = Q(M)_\nu$. If $\nu\colon K^* \to G$ is a valuation then K_ν is a valuation monoid resp. $\mathbb{F}_1$-algebra with $Q(K_\nu) = K$ and $K_\nu^* = \ker(\nu)$. The induced map $Q(K_\nu)^*/K_\nu^* \to G$ is an isomorphism if and only if ν is normed.

REMARK I.2.4.11. Since $\mathrm{PDiv}(M) = Q(M)/M^*$ holds, a cancellative monoid resp. integral $\mathbb{F}_1$-algebra M is a valuation monoid resp. $\mathbb{F}_1$-algebra if and only if $\mathrm{PDiv}(M)$ is totally ordered.

REMARK I.2.4.12. Let K be a simple monoid/$\mathbb{F}_1$-algebra, let $\nu\colon K^* \to G$ be a valuation and let $M \subseteq K_\nu$ be a submonoid/-$\mathbb{F}_1$-algebra. Then $G_{>0} \subseteq G_{\geq 0}$ is a maximal ideal and hence the intersection of $\nu^{-1}(G_{>0})$ resp. $\nu^{-1}(G_{>0}) \cup \{0\}$ with M is prime in M.

DEFINITION I.2.4.13. A valuation monoid/$\mathbb{F}_1$-algebra M is *discrete* if $\mathrm{PDiv}(M)$ is isomorphic to $\mathbb{Z}$ as an ordered group. A valuation ν to $G = \mathbb{Z}$ is a *discrete valuation.*

REMARK I.2.4.14. M is a discrete valuation monoid resp. $\mathbb{F}_1$-algebra if and only if M/M^* is isomorphic to $(\mathbb{N}_0, +)$ resp. the (additive) $\mathbb{F}_1$-algebra $(\mathbb{N}_0, +) \cup \{\infty\}$. In particular, such M is factorial and since $\mathrm{Div}(M) \cong \mathbb{Z}$, M is CIC. Its prime elements, called *uniformizers*, are pairwise associated to each other. They generate the only non-empty prime ideal which consequently is maximal.

REMARK I.2.4.15. We have seen that $\mathrm{Div}(\mathbb{N}_0) = \mathrm{PDiv}(\mathbb{N}_0) = \mathbb{Z}$ and since for every discrete valuation monoid M we have $\mathrm{Div}(M) = \mathrm{Div}(M/M^*) = \mathrm{Div}(\mathbb{N}_0) = \mathbb{Z}$, we obtain that M is in particular CIC.

PROPOSITION I.2.4.16. *Let M be a discrete valuation monoid. Then any further monoid $M \subseteq M' \subsetneq Q(M)$ equals M.*

PROOF. We have $\mathbb{N}_0 = M/M^* \subseteq M'/M^* \subsetneq Q(M)/M^* = Q(M/M^*) = \mathbb{Z}$. Since $\mathbb{N}_0$ is maximal among the proper submonoids of $\mathbb{Z}$ the assertion follows. □

DEFINITION I.2.4.17. Let M be a cancellative monoid resp. integral $\mathbb{F}_1$-algebra and let ν be a discrete valuation on $Q(M)$ with $M \subseteq Q(M)_\nu$. For each fractional ideal $\mathfrak{a}$ we define $\nu(\mathfrak{a})$ as the maximum of all values $\nu(f)$ where f ranges over all (non-zero) elements of $[M : \mathfrak{a}]$.

REMARK I.2.4.18. $\nu(\mathfrak{a})$ is well-defined because for each $a \in \mathfrak{a}$, $\nu(a)$ is an upper bound of the set $\{\nu(f) \mid f \in [M : \mathfrak{a}]\}$. Moreover, we have $\nu(\mathfrak{a}) = \nu(\operatorname{div}(\mathfrak{a}))$. If $\mathfrak{b}$ is a fractional ideal containing $\mathfrak{a}$ then $\nu(\mathfrak{b}) \geq \nu(\mathfrak{a})$.

I.2.5. The Krull property and its characterization. Monoids and $\mathbb{F}_1$-algebras of Krull type and their prime divisors, i.e. the basis elements of their divisor groups, generalize factorial monoids resp. $\mathbb{F}_1$-algebras and also form the basis for the geometric theory of Weil divisors and ultimately, Cox sheaves. We characterize them in terms of their divisor monoids. Moreover, we show that in the presence of noetherianity, e.g. finite generation, the Krull property is equivalent to normality (i.e. saturatedness in the quotient group resp. algebra).

DEFINITION I.2.5.1. A cancellative monoid resp. an integral $\mathbb{F}_1$-algebra M is said to be a *Krull monoid resp. $\mathbb{F}_1$-algebra*, to be *of Krull type* or to possess the *Krull property* if there exist a simple monoid/$\mathbb{F}_1$-algebra K containing M as a submonoid/-$\mathbb{F}_1$-algebra and a family of discrete valuations $\{\nu_i\}_{i\in I}$ on K such that

(i) $M = \bigcap_i K_{\nu_i}$,
(ii) for every (non-zero) $f \in M$ the number of $i \in I$ with $\nu_i(f) \neq 0$ is finite.

The family $\{\nu_i\}_{i\in I}$ is then said to define M in K.

REMARK I.2.5.2. An $\mathbb{F}_1$-algebra M is of Krull type if and only if $M \setminus 0$ is.

EXAMPLE I.2.5.3. If M is a simple monoid/$\mathbb{F}_1$-algebra then $\operatorname{Div}(M) = 0$ and M is of Krull type, defined by the empty family.

REMARK I.2.5.4. For a submonoid/$\mathbb{F}_1$-subalgebra M of Krull type in K defined by $\{\nu_i\}_{i\in I}$ the following hold:

(i) the restricted family $\{\nu_{i|Q(M)}\}_{i\in I}$ defines M in $Q(M)$,
(ii) we have $M^* = \ker(\sum_i \nu_i) = \bigcap_{i\in I} \ker(\nu_i)$ because $\ker(\sum_i \nu_i)$ is a subgroup of M, and conversely $\sum_i \nu_{i|M} \colon M \to \bigoplus_i \mathbb{N}_0$ maps units to units,
(iii) the localization $N^{-1}M$ by a submonoid $N \subseteq M$ is of Krull type and defined in K by those ν_i with $N \subseteq \ker(\nu_i)$. Indeed, if $\nu_i(f) \geq 0$ for all $i \in I$ with $N \subseteq \ker(\nu_i)$ then for a product s over suitable $s_i \in N \setminus \ker(\nu_i)$, where i is such that $\nu_i(f) < 0$, we have $sf \in M$ and $f = sf/s \in N^{-1}M$.

COROLLARY I.2.5.5. *For a family $M_i, i \in I$ of monoids/$\mathbb{F}_1$-algebras of Krull type defined in K by $\{\nu_{i,j}\}_{j\in J_i}$ such that each $f \in \bigcap_i M_i$ is a unit in all but finitely many M_i the intersection $M := \bigcap_i M_i$ is the monoid/$\mathbb{F}_1$-algebra of Krull type defined by $\{\nu_{i,j}\}_{i\in I, j\in J_i}$.*

The Krull property may be characterized in terms of divisor monoids as follows:

PROPOSITION I.2.5.6. *A cancellative monoid resp. integral $\mathbb{F}_1$-algebra M is of Krull type if and only if $\operatorname{Div}(M)$ is a group whose minimal positive elements form a basis.*

By positive elements of a partially ordered monoid we always mean elements which are greater than and different from the neutral element. During the remainder of the section we prove the above. We begin by reformulating the condition on $\operatorname{Div}(M)$.

LEMMA I.2.5.7. *In a partially ordered group G with elements a, b, c, d the following hold:*

(i) *If a is the infimum of $\{c, d\}$ then $a + b$ is the infimum of $\{b + c, b + d\}$.*

(ii) *If $c \in G_{>0}$ is a minimal element and $a, b \in G_{>0}$ then firstly* $\inf(c, a) \in \{0, c\}$ *and secondly, if $c \le a + b$ then $c \le a$ or $c \le b$.*
(iii) *If G is a lattice group (or a group in which each two elements have an upper bound) then $G_{>0}$ generates G.*

Proof. In (i) note that since $a \le c$ and $a \le d$ we have $a + b \le c + b$ and $a + b \le d + b$. If e is a lower bound of $b + c$ and $b + d$, then $e - b \le c$ and $e - b \le d$ which implies $e - b \le a$, i.e. $e \le a + b$.

In (ii) the first statement follows from the fact that a is non-negative and the only non-negative lower bounds of c are 0 and c. Secondly, we observe that if $\inf(c, a) = 0$ then $b = \inf(c + b, a + b) \ge c$.

For (iii) note that if $g \in G$ is not already positive and h is an upper bound of $\{g, 0\}$, then $g = h - (h - g)$ where $h - g > 0$ and $h > 0$. ☐

Proposition I.2.5.8. *For a partially ordered group G the following are equivalent:*

(i) *G has a basis of positive elements, i.e. G is as a partially ordered group isomorphic to $\bigoplus_{i\in I} \mathbb{Z}$ for some set I,*
(ii) *the minimal positive elements of G form a basis,*
(iii) *G is a lattice group and every non-empty set of positive elements of G has minimal elements.*

Proof. Assume that (iii) holds. Let P be the set of minimal elements of $G_{>0}$. If the set S of positive elements which are not a linear combination of elements of P were non-empty, it would have a minimal element g. Since $g \notin P$ there exists $p \in P$ with $p < g$. The element $g - p > 0$ would then be a linear combination of elements in P and so would $g = p + (g - p)$. This contradiction means that S is empty.

For linear independence consider a finite linear combination $a := \sum_{p\in P} \lambda_p p = 0$. Let $I, J \subseteq P$ be the subsets of those $p \in P$ with $\lambda_p > 0$ resp. $\lambda_p < 0$. Assume that I is non-empty. Then $0 < \sum_{q\in I} \lambda_q q = \sum_{p\in J} \lambda_p p$ and hence J is non-empty. For every $q \in I$ we then have $q \le \sum_{p\in J} \lambda_p p$ and hence $q \le p$ for some $p \in J$ by Lemma I.2.5.7 - a contradiction. Therefore, $I = \emptyset$ and in the same way $J = \emptyset$. ☐

In the injection defined below, we use the concept of valuations of fractional ideals from Definition I.2.4.17.

Proposition I.2.5.9. *Let M be a monoid/$\mathbb{F}_1$-algebra of Krull type defined by $\{\nu_i\}_{i\in I}$ in $Q(M)$. Then there is an injection of partially ordered sets*

$$\sum_{i\in I} \nu_i \colon \operatorname{Div}(M) \longrightarrow \bigoplus_{i\in I} \mathbb{Z}, \quad D \longmapsto (\nu_i(D))_{i\in I}$$

and the natural partial order on $\operatorname{Div}(M)$ is the partial order induced by $\bigoplus_i \mathbb{Z}$.

Lemma I.2.5.10. *Let M be a monoid/$\mathbb{F}_1$-algebra of Krull type defined in $Q(M)$ by $\{\nu_i\}_{i\in I}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of M, then $\mathfrak{a}$ is contained in all principal divisors containing $\mathfrak{b}$ if and only if $\nu_i(\mathfrak{a}) \ge \nu_i(\mathfrak{b})$ for all $i \in I$; in particular, we have*

$$\operatorname{div}(\mathfrak{b}) = \{w \in Q(M) | \nu_i(w) \ge \nu_i(\mathfrak{b}) \text{ for all } i \in I\}.$$

Proof. If $\mathfrak{a}$ is contained in all Mw containing $\mathfrak{b}$ then for all $i \in I$ we have

$$\nu_i(\mathfrak{b}) = \max_{\mathfrak{b}\subseteq Mw} \nu_i(w) \le \max_{\mathfrak{a}\subseteq Mw} \nu_i(w) = \nu_i(\mathfrak{a}).$$

Conversely, let $\sum_i \nu_i(\mathfrak{a}) \ge \sum_i \nu_i(\mathfrak{b})$. If $\mathfrak{b} \subseteq Mw$, then for all $a \in \mathfrak{a}$ and $i \in I$ we have $\nu_i(aw^{-1}) \ge \nu_i(a) - \nu_i(\mathfrak{a}) \ge 0$. Thus, $aw^{-1} \in M$ and we conclude $\mathfrak{a} \subseteq Mw$. ☐

Proof of Proposition I.2.5.9. Due to the above Lemma, $\sum_i \nu_i$ is injective, and the natural partial order on $\operatorname{Div}(M)$ is the induced one. ☐

Proposition I.2.5.11. *Let M be a monoid/$\mathbb{F}_1$-algebra of Krull type defined by $\{\nu_i\}_{i\in I}$ in $Q(M)$, let $N \subseteq M$ be a submonoid and let $\mathfrak{a} \leq_M Q(M)$ be a submodule. Then the following hold:*

(i) *For $f \in Q(M)$ we have $N^{-1}\mathfrak{a} \subseteq N^{-1}Mf$ if and only if there exists $s \in N$ with $\mathfrak{a} \subseteq Ms^{-1}f$, in particular $\mathfrak{a}$ is fractional if and only if $N^{-1}\mathfrak{a}$ is fractional.*

(ii) *If $\mathfrak{a}$ is fractional then we have $\nu_i(N^{-1}\mathfrak{a}) = \nu_i(\mathfrak{a})$ for each $i \in I$ with $N \subseteq \ker(\nu_i)$ and $N^{-1}\mathrm{div}_M(\mathfrak{a}) = \mathrm{div}_{N^{-1}M}(N^{-1}\mathfrak{a})$.*

Proof. In (i) suppose that $N^{-1}\mathfrak{a} \subseteq N^{-1}Mf$ and consider the set $J \subseteq I$ of those $j \in I$ such that $N \nsubseteq \ker(\nu_j)$ and $\nu_j(\mathfrak{a}) < \nu_j(f)$. For each $j \in J$ let $s_j \in N \setminus \ker(\nu_j)$ with $\nu_j(\mathfrak{a}) \geq \nu_j(f) - \nu_j(s_j)$. With $s := \prod_{j\in J} s_j$ we then have $\mathfrak{a} \subseteq Mf/s$ because for each $g \in \mathfrak{a}$ and $j \in I$ we obtain $\nu_j(g) \geq \nu_j(f/s)$.

For (ii) let $f \in \mathrm{div}_{N^{-1}M}(N^{-1}\mathfrak{a})$. Then there exists $s_j \in S \setminus \ker(\nu_j)$ for each j with $\nu_j(f) < \nu_j(N^{-1}\mathfrak{a})$. Then the product s over suitable powers of the s_j satisfies $sf \in \mathrm{div}(\mathfrak{a})$ and hence $f \in N^{-1}\mathrm{div}(\mathfrak{a})$. □

Proof of Proposition I.2.5.6, Part I. If M is of Krull type then it is CIC because it is an intersection of discrete valuation monoids/$\mathbb{F}_1$-algebras, meaning $\mathrm{Div}(M)$ is a group. Since $\mathrm{Div}(M)$ allows an injection of partially ordered sets into some $\bigoplus_{i\in I}\mathbb{Z}$ such that the natural partial order on $\mathrm{Div}(M)$ is the induced one, every subset of $\mathrm{Div}(M)_{>0}$ has minimal elements. □

Definition I.2.5.12. Let M be an cancellative monoid resp. integral $\mathbb{F}_1$-algebra whose divisor monoid is a group whose minimal positive elements form a basis. Then this basis is denoted $\mathfrak{P}(M)$ (or $\mathfrak{P}$ if no confusion can arise) and its members are the *prime divisors* of M.

For each $\mathfrak{p} \in \mathfrak{P}$ denote by $pr_{\mathfrak{p}}\colon \mathrm{Div}(M) = \bigoplus_{\mathfrak{p}\in\mathfrak{P}}\mathbb{Z}\mathfrak{p} \to \mathbb{Z}$ the projection onto the $\mathfrak{p}$-th coordinate. The *essential valuation* associated to $\mathfrak{p}$ is then the map $\nu_{\mathfrak{p}} := pr_{\mathfrak{p}} \circ \mathrm{div}$.

Remark I.2.5.13. In the in the above situation, consider a fractional ideal $\mathfrak{a}$ and a prime divisor $\mathfrak{p} \in \mathfrak{P}(M)$. Due to Proposition I.2.3.9 we have

$$\nu_{\mathfrak{p}}(\mathfrak{a}) = pr_{\mathfrak{p}}(\mathrm{div}(\mathfrak{a})) = \inf_{f\in\mathfrak{a}} pr_{\mathfrak{p}}(\mathrm{div}(f)) = \min_{f\in\mathfrak{a}} \nu_{\mathfrak{p}}(f).$$

The following concludes the proof of Proposition I.2.5.6.

Proposition I.2.5.14. *In the situation of Definition I.2.5.12 each $\nu_{\mathfrak{p}}$ is surjective and $\{\nu_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}(M)}$ defines M as a monoid resp. $\mathbb{F}_1$-algebra of Krull type in $Q(M)$.*

Proof. $\nu_{\mathfrak{p}}$ is surjective because there exists $f \in \mathfrak{p}$ with $\nu_{\mathfrak{p}}(f) = pr_{\mathfrak{p}}(\mathfrak{p}) = 1$. Furthermore, we have $M = \mathrm{div}^{-1}(\mathrm{Div}(M)_{\geq 0}) = \bigcap_{\mathfrak{p}} Q(M)_{\nu_{\mathfrak{p}}}$ and for every $f \in M$ the principal divisor $\mathrm{div}(f)$ is a finite sum over the $\mathfrak{p}$, meaning that only finitely many $\nu_{\mathfrak{p}}(f)$ are non-zero. □

Proposition I.2.5.15. *In Proposition I.2.5.9 the map $\sum_i \nu_i$ is an isomorphism if and only if $\{\nu_i\}_i$ are the essential valuations.*

Proof. If $\{\nu_i\}_{i\in I} = \{\nu_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}(M)}$ is the family of essential valuations then Remark I.2.5.13 tells us that the map $\phi\colon D \mapsto (\nu_i(D))_{i\in I}$ is a homomorphism which maps $\mathfrak{p}$ to the basis vector $e_{\mathfrak{p}}$, i.e. an isomorphism. Conversely, if ϕ is an isomorphism then it restricts to a bijection of the sets $\mathfrak{P}(M)$ and $\{e_i\}_{i\in I}$ of minimal positive elements. For $\mathfrak{p} \in \mathfrak{P}(M)$ with $\phi(\mathfrak{p}) = e_i$ we deduce that

$$\nu_i(D) = pr_i(\phi(D)) = pr_{\mathfrak{p}}(D) = \nu_{\mathfrak{p}}(D)$$

holds for every $D \in \mathrm{Div}(M)$. Therefore, $\{\nu_i\}_{i\in I} = \{\nu_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}(M)}$. □

By Proposition I.2.4.5 in presence of noetherianity the criterion of Proposition I.2.5.6 becomes the following.

Corollary I.2.5.16. *A noetherian monoid resp.* $\mathbb{F}_1$*-algebra is of Krull type if and only if it is normal.*

I.2.6. Prime divisors and maps of divisor monoids of monoids and $\mathbb{F}_1$-algebras of Krull type. Given a homomorphism $\phi\colon M \to N$ one may consider the map $\mathrm{Div}(M) \to \mathrm{Div}(N)$ of sets which sends D to $\mathrm{div}(N\phi(D))$. However, this need not be a homomorphism of semigroups. For the case that M and N are of Krull type we define a homomorphism $\mathrm{Div}(M) \to \mathrm{Div}(N)$ in terms of prime divisors. The (non-standard) results we give in Proposition I.2.6.9 on properties of this homomorphism and the respective essential valuations are a crucial preparation of later results on Cox sheaves and characteristic spaces.

Proposition I.2.6.1. *The set of prime divisors defined in Definition I.2.5.12 of a monoid resp.* $\mathbb{F}_1$*-algebra* M *of Krull type has the following descriptions:*

$$\begin{aligned}\mathfrak{P}(M) &= \{\textit{prime elements of } \mathrm{Div}(M)_{\geq 0}\} = \{\mathfrak{q} \trianglelefteq M \mid \mathfrak{q} \textit{ prime, } \mathrm{ht}(\mathfrak{q}) = 1\}\\ &= \{\mathfrak{q} \in \mathrm{Div}(M)_{\geq 0} \mid \mathfrak{q} \textit{ is a prime ideal of } M\}\end{aligned}$$

Remark I.2.6.2. Due to Remark I.2.4.7 each prime divisor $\mathfrak{p} = M \cap \nu_{\mathfrak{p}}^{-1}(\mathbb{Z}_{>0})$ of a Krull monoid M is a prime ideal. Since $M \setminus \mathfrak{p} = M \cap \ker(\nu_{\mathfrak{p}})$ and different prime divisors do not contained one another we have

$$M_{\mathfrak{p}} = \bigcap_{\substack{\mathfrak{q} \in \mathfrak{P}(M)\\ M\setminus\mathfrak{p} \subseteq \ker(\nu_{\mathfrak{q}})}} Q(M)_{\nu_{\mathfrak{q}}} = Q(M)_{\nu_{\mathfrak{p}}}.$$

In particular, $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}_{\mathfrak{p}}) = 1$.

Proof of Proposition I.2.6.1. If $\mathfrak{q} \trianglelefteq M$ is a prime ideal of height 1, then Remark I.2.5.4 gives

$$M_{\mathfrak{q}} = \bigcap_{M\setminus\mathfrak{q} \subseteq \ker(\nu_{\mathfrak{p}})} Q(M)_{\nu_{\mathfrak{p}}}.$$

Since $M_{\mathfrak{q}} \neq Q(M)$, there exists a $\mathfrak{p} \in \mathfrak{P}$ with $M \setminus \mathfrak{q} \subseteq M \cap \ker(\nu_{\mathfrak{p}}) = M \setminus \mathfrak{p}$, i.e. $\mathfrak{p} \subseteq \mathfrak{q}$ and by minimality of $\mathfrak{q}$ we deduce $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P} \subseteq \mathrm{Div}(M)_{>0}$.

If $\mathfrak{q} \in \mathrm{Div}(M)_{>0}$ considered as an ideal of M is prime, then we have

$$\mathfrak{q} = \sum_{\mathfrak{p}} pr_{\mathfrak{p}}(\mathfrak{q})\mathfrak{p} = \mathrm{div}\left(\prod_{\mathfrak{p}} \mathfrak{p}^{pr_{\mathfrak{p}}(\mathfrak{q})}\right) \supseteq \prod_{\mathfrak{p}} \mathfrak{p}^{pr_{\mathfrak{p}}(\mathfrak{q})} = \prod_{\nu_{\mathfrak{p}}(\mathfrak{q})>0} \mathfrak{p}^{pr_{\mathfrak{p}}(\mathfrak{q})},$$

which means that $\mathfrak{q}$ contains some $\mathfrak{p}$, and minimality yields $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}$. □

Proposition I.2.6.3. *The family of essential valuations* $\{\nu_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}(M)}$ *of a Krull monoid* M *is contained in each family of valuations defining* M *in* $Q(M)$.

Proof. Let $\{\nu_j\}_{j\in J}$ be any family defining M in $Q(M)$ and let $\mathfrak{p} \in \mathfrak{P}$. Then by Remark I.2.5.4 we have

$$Q(M)_{\nu_{\mathfrak{p}}} = M_{\mathfrak{p}} = \bigcap_{M\setminus\mathfrak{p}\subseteq\ker(\nu_j)} Q(M)_{\nu_j}$$

and since $M_{\mathfrak{p}} \neq Q(M)$ this cannot be the empty intersection, which means there exists $j \in J$ with $Q(M)_{\nu_{\mathfrak{p}}} \subseteq Q(M)_{\nu_j}$. Now, Proposition I.2.4.16 gives $Q(M)_{\nu_{\mathfrak{p}}} = Q(M)_{\nu_j}$ and hence $\nu_{\mathfrak{p}} = \nu_j$. □

REMARK I.2.6.4. For a submonoid N of a Krull monoid M the canonical maps restrict to a bijection between $\mathfrak{P}(N^{-1}M)$ and the set of those $\mathfrak{p} \in \mathfrak{P}(M)$ with $\mathfrak{p} \cap N = \emptyset$. For each $\mathfrak{q} \in \mathfrak{P}(S^{-1}M)$ we have $Q(M)_{\nu_\mathfrak{q}} = S^{-1}M_\mathfrak{q} = M_{\imath_S^{-1}(\mathfrak{q})} = Q(M)_{\nu_{\imath_S^{-1}(\mathfrak{q})}}$, meaning that $\nu_\mathfrak{q} = \nu_{\imath_S^{-1}(\mathfrak{q})}$.

Consequently, each prime ideal of M is the union of the prime divisors it contains.

REMARK I.2.6.5. Let M be a Krull monoid and let $\mathfrak{p} \in \mathfrak{P}(M)$. Then there exists $f \in M$ such that $\mathfrak{p}_f$ is principal in M_f. Indeed, let g/h be a uniformizer of $M_\mathfrak{p}$ and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \mathfrak{P}(M)$ be those prime divisors which differ from $\mathfrak{p}$ and belong to the support of $\operatorname{div}(g/h)$. Then there exist $f_i \in \mathfrak{p}_i \setminus \mathfrak{p}$ and we have $\mathfrak{p}_{hf_1\cdots f_n} = \langle g/hf_1 \cdots f_n\rangle$.

Next, we treat canonical homomorphisms between divisor monoids of Krull monoids.

REMARK I.2.6.6. Let $\phi\colon M \to N$ be a homomorphism between Krull monoids and let $\mathfrak{a} \leq_M Q(M)$ be a fractional ideal. Then there are only finitely many $\mathfrak{q} \in \mathfrak{P}(N)$ with $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q})$.

Indeed, if $\phi(\mathfrak{a}) \subseteq \mathfrak{q}$ then $\operatorname{div}(N\phi(\mathfrak{a})) \geq \mathfrak{q}$, because $N\phi(a) \geq \mathfrak{q}$ holds for every $a \in \mathfrak{a}$. Consequently, we have $\nu_\mathfrak{q}(\operatorname{div}(N\phi(\mathfrak{a}))) > 0$ and since all but finitely many coordinates of $\operatorname{div}(N\phi(\mathfrak{a}))$ are zero, the assertion follows.

CONSTRUCTION I.2.6.7. Let $\phi\colon M \to N$ be a homomorphism between Krull monoids and denote the induced map also by $\phi\colon Q(M) \to Q(N)$. The *natural homomorphism of divisor monoids* β_ϕ maps $\mathfrak{p} \in \mathfrak{P}(M)$ to the divisor whose $\mathfrak{q}$-th coordinate is $\nu_\mathfrak{q}(N_\mathfrak{q}\phi(\mathfrak{p}_{\phi^{-1}(\mathfrak{q})}))$ if $\operatorname{Cl}(M_{\phi^{-1}(\mathfrak{q})}) = 0$ and zero otherwise. Note that $\nu_\mathfrak{q}(\beta_\phi(\mathfrak{p}))$ is non-zero if and only if $\operatorname{Cl}(M_{\phi^{-1}(\mathfrak{q})}) = 0$ and $\mathfrak{p} \subseteq \phi^{-1}(\mathfrak{q})$.

PROPOSITION I.2.6.8. *For a homomorphism $\phi\colon M \to N$ between Krull monoids with canonical extension $\phi'\colon Q(M) \to Q(N)$, prime divisors $\mathfrak{p} \in \mathfrak{P}(M)$, $\mathfrak{q} \in \mathfrak{P}(N)$ and a fractional $\mathfrak{a} \leq_M Q(M)$ the following hold:*

(i) *If $\operatorname{Cl}(M_{\phi^{-1}(\mathfrak{q})}) = 0$ then we have*

$$\operatorname{pr}_\mathfrak{q}(\beta_\phi(\operatorname{div}(\mathfrak{a}))) = (\nu_\mathfrak{q} \circ \phi')(\mathfrak{a}) = \nu_\mathfrak{q}(N\phi'(\operatorname{div}(\mathfrak{a}))) = \operatorname{pr}_\mathfrak{q}(\operatorname{div}(N\phi'(\operatorname{div}(\mathfrak{a})))),$$

in particular, $\operatorname{pr}_\mathfrak{q} \circ \beta_\phi \circ \operatorname{div}_M = \operatorname{pr}_\mathfrak{q} \circ \operatorname{div}_N \circ \phi'$. Consequently, there exists $D \in \operatorname{Div}(M)$ with $\mathfrak{q} \in \operatorname{supp}(D)$ if and only if $\phi^{-1}(\mathfrak{q})$ is non-empty with $\operatorname{Cl}(M_{\phi^{-1}(\mathfrak{q})}) = 0$.

(ii) *If $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ and $\beta_\phi(\mathfrak{p}) = \mathfrak{q}$ then we have $\nu_\mathfrak{p} = \nu_\mathfrak{q} \circ \phi'$.*

(iii) *If $\nu_\mathfrak{p} = \nu_\mathfrak{q} \circ \phi'$ then we have $\nu_\mathfrak{p}(\mathfrak{a}) = \nu_\mathfrak{q}(N\phi'(\mathfrak{a}))$.*

PROOF. In (i) let $\mathfrak{P}'$ be the set of all $\mathfrak{p}' \in \operatorname{supp}(\operatorname{div}(\mathfrak{a}))$ with $\mathfrak{p}' \subseteq \phi^{-1}(\mathfrak{q})$. For each $\mathfrak{p}' \in \mathfrak{P}'$ fix $g_{\mathfrak{p}'} \in Q(M)$ with $\mathfrak{p}'_{\phi^{-1}(\mathfrak{q})} = M_{\phi^{-1}(\mathfrak{q})}g_{\mathfrak{p}'}$. Then the product g over all $g_{\mathfrak{p}'}^{\nu_{\mathfrak{p}'}(\mathfrak{a})}$ satisfies $\operatorname{div}(\mathfrak{a})_{\phi^{-1}(\mathfrak{q})} = M_{\phi^{-1}(\mathfrak{q})}g$ because we have $\nu_{\mathfrak{p}'}(\operatorname{div}(\mathfrak{a})_{\phi^{-1}(\mathfrak{q})}) = \nu_{\mathfrak{p}'}(g)$ for all $\mathfrak{p}' \in \mathfrak{P}'$ by Proposition I.2.5.11. Thus, we calculate

$$\begin{aligned}
\operatorname{pr}_\mathfrak{q}(\beta_\phi(\operatorname{div}(\mathfrak{a}))) &= \sum_{\mathfrak{p}' \in \mathfrak{P}'} \nu_{\mathfrak{p}'}(\mathfrak{a})\nu_\mathfrak{q}(\phi'(g_{\mathfrak{p}'})) = \nu_\mathfrak{q}(\phi'(g)) = (\nu_\mathfrak{q} \circ \phi')(M_{\phi^{-1}(\mathfrak{q})}g) \\
&= (\nu_\mathfrak{q} \circ \phi')(\operatorname{div}(\mathfrak{a})_{\phi^{-1}(\mathfrak{q})}) = (\nu_\mathfrak{q} \circ \phi')(\operatorname{div}(\mathfrak{a})) = (\nu_\mathfrak{q} \circ \phi')(\mathfrak{a}) \\
&= \nu_\mathfrak{q}(\phi'(g)) = \nu_\mathfrak{q}(N_\mathfrak{q}\phi'(M_{\phi^{-1}(\mathfrak{q})}g)) = \nu_\mathfrak{q}(N_\mathfrak{q}\phi'(\operatorname{div}(\mathfrak{a})_{\phi^{-1}(\mathfrak{q})})) \\
&= \min_{f \in \operatorname{div}(\mathfrak{a})_{\phi^{-1}(\mathfrak{q})}} \nu_\mathfrak{q}(\phi'(f)) = \min_{f \in \operatorname{div}(\mathfrak{a})} \nu_\mathfrak{q}(\phi'(f)) = \nu_\mathfrak{q}(N\phi'(\operatorname{div}(\mathfrak{a}))).
\end{aligned}$$

In (ii) we calculate $\operatorname{pr}_\mathfrak{q}(\operatorname{div}_N(\phi'(f))) = \operatorname{pr}_\mathfrak{q}(\beta_\phi(\operatorname{div}_M(f)) = \operatorname{pr}_\mathfrak{p}(\operatorname{div}_M(f))$. For (iii) we calculate

$$\nu_\mathfrak{p}(\mathfrak{a}) = \min_{g \in \phi'(\mathfrak{a})} \nu_\mathfrak{q}(g) = \min_{h \in N\phi'(\mathfrak{a})} \nu_\mathfrak{q}(h) = \nu_\mathfrak{q}(N\phi'(\mathfrak{a})).$$

$\square$

PROPOSITION I.2.6.9. *For a morphism $\phi\colon M \to N$ between monoids of Krull type and its extension $\phi'\colon Q(M) \to Q(N)$ the following hold:*

(i) *β_ϕ restricts to a bijection $\mathfrak{P}' := \mathfrak{P}(M) \setminus \ker(\beta_\phi) \to \mathfrak{P}(N)$ if and only if there exists a defining family $\{\nu_i\}_{i\in I}$ of N in $Q(N)$ such that all $\nu_i \circ \phi'$ are pairwise different essential valuations of M. In this situation, the inverse of $(\beta_\phi)_{|\langle \mathfrak{P}'\rangle}$ sends $E \in \mathrm{Div}(N)_{\geq 0}$ to the ideal $\phi^{-1}(E)$ and $E \in \mathrm{Div}(N)$ to $\sum_i \nu_i(E)\phi^{-1}(\nu_i^{-1}(\mathbb{N}))$. Moreover, we have $\nu_i = \nu_{\beta_\phi(\phi^{-1}(\nu_i^{-1}(\mathbb{N})))}$.*

(ii) *$\mathrm{Cl}(M_{\phi^{-1}(\mathfrak{q})}) = 0$ holds for every $\mathfrak{q} \in \mathfrak{P}(N)$ and β_ϕ restricts to an injection $\mathfrak{P}(M) \to \mathfrak{P}(N)$ if and only if there exists a defining family $\{\nu_i\}_{i\in I}$ of M in $Q(M)$ and an injection $d\colon I \to \mathfrak{P}(N)$ such that $\nu_i = \nu_{d(i)} \circ \phi'$ holds for each $i \in I$, and $M \subseteq \ker(\nu_\mathfrak{q} \circ \phi')$ holds for each $\mathfrak{q} \in \mathfrak{P}(N) \setminus \mathrm{im}(d)$.*

(iii) *β_ϕ restricts to a bijection $\mathfrak{P}(M) \to \mathfrak{P}(N)$ if and only if N has a defining family $\{\mu_\mathfrak{p}\}_{\mathfrak{p}\in\mathfrak{P}(M)}$ in $Q(N)$ such that $\nu_\mathfrak{p} = \mu_\mathfrak{p} \circ \phi'$.*

PROOF. Suppose the latter condition in (i) holds. Then the preimage of $E \in \mathrm{Div}(N)_{\geq 0}$ under ϕ is the set of $f \in Q(M)$ such that $\nu_{\mathfrak{q}\circ\phi'}(f) \geq \nu_\mathfrak{q}(E)$ holds for each $\mathfrak{q} \in \mathfrak{P}(N)$ and $\nu_\mathfrak{p}(f) \geq 0$ holds for each $\mathfrak{p} \in \mathfrak{P}(M) \cap \ker(\beta_\phi)$. In particular, we have $\phi^{-1}(\mathfrak{q}) = M \cap (\nu_\mathfrak{q} \circ \phi')^{-1}(\mathbb{N}) \in \mathfrak{P}(M)$ for each $\mathfrak{q} \in \mathfrak{P}(N)$, which gives the first condition.

Since ν_i is non-trivial, there exists a $\mathfrak{q} \in \mathfrak{P}(N)$ which lies in $\mathfrak{a}_i := N \cap \nu_i^{-1}(\mathbb{N})$. Then $\mathfrak{p}_\mathfrak{q} = \phi^{-1}(\mathfrak{q}) \subseteq \phi^{-1}(\mathfrak{a}_i) = M \cap (\nu_i \circ \phi')^{-1}(\mathbb{N}) =: \mathfrak{p}_i$ which means $\mathfrak{p}_\mathfrak{q} = \mathfrak{p}_i$ and hence $\nu_\mathfrak{q} \circ \phi' = \nu_i \circ \phi'$, i.e. $\nu_\mathfrak{q} = \nu_i$.

Suppose that in (ii) the latter condition holds. Let $\{\nu_i\}_i$ be the essential valuations of M. For $\mathfrak{p} \in \mathfrak{P}(M)$ we have $\phi^{-1}(d(\mathfrak{p})) = \phi^{-1}(N \cap \nu_{d(\mathfrak{p})}^{-1}(\mathbb{N})) = M \cap \nu_\mathfrak{p}^{-1}(\mathbb{N}) = \mathfrak{p}$. For $\mathfrak{q} \in \mathfrak{P}(N) \setminus \mathrm{im}(d)$ we have $\phi^{-1}(\mathfrak{q}) = \emptyset$ which shows the first condition. $\square$

PROPOSITION I.2.6.10. *Let $\phi\colon M \to M'$ be a homomorphism between Krull monoids and let $N \subseteq M$ and $N' \subseteq M'$ be submonoids with $\phi(N) \subseteq N'$ and let $\imath_N\colon M \to N^{-1}M$ be the localization map. Then the following hold:*

(i) *$\beta_{\imath_N}(D) = N^{-1}D$ holds for each $D \in \mathrm{Div}(M)$. In particular, the induced map $\mathrm{Cl}(M) \to \mathrm{Cl}(N^{-1}M)$ is surjective.*

(ii) *There is a commutative diagram of natural maps of divisor monoids*

$$\begin{tikzcd}
\mathrm{Div}(M) \arrow[rr, "\beta_\phi"] \arrow[d, "\beta_{\imath_N}"] \arrow[drr, "\beta_{\imath_{N'}\circ\phi}"] & & \mathrm{Div}(M') \arrow[d, "\beta_{\imath_{N'}}"] \\
\mathrm{Div}(N^{-1}M) \arrow[rr, "\beta_{N^{-1}(\imath_{N'}\circ\phi)}"] & & \mathrm{Div}(N'^{-1}M')
\end{tikzcd}$$

PROOF. In (i) note that since each $\mathfrak{q} \in \mathfrak{P}(N)$ satisfies $\mathrm{Cl}(M_{\imath_N^{-1}(\mathfrak{q})}) = 0$ we obtain $\beta_{\imath_N}(D) = \mathrm{div}(\langle \imath_N(D)\rangle_{N^{-1}M}) = N^{-1}D$ using Proposition I.2.5.11.

For (ii) let $\mathfrak{q} \in \mathfrak{P}(N'^{-1}M')$ and $\mathfrak{p} \in \mathfrak{P}(M)$ with $\mathfrak{p} \cap N = \emptyset$. Then the equalities $M_{\phi^{-1}(\imath_{N'}^{-1}(\mathfrak{q}))} = N^{-1}M_{N^{-1}(\imath_{N'}\circ\phi)(\mathfrak{q})}$ and $\mathfrak{p}_{\phi^{-1}(\imath_N^{-1}(\mathfrak{q}))} = N^{-1}\mathfrak{p}_{N^{-1}(\imath_{N'}\circ\phi)(\mathfrak{q})}$ give the assertion. $\square$

In light of the above, we will also write $pr_{N^{-1}M}$ instead of $\beta_{\imath_N}$.

REMARK I.2.6.11. Let $M_i, i \in I$ be a family of monoids/$\mathbb{F}_1$-algebras. Then the coproduct $M := \coprod_i M_i$ (for monoids, this is the direct sum) is of Krull type if and only if all M_i are. Indeed, extending the essential valuations of each M_i trivially to $Q(M) = \coprod_i Q(M_i)$ defines a family of valuations realizing M in $Q(M)$ and these are the essential valuations. Conversely, restricting a defining family of M to $Q(M_i)$ realizes $M_i = M \cap Q(M_i)$ as a monoid/$\mathbb{F}_1$-algebra of Krull type.

Moreover, we have a bijection $\bigsqcup_i \mathfrak{P}(M_i) \to \mathfrak{P}(M)$ sending $\mathfrak{p} \in \mathfrak{P}(M_i)$ to $M\mathfrak{p}$. The monomorphisms β_i of divisor monoids corresponding to the inclusions $M_i \subseteq M$ fit together to an isomorphism $\coprod_i \mathrm{Div}(M_i) \to \mathrm{Div}(M)$. Since we always have $\beta_i(\mathrm{PDiv}(M_i)) = \mathrm{im}(\beta_i) \cap \mathrm{PDiv}(M)$ the induced map $\coprod_i \mathrm{Cl}(M_i) \to \mathrm{Cl}(M)$ is an isomorphism.

Proposition I.2.6.12. *Let M be a cancellative monoid which is generated by its irreducible elements. Then each prime element $p \in M$ defines a discrete valuation ν_p on $Q(M)$ via $\nu_p(m) := \sup\{k \in \mathbb{N}_0 \mid p^k | m\}$ and $\nu_p(m/n) := \nu_p(m) - \nu_p(n)$.*

Proof. For $m = q_1 \cdots q_d$ with irreducible elements q_i, $\nu_p(m)$ is equal to the number of indices for which p divides q_i, i.e. is associated to q_i. □

Lemma I.2.6.13. *Let M be a monoid/$\mathbb{F}_1$-algebra such that each ascending chain of principal ideals is stationary. Then M is generated by its units and irreducible elements.*

Proof. Suppose that the set S of proper principal ideals generated by a (non-zero) element which is no product of irreducible elements is non-empty. Then, by assumption it has a maximal element Ma whose generator a is in particular not irreducible. So there are non-units b, c with $a = bc$ and we have $Ma \subsetneq Mb$ and $Ma \subsetneq Mc$. By maximality of Ma the elements b and c are products of irreducible elements. But then, so is a - a contradiction. □

Remark I.2.6.14. A Krull monoid M satisfies the ascending chain for principal ideals. Indeed, any ascending chain of principal ideals corresponds to a descending chain of positive principal divisors. The second chain becomes stationary and hence so does the first.

Proposition I.2.6.15. *Let M be a cancellative monoid and let $N \subseteq M$ be a submonoid generated by a set of prime elements of M and units of M. Then M is a Krull monoid if and only if M is generated by its units and irreducible elements and $N^{-1}M$ is a Krull monoid. Moreover, in these cases the canonical map $\mathrm{Cl}(M) \to \mathrm{Cl}(N^{-1}M)$ is an isomorphism.*

Proof. Let $P \subset N$ be a set of pairwise non-associated prime elements of M which together with a set of units of M generate N. We show that M is the intersection of $N^{-1}M$ and the Krull monoid defined by $\{\nu_p\}_{p \in P}$. Let $n = up_1^{k_1} \cdots p_d^{k_d} \in N$ with $p_i \in P$ and $u \in M^*$, and let $m \in M$. If $\nu_p(m/n) \geq 0$ holds for all $p \in P$ then $p_i | m$ and inductively we obtain $n | m$, i.e. $m/n \in M$.

With respect to the class groups, consider $D \in \mathrm{Div}(M)$ such that $\beta_{\iota_N}(D) = \mathrm{div}_{N^{-1}M}(f)$ with some $f \in Q(M)$. Then each $\mathfrak{p} \in \mathrm{supp}(D - \mathrm{div}_M(f))$ intersects N non-trivially and hence contains an element $p \in N$ which is prime in M. By minimality, we have $\mathfrak{p} = \langle p \rangle = \mathrm{div}(p)$ and thus, $D \in \mathrm{PDiv}(M)$. □

Proposition I.2.6.16. *A cancellative monoid M is factorial if and only if M is a Krull monoid with $\mathrm{Cl}(M) = 0$.*

Proof. If M is factorial, then the canonical map $\mathrm{Cl}(M) \to \mathrm{Cl}(Q(M))$ is an isomorphism by Proposition I.2.6.15. Conversely, if M is a Krull monoid with $\mathrm{Cl}(M) = 0$ then $M/M^* \cong \mathrm{PDiv}(M)_{\geq 0} = \mathrm{Div}(M)_{\geq 0}$ is factorial and hence M is factorial. □

Example I.2.6.17. Let $K \subset \mathbb{Z}^2$ be the subgroup generated by $(1,2), (2,1)$. Then $M := \mathbb{K} \cap \mathbb{N}^2$ is a Krull monoid and the divisor homomorphism is the inclusion $K \to \mathbb{Z}^2$. Thus, $\mathrm{Cl}(M)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ and hence M is not factorial. This shows that the Approximation Theorem for Krull rings has no analogon for Krull monoids.

REMARK I.2.6.18. If each ideal of a cancellative monoid resp. an integral $\mathbb{F}_1$-algebra M is principal, then each fractional ideal is also principal. In particular, we then have $\mathrm{Cl}(M) = 0$. If additionally, M is of Krull type, then M is factorial by the above Proposition.

PROPOSITION I.2.6.19. *For a cancellative monoid M the following are equivalent:*

(i) *M is a discrete valuation monoid,*
(ii) $\mathrm{Div}(M) = \mathbb{Z}$,
(iii) $\mathrm{PDiv}(M) = \mathbb{Z}$,
(iv) *M is factorial with $|\mathfrak{P}(M)| = 1$.*

PROOF. Equivalence of (i) and (ii) is due to the criterion for Krull monoids. If M is a discrete valuation monoid, then $M/M^* \cong \mathbb{N}_0$ is factorial with $|\mathfrak{P}(M)| = 1$. Moreover, (vi) implies (iii) because by the previous Proposition M is a Krull monoid and therefore $\mathrm{Div}(M)$ is generated by the unique element of $\mathfrak{P}(M)$, and again by factoriality we have $\mathrm{Div}(M) = \mathrm{PDiv}(M)$. If (iii) holds then $\mathrm{div}\colon Q(M) \to \mathrm{PDiv}(M) = \mathbb{Z}$ is a valuation and we have $M = \mathrm{div}^{-1}(\mathrm{PDiv}(M)_{\geq 0}) = \mathrm{div}^{-1}(\mathbb{Z}_{\geq 0})$, i.e. M is a discrete valuation monoid. □

I.2.7. Regular $\mathbb{F}_1$-algebras and the Auslander-Buchsbaum-Theorem. The classical Auslander-Buchsbaum-Theorem states that (noetherian) regular local rings are factorial. We show that for integral noetherian $\mathbb{F}_1$-algebras the analogous statements as well as its converse also hold. Geometrically, this is reflected in the fact that an affine quasi-toric variety is smooth if and only if it is (globally) factorial. Recall from Example I.1.1.15 that in the setting of modules over $\mathbb{F}_1$-algebras to form the quotient by a submodule is to contract that submodule to a single point.

REMARK I.2.7.1. Let A be an integral $\mathbb{F}_1$-algebra whose maximal ideal we denote $\mathfrak{m} := A \setminus A^*$. Then $\mathfrak{m}/\mathfrak{m}^2$ is a free $A/\mathfrak{m}$-module. More precisely, $\mathfrak{m}/\mathfrak{m}^2$ is the coproduct over principal $(A/\mathfrak{m})$-submodules $(A/\mathfrak{m})[v]$ for certain $v \in \mathfrak{m} \setminus \mathfrak{m}^2$, each of which is isomorphic to $A/\mathfrak{m}$ as an $(A/\mathfrak{m})$-module.

Indeed, consider $a, b \in A^*$ and $v, w \in \mathfrak{m} \setminus \mathfrak{m}^2$. If $[a][v] = [b][v]$ then $[a][v] \neq [0]$ implies $av = bv$ and cancellation gives $a = b$. Thus, the orbit map $A/\mathfrak{m} \to (A/\mathfrak{m})[v]$ is an isomorphism. Secondly, if $[a][v] = [b][w] \neq [0]$ then we have $v = a^{-1}bw$ and $w = b^{-1}av$ which gives $(A/\mathfrak{m})[v] = (A/\mathfrak{m})[w]$.

PROPOSITION I.2.7.2. *Let A be an integral $\mathbb{F}_1$-algebra with maximal ideal $\mathfrak{m}$. If $\{v_i\}_{i\in I}$ is minimal among the subsets of A which together with A^* generate A as an $\mathbb{F}_1$-algebra then $\{\overline{v_i}\}_{i\in I}$ is an $(A/\mathfrak{m})$-basis of $\mathfrak{m}/\mathfrak{m}^2$.*

PROOF. Due to minimality we have $\{v_i\}_{i\in I} \subseteq \mathfrak{m}$. Suppose v_j belongs to $\mathfrak{m}^2$, i.e. it is the product of two elements $u \prod_i v_i^{k_i}$ and $v \prod_i v_i^{l_i}$ where u, v are units and only a finite but non-zero number of each k_i and l_i are non-zero. If $k_j + l_j$ were greater than 0 we had $uvv_j^{k_j+l_j-1} \prod_{i\neq j} v_i^{k_i+l_i} = 1$ - a contradiction to the minimality requirement. But then v_j is a product of the other v_i and elements of A^* - again in contradiction to the minimality requirement. We conclude that each v_j belongs to $\mathfrak{m} \setminus \mathfrak{m}^2$. Each non-zero element of $\mathfrak{m} \setminus \mathfrak{m}^2$ is of the form $v_i u$ with $u \in A^*$. Thus $\mathfrak{m} \setminus \mathfrak{m}^2 = \bigcup_i A^* v_i$ and this union is disjoint due to the minimality requirement. □

PROPOSITION I.2.7.3. *Let A be an integral noetherian $\mathbb{F}_1$-algebra with maximal ideal $\mathfrak{m}$. If the elements $v_1, \ldots, v_n$ satisfy $\mathfrak{m} = Av_1 \cup \ldots \cup Av_n$ then together with A^* they generate A as an $\mathbb{F}_1$-algebra.*

PROOF. For $v \in \mathfrak{m}$ let M be the set of those principal ideals $\mathfrak{a}$ of A for which there exist $a \in \mathfrak{a}$ with $\mathfrak{a} = Aa$ as well as $k_1, \dots, k_n \in \mathbb{N}_0$ with $v = av_1^{k_1} \cdots v_n^{k_n}$. M is non-empty since v lies in some Av_i. By noetherianity M has a maximal element $\mathfrak{b}$. Let $a \in \mathfrak{b}$ and $k_1, \dots, k_n \in \mathbb{N}_0$ with $\mathfrak{b} = Aa$ and $v = av_1^{k_1} \cdots v_n^{k_n}$. By maximality a cannot lie in any Av_i, hence it is a unit. □

COROLLARY I.2.7.4. *For an integral noetherian $\mathbb{F}_1$-algebra A with maximal ideal $\mathfrak{m}$ there exists a finite minimal set $\{v_i\}_{i=1}^n$ of elements who together with A^* generate A as an algebra and whose classes form a basis of $\mathfrak{m}/\mathfrak{m}^2$.*

DEFINITION I.2.7.5. An integral noetherian $\mathbb{F}_1$-algebra A with maximal ideal $\mathfrak{m}$ is called *regular* if the Krull dimension $\dim(M)$ is equal to the number of non-zero $(A/\mathfrak{m})$-orbits of $\mathfrak{m}/\mathfrak{m}^2$.

PROPOSITION I.2.7.6. *An integral noetherian $\mathbb{F}_1$-algebra A with maximal ideal $\mathfrak{m}$ is regular if and only if it is factorial.*

PROOF. Set $B := A/\mathfrak{m}$ and suppose that A is regular. By Corollary I.2.7.4 there exist $v_1, \dots, v_r \in A$ which together with A^* generate A as an $\mathbb{F}_1$-algebra and whose classes form a basis of $\mathfrak{m}/\mathfrak{m}^2$. Since $\dim(A) = r$ there exists an ascending chain of length r of prime ideals of A. Taking complements produces a chain $A^* = \tau_0 \subsetneq \ldots \subsetneq \tau_r = A \setminus 0$ of faces of A. Each τ_i is generated as a monoid by A^* together with those v_j which lie in τ_i. After reordering of the elements $v_1, \dots, v_r$ we may thus assume that τ_i is generated as a monoid by $v_1, \dots, v_i$ and A^*. We claim that $v_1, \dots, v_r$ are irreducible. Let $v_i = u \prod_{i=1}^r v_i^{k_i} v \prod_{i=1}^r v_i^{l_i}$ with $u, v \in A^*$ and $k_i, l_i \in \mathbb{N}_0$. By minimality of $\{v_1, \dots, v_r\}$ we have $k_i + l_i > 0$. Since none of the v_j are units we must have $k_i + l_i = 1$ and $k_j + l_j = 0$ for $j \neq i$.

For uniqueness consider $u \prod_i v_i^{k_i} = v \prod_i v_i^{l_i}$. After full cancellation we may assume that at least one of k_i and l_i is zero for each i. Suppose that there exists j with $k_j + l_j \neq 0$ and that j is maximal with that property. We may assume that $k_j \neq 0$. Since v_j is no unit there also exists a maximal $h < j$ with $l_h \neq 0$. Then $u \prod_{i=1}^n v_i^{k_i} \in \tau_h$ gives $v_j \in \tau_h$ - a contradiction. Thus, after cancellation all k_i and l_i must be zero, i.e. we have $u = v$. This shows factoriality of A.

Conversely, if A is factorial consider a prime system P of A. Since the elements of P are pairwise non-associated, P is minimal among the subsets of A which together with A^* generate A as an $\mathbb{F}_1$-algebra. By Proposition I.2.7.2 $\{[p]\}_{p \in P}$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$. In particular, $|P|$ is finite. On the other hand, each proper face τ of A is generated as a monoid by A^* and those $p \in P$ with $p \in \tau$, and to faces coincide if and only if they contain the same elements of P. Therefore, the length $\dim(M)$ of a maximal properly ascending chain of proper faces of A is equal to $|P|$ and we conclude that A is regular. □

CHAPTER II

Graded algebra and divisibility theory of graded rings

In this chapter we give algebraic preparations for several geometric concepts like Veronesean good quotients as well as (invariant) Weil divisors and Cox sheaves on graded schemes resp. quasi-torus actions. We also study graded factoriality, which is an important property of Cox rings. The objects under investigation are graded monoids, $\mathbb{F}_1$-algebras and rings as well as graded algebras and modules over them. The general approach will be to globally fix a graded monoid/$\mathbb{F}_1$-algebra or ring A as a base, formulate statements on algebras or modules over A and differentiate between the cases in the proofs. Here, proofs for the monoid/$\mathbb{F}_1$-algebra cases may be entirely ommitted whenever the grading structure is irrelevant and the statement was proven in Chapter I. The results of the present chapter were partly published by the author in [**5**]

II.1. graded rings and their modules

After basic definitions and canonical constructions in Section II.1.1 we study morphisms and constructions which are peculiar to the graded setting in Section II.1.2. In particular, we introduce and characterize component-wise bijective graded morphisms which will later be a defining feature of Cox sheaves. We give preparations such as the appropriate notions of limits needed to define structure sheaves of graded schemes in Section II.1.3. Graded monoid algebras, colimits and tensor products are discussed in Sections II.1.4, II.1.5 and II.1.6, respectively. Next to further preparations on homogeneously prime and radical ideals and localization Section II.1.8 studies the behaviour of graded ideals under Veronese subalgebras in order to make sense of Veronesean good quotients of graded schemes. We also prove the graded version of Hilbert's basis theorem which states that polynomial rings over K-noetherian rings are K-noetherian, see Section II.1.7.

II.1.1. categories of graded monoids, $\mathbb{F}_1$-algebras, rings and their algebras and modules. In the definitions below, the coproduct is taken in the category of sets, pointed sets or abelian groups, respectively.

Definition II.1.1.1. A *graded monoid/$\mathbb{F}_1$-algebra/ring* is a monoid,$\mathbb{F}_1$-algebra or ring A together with a decomposition $A = \coprod_{w \in gr(A)} A_w$, indexed by an abelian group $gr(A)$, into subsets/pointed subsets/additive subgroups, such that we have $1_A \in A_0$ and $A_v A_w \subseteq A_{v+w}$ for all $v, w \in gr(A)$.

A *morphism* of such objects is a homomorphism $\phi\colon A \to B$ of monoids/$\mathbb{F}_1$-algebras/rings with an accompanying homomorphism $\psi\colon gr(A) \to gr(B)$ such that $\phi(A_w) \subseteq B_{\psi(w)}$ holds for all $w \in gr(A)$. The category thus defined is denoted $\mathbf{GrMon}/\mathbf{GrAlg}_{\mathbb{F}_1}/\mathbf{GrRing}$.

The category of objects under a fixed object A of $\mathbf{GrMon}$, $\mathbf{GrAlg}_{\mathbb{F}_1}$ or $\mathbf{GrRing}$ is denoted $\mathbf{GrAlg}_A$. The objects of $\mathbf{GrAlg}_A$ are called *graded A-algebras* or *graded algebras over A*.

Definition II.1.1.2. Let A be a graded monoid/$\mathbb{F}_1$-algebra/ring. A *graded A-module* is an A-module M together with an $gr(A)$-module structure γ_M on a set $gr(M)$ and a decomposition $M = \coprod_{w \in gr(M)} M_w$ into subsets/pointed subsets/subgroups such that we have $A_v M_w \subseteq M_{\gamma_M(v,w)}$ for $v \in gr(A), w \in gr(M)$.

A morphism from a graded A-module M to a graded A-module N is a morphism of A-modules $\phi\colon M \to N$ together with a morphism $\psi\colon gr(M) \to gr(N)$ of $gr(A)$-modules such that $\phi(M_w) \subseteq N_{\psi(w)}$ holds for $w \in gr(M)$. The category thus defined is denoted $\mathbf{GrMod}_A$.

Definition II.1.1.3. For a graded A-algebra/-module B, $gr(B)$ is the *grading object* of B and will also be called the *grading group/set* if $gr(A) = \{0\}$.

Example II.1.1.4. Let A be $\{1\}$, $\mathbb{F}_1$ or $\mathbb{Z}$. For an arbitrary A-algebra/-module R and an abelian group K we obtain a *trivial* K-grading by setting $R_{0_K} := R$ and defining each further R_w as the initial A-module.

Remark II.1.1.5. Let A be a graded algebra over $\{1\}$, $\mathbb{F}_1$ or $\mathbb{Z}$ and let B be the underlying algebra over $\{1\}$, $\mathbb{F}_1$ or $\mathbb{Z}$, respectively. Then the following hold:

(i) The forgetful functor from graded A-algebras/-modules to B-algebras/-modules preserves initial and final objects.
(ii) The functor sending a graded A-algebra/-module to its grading object preserves initial and final objects.
(iii) By Lemma A.0.0.4 the functor gr is right adjoint to the faithful functor tr sending K to the initial B-algebra/-module, endowed with the canonical K-grading.

Next, we turn to subcategories of the various categories defined at the beginning.

Definition II.1.1.6. A morphism in one of the above categories is called *degree-preserving* if the accompanying map is an identity map. For a fixed group K the objects of $\mathbf{GrMod}$, $\mathbf{GrAlg}_{\mathbb{F}_1}$ resp. $\mathbf{GrRing}$ with grading group K together with degree-preserving morphisms form subcategories $\mathbf{GrMon}^K$, $\mathbf{GrAlg}_{\mathbb{F}_1}^K$ and $\mathbf{GrRing}^K$, respectively.

For a fixed graded monoid/$\mathbb{F}_1$-algebra/ring A and $gr(A)$-algebra resp. -module γ the category of graded A-algebras/-modules accompanied by γ with degree-preserving morphisms is denoted $\mathbf{GrMod}_A^\gamma$ resp. $\mathbf{GrAlg}_A^\gamma$. If in the above γ is an inclusion of subgroups then we use the upper index K instead of γ.

Definition II.1.1.7. Let $\mathfrak{C}$ denote $\mathbf{GrMon}$, $\mathbf{GrAlg}_{\mathbb{F}_1}$ or $\mathbf{GrRing}$. Let A be a $\mathfrak{C}$-object. A *graded submonoid/$\mathbb{F}_1$-subalgebra/subring* is a $\mathfrak{C}$-object supported on a subset of A, such that the inclusion and $\mathrm{id}_{gr(A)}$ form a $\mathfrak{C}$-morphism. The category formed by these objects together with degree-preserving inclusion maps is denoted $\mathbf{GrSubMon}(A)/\mathbf{GrSubAlg}_{\mathbb{F}_1}(A)/\mathbf{GrSubRing}(A)$.

Let $\mathfrak{D}$ denote $\mathbf{GrAlg}_A$ resp. $\mathbf{GrMod}_A$. A *graded A-subalgebra/-module* of a $\mathfrak{D}$-object B is a $\mathfrak{D}$-object supported on a subset of B, such that the inclusion and $\mathrm{id}_{gr(B)}$ form a $\mathfrak{D}$-morphism. The category formed by these objects together with degree-preserving inclusion maps is denoted $\mathbf{GrSubAlg}_A(B)$ resp. $\mathbf{GrSubMod}_A(B)$.

Example II.1.1.8. A graded ring A is a graded module over itself. Its graded submodules are then called *graded ideals* and their category is denoted $\mathbf{GrId}(A)$.

Example II.1.1.9. For a graded submodule N of $M \in \mathbf{GrMod}_A$ the annihilator $\mathrm{Ann}(N)$ is a graded ideal of A.

Remark II.1.1.10. Let A denote i) $\mathbb{Z}$ or ii) $\mathbb{F}_1$ and let B denote i) $\mathbb{F}_1$ or ii) $\{1\}$. Let $\mathfrak{C}$ denote $\mathbf{GrAlg}_A$ resp. $\mathbf{GrMod}_A$ and let $\mathfrak{D}$ denote $\mathbf{GrAlg}_B$ resp. $\mathbf{GrMod}_B$. Let $\mathfrak{K}$ denote the category of i) abelian groups or ii) pointed sets and let $\mathfrak{L}$ denote the category of i) pointed set or ii) sets. Let $\mathfrak{f}\colon \mathfrak{K} \to \mathfrak{L}$ be the forgetful functor.

Then sending a $\mathfrak{C}$-object C to $\mathfrak{hom}(C) := \coprod_{w\in gr(C)} \mathfrak{f}(C_w)$ defines a functor to $\mathfrak{D}$. For a fixed graded A-algebra R this induces a functor $\mathbf{GrMod}_R \to \mathbf{GrMod}_{\mathfrak{hom}(R)}$. In case i) we also denote $\mathfrak{hom}$ by $(\cdot)^{\mathrm{hom}}$ and call it the functor of homogeneous elements.

Definition II.1.1.11. Let $B = \coprod_{w\in gr(B)} B_w$ be a graded monoid, $\mathbb{F}_1$-algebra or ring resp. a module over one of the former.

(i) B_w is the *w-homogeneous component* or *homogeneous component of degree w*, its (non-zero) elements are called *w-homogeneous* or *homogeneous of degree w*.
(ii) For each subset $K \subseteq gr(B)$ denote by $B_K = \coprod_{w\in gr(B)} B_w$ the coproduct (in the category of sets, pointed sets or abelian groups, respectively) over all homogeneous components of degree in K.
(iii) The union over all homogeneous components forms the set B^{hom} of homogeneous elements.
(iv) If B is a graded ring or a module over one then for an arbitrary element f the image f_w under the projection onto the w-homogeneous component is the *w-homogeneous part* of f.

Remark II.1.1.12. For a graded ring R we have $(R^{\mathrm{hom}})^* = R^{\mathrm{hom}} \cap R^*$, because multiplicatively inverse elements of homogeneous elements are homogeneous. Moreover, the requirement $1_R \in R_0$ is superfluous because $(1_R)_0 r = r$ holds for all homogeneous (and hence all) elements r of R.

Remark II.1.1.13. Let $\mathfrak{C}$ be the category of sets/pointed sets or abelian groups, let A be $\{1\}$, $\mathbb{F}_1$ or $\mathbb{Z}$, equipped with the trivial grading by $gr(A) = \{0\}$, let K be a $gr(A)$-module and let $w \in K$. Then due to Lemma A.0.0.2 the functor $(\cdot)_w \colon \mathbf{GrMod}_A^K \to \mathfrak{C}$ assigning the w-homogeneous component is left adjoint to the faithful functor sending G to the graded module whose w-homogeneous component is G and whose other homogeneous component are $\emptyset$ resp. $\{0\}$. Moreover, for a fixed K-graded A-module M the restriction of $(\cdot)_w$ to $\mathbf{GrSubMod}_A(M)$ maps to $\mathbf{SubMod}_A(M_w)$ sending a graded submodule $N \le M$ to N_w is left adjoint to the faithful functor sending $G \le M_w$ to the graded submodule of M defined by G. Furthermore, $(\cdot)_w$ commutes with intersections.

Remark II.1.1.14. Let A be a (0-graded) monoid/$\mathbb{F}_1$-algebra/ring. Let $\mathfrak{C}$ denote $\mathbf{Alg}_A$ or $\mathbf{Mod}_A$ and let $\mathfrak{D}$ denote the corresponding one of $\mathbf{GrAlg}_A$ and $\mathbf{GrMod}_A$. Then the following hold:

(i) $\mathfrak{C}$ embedds as a full subcategory into $\mathfrak{D}$, by endowing C with the trivial grading by the 0-algebra 0 resp. the 0-module $\{pt\}$. Due to Lemma A.0.0.2 this inclusion functor is right adjoint to the forgetful functor.
(ii) If $\mathfrak{C} = \mathbf{Alg}_A$ then equipping a $\mathfrak{C}$-object C with the trivial K-grading embedds $\mathfrak{C}$ as a full subcategory into $\mathbf{GrAlg}_A^K$.

Definition II.1.1.15. Let B be a graded monoid/$\mathbb{F}_1$-algebra/ring or a module over one of the former.

(i) The *degree map* deg, also written $\deg_K$ for $K = gr(B)$, sends each non-zero homogeneous element to its degree in $gr(B)$.
(ii) The set $\mathrm{degsupp}(B) := \mathrm{im}(\deg)$ of degrees with non-trivial homogeneous component is the *degree support* of B.

Remark II.1.1.16. Let $\mathfrak{C}$ be the category of inclusion morphisms $S \subseteq G$ of submonoids S of simple monoids G resp. of subsets S in groups G with $\mathfrak{C}$-morphisms being group homomorphisms $G \to G'$ which restrict to homomorphisms $S \subseteq S'$ of the submonoids resp. to maps of the subsets.

Then endowing $S \subseteq G$ with the G-grading obtained by taking the inclusion map as the degree map defines a faithful functor from $\mathfrak{C}$ to (simple) graded 1-algebras/-modules which by Lemma A.0.0.2 is right adjoint to the functor sending N to $\deg(N) \subseteq gr(N)$. The restrictions to (simple) subalgebras/-submodules of a fixed group K on the one hand and K-graded (simple) 1-algebras/-modules on the other again define an adjoint pair.

Remark II.1.1.17. Let A be a graded monoid/$\mathbb{F}_1$-algebra/ring and let B be an A-algebra/-module. Then multiplication with an element/unit $a \in A_v$ defines an homo-/isomorphism $B_w \to B_{vw}$ of A_0-modules. Moreover, deg restricts to a homomorphism on the set of homogeneous units of A.

Next, we deal with graded congruences and submodules. Congruences for monoids, $\mathbb{F}_1$-algebras and modules over them were discussed in Section I.1.1.

Definition II.1.1.18. A *congruence* on ring R is an equivalence relation $\sim$ on R such that $a \sim a'$ and $b \sim b'$ imply $a+b \sim a'+b'$ and $ab \sim a'b'$ for all $a, a', b, b' \in R$.

A *congruence* on an R-module M is an equivalence relation $\sim$ on M such that $u \sim u'$ and $v \sim v'$ imply $u + v \sim u' + v'$ and $ru \sim ru'$ for all $u, u', v, v' \in M$ and $r \in R$.

Remark II.1.1.19. For a congruence on an R-module M the elements which are equivalent to 0_M form a submodule of M. Conversely, the pairs of elements whose difference lies in a given submodule of M form a congruence of M. This constitutes an inclusion preserving bijection between congruences on M and submodules of M.

Remark II.1.1.20. Arbitrary intersections of congruences are again congruences. The intersection over all congruences containing a given set of pairs is the congruence *generated* by these pairs.

Definition II.1.1.21. Let A be a graded monoid/$\mathbb{F}_1$-algebra/ring and let M be an algebra/module over A. A congruence $\sim \subseteq M \times M$ is *graded* if it is generated as a congruence by pairs of homogeneous elements whose coordinates belong to the same homogeneous component.

Remark II.1.1.22. For a congruence $\sim$ on a graded A-algebra/-module M the following hold:

(i) If A is a graded monoid then $\sim$ is graded if and only if deg is constant on each equivalence class.
(ii) If A is a graded $\mathbb{F}_1$-algebra then $\sim$ is graded if and only if deg is constant on each non-zero equivalence class.
(iii) If A is a graded ring then $\sim \subseteq M \times M$ is graded if, whenever we have $\sum_w f_w \sim \sum_w g_w$ then $f_w \sim g_w$ holds for all the homogeneous parts.

Remark II.1.1.23. The kernel relation of each degree-preserving morphism is graded. Conversely, the quotient map associated to a graded congruence is a degree-preserving morphism.

Remark II.1.1.24. Let $N \subseteq A$ be a submonoid of homogeneous elements and let B be an A-algebra/-module. Let $\sim_N$ be the congruence generated by all pairs (b, nb) where $b \in B$ and $n \in N$. If $N \subseteq A_0$ then $\sim_N$ is graded. If A is a graded ring and, then the submodule corresponding to $\sim_N$ is generated by all terms $1 - n$ for $n \in N$.

Remark II.1.1.25. Subalgebras or -modules of an algebra or module over a graded monoid/$\mathbb{F}_1$-algebra A are canonically graded. If A is a graded ring and B is a graded A-algebra/-module then for a subalgebra resp. submodule C the following are equivalent:

(i) C carries the structure of a graded subring resp. a graded submodule,
(ii) C is the sum of its intersections with the homogeneous parts,
(iii) C contains the homogeneous parts of all its elements,
(iv) C is generated as a subring resp. submodule by homogeneous elements.

In all cases, C_w is the intersection of C with B_w.

Indeed, if (i) holds then the supplement follows from the fact that for elements of C the homogeneous parts with respect to the gradings of C and B are the same. (iii) implies (ii) because products of homogeneous elements are homogeneous.

Corollary II.1.1.26. *A submodule $\mathfrak{m}$ of a graded module M over a graded ring A carries the structure of a graded submodule if and only if the corresponding congruence $\sim_{\mathfrak{m}}$ is graded.*

Corollary II.1.1.27. *Sums of graded submodules are graded submodules. Products of graded ideals are graded ideals. For an R-algebra $R \to S$ the quotient $[\mathfrak{a}\colon \mathfrak{b}]$ of two graded R-submodules $\mathfrak{a}, \mathfrak{b}$ of S is again graded.*

Construction II.1.1.28. Let M be a graded module over a graded ring R. For an arbitrary R-submodule N of M the graded submodule

$$N^{\mathrm{gr}} := \langle N \cap M^{\mathrm{hom}} \rangle = \bigoplus_{w \in gr(M)} N \cap M_w$$

has the same homogeneous elements as N and is maximal among all graded submodules contained in N.

Remark II.1.1.29. For a morphism $\phi\colon M \to M'$ of graded R-modules and *any* R-submodule $N' \leq_R M'$ we have $\phi^{-1}(N^{\mathrm{gr}})^{\mathrm{gr}} = \phi^{-1}(N)^{\mathrm{gr}}$.

Definition II.1.1.30. An ideal $\mathfrak{a}$ of the $\mathbb{F}_1$-algebra R^{hom} is said to be *closed under partial addition* if for every $w \in K$ and all $r, r' \in R_w \cap \mathfrak{a}$ we have $r + r' \in \mathfrak{a}$. Such an ideal is also called a *sesquiad ideal* of R^{hom} or, more precisely an *ideal of the sesquiad* (R^{hom}, R).

Remark II.1.1.31. Intersections of sesquiad ideals of are again sesquiad ideals.

Construction II.1.1.32. Let $\mathfrak{a}_i, i \in I$ be a family of sesquiad ideals of R^{hom}.

(i) The sum $\sum_{i \in I} \mathfrak{a}_i$ is the intersection over all sesquiad ideals containing $\bigcup_{i \in I} \mathfrak{a}_i$.
(ii) In case I is finite, the product $\prod_{i \in I} \mathfrak{a}_i$ is the intersection over all sesquiad ideals containing each product $\prod_{i \in I} a_i$ where $a_i \in \mathfrak{a}_i$.

Remark II.1.1.33. By [**13**] a sesquiad is a pair (M, R) where M is an $\mathbb{F}_1$-subalgebra of the ring R which generates R as a ring. Sesquiads offer a unified theory for $\mathbb{F}_1$-algebras, rings and graded rings, and possibly many others. However, sesquiad homomorphisms between graded rings might not automatically be also graded homomorphisms.

Proposition II.1.1.34. *Let R be a K-graded ring. Then there are mutually inverse canonical bijections*

$$\begin{aligned} \{\textit{sesquiad-ideals of } R^{\mathrm{hom}}\} &\longleftrightarrow \mathbf{GrSubMod}_R(R) \\ \mathfrak{a} &\longmapsto \langle \mathfrak{a} \rangle \\ \mathfrak{b} \cap R^{\mathrm{hom}} &\longleftarrow\!\shortmid \mathfrak{b} \end{aligned}$$

Remark II.1.1.35. A graded monoid may also be conceptualised as a homomorphism $\delta\colon A \to K$ of monoids to a simple K such that A is the coproduct, i.e. the disjoint union, over all $\delta^{-1}(w)$ for all (non-absorbing) $w \in K$. Here, we denote K multiplicatively. Morphisms may be thought of as pairs of homomorphisms such that the resulting diagrams of monoids commute.

Remark II.1.1.36. If every non-trivial homogeneous component of a graded $\mathbb{F}_1$-algebra A contains a non-zero divisor then $\deg(A \setminus 0)$ is a monoid.

Remark II.1.1.37. Let $\pi\colon A \to B, \psi\colon gr(A) \to gr(B)$ be a morphism of graded rings. A B-module/-algebra N with accompanying map γ defines an A-module supported on N with accompanying map $\psi \circ \gamma$ by composing the scalar multiplication with $\pi \times \mathrm{id}_N$ resp. the map of graded rings with π. This defines a functor $(\pi, \psi)^*$ from (constantly) graded B-algebras/-modules to (constantly) graded A-algebras/-modules.

If π and ψ are surjective then by Lemma A.0.0.2 $(\pi, \psi)^*$ is right adjoint to the functor $(\pi, \psi)_*$ which sends a graded A-module/-algebra M with accompanying map γ the quotient $M/\langle \ker(\psi)M \rangle$ together with the induced map $\delta\colon gr(B) \to gr(M)/\gamma(\ker(\psi))$.

II.1.2. coarsenings and component-wise bijections. Recall that left adjoints preserve colimits, in particular coproducts and cokernels, and epimorphisms whereas right adjoints preserve limits, in particular categorial products and kernels, and monomorphisms. Throughout this entire section let A be a fixed graded monoid, $\mathbb{F}_1$-algebra or ring.

Construction II.1.2.1. Let $\psi\colon L \to K$ be a morphism of $gr(A)$-algebras/-modules whose $gr(A)$-algebra structure rep. scalar multiplications we denote by λ and κ. Let $\mathfrak{C}$ denote $\mathbf{GrAlg}_A^\kappa$ or $\mathbf{GrMod}_A^\kappa$ and let $\mathfrak{D}$ denote $\mathbf{GrAlg}_A^\lambda$ resp. $\mathbf{GrMod}_A^\lambda$. The *coarsening* functor $\mathrm{co}_\psi\colon \mathfrak{D} \to \mathfrak{C}$ sends D to $D = \coprod_{w \in K} D_{\psi^{-1}(w)}$. There is a canonical morphism (id_D, ψ) of graded A-algebras/-modules which has the property that each morphism $(\alpha, \psi)\colon D \to C$ factors uniquely into a composition of (id_D, ψ) and a $\mathfrak{C}$-morphism $\mathrm{co}_\psi(D) \to C$, the latter being (α, id_K).

The *augmentation* or *lifting* functor $\mathrm{aug}_\psi\colon \mathfrak{C} \to \mathfrak{D}$ sends C to $\coprod_{v \in L} C_{\psi(v)}$. There is an induced morphism $\mathrm{aug}_\psi(C) \to C$ with accompanyment ψ which on homogeneous components is given by the identity $\mathrm{aug}_\psi(C)_v \to C_{\psi(v)}$. It has the property that each morphism $(\alpha, \psi)\colon D \to C$ of graded A-algebras/-modules factors uniquely into a composition a the $\mathfrak{D}$-morphism $D \to \mathrm{aug}_\psi(C)$ and the canonical morphism $\mathrm{aug}_\psi(C) \to C$. Specifically, the first morphism is given on homogeneous components via $D_v \xrightarrow{\alpha} C_{\psi(v)} = \mathrm{aug}_\psi(C)_v$. The unique factoring properties give rise to an adjunction and we have an adjoint pair $(\mathrm{co}_\psi, \mathrm{aug}_\psi)$.

The above adjunction is due to [**24**].

Remark II.1.2.2. Let $\phi\colon R \to R', \psi\colon gr(R) \to gr(R')$ be a graded morphism and let $\mathfrak{a}'$ be a $gr(R')$-graded ideal of R'. Then $\phi^{-1}(\mathfrak{a}')$ is a $gr(R')$-graded ideal with respect to the $gr(R')$-grading of R induced by ψ.

Construction II.1.2.3. Let B be a graded A-algebra/-module. For a $gr(A)$-subalgebra/-module M of $gr(B)$ the *Veronese A-subalgebra/-module* is the coproduct B_M over all B_w where $w \in M$, endowed with the canonical M-grading. We also say that $B_M \subseteq B$ is Veronesean.

By Lemma A.0.0.2 the functor sending a $gr(A)$-subalgebra/-module M of $gr(B)$ to B_M is right adjoint to the functor sending an A-subalgebra/-module to the $gr(A)$-subalgebra/-module generated by its degree support.

Definition II.1.2.4. A morphism $(\phi, \psi)\colon B \to C, gr(B) \to gr(C)$ of graded A-algebras/-modules is called Veronesean if it defines an isomorphism onto $C_{\mathrm{im}(\psi)}$. A graded A-algebra $A \to B$ is a *Veronesean algebra* if its structure morphism is Veronesean.

Definition II.1.2.5. Let R be a graded A-algebra and let $G \subseteq gr(R)$ be a subgroup. Then R is called *G-associated* if for each $f \in R^{\mathrm{hom}}$ there exists $g \in (R^{\mathrm{hom}})^*$

with $fg \in R_G$, i.e. we have $\operatorname{degsupp}(R) \subseteq \deg((R^{\mathrm{hom}})^*) + G$, or equivalently, $\langle \operatorname{degsupp}(R) \rangle + G = \deg((R^{\mathrm{hom}})^*) + G$.

PROPOSITION II.1.2.6. *Let A be a graded monoid and let $K \leq gr(A)$ be a subgroup. Then the following hold:*

(i) *$A_K \subseteq A$ is a* pseudo-face *in the sense of I.2.2.9,*
(ii) *$A_K A^*$ is a face of resp. equal to A if and only if* $\deg(A_K) + \deg(A^*)$ *is a face of resp. equal to* $\deg(A)$.

PROOF. Assertion (i) follows from Remark I.2.2.11. For (ii), let $A_K A^*$ be a face of A. Let $w = \deg(r), w' = \deg(r') \in \deg(A)$ such that $w+w' \in \deg(A_K)+\deg(A^*)$. Let $v \in \deg(A^*)$ with $w + w' + v \in \deg(A_K)$ and let $a \in A^*$ with $\deg(a) = v$. Then $arr'A_{w+w'+v}$ and the face property implies $a, r, r' \in A_K A^*$. Thus, there exist $t, t' \in A_K$ and $s, s' \in A^*$ with $r = ts$ and $r' = t's'$, and we conclude $w = \deg(t) + \deg(s), w' = \deg(t') + \deg(s') \in \deg(A_K) + \deg(A^*)$.

For the converse, let $r, r' \in A$ with $rr' = ts$ where $t \in A_K$ and $s \in A^*$. Then $\deg(r) + \deg(r') \in \deg(A_K) + \deg(A^*)$ and the face property implies that $\deg(r) = w + v$ and $\deg(r') = w' + v'$ with $w, w' \in \deg(A_K)$ and $v, v' \in \deg(A^*)$. Let $a, a' \in A^*$ with $v = \deg(a)$ and $v' = \deg(a')$. Then $ar \in A_w$ and $a'r' \in A_{w'}$ which gives $r = ara^{-1} \in A_K A^*$ and $r' = a'r'a'^{-1} \in A_K A^*$.

If $\deg(A_K) + \deg(A^*) = \deg(A)$ holds then for $f \in A$ we have $\deg(f) = w + \deg(a)$ with certain $w \in K, a \in A^*$ and consequently, $f = (fa^{-1})a \in A_K A^*$. □

PROPOSITION II.1.2.7. *Let A be a graded monoid/$\mathbb{F}_1$-algebra, let $K \leq gr(A)$ be a subgroup and denote by M and M_K the monoids generated by* $\operatorname{degsupp}(A)$ *and* $\operatorname{degsupp}(A_K)$*, respectively.*

If $\tau := M_K + \deg(A^*)$ is a face of M then $0 \neq fg \in A_K A^*$ with $f, g \in A$ always implies $f, g \in A_K A^*$. Conversely, if $A_K A^* \preceq A$ then τ is a face of M.

PROOF. $\deg(f) + \deg(g) = \deg(fg) \in \deg(A_K A^* \setminus 0) = M_K + \deg(A^*)$ implies $\deg(f), \deg(g) \in M_K + \deg(A^*)$. Thus, there exist $a, b \in A^*$ such that $\deg(f) + \deg(a) \in M_K$ and $\deg(g) + \deg(b) \in M_K$ which implies $af, bg \in A_K$ and hence $f, g \in A_K A^*$.

Suppose that $A_K A^* \preceq A$. Let $b = \sum_w k_w w$ and $c = \sum_w l_w w$ be finite non-negative linear combinations of elements of $\operatorname{degsupp}(A)$, whose sum belongs to τ. For $w \in \operatorname{degsupp}(A)$ with $k_w + l_w \neq 0$ choose a non-zero $f_w \in A_w$. Then the product over all these f_w belongs to $A_K A^*$ and by assumption $f_w = g_v^{(w)} u^{(w)}$ holds with certain $g_v^{(w)} \in A_v \setminus \{0\}$ and $u^{(w)} \in A^*$, where $v \in K$. Thus, $w \in \tau$ and in particular, $a, b \in \tau$. □

REMARK II.1.2.8. Let R be a graded ring and let $K \leq gr(R)$ be a subgroup. If $g \in R_K$ and $h \in R^{\mathrm{hom}}$ satisfy $0 \neq gh \in R_K$, then $h \in R_K$.

DEFINITION II.1.2.9. A morphism $\pi\colon R \to S, \psi\colon gr(R) \to gr(S)$ of graded A-algebras/-modules is a *component-wise bijection* (CB) (resp. a *component-wise bijective epimorphism* (CBE)) if for each $w \in gr(R)$ the induced morphism

$$R_{gr(A)w} \xrightarrow{\pi} S_{gr(A)\psi(w)}, \quad gr(A)w \xrightarrow{\psi} gr(A)\psi(w)$$

of graded A-modules is an isomorphism (and ψ is surjective).

REMARK II.1.2.10. A graded morphism $\pi\colon R \to S, \psi\colon gr(R) \to gr(S)$ of graded A-algebras/-modules is a CB if and only if for each $w \in gr(R)$ the restrictions $gr(A)w \to gr(A)\psi(w)$ and $R_w \to S_{\psi(w)}$ are bijective.

REMARK II.1.2.11. Let $\pi\colon R \to S, \psi\colon gr(R) \to gr(S)$ be a CB of graded A-algebras/-modules.

(i) For each $w \in gr(R)$ we have $gr(A)_w = gr(A)_{\psi(w)}$ for the respective stabilizer subgroups.
(ii) For each $r \in R^{\text{hom}}$ we have $(A^{\text{hom}})_r = (A^{\text{hom}})_{\pi(r)}$ for the respective stabilizer submonoids. Moreover, the orbit map $\text{orb}_r \colon A^{\text{hom}} \to A^{\text{hom}}r$ is injective if and only if $\text{orb}_{\pi(r)}$ is injective.

REMARK II.1.2.12. Let $\imath\colon A \to R, \jmath\colon gr(A) \to gr(R)$ be a graded A-algebra and let $\pi\colon R \to S, \psi\colon gr(R) \to gr(S)$ be a CB of graded A-algebras. Then we have $(\psi \circ \jmath)^{-1}(\psi(w)) = \jmath^{-1}(w)$ and $(\pi \circ \imath)^{-1}_{|A^{\text{hom}}}(\psi(r)) = \imath^{-1}_{|A^{\text{hom}}}(r)$ for $w \in gr(R)$ and $r \in R^{\text{hom}}$, the second equation following from the first and bijectivity on homogeneous parts, the first from bijectivity on $gr(A)$-orbits. In particular, $(\imath, \jmath)$ is a monomorphism if and only if $(\pi \circ \imath, \psi \circ \jmath)$ is a monomorphism.

PROPOSITION II.1.2.13. *Let $\pi\colon S \to R, \psi\colon gr(S) \to gr(R)$ be a morphism of graded monoids/$\mathbb{F}_1$-algebras/rings and denote by N the set of homogeneous preimages of 1_R. Then (π, ψ) is a CB if and only if* $\text{im}(\pi) = R_{\text{im}(\psi)}$ *and there exists a subgroup $N' \subseteq N$ such that* $\deg\colon N' \to \ker(\psi)$ *is bijective and the kernel relation of ϕ is the congruence $\sim_{N'}$ defined by N'. Moreover, in this case N' equals N.*

PROOF. First, consider the cases of graded monoids/$\mathbb{F}_1$-algebras. If (π, ψ) form a CB then $\deg\colon N \to \ker(\psi)$ is bijective, and whenever $s \in S_v, t \in S_w$ satisfy $\pi(s) = \pi(t)$ the element $u \in N_{v-w}$ satisfies $\pi(s) = \pi(tu)$ and hence $s = tu$.

Conversely, suppose that $\deg\colon N' \to \ker(\psi)$ is surjection of groups and $\sim_{N'}$ equals the kernel relation of π. For $s, t \in S_v$ with $\pi(s) = \pi(t)$ we then have $sN' = tN'$. Then there exists $u \in N'_0$ with $s = tu$ and since $\deg(1) = \deg(u)$ we have $u = 1$. For the supplement, note that for $u \in N$ there exists $u' \in N'_{\deg(u)}$ and we conclude $u = u'$.

If (π, ψ) is a CB of graded rings then denote the element of N_v by u_v. For $s = \sum_{v \in \ker(\psi)} s_{w+v} \in \ker(\pi)$ with $s_{w+v} \in S_{w+v}$ we then have

$$s = \sum_{v \in \ker(\psi)} s_{w+v}(1_S - u_{-v}) + \sum_{v \in \ker(\psi)} s_{w+v}u_{-v} \quad \in \langle 1_S - u \mid u \in N \rangle$$

because the right-hand summand belongs to $S_w \cap \ker(\pi) = \{0\}$.

If (π, ψ) is a morphism of graded rings such that $\deg\colon N' \to \ker(\psi)$ is bijective and $\sim_{N'}$ equals the kernel relation of π, denote the element of N'_v by u_v. Let $s \in S_w \cap \ker(\pi)$. Then there exist $s^{(u)}_{w'} \in S_{w'}$ for $u \in N' \setminus \{1\}$ and $w' \in \psi^{-1}(\psi(w))$ such that $s = \sum_{u,w'} s^{(u)}_{w'}(1-u)$. Substituting each summand as

$$s^{(u)}_{w'}(1-u) = u_{w-w'}s^{(u)}_{w'}(1 - u^{-1}_{w-w'}u) - u_{w-w'}s^{(u)}_{w'}(1 - u^{-1}_{w-w'})$$

we obtain a sum $s = \sum_u a^{(u)}(1-u)$ where $a^{(u)} \in S_w$. w-Homogeneity of s then implies $0 = s_{w+\deg(u)} = -a^{(u)}u$, i.e. $a^{(u)} = 0$ holds for each u which gives $s = 0$. Now consider $r \in R_{\psi(w)}$. Then there are $s^{(v)} \in S_{w+v}$ for $v \in \ker(\psi)$ such that $\pi(\sum_v s^{(v)}) = r$ and we conclude $\sum_v s^{(v)}u_{-v} \in S_w \cap \pi^{-1}(r)$. $\square$

REMARK II.1.2.14. For a CB $\phi\colon S \to R, \psi\colon gr(S) \to gr(R)$ of graded monoids resp. $\mathbb{F}_1$-algebras the following hold:

(i) In the case of $\mathbb{F}_1$-algebras, 0 is the only element mapped to 0.
(ii) (ϕ, ψ) defines a CBE onto $R_{\text{im}(\psi)}$,
(iii) $\psi^{-1}(\text{degsupp}(R)) = \text{degsupp}(S)$, $\psi(\text{degsupp}(S)) = \text{im}(\psi) \cap \text{degsupp}(R)$.
(iv) the canonical map $S/S^* \to R/R^*$ is injective with image $R_{\text{im}(\psi)}R^*/R^*$ and we have $\psi^{-1}(\deg(R^*)) = \deg(S^*)$ and $\psi(\deg(S^*)) = \text{im}(\psi) \cap \deg(R^*)$.
(v) If R is simple then so is S. Conversely, if S is simple then so is $R_{\text{im}(\psi)}$.

REMARK II.1.2.15. Let $\pi\colon S \to R$ be a CBE accompanied by ψ. If $S' \subseteq S$ is a graded subring then $\pi_{|S'}\colon S' \to \pi(S')$ is again a CBE accompanied by ψ. If $R' \subseteq R$

is a graded subring then $\pi_{|\pi^{-1}(R')^{\mathrm{gr}}}\colon \pi^{-1}(R')^{\mathrm{gr}} \to R'$ is a CBE with accompanying map ψ.

PROPOSITION II.1.2.16. *Consider the commutative diagram of graded morphisms*

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, "\pi_R"'] & S \arrow[d, "\pi_S"] \\ R' \arrow[r, "\phi'"] & S' \end{tikzcd}$$

and suppose that π_R and π_S are CBs accompanied by the same map ψ and that ϕ and ϕ' are degree-preserving. Then the following hold:

(i) *Injectivity/surjectivity of ϕ' implies injectivity/surjectivity of ϕ. The converse holds if ψ is surjective.*
(ii) *For each $w \in gr(R) = gr(S)$ we have $\phi(R)_w = \pi_S^{-1}(\phi'(R')) \cap S_w$.*

PROOF. We use the induced commutative diagram:

$$\begin{tikzcd} R_w \arrow[r, "\phi"] \arrow[d, "\pi_R"', "\cong"] & S_w \arrow[d, "\pi_S", "\cong"'] \\ R'_{\psi(w)} \arrow[r, "\phi'"] & S'_{\psi(w)} \end{tikzcd}$$

Assertion (i) now follows from the fact that for degree-preserving morphisms injectivity resp. surjectivity conditions need only be checked for homogeneous elements. □

PROPOSITION II.1.2.17. *Let $\pi_R\colon R \to R'$ be a morphism of graded monoids/$\mathbb{F}_1$-algebras/rings, let S and S' be graded algebras/modules over R resp. R' and let $\pi_S\colon S \to S'$ be a morphism of graded algebras/modules over $\{1\}/\mathbb{F}_1/\mathbb{Z}$ such that $\pi_S(rs) = \pi_R(r)\pi_S(s)$ holds for all $r \in R, s \in S$, and we have $\psi_S \circ \gamma = \gamma' \circ \psi_R$ for the respective accompanying maps.*

Then by Lemma A.0.0.2 the functor α from graded R-subalgebras/-modules of S to graded R'-subalgebras/-modules of S' which sends $\mathfrak{a}$ to $\langle \pi_S(\mathfrak{a})\rangle_{R'} = \langle \pi_S(\mathfrak{a}^{\mathrm{hom}})\rangle_{R'}$ is left adjoint to the functor β which sends $\mathfrak{b}$ to $\pi_S^{-1}(\mathfrak{b})^{\mathrm{gr}}$, and we have $\mathfrak{a} \subseteq \beta(\alpha(\mathfrak{a}))$ and $\alpha(\beta(\mathfrak{b})) \subseteq \mathfrak{b}$. Moreover, the following hold:

(i) *In the case of graded algebras α commutes with multiplicative products of subalgebras/-modules.*
(ii) *If $S'^{\mathrm{hom}} = (R'^{\mathrm{hom}})^* \pi_S(S^{\mathrm{hom}})$ then $\alpha \circ \beta$ is the identity, in particular, α is surjective and β is injective.*
(iii) *Suppose that π_R and π_S are CBs of monoids/$\mathbb{F}_1$-algebras/rings, and that $\gamma'^{-1}(\mathrm{im}(\psi_S)) \subseteq \mathrm{im}(\psi_R)$ holds. Then with respect to subalgebras and submodules $\beta \circ \alpha$ is the identity, in particular, α is injective and β is surjective. Furthermore, β then commutes with sums of submodules.*
(iv) *If α and β are mutually inverse then both commute with intersections, sums and finite products.*

PROOF. For (ii) let $\mathfrak{a} \leq_R S$ be a graded subalgebra/-module and consider a (non-zero, homogeneous) element $s \in \pi_S^{-1}(\langle \phi(\mathfrak{a})\rangle_{R'})$. Then $\pi_S(s) = \sum_{i=1}^n r_i' \pi_S(a_i)$ holds with (non-zero, homogeneous) $a_i \in \mathfrak{a}$ and $r_i' \in R'$ which we may choose such that $\gamma'(\deg(r_i')) + \psi_S(\deg(a_i)) = \psi_S(\deg(s))$. In the case of monoids/$\mathbb{F}_1$-algebras we have $n = 1$. Thus, we have $r_i' \in R'_{\gamma'^{-1}(\mathrm{im}(\psi_S))} \subseteq R'_{\mathrm{im}(\psi_R)} = \mathrm{im}(\pi_R)$, i.e. there exist (homogeneous) $r_i \in R$ with $\pi_R(r_i) = r_i'$. Let $u_i \in S$ be homogeneous units such that $\pi_S(u_i) = 1$ and $\deg(\phi(r_i)u_i a_i) = \deg(s)$. Then $\sum_i \phi(r_i)u_i a_i \in \mathfrak{a}$ and s have the same degree and image under π_S and hence coincide.

To see that β preserves sums of submodules consider $\mathfrak{b}_i \leq_{R'} S'$. Let $s \in S$ be (non-zero, homogeneous) such that there exist $i_1, \ldots, i_n$ and $b_j \in \mathfrak{b}_{i_j}$ with $\pi_S(s) = \sum_{j=1}^n b_j$. The b_i may be chosen of degree $\psi_S(\deg(s))$ and thus each b_j has a unique preimage a_j in $S_{\deg(s)}$ which also belongs to $\beta(\mathfrak{b}_{i_j})$. Since $\sum_j a_j$ and s have the same image and degree they coincide. $\square$

II.1.3. localization and limits. Here, we construct limits for the different categories of graded A-algebras/-modules, where A denotes a graded monoid, $\mathbb{F}_1$-algebra or ring which is fixed for the entire section. Moreover, in Proposition II.1.3.10 we prepare the proof of the sheaf property of structure sheaves of graded spectra.

Construction II.1.3.1. Let $S \subseteq \bigcup_w A_w$ be a multiplicative submonoid of the homogeneous elements of the graded monoid/$\mathbb{F}_1$-algebra/ring A. For $w \in gr(A)$ let $(S^{-1}A)_w$ be the union over all $(1/s)A_v$ where $s \in S \cap A_u$ with $v - u = w$. This turns $S^{-1}A$ into a $gr(A)$-graded monoid/$\mathbb{F}_1$-algebra/ring called the *graded localization* of A by S. The canonical map $\imath_S \colon A \to S^{-1}A$ is a degree-preserving graded morphism and is the initial object in the category of graded A-algebras $\phi\colon A \to R$ such that $\phi(S) \subseteq R^*$.

The assignment $S^{-1}(-)$ sending an A-algebra $\phi\colon A \to R$ to the $S^{-1}A$-algebra $S^{-1}\phi\colon S^{-1}A \to \phi(S)^{-1}R, a/s \mapsto \phi(a)/\phi(s)$ is functorial.

Construction II.1.3.2. Let (M, γ) be a graded A-module, let $S \subseteq \bigcup_w A_w$ be a submonoid and denote the localization map by $\jmath\colon M \to S^{-1}M$. Then defining $(S^{-1}M)_w$ as the union over all $(1/s)\jmath(M_v)$ where $s \in S \cap A_u$ with $v - \gamma(u) = w$ turns $(S^{-1}M, \gamma)$ into a graded module over $S^{-1}A$ called the *graded localization* of (M, γ) by S. Note that for a graded submodule $N \leq_A M$ we have $S^{-1}N = \langle \jmath(N) \rangle_{S^{-1}M}$.

Construction II.1.3.3. For an A-algebra $\phi\colon A \to R, \psi\colon gr(A) \to gr(R)$ the *graded localization* is the $S^{-1}A$-algebra $S^{-1}\phi\colon S^{-1}A \to \phi(S)^{-1}R, a/s \mapsto \phi(a)/\phi(s)$ together with the accompanying map ψ.

Remark II.1.3.4. Sending a graded A-algebra/-module to its localization by S defines a functor $S^{-1}(-)$ which commutes with the $(-)^{\mathrm{hom}}$-functor, with $A[\cdot]$, coarsening and augmentation.

Remark II.1.3.5. Let $\pi\colon S \to R$ be a CB(E) accompanied by ψ and let $M \subseteq \bigcup_w S_w$ be a submonoid. Then $\pi\colon M^{-1}S \to \pi(M)^{-1}R$ is a CB(E).

If in the above $S = \{f^n\}_{n \in \mathbb{Z}_{\geq 0}}$ holds with $f \in A^{\mathrm{hom}}$, then we write $A_f := S^{-1}A$ and $M_f := S^{-1}M$.

Remark II.1.3.6. For a graded module M over A, a submonoid $S \subseteq A$ (of homogeneous elements) and a finite family of graded A-submodules $N_i, i \in I$ we have $S^{-1}(\bigcap_i N_i) = \bigcap_i S^{-1}N_i$.

Construction II.1.3.7. Consider a small diagram $D\colon I \to \mathfrak{C}, i \mapsto B_i$. If the objects of $\mathfrak{C}$ have no fixed global accompanying map then D defines a diagram of $gr(A)$-algebras whose limit we denote $\gamma\colon gr(A) \to K = \lim_j gr(B_j)$, with the associated maps written as $\mathrm{pr}_i\colon K \to gr(B_j)$. Otherwise, $\gamma\colon gr(A) \to K$ is given together with $\mathfrak{C}$ and we set $\mathrm{pr}_i := \mathrm{id}_K$. Then the limit of D is $\coprod_{w \in K} \lim_i (B_i)_{\mathrm{pr}_i(w)}$. The (scalar) multiplication of homogeneous elements are induced by the maps

$$\lim_i (B_i)_{\mathrm{pr}_i(w)} \times \lim_i (B_i)_{\mathrm{pr}_i(v)} \to (B_i)_{\mathrm{pr}_i(w)} \times (B_i)_{\mathrm{pr}_i(v)} \to (B_i)_{\mathrm{pr}_i(w+v)}$$

resp.

$$A_u \times \lim_i (B_i)_{\mathrm{pr}_i(v)} \to A_u \times (B_i)_{\mathrm{pr}_i(v)} \to (B_i)_{\mathrm{pr}_i(\gamma(u)+v)}.$$

PROPOSITION II.1.3.8. *Let $\mathfrak{C}$ and $\mathfrak{C}^\gamma$ denote the category of graded A-algebras/-modules, in the latter case with a fixed accompanying $gr(A)$-algebra/-module K with structure map γ. Let $\mathfrak{K}$ denote the appropriate category of algebras resp. modules over the monoid/$\mathbb{F}_1$-algebra/ring underlying A. Then the following hold for a small diagram $D\colon I \to \mathfrak{C}^\gamma, i \mapsto B_i$:*

(i) *If A is a graded monoid then omitting the grading of the $\mathfrak{C}$-limit gives the $\mathfrak{K}$-limit of the induced diagram $I \to \mathfrak{K}$.*
(ii) *If D is a $\mathfrak{C}^\gamma$-diagram then the $\mathfrak{C}^\gamma$-limit B is an A-subalgebra of the $\mathfrak{K}$-limit B'. If additionally A is a ring and I is finite then $B = B'$.*
(iii) *The $\mathfrak{C}$-limit equals the $\mathfrak{C}^\gamma$-limit if and only if γ equals the limit of the diagram $gr \circ D$ of $gr(A)$-algebras.*

PROOF. Throughout, let B' denote the $\mathfrak{K}$-limit of the induced diagram. In (i) let $b = (b^{(i)})_i \in B'$ and consider $i, j \in I$ and a morphism $\alpha_{i,j}\colon i \to j$. Then $gr(D(\alpha_{i,j}))(\deg(b^{(i)})) = \deg(D(\alpha_{i,j})(b^{(i)})) = \deg(b^{(j)})$ holds and we conclude that b belongs to the $\mathfrak{C}$-limit.

In (ii) consider $(b^{(i)})_{i\in I} \in B'$ and let $b^{(i)} = \sum_{w\in K} b^{(i)}_w$ be the compositions into homogeneous elements. Then we have $b = \sum_{w\in K} (b^{(i)}_w)_{i\in I} \in B$. □

REMARK II.1.3.9. Let A be a graded $\mathbb{F}_1$-algebra such that $B := A \setminus 0$ is a (graded) monoid and let D be a diagram of graded A-algebras/-modules such that substracting all 0-elements defines a diagram D' of graded B-algebras/-modules. Then $\lim D' \sqcup \{0\}$ is the limit of D.

PROPOSITION II.1.3.10. *Let $F \subseteq A^{\mathrm{hom}}$ be a subset which generates A as an ideal. Then each object B of $\mathbf{GrAlg}_A^\gamma$ resp. $\mathbf{GrMod}_A^\gamma$ is canonically isomorphic to the limit of the diagram defined by all the morphisms $B_f \to B_{fg}$ where $f, g \in F$.*

Moreover, B is finitely generated over A if and only if each B_f is finitely generated over A_f. In particular, B is a homogeneously noetherian A-module if and only if each B_f is a homogeneously noetherian A_f-module. Furthermore, if A is a graded R-algebra then A is finitely generated over R if and only if each A_f is finitely generated over R.

PROOF. First, we treat the case that $F = \{f_1, \dots, f_d\}$ is finite. The statement is obvious for graded $\mathbb{F}_1$-algebras because in that case one of the f_j is a unit. Now, consider the case that A is a graded ring. For $j = 1, \dots, d$ let $w_j \in gr(A)$ with $f_j \in A_{w_j}$. Let $I \subseteq \mathcal{P}(\{1, \dots, d\})$ be the set of subsets of cardinality one or two with its natural partial order. For $i \in I$ let f_i be the product of all f_j with $j \in i$ and let B_i be the localization of B by the monoid generated by all these f_j. Then the diagram of $gr(A)$-algebras given by the maps $gr(B) = gr(B_j) \to gr(B_i) = gr(B)$ has limit $gr(B)$. For injectivity of the canonical degree-preserving homomorphism $B \to \lim B_i, x \mapsto (x/1)_{i\in I}$ note that if $x/1 = 0/1$ in every B_i then every f_j lies in the radical of $\mathrm{Ann}(x)$. But then so does 1_A which implies $1 \in \mathrm{Ann}(x)$, i.e. $x = 1_A x = 0$.

For surjectivity, consider an element $(x_i/f_i^{n_i})_i \in (\lim_i B_i)_w$ for some $w \in gr(B)$. Set $n := \max_{i\in I} n_i$ and put $y_i := f_i^{n-n_i} x_i$ for $i \in I$. Then $x_i/f_i^{n_i} = y_i/f_i^n$ in B_i. For each two-element set $i := \{j, k\} \in I$ we have $y_j/f_j^n = y_k/f_k^n$ in B_i and hence there exists m_i with $f_k^n f_i^{m_i} y_j = f_j^n f_i^{m_i} y_k$. By setting $m := \max_i m_i$ we obtain $f_k^n (f_j f_k)^m y_j = f_j^n (f_j f_k)^m y_k$ for all such i.

By assumption, there exist $a_j \in A_{-w_j}$ with $1 = \sum_j a_j f_j$. Taking the $d(n+m)$-th power we obtain $c_j \in A_{-(n+m)w_j}$ for $j = 1, \dots, d$ such that $1 = \sum_j c_j f_j^{n+m}$. Now, $x := \sum_j c_j f_j^m y_j \in B_w$ is the desired element with $x/1 = y_i/f_i^n$ in B_i. Indeed, for $j = 1, \dots, d$ we calculate

$$f_j^{n+m} x = \sum_{k=1}^{d} c_k f_j^n (f_j f_k)^m y_k = \sum_{k=1}^{d} c_k f_k^{n+m} f_j^m y_j = f_j^m y_j$$

which gives $x/1 = y_j/f_j^n$ in B_j. For $i = \{j,k\} \in I$ this implies $y_i/f_i^n = y_j/f_j^n = x/1$ in B_i.

Concerning finite generation over A consider $b_1^{(i)}, \ldots, b_{d_i}^{(i)} \in B^{\mathrm{hom}}$ such that B_{f_i} is generated as an A_{f_i}-algebra/-module by $b_1^{(i)}/1, \ldots, b_{d_i}^{(i)}/1$. For a $b \in B^{\mathrm{hom}}$ we have $b/1 = x_i/f_i^{n_i}$ with certain n_i and a A-linear combination x_i of (products of) the elements $b_l^{(i)}$. Then we have $(x_i/f_i^{n_i})_i \in \lim_i B_i$ and in the above notation x is an element of the A-subalgebra/-module generated by all $b_l^{(i)}$. Moreover, injectivity gives $x = b$.

If B is an A-module such that B_{f_j} is a noetherian A_{f_j}-module then in particular B_{f_i} is a noetherian A_{f_i}-module. For a graded A-submodule $\mathfrak{b}$ of B, each $\mathfrak{b}_{f_i}$ is finitely generated over A_{f_i}. Since $\mathfrak{b}$ is the limit of all $\mathfrak{b}_{f_i}$, it is finitely generated over A by the previous assertion.

Lastly, suppose that A is an R-algebra such that each A_{f_j} (and hence each A_{f_i}) is finitely generated over R. Let $a_1^{(i)}, \ldots, a_{k_i}^{(i)} \in A^{\mathrm{hom}}$ such that $a_1^{(i)}/1, \ldots, a_{k_i}^{(i)}/1$ and $1/f_i$ generate A_{f_i} as a R-algebra. Then in the above notation, A is generated by all $a_l^{(i)}$, all f_j and all c_k.

Now, we treat the case of a general subset $F \subseteq A^{\mathrm{hom}}$. Then there exists a finite subset $F' \subseteq F$ with $A = \langle F' \rangle$. Denote the canonical localization maps by $\imath_{f,g} \colon B_{f,f} = B_f \to B_{f,g}$. Let $\phi_{f,g} \colon C \to B_{f,g}, f,g \in F$ be a family of morphisms which are compatible with the all $\imath_{f,g}$. Then there exists a unique degree-preserving homomorphism $\phi \colon C \to B$ with $\phi_{f,g} = \imath_{f,g} \circ \phi$ for all $f,g \in F'$. For $a,b \in F$ there exists a unique $\phi' \colon C \to B$ with $\phi_{f,g} = \imath_{f,g} \circ \phi'$ for all $f,g \in F' \cup \{a,b\}$. By uniqueness of ϕ we have $\phi = \phi'$. Now, let $\phi'' \colon C \to B$ be another degree-preserving morphism with $\phi_{f,g} = \imath_{f,g} \circ \phi''$ for all $f,g \in F$. By uniqueness of ϕ we then have $\phi'' = \phi$ which establishes the limit property. All other properties with respect to general F directly follow from the finite case. $\square$

Proposition II.1.3.11. *Let $D \colon I \to \mathfrak{C}, i \mapsto B_i$ be a diagram of graded algebras/modules over a simple 0-graded monoid/$\mathbb{F}_1$-algebra A. Then we have a canonical isomorphism $(\lim_i B_i)/A^* \cong \lim_i (B_i/A^*)$.*

II.1.4. Free graded A-algebras and -modules in graded monoids and sets. This section deals with the properties of free algebras and modules over a fixed graded monoid/$\mathbb{F}_1$-algebra/ring A. First, we describe the construction which will be applied on the level of grading objects.

Construction II.1.4.1. Let G be a monoid and let H be a monoid resp. a set. Then the *free G-algebra in H* is is the set $G[H] := G \times H$ equipped with the induced G-algebra/-module structure. Specifically, the scalar multiplication is the product of the multiplication map of G with id_H. If H is a monoid, then the map sending g to $(g, 1_H)$ defines a G-algebra. For a morphism $\phi \colon H \to H'$ the induced map $G[\phi] := \mathrm{id}_G \times \phi \colon G[H] \to G[H']$ is a morphism of G-algebras/-modules.

Remark II.1.4.2. By Lemma A.0.0.2 the forgetful functor $\mathfrak{i}$ from G-algebras/-modules to monoids resp. sets is right adjoint to the faithful functor $G[\cdot]$ defined above. This makes use of the fact that for an G-algebra/-module H we have a canonical morphism $G[\mathfrak{i}(H)] \to H$, and likewise, for a monoid resp. a set X we have a canonical morphism $X \to \mathfrak{i}(G[X])$.

In the following we perform the construction of free A-algebras/-modules, distinguishing between the different types of base objects.

Construction II.1.4.3. Let A be a graded monoid and let N be a graded algebra/-module over the monoid $\{1\}$. The associated *free graded A-algebra/-module* is the A-algebra/-module $A[N]$ from Construction II.1.4.1, equipped with the $gr(A)[gr(N)]$-grading given by $A[N]_{(v,w)} = A_v \times N_w$. A graded morphism

$\phi\colon M \to M', gr(\phi)\colon gr(M) \to gr(M')$ induces a morphism $A[\phi]$ with accompanying morphism of grading objects $gr(A)[gr(\phi)]$ as constructed before.

CONSTRUCTION II.1.4.4. Let A be a graded $\mathbb{F}_1$-algebra and let N be a graded 1-algebra/-module. The *free graded A-algebra/-module* associated to N is the set $A[N]$ of maps $N \to A$ which at most one element is not mapped to 0_A. The homogeneous component of degree $(v,w) \in gr(A)[gr(N)]$ is the set of all $f \in A[N]$ for which each $n \in N$ with $f(n) \neq 0$ satisfies $f(n) \in A_v$ and $n \in N_w$. Equipped with point-wise A-action, $A[N]$ carries the structure of a $A[N]$-graded A-module.

If M is an $\mathbb{F}_1$-algebra then the product of $f,g \in A[M]$ sends a product $kl \neq 0_M$ of elements with $f(k) \neq 0_A$ and $g(l) \neq 0_A$ to $f(k)g(l)$ and all other $m \in M$ to 0_A. This turns $A[N]$ into a $gr(A)[gr(N)]$-graded A-algebra.

If M is a graded module over the graded $\mathbb{F}_1$-algebra B then the scalar multiplication $B \times M \to M$ induces a graded $A[B]$-module structure $A[B] \times A[M] \to A[M]$ whose grading object is the $gr(A)[gr(B)]$-module $gr(A)[gr(M)]$. Explicitly, the product of $f \in A[B]$ and $g \in A[M]$ sends (a non-zero) $m \in M$ to the sum over all $f(k)g(l)$ where $kl = m$.

In all cases, a morphism $\phi\colon M \to M'$ induces a morphism $A[\phi]\colon A[M] \to A[M']$ sending $f \in A[M]$ with $f(m) \neq 0_A$ to the map which sends $\phi(m)$ to $f(m)$, with the morphism of grading objects $gr(A)[gr(\phi)]$ sending (u,v) to $(u, gr(\phi)(v))$.

CONSTRUCTION II.1.4.5. Let A be a graded ring and let M be a graded $\mathbb{F}_1$-module/-algebra. Then the *graded A-module/-algebra* is the set $A[M]$ of morphisms $M \to A$ of $\mathbb{F}_1$-modules which attain 0_A for all but finitely many $m \in M$, equipped with point-wise addition and scalar multiplication.

If M is an $\mathbb{F}_1$-algebra then the product of $f,g \in A[M]$ is the map sending $m \neq 0_M$ to the sum over all $f(k)g(l)$ where $kl = m$. $f \in A[M]$ is homogeneous of degree $(v,w) \in gr(A)[gr(M)]$ if each $m \in M$ with $f(m) \neq 0_A$ satisfies $m \in M_w$ and $f(m) \in A_v$.

If M is a graded module over the graded $\mathbb{F}_1$-algebra B then the scalar multiplication $B \times M \to M$ induces a graded $A[B]$-module structure $A[B] \times A[M] \to A[M]$ whose grading object is the $gr(A)[gr(B)]$-module $gr(A)[gr(M)]$. Explicitly, the product of $f \in A[B]$ and $g \in A[M]$ sends (a non-zero) $m \in M$ to the sum over all $f(k)g(l)$ where $kl = m$.

In all cases, a morphism $\phi\colon M \to M'$ induces a morphism $A[\phi]\colon A[M] \to A[M']$ sending $f \in A[M]$ to the map which sends a non-zero $m' \in M'$ to the sum over all terms $f(m)$ where $m \in \phi^{-1}(m')$, with the morphism of grading objects $gr(A)[gr(\phi)]$ sending (u,v) to $(u, gr(\phi)(v))$.

DEFINITION II.1.4.6. If A is a graded ring and M is a graded 1-algebra/-module then we define $A[M] := A[\mathbb{F}_1[M]]$.

REMARK II.1.4.7. In the setting of the above constructions we have an injection $\chi\colon M \to A[M], m \mapsto \chi^m$ where χ^m is $(1_A, m)$ if A is a graded monoid resp. the function $\chi^m \in A[M]$ which assigns 1_A to m and 0_A to all other $m' \in M$, in the other cases. Moreover, for each (non-zero) $m \in M$ the canonical map $A \to A\chi^m$ is injective. χ^m is called the *monomial in m* and $\chi^M := \operatorname{im}(\chi)$ is called the *set of monomials in M*.

Let B be a graded algebra over 1 resp. $\mathbb{F}_1$, and let A be a graded algebra over 1 resp. $\mathbb{F}_1$, or a graded ring. Let K be a $gr(B)$-algebra/-module which is simultaneously a $gr(A)$-algebra/-module, with the scalar multiplication denoted γ resp. δ, such that those structures commute with one another. Then we obtain an induced $gr(A)[gr(B)]$-algebra/-module structure $\gamma\delta$ on K.

Let $\mathfrak{C}$ denote one of the categories $\mathbf{GrMod}_{A[B]}$, $\mathbf{GrMod}^{\gamma\delta}_{A[B]}$, $\mathbf{GrAlg}_{A[B]}$, $\mathbf{GrAlg}^{\gamma\delta}_{A[B]}$ and let $\mathfrak{D}$ denote the corresponding one of the categories $\mathbf{GrMod}_B$,

$\mathbf{GrMod}_B^\gamma$, $\mathbf{GrAlg}_B$, $\mathbf{GrAlg}_B^\gamma$. We have a canonical functor $\mathfrak{h}\colon \mathfrak{C} \to \mathfrak{D}$ which in case A is a graded monoid is the forgetful functor and otherwise is the functor $\mathfrak{hom}$ from Remark II.1.1.10.

Proposition II.1.4.8. *In the above situation, the following hold:*

(i) *For a $\mathfrak{D}$-object M the map from Remark II.1.4.7 induces an injective $\mathfrak{D}$-morphism $M \to \mathfrak{h}(A[M])$.*
(ii) *For a $\mathfrak{C}$-object V sending $f \in A[\mathfrak{h}(V)]$ to $\sum_{v\in V} f(v)v$ defines a surjective $\mathfrak{C}$-morphism $A[\mathfrak{h}(V)] \to V$. The kernel relation is generated by all the relations $a\chi^f \sim \chi^{af}$ (and in case A is a graded ring $\chi^{f+g} \sim \chi^f + \chi^g$) for $a \in A_u, f, g \in V_w, u \in gr(A), w \in gr(V)$.*
(iii) *The assigments of (i) and (ii) are functorial and give rise to an adjunction via Lemma A.0.0.2, i.e. $\mathfrak{h}$ is right adjoint to the faithful functor $A[\cdot]$ sending M to $A[M]$.*

Proof. In (i) note that if M is an $\mathbb{F}_1$-algebra/-module then χ maps into $\mathfrak{h}(A[M])$. If M is an 1-algebra/-module then each χ^m is non-zero and hence defines a unique element of $\mathfrak{h}(A[M])$. For (ii) consider $f = \sum_{i=1}^n a_i\chi_i^v, g = \sum_{j=1}^m b_j w_j$ with $a_i, b_j \in A^{\mathrm{hom}} \setminus 0$ and $v_i, w_j \in V^{\mathrm{hom}} \setminus 0$ satisfying $v := \sum_i a_i v_i = \sum_j b_j w_j$ and $\deg(a_i)\deg(v_i) = \deg(b_j)\deg(w_j)$ for all i, j. Then under the relation $\sim$ specified via its generators we have $f \sim \chi^v \sim g$. Since the kernel relation is $gr(V)$-graded the assertion follows. □

Corollary II.1.4.9. *As an A-module $A[M]$ is the coproduct over all $A\chi^m$ (where $m \neq 0_M$).*

Remark II.1.4.10. We have a canonical surjective degree-preserving morphism $\mathfrak{hom}(A)\times N \to \mathfrak{hom}(A[N])$. If A is a graded $\mathbb{F}_1$-algebra, then this is an isomorphism. If A is a graded ring then the kernel relation is generated by all relations $(0_A, n) \sim (a, 0_N)$ for $a \in A^{\mathrm{hom}}$ and $n \in N$.

Remark II.1.4.11. $A[\cdot]$ commutes with coarsening and augmentation. Moreover, $A[\cdot]$ preserves CB(E)s.

Definition II.1.4.12. In the situation of Proposition II.1.4.8, the image $A[\mathfrak{D}]$ of the functor $A[-]\colon \mathfrak{D} \to \mathfrak{C}$ is called *the category of free graded A-algebras/-modules with arguments in* $\mathfrak{D}$. If $\mathfrak{D}$ is the category of graded monoids then $A[\mathfrak{D}]$ is also called *the category of graded monoid algebras over A.*

Remark II.1.4.13. Consider a graded morphism

$$\phi\colon A[M] \longrightarrow A[M], \quad \psi\colon gr(A)[gr(M)] \longrightarrow gr(A)[gr(N)]$$

such that $\phi(\chi^M) \subseteq \chi^N$ and $\psi(\chi^{gr(M)}) \subseteq \chi^{gr(N)}$. Then there exists a unique graded morphism $(\alpha, \beta)\colon M \to N, gr(M) \to gr(N)$ such that $(\phi, \psi) = A[\alpha, \beta]$.

Definition II.1.4.14. A *graded polynomial ring* over a graded ring A is an A-algebra $A[M]$ in a free abelian graded monoid M with generators $T_i, i \in I$ and arbitrary grading group $gr(M)$, where $A[M]$ carries the $gr(M)$-grading induced by a $gr(A)$-algebra structure on $gr(M)$. $T_i, i \in I$ are called the *homogeneous indeterminates* and we also write $A[T_i|i \in I] := A[M]$.

Remark II.1.4.15. For a graded 1- or $\mathbb{F}_1$-algebra B and a graded algebra/module M over B consider the canonical injection $\chi\colon M \to A[M]$. By Lemma A.0.0.2 the functor $A[\cdot]$ from graded subalgebras/-modules of M to graded subalgebras/-modules of $A[M]$ is left adjoint to the functor sending V to $\chi^{-1}(V \cap \chi(M))$. Moreover, we have $N = \chi^{-1}(\chi(M) \cap A[N])$ and $A[\chi^{-1}(\chi(M) \cap V)] \subseteq V$ for all graded submodules $N \leq_B M$ and $V \leq_{A[B]} A[M]$. Furthermore, $A[\cdot]$ commutes with intersections, and if M is a B-algebra then $A[\cdot]$ also with products of submodules.

REMARK II.1.4.16. Suppose $A = \mathbb{K}$ is a field and B is a graded $\mathbb{F}_1$-algebra. Then the functor $\mathfrak{f}\colon \mathfrak{D} \to \mathfrak{D}$ which sends M to $\mathbb{K}[M]/\mathbb{K}^*$ is naturally isomorphic to $\mathrm{id}_{\mathfrak{D}}$, the isomorphism at M being the canonical map $M \to \mathbb{K}[M]/\mathbb{K}^*$. In particular, $\mathfrak{f}$ and $\mathrm{id}_{\mathfrak{D}}$ are also mutually essentially inverse equivalences.

EXAMPLE II.1.4.17. Let A be a graded $\mathbb{F}_1$-algebra/ring, let M be a graded monoid/$\mathbb{F}_1$-algebra and let $\mathfrak{a}$ be a graded ideal of A. Then as a pointed set/group $\langle \mathfrak{a} \rangle_{A[M]}$ is the coproduct over all $\mathfrak{a}\chi^m$ where m runs through all (non-zero) elements of M. Consequently, we have $\langle \mathfrak{a} \rangle_{A[M]} \cap A = \mathfrak{a}$ and a canonical isomorphism $A[M]/\langle \mathfrak{a} \rangle_{A[M]} \cong (A/\mathfrak{a})[M]$.

II.1.5. colimits. The colimit of a small diagram of graded A-subalgebras/-submodules of a given graded A-algebra/-module is generated by the union over the respective family of A-subalgebras/-submodules. In this section, we construct colimits for the categories of graded A-algebras/-modules, with or without a fixed accompanyment. In the following, let $\mathfrak{C}$ denote one of the categories $\mathbf{GrAlg}_A$, $\mathbf{GrAlg}_A^\gamma$, $\mathbf{GrMod}_A$, $\mathbf{GrMod}_A^\gamma$ and let $\mathfrak{D}$ denote the corresponding one of the categories $\mathbf{GrAlg}_{\{1\}}$, $\mathbf{GrAlg}_{\{1\}}^K$, $\mathbf{GrMod}_{\{1\}}$, $\mathbf{GrMod}_{\{1\}}^K$.

CONSTRUCTION II.1.5.1. Let $\mathfrak{K}$ be the non-graded category corresponding to $\mathfrak{D}$. Let $D\colon I \to \mathfrak{D}, i \mapsto N_i$ be a small diagram. In case $\mathfrak{D}$ denotes a category whose objects have arbitrary grading groups, let $gr(N)$ be the colimit of the diagram of grading groups, otherwise set $gr(N) := K$. Then the $\mathfrak{K}$-coproduct N' of the induced diagram has a canonical $\bigoplus_i gr(N_i)$-grading which naturally coarsens to a $gr(N)$-grading. The kernel relation of the canonical map from N' to the $\mathfrak{K}$-colimit N is then $gr(N)$-graded. N equipped with the resulting $gr(N)$-grading is then the $\mathfrak{D}$-colimit of D.

CONSTRUCTION II.1.5.2. Let $\mathfrak{i}\colon \mathfrak{C} \to \mathfrak{D}$ and $\mathfrak{f}\colon \mathfrak{D} \to \mathfrak{C}$ be the canonical functors. Let $D\colon I \to \mathfrak{C}$ be a small diagram. Let C' be the colimit of the induced $\mathfrak{D}$-diagram $\mathfrak{i} \circ D$. Let $gr(C)$ be the range of the colimit of the diagram $gr \circ D$ of $gr(A)$-algebras. Then the colimit of D is the quotient C of $\mathfrak{f}(C')$, equipped with the $gr(C)$-grading obtained by coarsening, modulo the graded congruence generated by all relations $a\chi^{f_i} \sim \chi^{af_i}$, and in case A is a graded ring also all relations $\chi^{f_i+g_i} \sim \chi^{f_i} + \chi^{g_i}$, where $f_i, g_i \in D(i)_{w_i}, w_i \in gr(D(i)), a \in A^{\mathrm{hom}}, i \in I$.

REMARK II.1.5.3. Let A be a graded monoid/$\mathbb{F}_1$-algebra/ring and let $D\colon I \to \mathbf{GrAlg}_A, i \mapsto B_i$ with colimit B. Let $f_i \in B_i^{\mathrm{hom}}, i \in I$ be elements which are 1 for all but finitely many i, such that for $\alpha\colon i \to j$ we have $D(\alpha)(f_i) \in (B_i)^*_{f_i}$. Let f be the element of B defined by $(f_i)_i$. Then B_f is canonically isomorphic to the colimit B' of the diagram $D'\colon I \to \mathbf{GrAlg}_A, i \mapsto (B_i)_{f_i}$.

Indeed, the map $B \to B'$ given through the colimit property of B by all the maps $B_i \to (B_i)_{f_i} \to B'$ induces a map $B_f \to B'$ which is inverse to the canonical map $B' \to B_f$ obtained through the colimit property of B' from the maps $(B_i)_{f_i} \to B_f$.

EXAMPLE II.1.5.4. The localization by a submonoid $S \subseteq R^{\mathrm{hom}}$ is the colimit of all principal localizations at elements of S.

REMARK II.1.5.5. Suppose that A is a graded ring/$\mathbb{F}_1$-algebra and $B = \mathfrak{hom}(A)$ is the induced graded $\mathbb{F}_1$-algebra/monoid. Then the canonical functors from (K-)graded A-algebras resp. -modules to (K-)graded B-algebras resp. -modules preserve limits and directed colimits.

PROPOSITION II.1.5.6. *Let λ and κ be $gr(A)$-algebra/-module structures on L and K and let $\psi\colon L \to K$ be a surjective morphism of $gr(A)$-algebras/-modules. Let $D\colon I \to \mathfrak{C}^\lambda$ and $E\colon I \to \mathfrak{C}^\kappa$ be small diagrams and let $F\colon I \to \mathrm{Mor}(\mathfrak{C})$ be a small diagram such that $F(i)$ is a CBE from $D(i)$ to $E(i)$ accompanied by ψ. Then the induced map from the (co-)limit of D to the (co-)limit of E is a CBE accompanied by ψ.*

Proof. The statement on limits is shown via direct calculations. For the statement on colimits first suppose that A is a graded monoid. Then the canonical map $\bigsqcup_i D(i) \to \bigsqcup_i E(i)$ is a CBE accompanied by ψ and hence so is the canonical map $\operatorname{colim}_i D(i) \to \operatorname{colim}_i E(i)$. Now consider the case that A is a graded $\mathbb{F}_1$-algebra/ring. Let $\mathfrak{hom}^L\colon \mathfrak{C}^\lambda \to \mathfrak{D}^L$ and $\mathfrak{hom}^K\colon \mathfrak{C}^\kappa \to \mathfrak{D}^K$ be the canonical functors. By the above the induced map $p\colon \operatorname{colim}_i \mathfrak{hom}^L(D(i)) \to \operatorname{colim}_i \mathfrak{hom}^K(E(i))$ is a CBE accompanied by ψ and thus $A[\operatorname{colim}_i \mathfrak{hom}^L(D(i))] \to A[\operatorname{colim}_i \mathfrak{hom}^K(E(i))]$ is a CBE accompanied by $\mathrm{id}_{gr(A)} \times \psi$. Consequently, the canonical morphism $\operatorname{colim}_i D(i) \to \operatorname{colim}_i E(i)$ is surjective. Injectivity follows from the fact that for all $v, u_j \in \operatorname{colim}_i \mathfrak{hom}^L(D(i))$ and (homogeneous) $a_j \in A$ such that χ^v and $a_j\chi^{u_j}$ have the same L-degree, $p(v) = \sum_{j=1}^n a_j p(u_j)$ implies $v = \sum_{j=1}^n a_j u_j$, where in the case of $\mathbb{F}_1$-algebras we take n to be 1. $\square$

Proposition II.1.5.7. *Let $D\colon I \to \mathfrak{A}, i \mapsto A_i$ be a non-empty diagram of graded $\mathbb{F}_1$-algebras/rings (with constant accompanying group K) and let $E\colon I \to \mathfrak{C}, i \mapsto C_i$ be a diagram of graded algebras/modules over $\{1\}$ resp. $\mathbb{F}_1$ (with constant accompanying K-algebra/-module γ). Then the following hold:*

(i) *If $A \to A_i, i \in I$ and $C \to C_i, i \in I$ are cones such that canonical map from $A[C]$ to $\lim_i A_i[C_i]$ is an isomorphism then so are the canonical maps $C \to \lim_i C_i$ and - if C and C_i are algebras - $A \to \lim_i A_i$.*

(ii) *If the canonical map from $\lim_i A_i[\lim_j C_j]$ to $\lim_i A_i[C_i]$ is an isomorphim then so are the ones to $\lim_i (A_i[\lim_j C_j])$, to $\lim_j (\lim_i A_i)[C_j]$ and to $\lim_{i,j} A_i[C_j]$.*

(iii) *Suppose that D (and E) send morphisms α of I to morphisms under which preimages of zero are zero, and that for each $i, j \in I$ there exist $i = i_0, \dots, i_n = j \in I$ and morphisms α_k from i_{k-1} to i_k or vice versa. Then the canonical map from $(\lim_{i\in I} A_i)[\lim_{j\in I} C_j]$ to $\lim_i A_i[C_i]$ is an isomorphism*

(iv) *If the canonical morphism from $\operatorname{colim}_i A_i[C_i]$ to $(\operatorname{colim}_i A_i)[\operatorname{colim}_j C_j]$ is bijective then the ones from $\operatorname{colim}_i (A_i[\operatorname{colim}_j C_j])$, $\operatorname{colim}_j (\operatorname{colim}_i A_i)[C_j]$ and $\operatorname{colim}_{i,j} A_i[C_j]$ are, too.*

(v) *The canonical morphism from $\operatorname{colim}_i A_i[C_i]$ to $(\operatorname{colim}_i A_i)[\operatorname{colim}_j C_j]$ is an isomorphism if I is directed.*

Proof. For (i) note that we have natural embeddings $C \to A[C]$ and $\lim_i C_i \to \lim_i A_i[C_i]$ which together with $C \to \lim_i C_i$ and $A[C] \to \lim_i A_i[C_i]$ form a commutative diagram. Under the extra assumption we likewise have embeddings natural embeddings $A \to A[C]$ and $\lim_i A_i \to \lim_i A_i[C_i]$ which together with $A \to \lim_i A_i$ and $A[C] \to \lim_i A_i[C_i]$ form a commutative diagram.

In (ii) the second and third isomorphism are applications of the first where the second resp. first argument are constantly $\lim_j C_j$ resp. $\lim_i A_i$ with identity morphisms. Furthermore, keeping the second argument constantly at C_j we obtain that the canonical map $(\lim_i A_i)[C_j] \to \lim_i A_i[C_j]$ is an isomorphism and hence so is the canonical map $\lim_j (\lim_i A_i)[C_j] \to \lim_j \lim_i A_i[C_j]$. The same argument with arrows reversed shows (iv).

In (iii) the extra assumptions guarantee that elements of $\lim_i A_i$, $(\lim_i C_i)$ and thus also of $\lim_i A_i[C_i]$ are zero if and only if they have a zero coordinate. For surjectivity let $(a_i\chi^{w_i})_i \in \lim_i A_i[C_i]$ be non-zero. Then all coordinates are non-zero and hence we have $a_j = D(\alpha)(a_i)$ and $w_j = E(\alpha)(w_i)$ for each morphism $\alpha\colon i \to j$. For injectivity let $(a_i)_i\chi^{(w_j)_j}, (b_i)_i\chi^{(v_j)_j} \in (\lim_i A_i)[\lim_j C_j]$ with $a_i\chi^{w_i} = b_i\chi^{v_i}$ for all i. If $(a_i\chi^{w_i})_i = 0$ then we have $(a_i)_i = 0$ (or $(w_j)_j = 0$) as well as $(b_i)_i = 0$ (or $(v_j)_j = 0$). If $(a_i\chi^{w_i})_i \neq 0$ then we have $0 \neq a_i = b_i$ and $(0 \neq) w_i = v_i$ for all i.

In (v) denote the canonical maps by $\phi_i\colon A_i \to \operatorname{colim}_i A_i$, $\psi_i\colon C_i \to \operatorname{colim}_i C_i$ and $\theta_i\colon A_i[C_i] \to \operatorname{colim}_i A_i[C_i]$. For surjectivity consider $a_i \in A_i, w_j \in C_j$. There

exist $\alpha\colon i \to k$ and $\alpha'\colon j \to k$, so we have
$$\phi_i(a_i)\chi^{\psi_j(w_j)} = \phi_k(D(\alpha)(a_i))\chi^{\psi_k(E(\alpha')(w_j))}.$$
For injectivity consider $a_i \in A_i$, $w_i \in C_i$, $b_j \in A_j$, $v_j \in C_j$. First note that $\phi_i(a_i) = 0$ or $\psi_i(w_i) = 0$ suffice for $\theta_i(a_i\chi^{w_i})$ to be zero. Indeed, in both cases there exists $\alpha\colon i \to k$ with $D(\alpha)(a_i) = 0$ or $E(\alpha)(w_i) = 0$, meaning $D(\alpha)(a_i)\chi^{E(\alpha)(w_i)} = 0$ and hence $\theta_i(a_i\chi^{w_i}) = 0$.

Now suppose that $\phi_i(a_i)\chi^{\psi_i(w_i)} = \phi_j(b_j)\chi^{\psi_j(v_j)}$ holds. Due to the above considerations we are left to treat the case that this equation is non-zero, i.e. we have $\phi_i(a_i) = \phi_j(b_j)$ and $\psi_i(w_i) = \psi_j(v_j)$. Then there exists $\alpha\colon i \to k$ and $\alpha'\colon j \to k$ with $D(\alpha)(a_i) = D(\alpha')(b_j)$. Since $\psi_k(E(\alpha)(w_i)) = \psi_k(E(\alpha')(v_j))$ there exists $\beta\colon k \to l$ with $E(\beta \circ \alpha)(w_i) = E(\beta \circ \alpha')(v_j)$. Thus, we have
$$D(\beta \circ \alpha)(a_i)\chi^{E(\beta\circ\alpha)(w_i)} = D(\beta \circ \alpha')(b_j)\chi^{E(\beta\circ\alpha')(v_j)}$$
and hence $\theta_i(a_i\chi^{w_i}) = \theta_j(b_j\chi^{v_j})$. □

PROPOSITION II.1.5.8. *Let A be a graded $\mathbb{F}_1$-algebra and let $\mathfrak{C}$ denote $\mathbf{GrAlg}_A$ or $\mathbf{GrAlg}_A^\gamma$ where $\gamma\colon gr(A) \to K$ is a homomorphism. Let $D\colon i \mapsto B_i$ be a small filtered diagram of graded A-algebras and let B and C be its $\mathfrak{C}$-limit and colimit.*

(i) *If all morphisms $D(i \to j)$ are monomorphisms then all the canonical maps $\mathrm{pr}_i\colon B \to B_i$ and $\phi_i\colon B_i \to C$ are monomorphisms.*
(ii) *Suppose that all morphisms $D(i \to j)$ are monomorphisms and that each B_i is free of zero divisors resp. integral. Then B and C are free of zero divisors/integral.*

PROOF. In (i) the statement on limits holds in all categories. For injectivity of ϕ_i let $f, g \in B_i$ with $\phi_i(f) = \phi_i(g)$. Then there exists $l \in I$ and $\alpha_{i,l}\colon i \to l$ such that $D(\alpha_{i,l})(f) = D(\alpha_{i,l})(g)$ which gives $f = g$.

For (ii) let $\pi_k\colon gr(B) \to gr(B_k)$ denote the canonical map. For non-zero $(f_i)_i \in B_w$ and $(g_i)_i \in B_v$ there exist $i, j \in I$ with $f_i \neq 0$ and $g_j \neq 0$. Let $k \in I$ be an object with morphisms $\alpha_{i,k}\colon i \to k$ and $\alpha_{j,k}\colon j \to k$. Then $f_k = D(\alpha_{i,k})(f_i) \in B_{\pi_k(w)} \setminus 0$ and $g_k = D(\alpha_{j,k})(g_j) \in B_{\pi_k(v)}$ which implies $f_k g_k \in B_{\pi_k(w+v)} \setminus 0$, i.e. $fg \in B_{w+v} \setminus 0$. Moreover, if each B_i is integral then $B \to B_i \to Q(B_i)$ is a composition of injections and hence B is integral.

If $\phi_i(f)\phi_j(h) = 0$ holds with $h \in B_j$ then for $k \in I$ which allows morphisms $\alpha_{i,k}\colon i \to k$ and $\alpha_{j,k}\colon j \to k$ we have $0 = \phi_k(D(\alpha_{i,k})(f)D(\alpha_{j,k})(h))$. By injectivity this means $D(\alpha_{i,k})(f)D(\alpha_{j,k})(h) = 0$ and by integrality, we may assume $D(\alpha_{i,k})(f) = 0$, which implies $f = 0$ and hence $\phi_i(f) = 0$. Moreover, if each B_i is integral then the canonical map $C \to \mathrm{colim}_i\, Q(B_i)$ is injective because each of the maps $B_i \to Q(B_i) \to \mathrm{colim}_i\, Q(B_i)$ is a composition of injections. □

II.1.6. tensor and symmetric products.

REMARK II.1.6.1. Due to Lemma A.0.0.4 the inclusion functor from simple graded monoids/$\mathbb{F}_1$-algebras to $\mathbf{GrMon}$ resp. $\mathbf{GrAlg}_{\mathbb{F}_1}$ is left adjoint to the functor sending a graded $\mathbb{F}_1$-algebra M to M^* resp $M^* \sqcup \{0_M\}$.

CONSTRUCTION II.1.6.2. Let $H_i, i \in I$ be a family of G-algebras/-modules. Let H be the product over all H_i in the category of 1-algebras/-modules. Let $\sim$ be the congruence on $G[H]$ generated by $(g, (h_i)_i) \sim (e_G, \{gh_j\} \times (h_i)_{i\neq j})$. Then the tensor product $\bigotimes_{i\in I} H_i$ in the category of G-algebras/-modules is $G[H]/\sim$. The induced map $H \to \bigotimes_{i\in I} H_i$ of 1-algebras/-modules is G-multilinear.

DEFINITION II.1.6.3. Let $\mathfrak{C}$ denote $\mathbf{GrAlg}_A$ or $\mathbf{GrMod}_A$. A *A-multilinear map* of $\mathfrak{C}$-objects consists of a set-theoretical map $\theta\colon \prod_{i\in I} N_i \to N$ from a $\mathfrak{C}$-product to a $\mathfrak{C}$-object and a map $\psi\colon \prod_{i\in I} gr(N_i) \to gr(N)$ such that

(i) ψ is a $gr(A)$-multilinear map,

(ii) for $j \in I$ and $f_i \in (N_i)_{w_i}$ the induced maps $N_j \to (f_i)_{i \neq j} \times N_j \xrightarrow{\theta} N$ and $gr(N_j) \to (w_i)_{i \neq j} \times gr(N_j) \xrightarrow{\psi} gr(N)$ together form a $\mathbf{GrMod}_A$-morphism,

(iii) and, in case $\mathfrak{C} = \mathbf{GrAlg}_A$, the pair (θ, ψ) forms a homomorphism of graded monoids/$\mathbb{F}_1$-algebras/rings.

A *morphism* from (θ, ψ) to (θ', ψ') consists of a $\mathfrak{C}$-morphism (ϕ, γ) such that $\theta' = \phi \circ \theta$ and $\psi' = \gamma \circ \psi$ hold. This forms the category of A-multilinear maps of $\mathfrak{C}$-objects.

Construction II.1.6.4. Let A be a graded monoid/$\mathbb{F}_1$-algebra/ring. Let $\mathfrak{C}$ be one of the categories $\mathbf{GrAlg}_A$ or $\mathbf{GrMod}_A$ and let $\mathfrak{D}$ be the corresponding one of the categories $\mathbf{GrAlg}_{\{1\}}$ and $\mathbf{GrMod}_{\{1\}}$. Let $\mathfrak{i}\colon \mathfrak{C} \to \mathfrak{D}$ and $A[\cdot]\colon \mathfrak{D} \to \mathfrak{C}$ be the canonical functors.

Let I be a set and let $C_i, i \in I$ be objects of $\mathfrak{C}$. The canonical $gr(A)$-multilinear map $\prod_i gr(C_i) \to \bigotimes_i gr(C_i)$ induces a $\bigotimes_i gr(C_i)$-grading of $\mathfrak{i}(\prod_i C_i)$ with the w-homogeneous component being the union over all $\mathfrak{i}(\prod_i C_i)_{w_i}$ with $\otimes_i w_i = w$. The $gr(A)$-algebra/-module structure then extends this grading to $A[\mathfrak{i}(\prod_i C_i)]$.

We now obtain a $\bigotimes_i gr(C_i)$-graded congruence $\sim$ defined by the relations $a\chi^{(c_i)_i} \sim \chi^{(ac_j)\times(c_i)_{i\neq j}}$ and $\chi^{(c_j+c_j')\times(c_i)_{i\neq j}} \sim \chi^{(c_i)_i} + \chi^{(c_j')\times(c_i)_{i\neq j}}$ - the latter only if A is a ring, where $j \in I$, $a \in A, c_j, c_j' \in C_j$ are arbitrary/homogeneous elements and c_j, c_j' belong to the same homogeneous part. The *tensor product* of $C_i, i \in I$ in $\mathfrak{C}$ is the quotient $\bigotimes_i C_i := A[\mathfrak{i}(\prod_i C_i)]/\sim$. The class of an element $\chi^{(c_i)_i}$ is denoted $\otimes_i c_i$. If A is a ring then for $c := (c_i)_i \in \prod_i C_i$ we write $\otimes_i c_i := \sum_{i,(w_i)_i} \otimes_i (c_i)_{w_i}$ and call elements of this form *pure tensors*. In the case $\mathfrak{C} = \mathbf{GrAlg}_A$, $\otimes_i 1_{C_i}$ is the neutral element of $\bigotimes_i C_i$.

Remark II.1.6.5. The *universal property of tensor products* is the observation that in the above situation, the canonical maps $\prod_i N_i \to A[\mathfrak{i}(\prod_i N_i)] \to \bigotimes_i N_i$ and $\prod_i gr(N_i) \to gr(A)[gr(\prod_i N_i)] \to \bigotimes_i gr(N_i)$ form the initial object of the subcategory of A-multilinear maps from $\prod_i N_i$.

Remark II.1.6.6. The forgetful functor $\mathbf{GrAlg}_A \to \mathbf{GrMod}_A$ preserves tensor products because in the above construction the congruences with respect to A-algebra and -module structure define the same relation.

Remark II.1.6.7. For finitely many graded A-algebras/-modules $N_i, i \in I$ with fixed accompanying A-algebra $\gamma\colon gr(A) \to K$ the map $\prod_i K \to K, (w_i)_i \mapsto \sum_i w_i$ is $gr(A)$-multilinear and thus induces a homomorphism $\psi\colon \bigotimes_i K \to K$ of $gr(A)$-algebras. Coarsening by ψ now yields the structure of a graded A-algebra/-module with accompanyment γ on $\bigotimes_i N_i$.

Remark II.1.6.8. For each graded A-algebra/-module B the scalar multiplication is A-bilinear and hence induces a morphism $A \otimes_A B \to B$ which is an isomorphism whose inverse sends b to $1 \otimes b$.

Remark II.1.6.9. For a submonoid $S \subseteq A^{\text{hom}}$ the functor $S^{-1}(-)$ is canonically isomorphic to $S^{-1}A \otimes_A (-)$.

Remark II.1.6.10. Let B be a graded A-algebra and let C be a graded B-algebra. Then we have a canonical isomorphism $C \otimes_B (B \otimes_A -) \cong C \otimes_A -$. In case $C = S^{-1}B$ holds with a submonoid $S \subseteq B^{\text{hom}}$ we thus have a canonical isomorphism $(S^{-1}B) \otimes_A - \cong S^{-1}(B \otimes_A -)$.

Remark II.1.6.11. Let R be a graded A-algebra with accompanyment γ. Let K be a $gr(A)$-algebra/-module, with structure homomorphism resp. scalar multiplication denoted ψ. Then $gr(R) \otimes_{gr(A)} K$ is a $gr(R)$-algebra/-module with structure homomorphism resp. scalar multiplication being $\gamma\colon gr(R) \to gr(R) \otimes K, u \mapsto u \otimes 1_K$ resp. $\gamma\colon gr(R) \times (gr(R) \otimes K) \to gr(R) \otimes K, (u, v \otimes w) \mapsto uv \otimes w$. Let $\mathfrak{C}$ be one of the categories $\mathbf{GrAlg}_R$, $\mathbf{GrAlg}_R^\gamma$, $\mathbf{GrMod}_R$, $\mathbf{GrMod}_R^\gamma$ and let $\mathfrak{D}$ be the corresponding one of the categories $\mathbf{GrRing}_A$, $\mathbf{GrRing}^\psi$, $\mathbf{GrMod}_A$, $\mathbf{GrMod}_A^\psi$. Then due to Lemma A.0.0.2 the forgetful functor $\mathfrak{C} \to \mathfrak{D}$ is canonically right adjoint to the functor $S \longmapsto R \otimes_A S$.

PROPOSITION II.1.6.12. *Let J be a set, let I_j, $j \in J$ be pairwise disjoint sets and let M_{i_j}, $i_j \in I_j$ be graded modules over A. Then we have canonical graded morphisms*

$$\bigoplus_{(i_j)_j \in \prod_{j\in J} I_j} \bigotimes_{j\in J} M_{i_j} \longrightarrow \bigotimes_{j\in J} \bigoplus_{i_j \in I_j} M_{i_j}$$

and

$$\bigotimes_{j\in J} \bigoplus_{i_j \in I_j} M_{i_j} \longrightarrow \prod_{(i_j)_j \in \prod_{j\in J} I_j} \bigotimes_{j\in J} M_{i_j}.$$

The second defines a left-inverse of the first, and if J is finite then these are mutually inverse isomorphisms.

PROOF. For a fixed $(i'_j)_j$ the inclusions $M_{i'_j} \to \bigoplus_{i_j\in I_j} M_{i_j}$ give rise to a morphism $\prod_j M_{i'_j} \to \prod_j \bigoplus_{i_j\in I_j} M_{i_j}$ whose composition with the canonical map to $\bigotimes_j \bigoplus_{i_j\in I_j} M_{i_j}$ is A-multilinear and hence induces a morphism from $\bigotimes_j M_{i'_j}$. The direct sum of these is the desired inverse.

Secondly, for each fixed $(i'_j)_j$ the projection $\prod_j \bigoplus_{i_j\in I_j} M_{i_j} \to \prod_j M_{i'_j}$ composed with the canonical map $\prod_j M_{i'_j} \to \bigotimes_j M_{i'_j}$ is A-multilinear and thus induces a morphism from $\bigotimes_j \bigoplus_{i_j\in I_j} M_{i_j}$. These morphisms induce a morphism to $\prod_{(i_j)_j} \bigotimes_j M_{i_j}$ whose image lies in $\bigoplus_{(i_j)_j} \bigotimes_j M_{i_j}$ if J is finite. The equations need only be checked for pure tensors, which may be done in direct calculations. $\square$

COROLLARY II.1.6.13. *Let $M_i, i \in I$ be finitely many graded A-modules. If $f_{i,j} \in M_i^{\mathrm{hom}}, j \in J_i$ form A-linearly independent systems resp. bases of M_i then so does the family of all $\otimes_{i,j} f_{i,j}$ with respect to $\bigotimes_i M_i$.*

COROLLARY II.1.6.14. *Let M and N be graded A-modules and suppose that $\{f_i\}_{i\in I} \subseteq M^{\mathrm{hom}}$ is an A-basis of M. Then we have*

$$M \otimes_A N \cong \bigoplus_{i\in I} (Af_i \otimes_A N) \cong \bigoplus_{i\in I} (A \otimes_A N) \cong \bigoplus_{i\in I} N$$

in particular, for a family $\{g_i\}_{i\in} \subseteq N$ of whose elements only finitely many are non-zero we have $\sum_{i\in I} f_i \otimes g_i = 0$ if and only if $g_i = 0$ holds for all $i \in I$.

CONSTRUCTION II.1.6.15. For a graded module M over A set $M_i := M$ for all elements i of a set I. Let L be the colimit of the diagram given by $\mathrm{id}_{gr(A)}$ and all the maps $gr(A) \to gr(M_i)$ and $gr(M_i) \to gr(M_j)$ for $i, j \in I$. Then $M^{\otimes_I} := (\bigotimes_{i\in I}) M_i$ and the congruence $\sim$ generated by all relations $\otimes_{i\in I} r_i \sim \otimes_{i\in I} r_{\sigma(i)}$ for $r_i \in M^{\mathrm{hom}}, \sigma \in \mathrm{Per}(I)$ are canonically L-graded, and $\mathrm{Sym}^I(M) := M^{\otimes_I}/\sim$ is the *I-fold symmetric product* in $\mathbf{GrMod}_A$. If we have $I := \{1, \dots, n\}$ for some $n \in \mathbb{N}$ then $\mathrm{Sym}^n(M) := \mathrm{Sym}^I(M)$ is the *n-fold symmetric product*.

REMARK II.1.6.16. For a graded monoid/$\mathbb{F}_1$-algebra/ring A and a homomorphism $\psi\colon gr(A) \to K$ the following hold:

(i) The forgetful functor $\mathbf{GrAlg}_A \to \mathbf{GrMod}_A$ is canonically right adjoint to the functor $\mathrm{Sym}_A\colon M \mapsto \bigoplus_{n\in\mathbb{N}_0} \mathrm{Sym}_A^n(M)$ where $\mathrm{Sym}_A(M)$ carries the canonical grading.

(ii) The forgetful functor $\mathbf{GrAlg}_A^\psi \to \mathbf{GrMod}_A^\psi$ is canonically right adjoint to the functor $\mathrm{Sym}_A^\psi\colon M \mapsto \bigoplus_{n\in\mathbb{N}_0} \mathrm{Sym}_A^n(M)$ where $\mathrm{Sym}_A^\psi(M)$ carries the canonical K-grading.

(iii) For a graded A-algebra R the forgetful functor from graded A-subalgebras to graded A-submodules of R is canonically right adjoint to the functor sending $M \leq_A R$ to the graded A-subalgebra of R generated by M.

PROPOSITION II.1.6.17. *A morphism $\phi\colon R \to S$, $\psi\colon gr(R) \to gr(S)$ is a mono-/isomorphism of graded A-algebras/-modules (with with constant accompanyment) if and only if ϕ and ψ are in-/bijective.*

PROOF. If (ϕ, ψ) is a monomorphism then ψ is a monomorphism because the functor gr and the forgetful functor from $gr(A)$-algebras/-modules to sets both have a left adjoint. Likewise, the functor $\mathfrak{h}$ from graded A-algebras/-modules (with fixed accompanyment) to graded monoids/sets (with fixed accompanyment) has a left adjoint, as does the forgetful functor from graded monoids/sets to sets. Thus, ϕ is injective on each homogeneous component, and by injectivity of ψ, on all of R. $\square$

II.1.7. graded noetherianity and principality. Here, we introduce terminology required in subsequent results and prove the graded version of Hilbert's basis theorem in Theorem II.1.7.5.

DEFINITION II.1.7.1. A K-graded module M over a graded ring R is *K-noetherian* or has a *noetherian grading*, if every graded submodule is finitely generated. R is $gr(R)$-noetherian if it is a $gr(R)$-noetherian module over itself.

PROPOSITION II.1.7.2. *The following are equivalent:*

(i) *M is K-noetherian,*
(ii) *every ascending chain of K-graded submodules of M becomes stationary,*
(iii) *every non-empty set of K-graded submodules of M has maximal elements.*

LEMMA II.1.7.3. *Let $\mathfrak{a}_i, i \in I$ be an ascending chain of K-graded submodules of the K-graded R-module M. Then $\mathfrak{a} := \bigcup_{i\in I} \mathfrak{a}_i$ is again a K-graded submodule.*

PROOF. Clearly, $\mathfrak{a}$ contains 0. If a, a' are elements of $\mathfrak{a}$ then there exist $i, j \in I$ with $a \in \mathfrak{a}_i$ and $a' \in \mathfrak{a}_j$. We may assume that $i < j$ and thus $a \in \mathfrak{a}_j$ and $a + a' \in \mathfrak{a}_j \subseteq \mathfrak{a}$. If $r \in R$ then $ra \in \mathfrak{a}_i \subseteq \mathfrak{a}$ and thus, $\mathfrak{a}$ is a submodule. In order to see that $\mathfrak{a}$ is K-graded consider $a = \sum_{w\in K} a_w \in \mathfrak{a}$. Then there exists $i \in I$ with $a \in \mathfrak{a}_i$ and hence $a_w \in \mathfrak{a}_i \subseteq \mathfrak{a}$ for every $w \in K$. $\square$

PROOF OF PROPOSITION II.1.7.2. If (i) holds, consider an ascending chain $\{\mathfrak{a}_i\}_{i\in I}$ of K-graded submodules. Then the submodule $\mathfrak{a} := \bigcup_i \mathfrak{a}_i$ is K-graded and is generated by some $r_1, \ldots, r_s$. Let $i \in I$ be an index such that $r_1, \ldots, r_s \in \mathfrak{a}_i$. Then $\mathfrak{a}_k \subseteq \mathfrak{a} \subseteq \mathfrak{a}_i$ for all $k \geq i$. The direction from (ii) to (iii) holds in any partially ordered set and is due to the axiom of choice.

If (iii) holds, let M be the set of all finitely generated K-graded submodules contained in a given K-graded $\mathfrak{a} \trianglelefteq R$. Let $\mathfrak{b} \in M$ be a maximal element. For any $r \in \mathfrak{a} \cap R^+$ we have $r \in \mathfrak{b} + \langle r\rangle \subseteq \mathfrak{b}$ by maximality, and thus $\mathfrak{a} = \mathfrak{b}$ is finitely generated. $\square$

REMARK II.1.7.4. Let (ϕ, ψ) be a morphism of graded rings/R-modules. If the assigment $\mathfrak{a} \mapsto \langle\phi(\mathfrak{a})\rangle$ between the sets of graded submodules is surjective then graded noetherianity of the domain implies graded noetherianity of the range. If $\mathfrak{a} = \phi^{-1}(\langle\phi(\mathfrak{a})\rangle)^{\mathrm{gr}}$ holds for every graded submodule $\mathfrak{a}$ then graded noetherianity of the range implies graded noetherianity of the domain.

THEOREM II.1.7.5. *Let R be a K-graded ring and let $R[T]$ be a K-graded polynomial ring. If R is K-noetherian then so is $R[T]$.*

DEFINITION II.1.7.6. A graded A-algebra R is of *finite type* over A if $gr(R)$ is of finite type over $gr(A)$, i.e. $gr(R)$ is a finitely generated $gr(A)$-algebra, and the underlying ring of R is of finite type over the underlying ring of A.

Corollary II.1.7.7. *Let $\phi\colon R \to S$ be a degree-preserving graded morphism such that S is a graded algebra of finite type over R. If R is K-noetherian then so is S.*

Proof of Theorem II.1.7.5. Let $\mathfrak{b}$ be a K-graded ideal of $R[T]$. For $n \geq 0$ let M_n be the union of $\{0\}$ and the set of $a \in R^+$ occuring as the leading coefficient of some K-homogeneous $f \in \mathfrak{b}$. Then M_n is closed under multiplication with R^{hom} and under addition of elements of the same K-degree. Thus, the K-graded ideal $\mathfrak{a}_n := \langle M_n \rangle$ satisfies $\mathfrak{a}_n \cap R^{\mathrm{hom}} = M_n$. Moreover, we have $M_n \subseteq M_{n+1}$ and hence $\{\mathfrak{a}_n\}_{n\geq 0}$ is an ascending chain of K-graded ideals of R. By K-noetherianity there exists $n_0 \geq 0$ with $\mathfrak{a}_n = \mathfrak{a}_{n_0}$ for all $n \geq n_0$. Moreover, we know that for $n = 0, \ldots, n_0$ we have $\mathfrak{a}_n = \langle a_{n,j}, j = 1, \ldots, d_n \rangle$ with certain $d_n \geq 0$, $a_{n,j} \in M_n \cap R^+$. For each n and j fix $f_{n,j} \in \mathfrak{b} \cap R[T]^+$ of maximal-degree n with leading coefficient $a_{n,j}$.

We claim that $\mathfrak{b}$ is generated by the (finite) set of all $f_{n,j}$, $n = 0, \ldots, n_0$, $j = 1, \ldots, d_n$. Otherwise, there were K-homogeneous elements in $\mathfrak{b}$ which are no $R[T]$-linear combination of the $f_{n,j}$. Among these consider an element g with minimal maximal-degree m. Then the leading coefficient a of g belongs to $\mathfrak{a}_m$. With $k := \min(m, n_0)$ we have $a \in \mathfrak{a}_k$ and hence there exist $r_{k,j} \in R_{\deg(a)-\deg(a_{k,j})}$ with $a = \sum_{j=1}^{d_k} r_{k,j} a_{k,j}$. But then $h = g - T^{m-k} \sum_{k,j} r_{k,j} f_{k,j} \in \mathfrak{b}$ has maximal-degree smaller than m and hence h is a linear combination of the $f_{n,j}$ which implies that g is a linear combination of the $f_{n,j}$ - a contradiction. □

Definition II.1.7.8. A graded submodule $\mathfrak{a}$ of a (K-)graded module M over R is *principally graded* or *K-principal* if there exists $f \in M^{\mathrm{hom}}$ with $\mathfrak{a} = \langle f \rangle$. R itself is a prinicipally graded A-algebra if each of its ideals is prinicipally graded.

Remark II.1.7.9. Let $\pi\colon S \to R$ be a CBE. Then R satisfies the ascending chain condition on $gr(R)$-principal ideals if and only if S satisfies the ascending chain condition on $gr(S)$-principal ideals.

Proposition II.1.7.10. *Let $\pi\colon S \to R, \psi\colon gr(S) \to gr(R)$ be a CBE of A-algebras, let $\{e_j\}_{j\in J}$ be generators for $\ker(\psi)$ and for $e \in \ker(\psi)$ let $\kappa(e) \in S_e$ be the preimage of 1_R. Then for a family $\{s_i\}_{i\in I} \subseteq S^{\mathrm{hom}}$ the family $\{\pi(s_i)\}_{i\in I}$ generates R over A if and only if $\{s_i\}_i \cup \{\kappa(e_j^{\pm 1})\}_j$ generate S over A. In particular, if $\ker(\psi)$ is finitely generated then R is of finite type over A if and only if S is so.*

Proof. For $s \in S_e$ we have $\pi(s) = \sum_{\nu \in \bigoplus_i \mathbb{N}_{\geq 0}} a_\nu \prod_i r_i^{\nu_i}$ with $a_\nu \in A^{\mathrm{hom}}$ where for each ν with $a_\nu \prod_i r_i^{\nu_i} \neq 0$ we have $\deg(a) + \sum_i \deg(r_i) = \deg(\pi(s))$ and hence there exists $u^{(\nu)} \in \bigoplus_j \mathbb{Z}$ with $e - \deg(a) - \sum_i \deg(s_i) = \sum_j u_j^{(\nu)} e_j$. Then $\sum_{\nu \in \bigoplus_i \mathbb{N}_{\geq 0}} a_\nu \prod_{i,j} \kappa(e_j)^{u_j^{(\nu)}} s_i^{\nu_i} \in S_e$ is mapped to $\pi(s)$ and thus equals s. □

II.1.8. graded primality and maximality. The notion of K-prime ideals is due to [**17**].

Definition II.1.8.1. A graded ideal $\mathfrak{p}$ of a K-graded ring R is

(i) *K-prime/-radical* or *homogeneously prime/radical* if $R^{\mathrm{hom}} \cap \mathfrak{p}$ is prime resp. radical,
(ii) *K-maximal* or *homogeneously maximal* if it is maximal among proper graded ideals of R.

For a K-prime ideal $\mathfrak{p}$ of R and a graded R-module M (e.g. $M = R$) the *localization of M at $\mathfrak{p}$* is $M_{\mathfrak{p}} := (R^{\mathrm{hom}} \backslash \mathfrak{p})^{-1} M$ and the localization map is denoted $\imath_{\mathfrak{p}}\colon M \to M_{\mathfrak{p}}$.

Remark II.1.8.2. Under the bijection of graded ideals of R and ideals of the *sesquiad* (R^{hom}, R) from Proposition II.1.1.34, homogeneously prime/maximal ideals correspond to *prime/maximal* ideals of (R^{hom}, R). The latter notions are defined for general sesquiads (also called *blueprints*) in [**22**], where they form the basis of a corresponding theory of *blue schemes*, which encompasses (graded) schemes over $\mathbb{F}_1$ and $\mathbb{Z}$.

Remark II.1.8.3. By Proposition I.1.4.6 a graded ideal $\mathfrak{p}$ of a graded ring R is homogeneously prime if and only if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ implies $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$ for all graded ideals $\mathfrak{a}, \mathfrak{b}$ of R.

Remark II.1.8.4. A graded ideal $\mathfrak{a}$ is homogeneously radical if it equals $\sqrt{\mathfrak{a}}^{\mathrm{gr}}$. Moreover, an arbitrary ideal $\mathfrak{a}$ satisfies $\sqrt{\mathfrak{a}}^{\mathrm{gr}} = \sqrt{\mathfrak{a}^{\mathrm{gr}}}^{\mathrm{gr}}$, because it suffices to check the equality for homogeneous elements and we have $\mathfrak{a} \cap R^{\mathrm{hom}} = \mathfrak{a}^{\mathrm{gr}} \cap R^{\mathrm{hom}}$

Remark II.1.8.5. Let $\mathfrak{p}$ be a graded-prime ideal of the graded ring R. Then $\deg(R^{\mathrm{hom}} \setminus \mathfrak{p}) = \operatorname{degsupp}((R/\mathfrak{p}))$ is a submonoid of $gr(R)$ and we have

$$\deg(((R_{\mathfrak{p}})^{\mathrm{hom}})^*) = \deg(Q(R^{\mathrm{hom}} \setminus \mathfrak{p})) = Q(\deg(R^{\mathrm{hom}} \setminus \mathfrak{p})) = \operatorname{degsupp}(Q_{\mathrm{gr}}(R/\mathfrak{p})).$$

Moreover, if $R = R_0[f_i | i \in I]$ holds with certain $f_i \in R^{\mathrm{hom}}$ then $\deg(R^{\mathrm{hom}} \setminus \mathfrak{p})$ is the submonoid of $gr(R)$ generated by all $\deg(f_i)$ with $f_i \notin \mathfrak{p}$. Indeed, each $g \in R_w$ is an R_0-linear combination of monomials $f_{i_1}^{n_1} \cdots f_{i_d}^{n_d}$ such that $\sum_j n_j \deg(f_{i_j}) = w$. If $g \notin \mathfrak{p}$ then one of these monomials does not belong to $\mathfrak{p}$ and hence none of the respective f_{i_j} do.

Remark II.1.8.6. Under graded homomorphisms ϕ, *homogenized* preimages $\phi^{-1}(\mathfrak{a})^{\mathrm{gr}}$ of homogeneously prime/radical ideals $\mathfrak{a}$ are again homogeneously prime resp. radical. Specifically, we always have

$$\phi^{-1}(\sqrt{\mathfrak{a}}^{\mathrm{gr}})^{\mathrm{gr}} = \phi^{-1}(\sqrt{\mathfrak{a}})^{\mathrm{gr}} = \sqrt{\phi^{-1}(\mathfrak{a})}^{\mathrm{gr}} = \sqrt{\phi^{-1}(\mathfrak{a})^{\mathrm{gr}}}^{\mathrm{gr}}.$$

Lemma II.1.8.7. *For a graded A-module M the following hold:*

(i) *$M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all $gr(A)$-prime/-maximal ideals $\mathfrak{m} \trianglelefteq A$. In particular, this holds if $M = \langle f \rangle$ with $f \in M^{\mathrm{hom}}$. In case A is a ring, $f, g \in M_w$ thus coincide if and only if $f/1 = g/1$ holds in $M_{\mathfrak{m}}$ for all $gr(A)$-prime/-maximal ideals $\mathfrak{m} \trianglelefteq A$.*

(ii) *A graded submodule $N \leq_A M$ is precisely the set of those $f \in M$ with $f/1 \in N_{\mathfrak{p}}$ for all $gr(A)$-prime/-maximal ideals $\mathfrak{m} \trianglelefteq A$.*

(iii) *For graded submodules $N, N' \leq_A M$ we have $N \subseteq N'$ if and only if $N_{\mathfrak{m}} \subseteq N'_{\mathfrak{m}}$ holds for all $gr(A)$-prime/-maximal ideals $\mathfrak{m} \trianglelefteq A$.*

Proof. In all instances the assertion with respect to $gr(A)$-maximal ideals implies that with respect to $gr(A)$-prime ideals via localization. In (i) note that if there exists $0 \neq m \in M^{\mathrm{hom}}$ then $\operatorname{Ann}(m)$ is proper and thus lies in some $gr(A)$-maximal ideal $\mathfrak{m}$. Then $m/1 \in M_{\mathfrak{m}}$ is non-zero. The supplement is an application of (i) to the submodule $\langle f - g \rangle$.

In (iii) we use the canonical isomorphisms $M_{\mathfrak{m}}/N_{\mathfrak{m}} \cong (M/N)_{\mathfrak{m}}$ to obtain that $f/1$ lies in $N_{\mathfrak{m}}$ if and only if $[f]/1 = 0_{M/N}/1$ holds in $(M/N)_{\mathfrak{m}}$. By (ii) we thus have $f/1 \in N_{\mathfrak{m}}$ for all $gr(A)$-maximal/-prime ideals $\mathfrak{m}$ of A if and only if $[f] = 0_{M/N}$, i.e. $f \in N$. Assertion (iv) is a consequence of (iii). □

Definition II.1.8.8. The *graded height* or *K-height* of a K-prime ideal $\mathfrak{p}$ of a K-graded ring R is the supremum $\mathrm{ht}_{\mathrm{gr}}(\mathfrak{p})$ over all $n \in \mathbb{N}_0$ which allow a chain $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ of K-prime ideals $\mathfrak{p}_i$ of R.

The *graded dimension* or *K-dimension* of R is the supremum $\dim_{\mathrm{gr}}(R)$ over all $n \in \mathbb{N}_0$ which allow a chain $\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_n$ of K-prime ideals of R.

Remark II.1.8.9. Let $S \subseteq R^{\mathrm{hom}}$ be a submonoid. Then by Proposition I.1.4.10 the assignments $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ and $\mathfrak{q} \mapsto \imath_S^{-1}(\mathfrak{q})$ define mutually inverse inclusion preserving bijections between the set of K-prime ideals of R whose intersection with S is empty and the set of K-prime ideals of $S^{-1}R$. In particular, the map $\mathfrak{q} \mapsto \imath_S^{-1}(\mathfrak{q})$ preserves the K-height of K-prime ideals. Moreover, we have $S^{-1}(\bigcap_i \mathfrak{p}_i) = \bigcap_i S^{-1}\mathfrak{p}_i$ for each family $\{\mathfrak{p}_i\}_{i \in I}$ of K-prime ideals of R whose intersection with S is empty.

Definition II.1.8.10. R is *K-local*, *homogeneously local* or *locally graded* if it has precisely one homogeneously maximal ideal. A graded morphism $\phi\colon R \to S$ of homogeneously local rings with homogeneously maximal ideals $\mathfrak{m}_R$ resp. $\mathfrak{m}_S$ is *homogeneously local* if $\mathfrak{m}_R = \phi^{-1}(\mathfrak{m}_S)^{\mathrm{gr}}$.

Remark II.1.8.11. By Remark II.1.8.9 graded localizations of graded rings at prime ideals are locally graded. If $\phi\colon R \to S$ is a graded homomorphism and $\mathfrak{p} \trianglelefteq S$ is homogeneously prime then the canonical map $R_{\phi^{-1}(\mathfrak{p})} \to S_{\mathfrak{p}}$ is homogeneously local.

Proposition II.1.8.12. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be graded ideals of a K-graded ring R. Then the following hold:*

(i) *Every intersection of K-prime ideals is K-radical.*
(ii) *$\sqrt{\mathfrak{a}}^{\mathrm{gr}}$ is the intersection over all graded-prime ideals containing $\mathfrak{a}$.*
(iii) *We have $\sqrt{\mathfrak{a}\mathfrak{b}}^{\mathrm{gr}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}}^{\mathrm{gr}} = \sqrt{\mathfrak{a}}^{\mathrm{gr}} \cap \sqrt{\mathfrak{b}}^{\mathrm{gr}}$.*

Proof. In (ii) let $r \in R^{\mathrm{hom}}$ and set $M := \{r^n\}_{n\geq 0}$. If M intersects $\mathfrak{a}$ non-trivially than every K-prime ideal which contains $\mathfrak{a}$ also contains r. If $M \cap \mathfrak{a}$ is empty then $M^{-1}\mathfrak{a}$ is proper and hence contained in some K-maximal ideal $\mathfrak{q}$. Then $\imath_M^{-1}(\mathfrak{q})$ contains $\mathfrak{a}$ but not r because it intersects M trivially. For (iii) we use Remark II.1.8.3 to calculate $\sqrt{\mathfrak{a} \cap \mathfrak{b}}^{\mathrm{gr}} \subseteq \sqrt{\mathfrak{a}}^{\mathrm{gr}} \cap \sqrt{\mathfrak{b}}^{\mathrm{gr}} = \sqrt{\mathfrak{a}\mathfrak{b}}^{\mathrm{gr}} \subseteq \sqrt{\mathfrak{a} \cap \mathfrak{b}}^{\mathrm{gr}}$. □

Proposition II.1.8.13. *Let R be a K-graded ring and let $S \subseteq R^{\mathrm{hom}}$ be a submonoid.*

(i) *For a graded ideal $\mathfrak{a}$ of R we have $S^{-1}\sqrt{\mathfrak{a}}^{\mathrm{gr}} = \sqrt{S^{-1}\mathfrak{a}}^{\mathrm{gr}}$.*
(ii) *If S is generated by a single element $f \in R^{\mathrm{hom}}$ then we have $S^{-1}\bigcap_i \mathfrak{p}_i = \bigcap_i S^{-1}\mathfrak{p}_i$ for any family $\{\mathfrak{p}_i\}_{i\in I}$ of K-prime ideals.*

Proof. For (i) consider $a/s \in S^{-1}R^{\mathrm{hom}}$. If $a^n \in \mathfrak{a}$ then $(a/s)^n \in S^{-1}\mathfrak{a}$, which shows $S^{-1}\sqrt{\mathfrak{a}}^{\mathrm{gr}} \subseteq \sqrt{S^{-1}\mathfrak{a}}^{\mathrm{hom}}$. If $(a/s)^n = b/t \in S^{-1}\mathfrak{a}$ holds with $b \in \mathfrak{a}$ and $t \in S$ then there exists $u \in S$ with $(tua)^n \in \mathfrak{a}$ and hence $a/s = tua/stu \in S^{-1}\sqrt{\mathfrak{a}}^{\mathrm{gr}}$. In (ii) note that if a/f^n belongs to the right-hand intersection then af lies in every $\mathfrak{p}_i$ and hence $af/f^{n+1} \in S^{-1}\bigcap_i \mathfrak{p}_i$. □

The following statements will be used for the verification of elementary properties of Veronesean good quotients of $\mathbb{F}_1$-schemes and graded schemes.

Proposition II.1.8.14. *Let R be a K-graded $\mathbb{F}_1$-algebra (resp. ring) and let $G \subseteq K$ be a subgroup. For a G-graded $\mathfrak{q} \leq_{R_G} R_G$ the ideal $\mathfrak{m}(\mathfrak{q})$ generated by all K-graded $\mathfrak{b} \leq_R R$ with $\mathfrak{b} \cap R_G \subseteq \mathfrak{q}$, called the* special ideal over $\mathfrak{q}$, *has the following properties:*

(i) *a (homogeneous) element $r \in R$ belongs to $\mathfrak{m}(\mathfrak{q})$ if and only if $Rr \cap R_G \subseteq \mathfrak{q}$.*
(ii) *We have $\mathfrak{m}(\mathfrak{q}) \cap R_G = \mathfrak{q}$, and consequently, for every K-graded ideal $\mathfrak{a}$ of R we have $\mathfrak{a} \subseteq \mathfrak{m}(\mathfrak{q})$ if and only if $\mathfrak{a} \cap R_G \subseteq \mathfrak{q}$.*
(iii) *For each (G-graded) $\mathfrak{p} \leq_{R_G} R_G$ we have $\mathfrak{q} \subseteq \mathfrak{p}$ if and only if $\mathfrak{m}(\mathfrak{q}) \subseteq \mathfrak{m}(\mathfrak{p})$. Consequently, we have $\mathfrak{m}(\bigcap_i \mathfrak{q}_i) = \bigcap_i \mathfrak{m}(\mathfrak{q}_i)$ for each family $\{\mathfrak{q}_i\}_i$ of G-graded ideals of R_G.*
(iv) *$\mathfrak{m}(\mathfrak{q})$ is proper resp. (homogeneously) prime, radical or maximal if and only if $\mathfrak{q}$ is so, and we have $\sqrt{\mathfrak{m}(\mathfrak{q})}^{\mathrm{gr}} = \mathfrak{m}(\sqrt{\mathfrak{q}}^{\mathrm{gr}})$.*
(v) *If $\mathfrak{q}$ is (homogeneously) prime then the canonical map $(R_G)_{\mathfrak{q}} \to (R_{\mathfrak{m}(\mathfrak{q})})_G$ is surjective. In the ring case this map is also injective if no element of $r \in R_G$ allows a (homogeneous) element $s \in R \setminus \mathfrak{m}(\mathfrak{q})$ with $rs = 0$.*
(vi) *For each K-maximal ideal $\mathfrak{a}$ of R, $\mathfrak{a} \cap R_G$ is G-maximal and we have $\mathfrak{a} = \mathfrak{m}(\mathfrak{a} \cap R_G)$.*

(vii) *For each submonoid $S \subseteq R_G$ (of homogeneous elements) we have a canonical isomorphism $\jmath\colon S^{-1}(R_G) \cong (S^{-1}R)_G$ and if $\mathfrak{q}$ is (homogeneously) prime then $S^{-1}\mathfrak{m}(\mathfrak{q}) = \mathfrak{m}(\jmath(S^{-1}\mathfrak{q}))$.*

Proof. First note that since intersecting with R_G commutes with sums of K-graded ideals we have $\mathfrak{m}(\mathfrak{q}) \cap R_G \subseteq \mathfrak{q}$. For each (homogeneous) $r \in \mathfrak{m}(\mathfrak{q})$ we thus have $Rr \cap R_G \subseteq \mathfrak{m}(\mathfrak{q}) \cap R_G \subseteq \mathfrak{q}$, which gives (i). For assertion (ii) note that each $f \in \mathfrak{q}$ satisfies $Rf \cap R_G \subseteq \langle \mathfrak{q} \rangle_R \cap R_G = \mathfrak{q}$ and hence lies in $\mathfrak{m}(\mathfrak{q})$. Assertion (iii) follows from (ii).

In (iv) note that $\mathfrak{m}(\mathfrak{q}) \cap R_G = \mathfrak{q}$ implies that if $\mathfrak{m}(\mathfrak{q})$ is not proper, or (homogeneously) prime/radical than so is $\mathfrak{q}$. If $\mathfrak{q} = R_G$ then $\mathfrak{m}(\mathfrak{q}) = R$ by definition. Now, let $\mathfrak{q}$ be (homogeneously) prime. For (K-graded) $\mathfrak{a}, \mathfrak{b} \leq_R R$ with $\mathfrak{ab} \subseteq \mathfrak{m}(\mathfrak{q})$ we then have $(\mathfrak{a} \cap R_G)(\mathfrak{b} \cap R_G) \subseteq \mathfrak{ab} \cap R_G \subseteq \mathfrak{q}$ which implies $\mathfrak{a} \cap R_G \subseteq \mathfrak{q}$ or $\mathfrak{b} \cap R_G \subseteq \mathfrak{q}$, i.e. $\mathfrak{a} \subseteq \mathfrak{m}(\mathfrak{q})$ or $\mathfrak{b} \subseteq \mathfrak{m}(\mathfrak{q})$.

If $\mathfrak{q}$ is (homogeneously) maximal, and $\mathfrak{m}(\mathfrak{q}) \subsetneq \mathfrak{n}$ holds with a (K-graded) ideal $\mathfrak{n}$ then $\mathfrak{q} \subsetneq \mathfrak{n} \cap R_G$ holds by (iii), and the assumption gives $\mathfrak{n} \cap R_G = R_G$ and hence $\mathfrak{n} = R$. Conversely, if $\mathfrak{m}(\mathfrak{q})$ is (homogeneously) maximal, and $\mathfrak{q} \subsetneq \mathfrak{n}$ holds with a (G-graded) ideal $\mathfrak{n}$ then $\mathfrak{m}(\mathfrak{q}) \subsetneq \mathfrak{m}(\mathfrak{n})$ holds by (iii). By assumption, we have $\mathfrak{m}(\mathfrak{n}) = R$ and hence $\mathfrak{n} = R_G$ by (ii).

Concerning graded radicals, we use that by Proposition II.1.8.12 and assertion (iii) $\mathfrak{m}(\sqrt{\mathfrak{q}}^{\mathrm{gr}})$ is the intersection of (homogeneously) prime ideals $\mathfrak{m}(\mathfrak{p})$ where $\mathfrak{p}$ ranges over all G-prime ideals containing $\mathfrak{q}$.

In (v) consider (homogenous) $a \in R \setminus 0, s \in R \setminus \mathfrak{m}(\mathfrak{q})$ with $\deg(a) - \deg(s) \in G$. By definition of $\mathfrak{m}(\mathfrak{q})$ there exists a (homogeneous) $b \in R$ with $bs \in R_G \setminus \mathfrak{q}$, in particular $b \in R \setminus \mathfrak{q}$, and we conclude that a/s is the image of $(ab)/(sb) \in (R_G)_{\mathfrak{q}}$. Assertion (vi) follows from (iv).

For (vii) consider $r \in R_w$ and $s \in S_v$. If $Rr \cap R_G \subseteq \mathfrak{q}$ then for each $a \in R_{w'}$ and $t \in S_{v'}$ with $w + w' - v - v' \in G$ we have $w + w' \in G$, i.e. $ar \in \mathfrak{q}$ and $(a/t)(r/s) \in S^{-1}\mathfrak{q}$. Conversely, if $S^{-1}R(r/s) \cap (S^{-1}R)_G \subseteq \jmath(S^{-1}\mathfrak{q})$ then for each $a \in R_{w'}$ with $w + w' \in G$ we have $ar/1 = b/t$ with $b \in \mathfrak{q}$ and $t \in S$. Thus there exists $u \in S$ with $tuar = ub \in \mathfrak{q}$ and (homogeneous) primality gives $ar \in \mathfrak{q}$. □

Proposition II.1.8.15. *Let R be a K-graded ring, let $\psi\colon K \to G$ be an epimorphism and let $\mathfrak{p}$ be a K-prime/-maximal ideal. Then there exists a G-prime/-maximal ideal $\mathfrak{q}$ such that $\mathfrak{p}$ is the ideal $\mathfrak{q}_K$ generated by all K-homogeneous elements of $\mathfrak{q}$.*

Proof. First consider the case that $\mathfrak{p}$ is K-maximal. Let M be the set of G-homogeneous ideals $\mathfrak{a}$ with $\mathfrak{a}_K = \mathfrak{p}$. Then M is non-empty because it contains $\mathfrak{p}$. Moreover, for every chain $\mathfrak{a}_i, i \in I$ in M the union $\mathfrak{a} := \bigcup_{i \in I} \mathfrak{a}_i$ is again an element of M. Indeed, $\mathfrak{a}$ is a G-graded ideal by Lemma II.1.7.3. By Zorns Lemma M has maximal elements. Let $\mathfrak{q}$ be one of these. We claim that $\mathfrak{q}$ is G-maximal. Indeed, if $\mathfrak{b}$ is a G-graded ideal properly containing $\mathfrak{q}$, then $\mathfrak{p} \subseteq \mathfrak{b}_K$. But $\mathfrak{b}$ cannot belong to M due to maximality of $\mathfrak{q}$ in M and thus we have $\mathfrak{p} \subsetneq \mathfrak{b}_K$ and K-maximality of $\mathfrak{p}$ implies $\mathfrak{b}_K = R$, i.e. $\mathfrak{b} = R$.

Now, consider the general case where $\mathfrak{p}$ is K-prime. Let $\imath_{\mathfrak{p}}\colon R \to R_{\mathfrak{p}}$ be the canonical map. By the above, there exists a G-maximal ideal $\mathfrak{a}$ of $R_{\mathfrak{p}}$ with $\mathfrak{a}_K = \mathfrak{p}_{\mathfrak{p}}$. We claim that the G-prime ideal $\mathfrak{q} := \imath_{\mathfrak{p}}^{-1}(\mathfrak{a})$ satisfies $\mathfrak{q}_K = \mathfrak{p}$. Indeed, if f is a K-homogeneous element of $\mathfrak{q}_K$ then $\imath_{\mathfrak{p}}(f) \in \mathfrak{a}$ is K-homogeneous and thus belongs to $\mathfrak{p}_{\mathfrak{p}}$ which implies that $f \in \imath_{\mathfrak{p}}^{-1}(\mathfrak{p}_{\mathfrak{p}}) = \mathfrak{p}$. Conversely, if f is a K-homogeneous element of $\mathfrak{p}$ then $\imath_{\mathfrak{p}}(f) \in \mathfrak{p}_{\mathfrak{p}} = \mathfrak{a}_K \subseteq \mathfrak{a}$ and hence $f \in \imath_{\mathfrak{p}}^{-1}(\mathfrak{a}) = \mathfrak{q}$. □

Proposition II.1.8.16. *Let F be a totally ordered group. A $K \oplus F$-graded ideal $\mathfrak{a}$ of a $K \oplus F$-graded ring R is $K \oplus F$-prime if and only if it is K-prime. Moreover, the K-radical of $\mathfrak{a}$ equals the $K \oplus F$-radical of $\mathfrak{a}$.*

PROOF. Let $\mathfrak{p}$ be $K \oplus F$-prime. Let g and h be K-homogeneous with $gh \in \mathfrak{p}$. Let $g = g_{n_1} + \ldots + g_{n_2}$ and $h = h_{m_1} + \ldots + h_{m_2}$ be the decompositions into F-homogeneous parts, written in ascending order with respect to their F-degree. F-homogeneity of $\mathfrak{p}$ yields $g_{n_2}h_{m_2} \in \mathfrak{p}$, so $g_{n_2} \in \mathfrak{p}$ or $h_{m_2} \in \mathfrak{p}$ by $K \oplus F$-primality of $\mathfrak{p}$. Now we have $(g - g_{n_2})h \in \mathfrak{p}$ or $g(h - h_{m_2}) \in \mathfrak{p}$ and by induction on the sum of the numbers of F-homogeneous parts of g and h, we get $g \in \mathfrak{p}$ or $h \in \mathfrak{p}$.

Now, suppose that for h as above there exists $n \in \mathbb{N}$ with $h^n \in \mathfrak{a}$. Since $h_{m_1}^n$ is the (nm_1)-homogeneous part of h^n, h_{m_1} belongs to the $K \oplus F$-radical. In particular, we have $h_{m_1} \in \sqrt{\mathfrak{a}}$ and hence $h - h_{m_1} \in \sqrt{\mathfrak{a}}$. By induction on the number of $K \oplus F$-homogeneous parts of h we obtain $h_{m_i} \in \sqrt{\mathfrak{a}}$ for each i. $\square$

PROPOSITION II.1.8.17. *Let M be an integral graded $\mathbb{F}_1$-algebra such that $gr(M)$ is a totally ordered group and let R be a K-graded ring. Consider $R[M]$ equipped with a K-grading that coarsens the canonical $K \oplus gr(M)$-grading and extends the K-grading of R. Then an ideal $\mathfrak{a} \leq_R R$ is K-prime/-radical if and only if $\mathfrak{b} := \langle \mathfrak{a} \rangle_{R[M]}$ is K-prime/-radical. Moreover, we have $\sqrt{\langle \mathfrak{a} \rangle_{R[M]}}^{\mathrm{gr}} = \langle \sqrt{\mathfrak{a}}^{\mathrm{gr}} \rangle_{R[M]}$.*

PROOF. First note that K-primality/-radicality of $\mathfrak{b}$ implies K-primality resp. -radicality of $\mathfrak{a} = \mathfrak{b} \cap R$. Now, consider the *induced* $K \oplus gr(M)$-grading of $R[M]$ given by $\deg(a\chi^m) := \deg_K(a) + \deg_K(\chi^m) + \deg(m)$. Note that this grading has the same homogeneous elements as the canonical $K \oplus gr(M)$-grading.

If $\mathfrak{a}$ is K-prime consider $f, g \in R^{\mathrm{hom}}$ and $m, n \in M$ with $f\chi^m g\chi^n \in \mathfrak{b}$. Then $fg \in \mathfrak{a}$ by Example II.1.4.17 and we may assume $f \in \mathfrak{a}$ which means $f\chi^m \in \mathfrak{b}$. This shows $K \oplus gr(M)$-primality with respect to the induced grading, and hence K-primality by Proposition II.1.8.16. Lastly, let $f\chi^m \in R[M]$ be $K \oplus gr(M)$-homogeneous and let $n \in \mathbb{N}$. If $f^n\chi^{nm} \in \mathfrak{b}$ then $f^n \in \mathfrak{a}$ by Example II.1.4.17, which means $f\chi^m \in \langle \sqrt{\mathfrak{a}}^{\mathrm{gr}} \rangle_{R[M]}$, and conversely, if $f^n \in \mathfrak{a}$ then $f^n\chi^{nm} \in \mathfrak{b}$. This shows the $K \oplus gr(M)$-radical (i.e. the K-radical) of $\mathfrak{b}$ is $\langle \sqrt{\mathfrak{a}}^{\mathrm{gr}} \rangle_{R[M]}$. $\square$

EXAMPLE II.1.8.18. Each torsion-free abelian group allows the structure of a totally ordered group. Thus, the above Proposition holds in particular for graded polynomial rings $R[T_i | i \in I] = R[\bigoplus_{i \in I} \mathbb{N}_0]$ with canonical $gr(R) \oplus \bigoplus_{i \in I} \mathbb{Z}$-grading.

II.2. divisibility theory of graded rings

In this section we develop the divisibility theory of graded rings ananogously to the divisibility theory of their sets of homogeneous elements, thereby laying the foundation for discussions on (invariant) divisors and Cox sheaves in later chapters. As a class of examples of *K-Krull rings* which occur as sections of Cox sheaves we treat *natural algebras* over *(graded) rings of Krull type* in Subsection II.2.6. In Subsection II.2.2 we prove a criterion for *graded factoriality* of a given ring R in terms of factoriality of Veronese subalgebras of suitable localizations of R. The latter condition may in turn be verified in terms of exponent vectors occuring in relations among generators of R.

For $K \oplus F$-graded rings, where F is free, we show that behaviour with respect to coarsening to the induced K-grading is very well, with *$K \oplus F$-integrality/-factoriality/-normality* being equivalent to K-integrality/-factoriality/-normality, see Subsections II.2.1 and II.2.7, respectively. Moreover, *$K \oplus F$-simple* rings are K-factorial, and R is a $K \oplus F$-Krull ring if and only if it is a K-Krull ring, in which case we have $\mathrm{Cl}_{K \oplus F}(R) \cong \mathrm{Cl}_K(R)$, see Theorem II.2.5.15.

II.2.1. graded integrality and simplicity. A ring is a field if and only if it is simple as a module over itself, meaning it has only trivial ideals. In order to avoid an ambiguous term like graded field we instead speak of *simply graded rings*.

This fits into the terminology of simple monoids resp. $\mathbb{F}_1$-algebras as those with only trivial ideals, see Definition I.2.1.2.

Definition II.2.1.1. A K-graded ring R such that R^{hom} is integral/simple is called *K-integral/-simple* or *integrally/simply graded.*

Remark II.2.1.2. The homogeneous zero divisors of a graded ring R coincide with the zero divisors of R^{hom}.

Remark II.2.1.3. A graded ring R is integrally graded if and only if R^{hom} has no zero divisors. Moreover, under these conditions if $a \in R^{\mathrm{hom}} \setminus 0$ satisfies $ab = ac$ with arbitrary b, c then $b = c$.

Indeed, if in the latter case $ab = ac$ holds with $a \in R^{\mathrm{hom}} \setminus 0$ and arbitrary $b, c \in R$ then $b = c$ because $a(b_w - c_w) = 0$ and hence $b_w = c_w$ holds for all $w \in gr(R)$.

Remark II.2.1.4. A K-graded ideal $\mathfrak{a}$ of a K-graded ring R is K-prime/-maximal if and only if $R/\mathfrak{a}$ is K-integral/-simple. In particular, K-maximal ideals are K-prime.

Remark II.2.1.5. A graded ring R is K-simple if and only if $\{0\}$ and R are the only graded ideals.

Definition II.2.1.6. The *homogeneous fraction ring* of an integrally graded ring R is $Q_{\mathrm{gr}}(R) := (R^{\mathrm{hom}} \setminus 0)^{-1}R$. For a K-graded ring R we will also write $Q_K(R)$ instead of $Q_{\mathrm{gr}}(R)$ to distinguish the *K-homogeneous fraction ring* from the fraction rings defined by suitable gradings of other groups on R.

Example II.2.1.7. Let M be a graded $\mathbb{F}_1$-algebra and let A be a graded ring. Then the $Q(M)^*$-graded ring $R = A[M]$ is integrally graded if and only if A is integrally graded and M is integral. Moreover, if A is a field then all ideals of R^{hom} are sesquiad ideals and the natural maps between ideals of M and graded ideals of R resp. M-modules of $Q(M)$ and graded R-modules of $Q_{\mathrm{gr}}(R) = \mathbb{K}[Q(M)]$ are mutually inverse bijections.

Remark II.2.1.8. Let R be a $K \oplus F$-integral ring, where F is a totally ordered abelian group. Then the following hold:

(i) R is K-integral.
(ii) all K-homogeneous units are $K \oplus F$-homogeneous.
(iii) The monoid of non-zero $K \oplus F$-homogeneous elements is a face of the monoid of non-zero K-homogeneous elements. In particular, if a is a non-zero K-homogeneous element such that Ra contains a non-zero $K \oplus F$-homogeneous element then a is $K \oplus F$-homogeneous.
(iv) If a/b is a fraction of K-homogeneous elements such that $R(a/b)$ is $K \oplus F$-homogeneous then there exists a fraction of $K \oplus F$-homogeneous elements c/d with $a/b = c/d$. Indeed, we have decomposition $a/b = (1/d)\sum_{u \in F} c_u$ where d, c_u are $K \oplus F$-homogeneous. But then $c := \sum_u c_u$ is $K \oplus F$-homogeneous because $Rc = RdR(a/b)$ is $K \oplus F$-homogeneous.

Remark II.2.1.9. A graded polynomial ring $R' := R[T_i | i \in I]$ has the same homogeneous units as R, and is integrally graded ring if and only if R is. Moreover, if R is integrally graded then $R^{\mathrm{hom}} \setminus 0$ is a face of $R'^{\mathrm{hom}} \setminus 0$.

Proposition II.2.1.10. *Let R be a $K \oplus F$-simple ring, where F is free, and denote by $\mathrm{pr}_F \colon K \oplus F \to F$ the canonical map. Then for a basis $v_i, i \in I$ of $G := \mathrm{pr}_F(\deg(R^{\mathrm{hom}} \setminus 0))$ and elements r_i with $\mathrm{pr}_F(\deg(r_i)) = v_i$ we obtain a degree-preserving isomorphism $\phi \colon R_{K\oplus 0}[G] \to R$ by sending χ^{v_i} to r_i.*

Proof. For surjectivity note that for a $r \in R^{\mathrm{hom}} \setminus \{0\}$ there exist l_i with $\deg(r) - \sum_i l_i v_i \in K$ and thus $r = \phi(r \prod_i r_i^{-l_i} \prod_i \chi^{l_i})$. For injectivity note that if $\sum_j a_j \prod_i r_i^{n_{i,j}} = 0$, where $a_j \in Q_{\mathrm{gr}}(R)_K$, then $a_j \prod_i r_i^{n_{i,j}} = 0$ and hence $a_j = 0$. □

Example II.2.1.11. Let A be an integral domain, let $M \subseteq K$ and $N \subseteq L$ be submonoids of abelian groups K and L. Let $Q\colon K \to L$ be a homomorphism of groups such that $Q(M) \subseteq N$. Then the induced homomorphism $\phi\colon A[M] \to A[N]$ has kernel

$$\ker(\phi) = \langle \chi^v - \chi^w \mid v, w \in M, v - w \in \ker(Q) \rangle.$$

Indeed, if $f = a_1\chi^{w_1} + \ldots + a_d\chi^{w_d}$ is an element of the kernel of ϕ, then we may suppose that all w_i have the same image under Q. Thus $\sum_{j=1}^{d} a_j = 0$ and we may write f as

$$f = \sum_{j=1}^{d-1} \left(\sum_{l=1}^{j} a_l \right) (\chi^{w_j} - \chi^{w_{j+1}}).$$

Since $A[N]$ is L-integral, so is $\mathrm{im}(\phi)$ and therefore the ideal $\ker(\phi)$ is L-prime. If M is a subgroup of K then $\ker(\phi) = \langle \chi^v - 1 \mid v \in \ker(Q) \cap M \rangle$.

Remark II.2.1.12. Each graded module M over a simply graded ring R is free. Moreover, each maximal homogeneous R-linearly independent family in M and each minimal homogeneous generating family of M is a basis.

Definition II.2.1.13. A K-noetherian K-local ring $(R, \mathfrak{m})$ is K-regular or *regularly graded* if its graded dimension (in the sense of Definition II.1.8.8) equals the rank of the free $R/\mathfrak{m}$-module $\mathfrak{m}/\mathfrak{m}^2$ (i.e. the length of a $R/\mathfrak{m}$-basis of $\mathfrak{m}/\mathfrak{m}^2$).

II.2.2. graded factoriality. The notion of graded factoriality is due to [**17**] where it was proven to be a characterizing property of Cox rings. We reduce graded factoriality to factoriality of rings in Theorem II.2.2.4 and give a sufficient condition for the latter via Proposition II.2.2.6.

Definition II.2.2.1. Let R be a K-integral ring.

(i) $r \in R^{\mathrm{hom}}$ is *K-prime/-irreducible* or *homogeneously prime/irreducible* if r is prime/irreducible in R^{hom}.
(ii) R is *K-factorial* or *factorially graded* if R^{hom} is factorial.

Proposition II.2.2.2. *Principally graded rings in the sense of II.1.7.8 are factorially graded.*

Proof. If the grading of R is principal it is noetherian and Lemma I.2.2.15 reduces the problem to showing that every irreducible $p \in R^{\mathrm{hom}}$ is prime. By definition, $\langle p \rangle$ is maximal among all the principal ideals of homogeneous elements - so in our case among all homogeneous ideals. Thus, $R/\langle p \rangle$ is simply and hence integrally graded. □

Remark II.2.2.3. For a K-factorial ring R which is finitely generated over a graded subring A each system P of K-primes of R and each system of homogeneous generators $\{f_i\}_{i \in I}$ determine a system $\{g_j\}_{j \in J}$ of pairwise non-associated K-primes and homogeneous units of R which generates R over A. Specifically, each f_i is a unique product of a homogeneous unit and elements of P, and all the factors occuring in such products make up $\{g_j\}_j$.

The behaviour of factoriality of monoids with respect to localization was discussed in Propositions I.2.6.10 and I.2.6.15. Using theses statements we prove the following criterion for graded factoriality.

THEOREM II.2.2.4. *Let $R = \mathbb{K}[f_1, \dots, f_r]$ be a K-graded algebra over the field $\mathbb{K}$ which is generated by primes $f_1, \dots, f_s$ and units $f_{s+1}, \dots, f_r$ of R^{hom}. Let $S \subseteq R^{\mathrm{hom}}$ be the multiplicative submonoid generated by $f_1, \dots, f_s$. Then R is K-factorial if and only if $(S^{-1}R)_0$ is factorial.*

PROOF. By Propositions I.2.6.15 and I.2.6.16 K-factoriality of R is equivalent to K-factoriality of $S^{-1}R$. Since $(S^{-1}R)^{\mathrm{hom}} = ((S^{-1}R)^{\mathrm{hom}})^*(S^{-1}R)_0$, Proposition I.2.2.13 gives that $S^{-1}R$ is K-factorial if and only if $(S^{-1}R)_0$ is factorial. □

In the situation of Theorem II.2.2.4 we have a degree-preserving epimorphism $\mathbb{K}[T_1, \dots, T_s, T_{s+1}^{\pm 1}, \dots, T_r^{\pm 1}] \to R$ whose kernel is generated by certain homogeneous elements $h_i \in \mathbb{K}[T_1, \dots, T_s, T_{s+1}^{\pm 1}, \dots, T_r \pm 1]$. Fix Laurent monomials T^{v_i} such that $\deg(T^{v_i} h_i) = 0_K$ and an isomorphism $\imath\colon \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \to \mathbb{K}[T_1^{\pm 1}, \dots, T_r^{\pm 1}]_0$. Let g_i be the unique Laurent polynomial with $\imath(g_i) = T^{v_i} h_i$. Then we have an induced isomorphism $(S^{-1}R)_0 \cong \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm}]/\langle g_1, \dots, g_m\rangle$.

REMARK II.2.2.5. If for a set $g_1, \dots, g_m$ of Laurent polynomials we have $g_i - T_{n-m+i} \in \mathbb{K}[T_1^{\pm 1}, \dots, T_{n-m}^{\pm 1}]$ then

$$\mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\langle g_1, \dots, g_m\rangle \cong \mathbb{K}[T_1^{\pm 1}, \dots, T_{n-m}^{\pm 1}]_{(g_1 - T_{n-m+1}) \cdots (g_m - T_n)}$$

is factorial.

Recall that the *support* of a Laurent polynomial $g \in \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, denoted $\mathrm{Supp}(g)$, is the set of exponent vectors $u \in \mathbb{Z}^n$ such that T^u occurs with a nonzero coefficient in g. By the *primitive span* of a set of vectors in $\mathbb{Z}^n$ we mean the saturation of their linear span.

PROPOSITION II.2.2.6. *Let $g_1, \dots, g_s \in \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$. Suppose there exist $u_i \in \mathrm{Supp}(g_i)$ for $i = 1, \dots, m$ such that for the primitive span N of the set of vectors $\bigcup_{i=1}^m (\mathrm{Supp}(g_i) \setminus \{u_i\})$ the sublattice $\sum_{i=1}^m \mathbb{Z}u_i + N$ is primitive in $\mathbb{Z}^n$ and equals $\bigoplus_{i=1}^m \mathbb{Z}u_i \oplus N$. Then $\mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\langle g_1, \dots, g_m\rangle$ is factorial.*

PROOF. Let $u_{m+1}, \dots, u_k \in N$ be a basis of N. Then $u_1, \dots, u_k$ is a basis of $\bigoplus_{i=1}^m \mathbb{Z}u_i \oplus N$ which due to primitivity in $\mathbb{Z}^n$ we may complete to a basis $u_1, \dots, u_n$ of $\mathbb{Z}^n$. Then $h_i := \alpha_{u_i}^{-1} g_i - T^{u_i}$ is a Laurent polynomial in $T^{u_{m+1}}, \dots, T^{u_k}$. Let ϕ be the automorphism of $\mathbb{Z}^n$ mapping u_i to the i-th standard basis vector e_i. Under the corresponding automorphism ϕ of $\mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]$, $\phi(h_i)$ is a Laurent polynomial in $T_{m+1}, \dots, T_k$ and by Remark II.2.2.5

$$\mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}]/\langle g_1, \dots, g_m\rangle \cong \mathbb{K}[T_1^{\pm 1}, \dots, T_{n-m}^{\pm 1}]_{\phi(h_1) \cdots \phi(h_m)}$$

is factorial. □

In the following we indicate how to check the condition of Proposition II.2.2.6 via Smith Normal Form calculations. Recall that the *Smith Normal Form* of an integer matrix A is the unique matrix $\mathrm{SNF}(A)$ of form

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

such that

(i) D is diagonal with positive diagonal entries satisfying $D_{i,i} | D_{i+1,i+1}$,
(ii) there exist unimodular matrices S and T with $S \cdot A \cdot T = \mathrm{SNF}(A)$.

REMARK II.2.2.7. In the Setting of Proposition II.2.2.6 let A be the matrix whose columns are the elements of $\bigcup_{i=1}^m \mathrm{Supp}(g_i) \setminus \{u_i\}$ and let B be the matrix whose columns are the vectors $u_1, \dots, u_m$. Let S, T be unimodular matrices such that SAT is in Smith Normal Form, set $l := \mathrm{rank}(A)$ and let C be the matrix whose rows are rows $l+1$ to n of the matrix SB. Then the condition of Proposition II.2.2.6 holds if and only if

$$\mathrm{SNF}(C) = \begin{pmatrix} E_s \\ 0_{n-l-m} \end{pmatrix}.$$

The condition that $\sum_{i=1}^{m} \mathbb{Z}u_i + N$ is a direct sum and primitive in $\mathbb{Z}^n$ may be checked by calculating a set $v_1, \ldots, v_k$ of generators of N and ascertaining that the non-zero part of the Smith Normal Form of the matrix

$$U := [v_1, \ldots, v_k, u_1, \ldots, u_m]$$

is E_{l+m}. Choose $k = l$ and $v_i = S^{-1}e_i$ for $i = 1, \ldots, l$. Then our task amounts to calculating

$$\mathrm{SNF}(U) = \mathrm{SNF}(SU) = \mathrm{SNF}([e_1, \ldots, e_l, SB]) = SNF \begin{pmatrix} E_l & 0 \\ 0 & C \end{pmatrix}$$

and the non-zero part of this Smith Normal Form is E_{l+m} if and only if the non-zero part of $\mathrm{SNF}(C)$ is E_m.

II.2.3. graded divisors of integrally graded rings. Here, we introduce the monoid of graded divisors in terms of which we later characterize homogeneously completely integrally closed rings, graded valuation rings and graded rings of Krull type.

Definition II.2.3.1. Let R be a K-integral ring. A non-zero K-principal R-submodule of $Q_{\mathrm{gr}}(R)$, i.e. a submodule of the form Rf with some non-zero $f \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}}$, is also called a *$K$-principal divisor* or *graded principal divisor* of R. A graded R-submodule of $Q_{\mathrm{gr}}(R)$ which is contained in a graded principal divisor is a *graded fractional ideal.*

Remark II.2.3.2. If $\mathfrak{a}$ is a non-zero K-graded R-submodule of $Q_{\mathrm{gr}}(R)$ then $(R^{\mathrm{hom}} \setminus 0)^{-1}\mathfrak{a} = Q_{\mathrm{gr}}(R)$. Indeed, if $a, b \in R^{\mathrm{hom}} \setminus 0$ are given and $f/g \in \mathfrak{a}$ is a non-zero homogeneous fraction, then $af = (ag)(f/g) \in \mathfrak{a}$ and hence $a/b = (1/bf)(af) \in (R^{\mathrm{hom}} \setminus 0)^{-1}\mathfrak{a}$.

Remark II.2.3.3. Every finite intersection of non-zero graded R-submodules of $Q_{\mathrm{gr}}(R)$ is again non-zero. Indeed, if $\mathfrak{a}, \mathfrak{b}$ are submodules containing non-zero homogeneous elements $a = vw^{-1}$ resp. $b = xy^{-1}$ where $v, w, x, y \in M$, then $vx = wxa = vyb$ is contained in their intersection.

Moreover, products and finite sums of fractional ideals are again fractional. Indeed, if $\mathfrak{a} \subseteq Rf$ and $\mathfrak{b} \subseteq Rg$ with homogeneous fractions $f = uv^{-1}$ and $g = xy^{-1}$, then $\mathfrak{ab} \subseteq Mfg$ and $\mathfrak{a} + \mathfrak{b} \subseteq Mv^{-1}y^{-1}$.

Remark II.2.3.4. Let $\phi\colon R \to S$ be a morphism between integrally graded rings with $\ker(\phi) \cap R^{\mathrm{hom}} = \{0_R\}$. Let $\phi\colon Q_{\mathrm{gr}}(R) \to Q_{\mathrm{gr}}(S), r/t \mapsto \phi(r)/\phi(t)$ be the induced morphism. Then for a fractional graded ideal $\mathfrak{a} \leq_R Q_{\mathrm{gr}}(R)$, $\langle \phi(\mathfrak{a}) \rangle_S$ is fractional.

Construction II.2.3.5. A *graded divisor* is a non-zero intersection over a (non-empty) family of graded principal divisors. For each graded fractional ideal $\mathfrak{a}$ the intersection $\mathrm{div}^{\mathrm{gr}}(\mathfrak{a})$ over all graded principal divisors containing $\mathfrak{a}$ is a graded divisor. The set $\mathrm{Div}^{\mathrm{gr}}(R)$ of graded divisors endowed with the operation sending $D, D' \in \mathrm{Div}^{\mathrm{gr}}(R)$ to $D + D' := \mathrm{div}(DD')$ is a monoid with neutral element $0_{\mathrm{Div}^{\mathrm{gr}}(R)} := R$. Setting $D \leq D'$ if and only if $D \supseteq D'$ turns $Div^{\mathrm{gr}}(R)$ into a partially ordered monoid, called the *graded divisor monoid* of R. It comes with the *divisor homomorphism*

$$\mathrm{div}^{\mathrm{gr}}\colon Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0 \longrightarrow \mathrm{Div}^{\mathrm{gr}}(R), \quad f \longmapsto \mathrm{div}^{\mathrm{gr}}(f) := \mathrm{div}^{\mathrm{gr}}(Rf) = Rf$$

whose image $\mathrm{PDiv}^{\mathrm{gr}}(R)$ is the subgroup of graded principal divisors. The factor monoid $Cl^{\mathrm{gr}}(R) = \mathrm{Div}^{\mathrm{gr}}(R)/\,\mathrm{PDiv}^{\mathrm{gr}}(R)$ is the *graded divisor class monoid* of R.

Proposition II.2.3.6. *There is a canonical isomorphism*

$$\begin{aligned} \mathrm{Div}^{\mathrm{gr}}(R) &\longleftrightarrow \mathrm{Div}(R^{\mathrm{hom}}) \\ D &\longmapsto D \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \\ \langle E \rangle_R &\longleftarrow E \end{aligned}$$

which restricts to an isomorphism $\mathrm{PDiv}^{\mathrm{gr}}(R) \to \mathrm{PDiv}(R^{\mathrm{hom}})$.

Proof. Evidently, each $f \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0$ satisfies $Rf \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}} = R^{\mathrm{hom}} f$ and $R(R^{\mathrm{hom}} f) \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}} = R^{\mathrm{hom}} f$. This establishes the claim on principal divisors and for $D \in \mathrm{Div}^{\mathrm{gr}}(R)$ we see that the set $D \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}} = \bigcap_{D \subseteq Rf} R^{\mathrm{hom}} f$ is indeed a divisor of R^{hom}.

On the other hand, for a divisor $E = \bigcap_{i \in I} R^{\mathrm{hom}} f_i$ of R^{hom} given by a family $f_i \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0, i \in I$ we claim that $\langle E \rangle_R = \bigcap_{i \in I} Rf_i$. If $r \in R^{\mathrm{hom}} \setminus 0$ and $g \in E$ then $rg \in R^{\mathrm{hom}} f_i$ for every i and hence $rg \in \bigcap_i Rf_i$. Conversely, if $g \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0$ lies in every Rf_i, then for every i there exists $r_i \in R^{\mathrm{hom}}$ such that $g = r_i f_i$ which means that $g \in E \subseteq \langle E \rangle_R$. Thus, $\langle E \rangle_R$ is a graded divisor of R.

To see that $R(D \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}}) = D$ note that the left hand side is by definition a graded R-submodule of D. For the other inclusion let $f \in D$ be homogeneous. Then $1f \in R(D \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}})$.

For the equation $E = (\langle E \rangle_R) \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}}$ consider a product $rf \in \langle E \rangle_R$ of homogeneous elements. Then $rf \in E$ because $r \in R^{\mathrm{hom}}$. □

Corollary II.2.3.7. *The graded divisor monoid of R has the following properties:*

(i) *For $f, g \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0$ with $f + g \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0$ we have*

$$\mathrm{div}^{\mathrm{gr}}(f+g) \geq \mathrm{div}^{\mathrm{gr}}(Rf + Rg) = \inf\{\mathrm{div}^{\mathrm{gr}}(f), \mathrm{div}^{\mathrm{gr}}(g)\}.$$

(ii) *For each $D \in \mathrm{Div}^{\mathrm{gr}}(R)$ we have*

$$D = \left\{ \sum_{w \in K} f_w \mid f_w = 0 \text{ or } \mathrm{div}^{\mathrm{gr}}(f_w) \geq D \text{ for all } w \in K \right\}.$$

Proof. For (ii) let $D \in \mathrm{Div}^{\mathrm{gr}}(R)$. For $g \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0$ we have $g \in D$ if and only if $Rg \subseteq D$, i.e. $\mathrm{div}^{\mathrm{gr}}(g) \geq D$. If $f = \sum_{w \in K} f_w$ is the decomposition into homogeneous parts then $f \in D$ if and only if for every $w \in K$ with $f_w \neq 0$ we have $f_w \in D$, i.e. $\mathrm{div}^{\mathrm{gr}}(f_w) \geq D$. □

Remark II.2.3.8. Let R be an integrally $K \oplus F$-graded rings where F is totally ordered. Then coarsening via the projection map $K \oplus F \to K$ defines a monomorphism $\mathrm{Div}_{K \oplus F}(R) \to \mathrm{Div}_K(R)$ which respects principality and induces a homomorphism $\mathrm{Cl}_{K \oplus F}(R) \to \mathrm{Cl}_K(R)$ which is injective due to Remark II.2.1.8.

II.2.4. graded valuation rings and graded Euclidean rings. As a preparation for the notion of graded rings of Krull type we introduce graded valuations. Moreover, we discuss graded Euclidean rings and in particular, polynomial rings of simply graded rings.

Definition II.2.4.1. A K-graded ring R is a *(discrete) graded valuation ring* if R^{hom} is a (discrete) valuation $\mathbb{F}_1$-algebra. A K-prime element of a discrete graded valuation ring R is called a *homogeneous uniformizer* of R.

Definition II.2.4.2. A *(disrecte) graded valuation* or *K-valuation* on a K-simple ring S is a homomorphism of semigroups $\nu\colon (S^{\mathrm{hom}})^* \to G$ to a totally ordered group G (resp. to $\mathbb{Z}$) such that $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$ holds for all $a, b \in (S^{\mathrm{hom}})^*$ with $a + b \in (S^{\mathrm{hom}})^*$. A surjective graded valuation is called *normed*.

Proposition II.2.4.3. *Let R be an simply $K \oplus F$-graded ring where F is a totally ordered group. Then each graded valuation ν on R extends to $Q_K(R)$ via $\nu(a) := \min\{\nu(a_u) | u \in F, a_u \neq 0\}$ and $\nu(a/b) := \nu(a) - \nu(b)$ for non-zero K-homogeneous $a, b \in R$.*

Proof. For K-homogeneous non-zero $a, b \in R$ let u and v be minimal with respect to the chosen order such that a_u and b_v are non-zero and we have $\nu(a) = \nu(a_u)$ and $\nu(b) = \nu(b_v)$. Using that ig $u < u'$ and $v < v'$ hold then $u + v < u' + v'$ we obtain $\nu(ab) = \nu(a_u b_v) = \nu(a_u) + \nu(b_v) = \nu(a) + \nu(b)$. This shows that ν is a graded valuation on $Q_K(R)$. $\square$

Definition II.2.4.4. A *euclidean graded ring* is an integrally graded ring R with *degree function* or *euclidean function* $\delta\colon R^{\text{hom}} \setminus 0 \to \mathbb{N}_0$ such that for each $f \in R^{\text{hom}}$ and $g \in R^{\text{hom}} \setminus 0$ there exist $q, r \in R^{\text{hom}}$ with $f = qg + r$ and $\delta(r) < \delta(g)$ or $r = 0$.

Remark II.2.4.5. A K-valuation ν on a K-simple ring S defines a K-valuation ring S_ν such that $S_\nu^{\text{hom}} = \nu^{-1}(G_{\geq 0}) \sqcup \{0\}$, $\nu^{-1}(0) = (S_\nu^{\text{hom}})^*$ and $S = Q_{\text{gr}}(R)$. Conversely, for a K-valuation ring R the map $\text{div}\colon (Q_{\text{gr}}(R)^{\text{hom}})^* \to \text{PDiv}_{gr}(R)$ is a normed K-valuation and we have $R = Q_{\text{gr}}(R)_{\text{div}}$.

Remark II.2.4.6. A discrete graded valuation ring R is locally and factorially graded and each homogeneous uniformizer generates the unique maximal graded ideal of R. Furthermore, R together with $(\text{div}_{gr})_{|R^{\text{hom}} \setminus 0}$ is euclidean.

Remark II.2.4.7. In a euclidean graded ring R each non-zero graded ideal $\mathfrak{a} \trianglelefteq R$ is generated by one of its homogeneous elements $f \in \mathfrak{a}$ with minimal degree $\delta(f) \in \mathbb{N}_0$.

Proposition II.2.4.8. *For a graded polynomial ring $R[T]$ the following are equivalent:*

(i) *$R[T]$ is principally and integrally graded,*
(ii) *$R[T]$ is euclidean,*
(iii) *R is simply graded.*

Proof. If R is simply graded then let $f = a_0 T^0 + \ldots + a_m T^m \in R[T]^{\text{hom}}$ and $g = b_0 T^0 + \ldots + b_n T^n \in R[T]^{\text{hom}} \setminus 0$ with $a_i, b_j \in R^{\text{hom}}$ and $a_m, b_n \neq 0$. If $m = 0$, then $f = 0 \cdot g + f$ is as wanted. Let now $m > 0$. We only need to consider the case $m \geq n$. Then $f' := f - b_n^{-1} a_m T^{m-n} g \in R[T]_{\deg(f)}$ and by induction we find $q', r \in R[T]^{\text{hom}} \setminus 0$ with $f' = q'g + r$ and $r = 0$ or $\deg_{\max}(r) < \deg_{\max}(g)$. Thus, we get $f = qg + r$ where $q := q' + b_n^{-1} a_m T^{m-n}$.

If R is principally and integrally graded then T is K-prime and hence K-irreducible. Thus, $\langle T \rangle$ is maximal among principal ideals of K-homogeneous elements, i.e. among all graded ideals. Consequently, all non-zero K-homogeneous elements of $R \cong R[T]/\langle R \rangle$ are units. $\square$

Remark II.2.4.9. For a graded homomorphism $\phi\colon R \to S$ between simply graded rings the following hold:

(i) Each graded valuation ν on S defines a graded valuation $\mu := \nu \circ \phi_{|R^{\text{hom}} \setminus 0}$ on R which valuates $\phi^{-1}(1_S) \cap R^{\text{hom}}$ trivially, and the associated graded valuation rings satisfy $\phi(R_\mu) \subseteq S_\nu$. If ϕ is a CB then $R_\mu \to S_\nu$ is a CB.
(ii) If μ is a graded valuation on R and $S^{\text{hom}} = A\phi(R^{\text{hom}})$ holds with a subgroup $A \subseteq S^{\text{hom}} \setminus 0$ such that $\phi^{-1}(A) \cap R^{\text{hom}} \subseteq \ker(\mu)$ then setting $\nu(a\phi(r)) := \mu(r)$ for $a \in A$ and $r \in R^{\text{hom}} \setminus 0$ defines a valuation on S, and we have $\phi(R_\mu) \subseteq S_\nu$. If ϕ is a CBE then $R_\mu \to S_\nu$ is a CBE.

II.2.5. graded rings of Krull type. Now, we define and characterize graded rings of Krull type and introduce homomorphisms of monoids of graded divisors. Moreover, we study Krull property and class group under coarsening from a $K\oplus F$-grading to a K-grading, where F is free.

Definition II.2.5.1. An integrally graded ring R is *of Krull type* if it is the graded subring of a simply graded ring S and there is a (possibly empty) family of discrete graded valuations ν_i, $i \in I$ such that

(i) $R = \bigcap_{i\in I} S_{\nu_i}$,

(ii) for each $0 \neq a \in R^{\mathrm{hom}}$ there are only finitely many $i \in I$ with $\nu_i(a) \neq 0$.

The family $\{\nu_i\}_{i\in I}$ is then said to define R in S.

Remark II.2.5.2. If a family $\{\nu_i\}_{i\in I}$ defines R in S, then the restricted family $\{\nu_{i|(Q_{\mathrm{gr}}(R)^{\mathrm{hom}})^*}\}_i$ defines R in $Q_{\mathrm{gr}}(R)$.

If R^{hom} is of Krull type then the minimal positive elements $\mathfrak{P}(R)$, called the *K-prime divisors*, form a basis of $\mathrm{Div}_{\mathrm{gr}}(R) \cong \mathrm{Div}(R^{\mathrm{hom}})$. If no confusion can arise we write or $\mathfrak{P}$ for $\mathfrak{P}(R)$. Due to Corollary II.2.3.7 for each $\mathfrak{p} \in \mathfrak{P}$ the composition of $\mathrm{div}^{\mathrm{gr}}$ with the projection $\mathrm{Div}_{\mathrm{gr}}(R) \to \mathbb{Z}\mathfrak{p}$ is a graded valuation, called the *essential* graded valuation corresponding to $\mathfrak{p}$. It restricts to the essential valuation $\nu_{\mathfrak{p}\cap R^{\mathrm{hom}}}$ of the prime divisor $\mathfrak{p} \cap R^{\mathrm{hom}} \in \mathfrak{P}(R^{\mathrm{hom}})$. Since $\{\nu_{\mathfrak{p}\cap R^{\mathrm{hom}}}\}_{\mathfrak{p}}$ define R^{hom} as an $\mathbb{F}_1$-algebra of Krull type $\{\nu_{\mathfrak{p}}\}_{\mathfrak{p}}$ define R as a graded ring of Krull type. Thus, we have the following.

Proposition II.2.5.3. *A graded ring R is of Krull type if and only if R^{hom} is of Krull type.*

Proposition II.2.5.4. *For a K-Krull ring R the set of K-prime divisors has the following descriptions:*

$$\begin{aligned}\mathfrak{P}(R) &= \{D \in \mathrm{Div}_{\mathrm{gr}}(R)_{\geq 0} \mid D \textit{ is prime}\}\\ &= \{\mathfrak{q} \in \mathrm{Div}_{\mathrm{gr}}(R)_{\geq 0} \mid \mathfrak{q} \textit{ is a } K\textit{-prime ideal of } R\}\\ &= \{\mathfrak{q} \trianglelefteq R \mid \mathfrak{q} \textit{ is } K\textit{-prime with } \mathrm{ht}_{gr}(\mathfrak{q}) = 1\}\end{aligned}$$

Proof. From $\mathrm{Div}_{gr}(R) \cong \mathrm{Div}(R^{\mathrm{hom}})$ and Chapter I we obtain all but the last equality. If $D \in \mathrm{Div}_{gr}(R)$ is K-prime and $\mathfrak{p} \subseteq D$ is a non-zero K-prime ideal then $\mathfrak{p} \cap R^{\mathrm{hom}} \subseteq D \cap R^{\mathrm{hom}}$ is an inclusion of non-empty prime ideals which implies $\mathfrak{p} \cap R^{\mathrm{hom}} = D \cap R^{\mathrm{hom}}$ and hence $\mathfrak{p} = D$. Conversely, if $\mathfrak{q}$ is a minimal non-zero K-prime ideal then $R^{\mathrm{hom}}_{\mathfrak{q}}$ is the intersection over all $R_{\mathfrak{p}}$ where $\mathfrak{p}$ is a K-prime divisor lying in $\mathfrak{q}$. Since $R_{\mathfrak{q}} \neq Q_{\mathrm{gr}}(R)$ such a $\mathfrak{p}$ exists and minimality gives $\mathfrak{q} = \mathfrak{p} \in \mathfrak{P}$. □

Proposition II.2.5.5. *The family of essential K-valuations $\{\nu_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}(M)}$ of a K-Krull ring R is minimal among all families of valuations defining R in $Q_{\mathrm{gr}}(R)$. Moreover, for each $\mathfrak{p} \in \mathfrak{P}$ we have $Q_{\mathrm{gr}}(R)_{\nu_{\mathfrak{p}}} = R_{\mathfrak{p}}$ and $\mathfrak{p} \cap R^{\mathrm{hom}} \setminus 0 = \nu_{\mathfrak{p}}^{-1}(\mathbb{N})$.*

Remark II.2.5.6. Each K-prime ideal $\mathfrak{q}$ in a graded ring R of Krull type ring satisfies

$$R_{\mathfrak{q}} = \bigcap_{\substack{\mathfrak{p}\in\mathfrak{P}(R)\\ \mathfrak{p}\subseteq\mathfrak{q}}} Q_{\mathrm{gr}}(R)_{\nu_{\mathfrak{p}}} = \bigcap_{\substack{\mathfrak{p}\in\mathfrak{P}(R)\\ \mathfrak{p}\subseteq\mathfrak{q}}} R_{\mathfrak{p}}.$$

In particular, if $\mathfrak{q}'$ is another K-prime ideal then $\mathfrak{q} \subseteq \mathfrak{q}'$ if and only if every K-prime divisor $\mathfrak{p}$ inside $\mathfrak{q}$ is also contained in $\mathfrak{q}'$.

Remark II.2.5.7. Due to Remark I.2.6.5, for every homogeneously prime divisor $\mathfrak{p}$ of a K-Krull ring R there exists $f \in R^{\mathrm{hom}}$ such that $\mathfrak{p}_f$ is K-principal in R_f.

Remark II.2.5.8. Let $R_i, i \in I$ be a family of K-Krull rings in the K-simple ring S. If every non-zero homogeneous element f of $R := \bigcap_i R_i$ is a unit in all but finitely many R_i then R is a K-Krull ring.

Definition II.2.5.9. Let R be a K-Krull ring defined by $\{\nu_i\}_{i\in I}$ in $Q_{\mathrm{gr}}(R)$. For each graded fractional ideal $\mathfrak{a}$ set $\nu_i(\mathfrak{a}) = \max_{\mathfrak{a}\subseteq Rf} \nu_i(f) = \nu_i(\mathfrak{a} \cap Q_{\mathrm{gr}}(R)^{\mathrm{hom}})$.

Construction II.2.5.10. Let $\phi\colon R \to S$ be a morphism between graded rings of Krull type. Then the composition $\beta_\phi\colon \mathrm{Div}_{\mathrm{gr}}(R) \cong \mathrm{Div}(R^{\mathrm{hom}}) \to \mathrm{Div}(S^{\mathrm{hom}}) \cong \mathrm{Div}_{\mathrm{gr}}(S)$ of the canonical maps is the *natural homomorphism of graded divisor monoids* induced by ϕ.

Proposition II.2.5.11. *Let $\phi\colon R \to S$ be a morphism between simply graded rings. Let $\{\nu_i\}_{i\in I}$ be a family of graded valuations on S and let $\{\mu_i\}_{i\in I} := \{\nu_i \circ \phi_{|(R^{\mathrm{hom}})^*}\}_{i\in I}$ be the corresponding family of graded valuations on R. Let R' and S' be the intersections of all the rings R_{μ_i} resp. S_{ν_i}.*

(i) *If $\{\nu_i\}_i$ defines S' as a graded ring of Krull type in S then $\{\mu_i\}_i$ defines R' as a graded ring of Krull type in R, and if $S^{\mathrm{hom}} = (S'^{\mathrm{hom}})^*\phi(R^{\mathrm{hom}})$ then the converse is true. Moreover, in the latter case $\{\nu_i\}_{i\in I}$ are then the essential graded valuations of S' if and only if $\{\mu_i\}_{i\in I}$ are the essential graded valuations of R'.*

(ii) *If ϕ is a CB(E) then so is $R' \to S'$.*

Remark II.2.5.12. Let $R' := R[T_i | i \in I]$ be a graded polynomial ring over a K-Krull ring R and consider the grading by $L := K \oplus \bigoplus_i \mathbb{Z}$ assigning to T_i the degree $(\deg_K(T_i), e_i)$. Then the monoid of non-zero L-homogeneous elements of R' is the coproduct (in this case, also the Cartesian product) of the abelian monoids $R^{\mathrm{hom}} \setminus 0$ and $\bigoplus_i \mathbb{N}_0$. Thus, by Remark I.2.6.11 R' is an L-Krull ring and we have $\mathrm{Cl}_L(R') \cong \mathrm{Cl}_K(R)$.

Remark II.2.5.13. Let F be a totally ordered abelian group. Let $\{\nu_i\}_{i\in I}$ be a family of $K \oplus F$-valuations defining a $K \oplus F$-Krull ring R in S such that for non-zero $s \in S^{\mathrm{hom}}$ the set of i with $\nu_i(s) \neq 0$ is finite. Then the extended family $\{\mu_i\}_i$ from Proposition II.2.4.3 defines a K-Krull ring R' in $Q_K(S)$.

Indeed, if $f = f_1 + \ldots + f_m$ is a decomposition into $K \oplus F$-homogeneous parts of the same K-degree then the set of all i with $\mu_i(f) \neq 0$ is contained in the union of all i with $\nu_i(f_j) \neq 0$ where $j = 1, \ldots, m$.

Lemma II.2.5.14. *Let F be a totally ordered abelian group. For a $K \oplus F$-graded K-Krull ring R the following hold:*

(i) *The prime $K\oplus F$-graded divisors $\mathfrak{p}$ of the $K\oplus F$-Krull ring R are precisely the prime K-graded divisors of R which are $K \oplus F$-homogeneous, i.e. those which contain a non-zero $K \oplus F$-homogeneous element.*

(ii) *the canonical map $\beta\colon \mathrm{Div}_{K\oplus F}(R) \to \mathrm{Div}_K(R)$ induced by the map from the $K \oplus F$-graded ring R to the K-graded ring R coincides with the map from Remark II.2.3.8. Consequently, it preserves primality of graded divisors and the induced map $\mathrm{Cl}_{K\oplus F}(R) \to \mathrm{Cl}_K(R)$ is a monomorphism.*

Proof. If $\mathfrak{p} \in \mathfrak{P}_K(R)$ contains a non-zero $K \oplus F$-homogeneous element r then we have

$$\mathfrak{b} := \prod_{\mathfrak{q}\in\mathfrak{P}_{K\oplus F}(R)} \mathfrak{q}^{\nu_{\mathfrak{q}}(r)} \subseteq \mathrm{div}_{K\oplus F}(\mathfrak{b}) = Rr \subseteq \mathfrak{p}.$$

Thus, we have $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{q} \in \mathfrak{P}_{K\oplus F}(R)$, and K-primality of $\mathfrak{q}$ gives $\mathfrak{q} = \mathfrak{p}$.

For $\mathfrak{q} \in \mathfrak{P}_{K\oplus F}(R)$ the $K \oplus F$-graded localization R' at $\mathfrak{q}$ is again a K-Krull ring defined by the K-valuations of all $\mathfrak{p} \in \mathfrak{P}_K(R)$ contained in $\mathfrak{q}$. If $r \in \mathfrak{q} \setminus 0$ is $K \oplus F$-homogeneous it cannot be a unit in R' and thus there exists $\mathfrak{p} \in \mathfrak{P}_K(R)$ with $r \in \mathfrak{p} \subseteq \mathfrak{q}$. Since $\mathfrak{p}$ is then $K \oplus F$-graded we conclude $\mathfrak{p} = \mathfrak{q}$.

For (ii) note that by (i) for each K-graded prime divisor $\mathfrak{q}$ the $K \oplus F$-graded ideal $\mathfrak{q}_{K\oplus F}$ generated by the $K \oplus F$-homogeneous elements of $\mathfrak{q}$ is $\{0\}$ or $\mathfrak{q}$, in particular $\mathrm{Cl}_{K\oplus F}(R_{\mathfrak{q}_{K\oplus F}}) = 0$ holds. Thus, we have $\beta(D) = \mathrm{div}_K(D) = D$ by Proposition I.2.6.8. □

Theorem II.2.5.15. *Let R be a $K \oplus F$-graded ring, where F is free. Then R is a $K \oplus F$-Krull ring if and only if it is a K-Krull ring. Moreover, the canonical map $\mathrm{Cl}_{K\oplus F}(R) \to \mathrm{Cl}_K(R)$ is an isomorphism.*

Proof. Suppose that R is a $K \oplus F$-Krull ring. Let R' be the K-Krull ring defined by the extensions of the essential $K \oplus F$-valuations of R to $Q_K(R)$.

We claim that $Q_{K\oplus F}(R)$, which by Proposition II.2.1.10 is $Q_{K\oplus\mathbb{Z}}(R)_K[G]$ for a subgroup G of F, is K-factorial. Then $R = R' \cap Q_{K\oplus F}(R)$ is again a K-Krull ring, and due to K-factoriality of $Q_{K\oplus F}(R)$, $\mathrm{Cl}_K(R)$ is generated by the classes of K-prime divisors containing a non-zero $K \oplus F$-homogeneous element. By Lemma II.2.5.14 these are the $K \oplus F$-graded prime divisors of R and the canonical homomorphism $\mathrm{Cl}_{K\oplus F}(R) \to \mathrm{Cl}_K(R)$ is bijective.

If $\mathrm{rk}(F) = 1$ then K-factoriality of $Q_{K\oplus F}(R)$ follows from Propositions II.2.4.8 and II.2.2.2. Therefore $K\oplus\mathbb{Z}$-factorial rings, and inductively $K\oplus\mathbb{Z}^m$-factorial ones, are K-factorial. By Remark II.2.5.12 (Laurent) polynomial rings in m variables over a K-simple ring S are $K \oplus \mathbb{Z}^m$-factorial and hence K-factorial by the above. This shows the claim for the case that F is finitely generated. For an arbitrary Laurent polynomial ring $S' := S[T_i^{\pm 1} | i \in I]$ note that an element h is K-prime if and only if it is K-prime in one (and hence each) Laurent polynomial ring in finitely many variables including those occuring in h. Thus, each K-homogeneous non-zero non-unit of S' is a product of K-primes. This proves the claim. □

We have now generalized the factoriality criterion from [**2**] the following.

Corollary II.2.5.16. *A $K \oplus F$-graded ring, where F is free is $K \oplus F$-factorial if and only if it is K-factorial.*

In the above proof we also showed the following two statements:

Corollary II.2.5.17. *Graded polynomial rings over factorially graded rings are factorially graded.*

Corollary II.2.5.18. *$K \oplus F$-simple rings, where F is free, are K-factorial.*

Remark II.2.5.19. Let R be factorially graded, let $G \leq gr(R)$ be a subgroup with $R = R_G[f_i | i \in I]$ for certain primes f_i of R^{hom}. Then by Remark II.1.8.5 the localizations $R_{\mathfrak{p}}$ at all $\mathfrak{p} \in \mathfrak{P}(R)$ satisfy $\deg((R_{\mathfrak{p}}^{\mathrm{hom}})^*) + G = \langle \deg(R^{\mathrm{hom}} \setminus 0)\rangle + G$ if and only if for all $i \in I$ we have $\langle \deg(R^{\mathrm{hom}} \setminus 0)\rangle + G = \langle \deg(f_j) | j \in I \setminus \{i\}\rangle + G$.

II.2.6. natural divisorial algebras over $\mathbb{F}_1$-algebras and graded rings. In this section, A denotes a graded $\mathbb{F}_1$-algebra or ring. The graded A-algebras of Krull type discussed here are modeled after divisorial $\mathcal{O}_X$-algebras and have similar properties.

Definition II.2.6.1. Let A be an integrally graded $\mathbb{F}_1$-algebra resp. ring. A Veronesean algebra $A \subseteq R$ (in the sense of Definition II.1.2.4) is *natural* if R is integrally graded and the assignment $(Q_{\mathrm{gr}}(R)^{\mathrm{hom}})^* \to \mathrm{Div}^{\mathrm{gr}}(A), f \mapsto Rf \cap Q_{\mathrm{gr}}(A)$ defines a homomorphism.

In the following, we consider algebras $\imath\colon A \subseteq R$ over a $\mathbb{F}_1$-algebra/graded ring A with R being integrally graded. We then denote $Q_{\mathrm{gr}}(A)$ by A' and $Q_{\mathrm{gr}}(R)$ by R'. Lastly, we denote $gr(A)$ by K.

Remark II.2.6.2. For a natural Veronesean algebra $A \subseteq R$ the following hold:

(i) $\mathrm{degsupp}(R) + K$ equals $\langle \mathrm{degsupp}(R) \rangle + K$ because $R_{w+K} \neq 0$ holds for each $w \in \langle \mathrm{degsupp}(R) \rangle$.
(ii) R' coincides with the localization of R by the non-zero (homogeneous) elements of A because for each (homogeneous) $f \in R'$ there exists $g \in R$ such that $gf^{-1} \in A' \setminus 0$. Consequently, we have $R'_K = A'$, and $Rf \cap A' = R_{-w+K}f$ holds for each $f \in R'_w$.
(iii) The special ideal over $\{0\}$ is $\mathfrak{m}(\{0\}) = \{0\}$, because $Rf \cap A$ is non-zero for each (homogeneous) $f \in R$.
(iv) In the case of rings we have $R_{-w+K}(f+g) \geq \inf\{R_{-w+K}f, R_{-w+K}g\}$ for $f, g \in R'_w$ with $f + g \neq 0$ because $R_{-w+K}(f+g) \subseteq R_{-w+K}f + R_{-w+K}g$ holds.

Example II.2.6.3. Let A be an integrally graded $\mathbb{F}_1$-algebra/ring, let M be a simple graded monoid and let ϕ be a monoid homomorphism from M to $\mathrm{Div}(A)$ resp. $\mathrm{Div}^{\mathrm{gr}}(A)$. Let R be the graded A-subalgebra of the $gr(A) \oplus gr(M)$-graded A'-algebra $A'[M]$ with $R_{v+w} := \phi(-w)_v \chi^w$ for $v \in gr(A), w \in G$. Then $A \subseteq R$ is natural and is called the *divisorial A-algebra* associated to ϕ. Indeed, the map $a\chi^w \mapsto Ra\chi^w \cap Q_{\mathrm{gr}}(A) = \phi(w) + \mathrm{div}(a)$ constitutes the required a group homomorphism.

Proposition II.2.6.4. *An A-algebra $A \to R$ is Veronesean and natural if and only if there exists a divisorial A-algebra S and a CBE $\pi\colon S \to R$ of A-algebras.*

Proof. Let $\phi\colon L \to gr(R)$ be a map from a free abelian group L such that the composition with the canonical map $gr(R) \to gr(R)/K$ is surjective. Then the map $\psi\colon K \oplus L \to gr(R), w + v \mapsto w + \phi(v)$ is also surjective. Let $L' := \phi^{-1}(\mathrm{degsupp}(R'))$ and let S' be the Veronese subalgebra $A'[L]_{K \oplus L'}$ equipped with the $K \oplus L$-grading.

Let $\{e_i\}_{i \in I}$ be a basis of L' and choose a non-zero $f_i \in R'_{\phi(e_i)}$ for each $i \in I$. For $v = \sum_i \lambda_i e_i$ set $f^v = \prod_i f_i^{\lambda_i}$. Sending χ^v to f^v then defines a CBE $\pi\colon S' \to R'$ of A'-algebras with accompanying map ψ. Each $\pi_v\colon S'_{v+K} \to R'_{\phi(v)+K}$ restricts to an isomorphism $S_{v+K} := (Rf^{-v} \cap A')\chi^v \to R_{\phi(v)+K}$ because $Rf^{-v} \cap A' = R_{\phi(v)+K} f^{-v}$. The A-subalgebra S generated by all S_{v+K} is then divisorial and π restricts to a CBE $\pi\colon S \to R$. $\square$

Proposition II.2.6.5. *Let $\imath\colon A \subseteq R$ be a Veronesean algebra with A of Krull type. Then $\imath$ is natural if and only if R is of Krull type and the canonical map $\beta_\imath$ of divisor monoids is an isomorphism of partially ordered groups. Moreover, in these cases with p denoting the composition of* deg *with the quotient map $gr(R) \to gr(R)/K$ the following hold:*

(i) *For each $\mathfrak{p} \in \mathfrak{P}(R_K)$ the corresponding $gr(R)$-prime ideal $\mathfrak{m}(\mathfrak{p})$ from Proposition II.1.8.14 equals $\beta(\mathfrak{p})$. In particular, R_K is K-local if and only if R_K is $gr(R)$-local.*
(ii) *For $\mathfrak{q} \in \mathfrak{P}(R)$ each homogeneous uniformizer of $A_{\mathfrak{q} \cap A}$ is a homogeneous uniformizer of $R_{\mathfrak{q}}$, i.e.* $\deg(((R_{\mathfrak{q}})^{\mathrm{hom}})^*) + K = \langle \deg(R^{\mathrm{hom}} \setminus 0) \rangle + K$.
(iii) *Denote by $D(-)$ and $C(-)$ the monoids of (graded) divisors resp. their classes. Then in the following diagram of abelian groups the rows, columns*

and the dashed sequence are exact:

$$
\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d, dashed] & \\
1 \arrow[r] & (\bigcup_w A_w)^* \arrow[r] \arrow[d] & (\bigcup_v R_v)^* \arrow[r] \arrow[d] & p((\bigcup_v R_v)^*) \arrow[r] \arrow[d, dashed] & 0 \\
1 \arrow[r] & (\bigcup_w A'_w)^* \arrow[r] \arrow[d] & (\bigcup_v R'_v)^* \arrow[r] \arrow[d] & p((\bigcup_v R'_v)^*) \arrow[r] \arrow[ddll, dashed, "\overline{\delta}"] & 0 \\
 & D(A) \arrow[r, bend left, "\beta"] \arrow[r, phantom, "\cong"] \arrow[d] & D(R) \arrow[l, bend left, "\alpha"] \arrow[d] & & \\
 & C(A) \arrow[r, dashed] \arrow[d] & C(R) \arrow[r, dashed] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
$$

PROOF. If A is of Krull type then for every $\mathfrak{p} \in \mathfrak{P}(A)$ composing the $\mathfrak{p}$-th projection $pr_{\mathfrak{p}}$ with the canonical map $f \mapsto Rf \cap A'$ defines a (graded) valuation $\mu_{\mathfrak{p}}$ on R'. The family $\{\mu_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathfrak{P}(A)}$ then realizes R as an $\mathbb{F}_1$-algebra/graded ring of Krull type in R' and thus Proposition I.2.6.9 implies that $\beta_\imath$ is an isomorphism.

If $\beta_\imath$ is an isomorphism then its inverse α sends $\mathfrak{p} \in \mathfrak{P}(R)$ to $\mathfrak{p} \cap R_K$ and direct calculations show that sending f to $Rf \cap A' = \alpha(Rf)$ defines a homomorphism.

In (i) note that for $\mathfrak{p} \in \mathfrak{P}(A)$ each $\mathfrak{q} \in \mathfrak{P}(R)$ with $\mathfrak{q} \subseteq \mathfrak{m}(\mathfrak{p})$ satisfies $\alpha(\mathfrak{q}) = A \cap \mathfrak{q} \subseteq \mathfrak{p}$ by Proposition I.2.6.9, and hence $\alpha(\mathfrak{q}) = \mathfrak{p}$. Thus, $\mathfrak{m}(\mathfrak{p})$ contains no $\mathfrak{q} \in \mathfrak{P}(R)$ apart from $\beta(\mathfrak{p})$ and hence equals $\beta(\mathfrak{p})$.

Assertion (ii) follows from (i) and the fact that $\beta_\imath$ preserves primality. □

REMARK II.2.6.6. Let $R_K \subseteq R$ be a natural algebra of Krull type and let $S \subseteq R_K^{\mathrm{hom}} \setminus 0$ be a submonoid. Then $S^{-1}R_K \subseteq S^{-1}R$ is natural.

II.2.7. graded normality. Throughout this section, R is a K-integral ring. We introduce graded normality and show that in presence of graded noetherianity it is equivalent to the graded Krull property.

DEFINITION II.2.7.1. Let $R \subseteq S$ be a degree-preserving inclusion of integrally graded rings. The *(complete) integral graded closure of* R *in* S is the graded ring $\mathrm{Int}_{\mathrm{gr}}(R,S)$ (resp. $\mathrm{CInt}(R,S)$) generated by all homogeneous elements of S which are integral over R (resp. almost integral over R in the sense of Definition I.2.4.1). For $S = Q_{\mathrm{gr}}(R)$ we use the notation $\mathrm{Int}_{\mathrm{gr}}(R)$. To clarify the grading in question we will denote the grading group K as an index when neccessary. If R equals $\mathrm{Int}_{\mathrm{gr}}(R)$ (resp. $\mathrm{CInt}_{\mathrm{gr}}(R)$) then R is *normally graded* or *K-normal* (resp. K-completely integrally closed, or K-CIC).

REMARK II.2.7.2. The classical theory on integral closures already provides the fact that sums and products of integral elements are integral. Moreover, we always have $\mathrm{Int}(R^{\mathrm{hom}}, S^{\mathrm{hom}}) \subseteq \mathrm{Int}_{\mathrm{gr}}(R,S)^{\mathrm{hom}}$ and $\mathrm{CInt}(R^{\mathrm{hom}}, S^{\mathrm{hom}}) = \mathrm{CInt}_{\mathrm{gr}}(R,S)^{\mathrm{hom}}$.

PROPOSITION II.2.7.3. *We have* $\mathrm{Int}_{\mathrm{gr}}(R, Q_{\mathrm{gr}}(R)) \subseteq \mathrm{CInt}_{\mathrm{gr}}(R, Q_{\mathrm{gr}}(R))$. *If the grading is noetherian, then equality holds.*

PROOF. If $f^n = \sum_{i=0}^{n-1} a_i f^i$ holds with $f = g/h \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}}, a_i \in R^{\mathrm{hom}}$ and $n \in \mathbb{N}$ then $h^{n-1}f^m \in R$ for $m = 1, \ldots, n-1$ and hence $h^{n-1}f^n \in R$. For $m > n$ and $b = h^{n-1}$ we have $bf^m = \sum_{i=0}^{n-1} a_i b f^{m-n+i}$ and obtain $bf^m \in R$ by induction over m.

For the second part let $f \in Q_{\mathrm{gr}}(R)^{\mathrm{hom}} \setminus 0$ such that there exists $g \in R^{\mathrm{hom}} \setminus 0$ with $gf^n \in R$ for all $n \in \mathbb{Z}_{>0}$. Then the chain of ideals defined by $\mathfrak{a}_n := \langle gf^k \mid 1 \le k \le n \rangle$ is stationary, so there exists an n with $gf^{n+1} = \sum_{i=1}^{n} r_i g f^i$ for certain $r_i \in R$. Dividing by gf gives an integrality equation for f. □

Corollary II.2.7.4. *A K-noetherian ring is a K-Krull ring if and only if it is K-normal.*

Proof. By Proposition II.2.7.3 we know that R is K-cic. By K-noetherianity, maximal elements exist in all sets of K-homogeneous ideals of R, in particular, in sets of non-negative graded divisors. □

Proposition II.2.7.5. *Let $\phi\colon R \to S$ be a graded homomorphism between integrally graded rings with $R^{\mathrm{hom}} \cap \ker(\phi) = \{0\}$. Denote the induced map by $\phi'\colon Q_{\mathrm{gr}}(R) \to Q_{\mathrm{gr}}(S)$. Then the following hold:*

(i) *We have $\phi'(\mathrm{Int}_{\mathrm{gr}}(R)) \subseteq \mathrm{Int}_{\mathrm{gr}}(S)$. If $\phi'^{-1}(S)^{\mathrm{gr}} = R$ then graded normality of S implies graded normality of R.*

(ii) *If ϕ is CB then $\phi'^{-1}(\mathrm{Int}_{\mathrm{gr}}(S))^{\mathrm{gr}} = \mathrm{Int}_{\mathrm{gr}}(R)$. If we additionally have $S^{\mathrm{hom}} = (S^{\mathrm{hom}})^* \phi(R^{\mathrm{hom}})$ then $\mathrm{Int}_{\mathrm{gr}}(S)^{\mathrm{hom}} = (S^{\mathrm{hom}})^* \phi'(\mathrm{Int}_{\mathrm{gr}}(R)^{\mathrm{hom}})$, in particular, graded normality of R then implies graded normality of S.*

(iii) *If $\phi = \imath_M$ is a localization map then $\mathrm{Int}_{\mathrm{gr}}(M^{-1}R) = M^{-1}\mathrm{Int}_{\mathrm{gr}}(R)$. In particular, graded normality of R implies graded normality of $M^{-1}R$.*

Proof of Proposition II.2.7.5. In (ii) first note that if the image of a homogeneous fraction a/b satisfies an integrality equation with homogeneous coefficients then for degree reason these may be chosen to lie in $\phi(R^{\mathrm{hom}})$. For the second statement let $f/g \in Q_{\mathrm{gr}}(S)^{\mathrm{hom}} \setminus 0$ be integral over S with $f, g \in R^{\mathrm{hom}} \setminus 0$. Then $(f/g)^n = \sum_{i=0}^{n-1} a_i (f/g)^i$ holds with certain $a_i \in S_{(n-i)\deg(f/g)}$. Let $s_f, s_g \in (S^{\mathrm{hom}})^*$ with $fs_f = \phi(b), gs_g = \phi(c)$ where $b, c \in R^{\mathrm{hom}}$. Then $s_f^n s_g^{-n} a_i = \phi(d_i)$ holds with $d_i \in R_{(n-i)\deg(a/b)}$. Multiplying the above equation with $s_f^n s_g^{-n}$ yields $\phi((b/c)^n) = \phi\big(\sum_{i=0}^{n-1} d_i (b/c)^i\big)$ and hence $(b/c)^n = \sum_{i=0}^{n-1} d_i (b/c)^i$.

In (iii) note that since $\imath_M(\mathrm{Int}_{\mathrm{gr}}(R)) \subseteq \mathrm{Int}_{\mathrm{gr}}(M^{-1}R)$ we have $M^{-1}\mathrm{Int}_{\mathrm{gr}}(R) \subseteq \mathrm{Int}_{\mathrm{gr}}(M^{-1}R)$. For the converse let $f/g \in \mathrm{Int}_{\mathrm{gr}}(M^{-1}R)^{\mathrm{hom}} \setminus 0$. Then there are $n \in \mathbb{N}$ and $a_i/m_i \in M^{-1}R^{\mathrm{hom}}$ with $(f/g)^n = \sum_{i=0}^{n-1} (a_i/m_i)(f/g)^i$. Set $m := m_0 \cdots m_{n-1}$. Multiplying the above equation with m^n turns gives $mf/g \in \mathrm{Int}_{\mathrm{gr}}(R)$, so $f/g \in M^{-1}\mathrm{Int}_{\mathrm{gr}}(R)$. □

Remark II.2.7.6. For degree-preserving inclusions $A \subseteq R \subseteq S$ of integrally graded rings we have

$$\mathrm{Int}_{\mathrm{gr}}(A, S) = \mathrm{Int}_{\mathrm{gr}}(A, \mathrm{Int}_{\mathrm{gr}}(R, S)), \quad \mathrm{CInt}_{\mathrm{gr}}(A, S) = \mathrm{CInt}_{\mathrm{gr}}(A, \mathrm{CInt}_{\mathrm{gr}}(R, S)).$$

Proposition II.2.7.7. [**25**] *Let S be integrally $K \oplus F$-graded where F is totally ordered and let $R \subseteq S$ be a graded subring. Then we have*

$$\mathrm{Int}_K(R, S) = \mathrm{Int}_{K\oplus F}(R, S), \quad \mathrm{CInt}_K(R, S) = \mathrm{CInt}_{K\oplus F}(R, S).$$

Proof. For $w \in K$ let $h \in S_w$ satisfy $h^n = \sum_{i=1}^{n-1} a_i h^i$ with $a_i \in R_{(n-i)w}$ and let $h = \sum_{j=1}^{d} h_{u_j}$ be a decomposition into $K \oplus F$-homogeneous parts with $u_1 < \ldots < u_d$. Then $h_{u_1}^n$ is the term of lowest F-degree in h^n and hence equals the nu_1-homogeneous part of the right-hand side of the equation, i.e. h_{u_1} is integral over R. Thus, $h - h_{u_1}$ is integral over R and by induction, each h_{u_j} is.

If $h \in S_w$ satisfies $gh^n \in R$ for some $g \in R_v$ then consider homogeneous decompositions h as before and $g = \sum_{k=1}^{l} g_{m_k}$ into $K \oplus F$-homogeneous parts with ascending F-degrees. Then $g_{m_1} h_{u_1}^n$ is the $m_1 + nu_1$-homogeneous part of gh^n and hence belongs to R. Thus, h_{u_1} and thereby $h - h_{u_1}$ belong to $\mathrm{CInt}_{\mathrm{gr}}(R, S)$ and inductively, every h_{u_i} does. □

Corollary II.2.7.8. *A $K \oplus F$-graded ring R, where F is free, is $K \oplus F$-normal/-CIC if and only if it is K-normal/-CIC.*

Proof. Set $S := Q_{K\oplus F}(R)$. K-factoriality of S, which is due to Corollary II.2.5.18, gives $\mathrm{Int}_K(S, Q_K(S)) = S$ and $\mathrm{CInt}_K(S, Q_K(S)) = S$. Consequently, we have $\mathrm{Int}_K(R, Q_K(S)) = \mathrm{Int}_{K\oplus F}(R, S)$ and $\mathrm{CInt}_K(R, Q_K(S)) = \mathrm{CInt}_{K\oplus F}(R, S)$ by Remark II.2.7.6 and Proposition II.2.7.7. □

Proposition II.2.7.9. *Let M be an integral $\mathbb{F}_1$-algebra. With respect to the canonical $Q(M)^*$-grading we then have $\mathrm{Int}_{\mathrm{gr}}(\mathbb{K}[M]) = \mathbb{K}[\mathrm{Int}(M)]$. In particular, M is normal if and only if $\mathbb{K}[M]$ is normally graded.*

Proof. For $f = a\chi^w \in \mathrm{Int}_{\mathrm{gr}}(\mathbb{K}[M])_w$ there exist $n \geq 0$ and $g_i \in \mathbb{K}[M]_{(n-i)w}$ with $f^n = \sum_{i=0}^{n-1} g_i f^i$. Since $f^n \neq 0$ there exists an i such that $g_i \neq 0_{\mathbb{K}}$. But then $(n-i)w \in M$ and we conclude $w \in \mathrm{Int}(M)$.

Conversely, for $w \in \mathrm{Int}(M)$ there exits $n \geq 1$ with $nw = u \in M$ and hence $(\chi^w)^n = chi^u \in \mathbb{K}[M]$ which means $\chi^w \in \mathrm{Int}_{\mathrm{gr}}(\mathbb{K}[M])$. □

Theorem II.2.7.10. *The following are equivalent:*

(i) *R is normal,*
(ii) *every graded localization $R_{\mathfrak{p}}$ at a K-prime ideal $\mathfrak{p}$ is K-normal,*
(iii) *every graded localization $R_{\mathfrak{m}}$ at a K-maximal ideal $\mathfrak{m}$ is K-normal.*

Proof. If (iii) holds then for every K-maximal $\mathfrak{m} \trianglelefteq R$ we have

$$(\mathrm{Int}_{\mathrm{gr}}(R)/R)_{\mathfrak{m}} \cong \mathrm{Int}_{\mathrm{gr}}(R)_{\mathfrak{m}}/R_{\mathfrak{m}} \cong \mathrm{Int}_{\mathrm{gr}}(R_{\mathfrak{m}})/R_{\mathfrak{m}} = 0$$

using in turn the fact that localization commutes with factor modules and integral closures, and the assumption. Lemma II.1.8.7 now gives $\mathrm{Int}_{\mathrm{gr}}(R)/R = 0$. □

CHAPTER III

Sheaves of Krull type

This chapter assembles sheaf-theoretic preparation for later chapters. Section III.1 features criteria for the sheaf property which will be applied in the context of structure sheaves of graded schemes as well as a criterion which captures in which sense the sheaf property is a continuity property, see Proposition III.1.0.7. Structure sheaves of graded schemes fall into one of several classes of sheaves of graded objects which will be juxtaposed in Section III.2 where we also treat the various sheafifications. Section III.3 discusses spaces with structure sheaves as well as modules over them. In Section III.4 we introduce valuation sheaves which assign $\mathbb{Z}$ or 0, valuations on sheaves, leading to the concept of sheaves of Krull types, which is the sheaf-theoretic analogon of graded monoids or rings of Krull type. The existence of a canonical Krull structure will later be a key property of Cox sheaves and structure sheaves in the setting of graded schemes (over $\mathbb{Z}$ or $\mathbb{F}_1$) of Krull type where we have a suitable notion of Weil and principal divsiors, see Chapter V. In order to study the relation between Cox sheaves and divisorial $\mathcal{O}_X$-algebras we treat componentwise bijective epimorphisms of graded presheaves in Section III.5 in general and the behaviour of the Krull property under them in particular. Parts of the last two sections were published by the author in [**6**]. Throughout, we work with a topological space (X, Ω_X) but usually write only X with the topology Ω_X understood.

III.1. $\mathfrak{C}$-sheaves on bases and continuity properties of $\mathfrak{C}$-sheaves

Recall that a $\mathfrak{C}$-presheaf $\mathcal{F}$ on a topological space (X, Ω_X), i.e. a contravariant functor $\Omega_X \to \mathfrak{C}$, is a $\mathfrak{C}$-*sheaf* if for every open $U \subseteq X$ and every cover $U = \bigcup_{i \in I} U_i$ the diagram given by all the morphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$ has limit $\mathcal{F}(U)$ in $\mathfrak{C}$. For later application we list two criteria for a $\mathfrak{C}$-presheaf $\mathcal{F}$ to be a $\mathfrak{C}$-sheaf. The first clarifies in what sense the sheaf property is a continuity-property, see Remark III.1.0.9. The second is a well-known criterion in terms of a basis of Ω_X which is stable under finite intersections. We start by recalling the concepts of basis of a topology and stalk of a presheaf at an irreducible closed subset.

Construction III.1.0.1. Let (X, Ω_X) be a topological space and let $\mathcal{B}$ be a basis for Ω_X, e.g. $\mathcal{B} = \Omega_X$. Then defining $\mathcal{B}(U)$ as the subset of those $V \in \mathcal{B}$ with $V \subseteq U$ constitutes a $\mathbf{Set}^{\mathrm{op}}$-presheaf which is also denoted $\mathcal{B}$.

Remark III.1.0.2. A non-empty topological space X with basis $\mathcal{B}$ is irreducible if and only if for all non-empty $U, V \in \mathcal{B}$ there exists a non-empty element of $\mathcal{B}(U) \cap \mathcal{B}(V)$.

Definition III.1.0.3. Let $\mathfrak{C}$ be a category with all directed (i.e. upward-directed) colimits. The *stalk* $\mathcal{R}_Y$ of a $\mathfrak{C}$-presheaf $\mathcal{R}$ on X at a closed irreducible subset $Y \subseteq X$ is defined as the colimit of the diagram given by all $\mathcal{R}(U)$, where runs through the inclusion ordered, downward-directed set of all $U \in \Omega_X$ with $U \cap Y \neq \emptyset$. For a point $x \in X$ the *stalk at* x is $\mathcal{R}_x := \mathcal{R}_{\overline{\{x\}}}$.

Example III.1.0.4. For a topological space (X, Ω_X) with a basis $\mathcal{B}$ and a closed irreducible subset $Y \subseteq X$ the stalk $\mathcal{B}_Y$ of $\mathcal{B}$, considered as a presheaf in the sense of Construction III.1.0.1, is (canonically isomorphic to) the set of all $U \in \mathcal{B}$ with $U \cap Y \neq \emptyset$.

REMARK III.1.0.5. If $\mathcal{B}$ is a basis of the topology Ω_X on X and the category $\mathfrak{C}$ has directed colimits then for a $\mathfrak{C}$-presheaf $\mathcal{F}$ on X and an irreducible closed $Y \subseteq X$, the canonical morphism $C := \mathrm{colim}_{U \in \mathcal{B}_Y} \mathcal{F}(U) \to \mathcal{F}_Y$ is an isomorphism.

For the inverse choose $W_U \in \mathcal{B}_Y \cap \mathcal{B}(U)$ for each $U \in \Omega_Y$ and consider the morphism $\mathcal{F}(U) \to \mathcal{F}(W_U) \to C$. These morphisms are compatible with restricition morphisms because $\mathcal{B}$ is directed and hence induce a morphism $\mathcal{F}_Y \to C$ which is the required inverse.

DEFINITION III.1.0.6. Let $\mathcal{F}$ be a $\mathfrak{C}$-presheaf on X. Let $\mathcal{B}$ be a $\cap$-stable basis of Ω_X. Then $\mathcal{F}$ is *a $\mathfrak{C}$-sheaf with respect to* $\mathcal{B}$ if whenever $U \in \mathcal{B}$ has a cover by $\{U_i\}_{i \in I} \subseteq \mathcal{B}$ then $\mathcal{F}(U)$ is the limit in $\mathfrak{C}$ of the diagram given by all the morphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$.

In the following the equivalence of (i) and (ii) appears to be a new result, while the equivalence of (i) and (iii) is well-known.

PROPOSITION III.1.0.7. *Let $\mathcal{F}$ be a $\mathfrak{C}$-presheaf on X. Let $\mathcal{B}$ be a $\cap$-stable basis of Ω_X. Then the following are equivalent:*

(i) *$\mathcal{F}$ is a $\mathfrak{C}$-sheaf with respect to $\mathcal{B}$,*
(ii) *$\mathcal{F}$ sends small $\mathcal{B}$-colimits which are also* **Set**-*colimits to $\mathfrak{C}$-limits,*
(iii) *$\mathcal{F}$ sends small codirected (i.e. downward-directed) $\mathcal{B}$-colimits to $\mathfrak{C}$-limits.*

COROLLARY III.1.0.8. *A $\mathfrak{C}$-presheaf on X is a sheaf if and only if it takes Ω_X-colimits which are also* **Set**-*colimits to $\mathfrak{C}$-limits.*

REMARK III.1.0.9. For a $\mathfrak{C}$-presheaf $\mathcal{F}$ on X consider the associated covariant functor $\mathcal{F}' \colon \Omega_X^{op} \to \mathfrak{C}$. Then Proposition III.1.0.7 relates the sheaf property of $\mathcal{F}$ with respect to a basis $\mathcal{B}$ to preservation of certain limits under $\mathcal{F}'_{|\mathcal{B}^{op}}$. In that sense, the sheaf property may be viewed as a continuity property.

PROOF OF PROPOSITION III.1.0.7. Suppose that $\mathcal{F}$ is a sheaf with respect to $\mathcal{B}$. Let $\mathcal{D} \colon I \to \mathcal{B}, i \mapsto U_i$ be a diagram where I is a small category and the morphisms of $\mathcal{B}$ are the inclusions of open sets. Suppose that $U \in \mathcal{B}$ is a colimit of the diagram obtained by composing $\mathcal{D}$ with the inclusion functor from $\mathcal{B}$ to **Set**. Then for every $x \in U$ there exists $i \in I$ with $x \in U_i$. Moreover, for every $i, j \in I$ with $x \in U_i \cap U_j$ there exist $i = i_1^x, \ldots, i_{n_x}^x = j \in I$ such that for each l we have $x \in U_{i_l}$ and for $l = 1, \ldots, n-1$ there exists a morphism $i_l \to i_{l+1}$ or $i_{l+1} \to i_l$. We consider the choice of i_l^x to be fixed for the remainder of this proof and set $U_{i,j}^x = U_{i_1^n} \cap \ldots \cap U_{i_{n_x}^x} \in \mathcal{B}$.

Let K be the category defined as follows. The objects of K are the objects of I plus one object $k_{i,j}$ for each unordered pair $i, j \in I$, i.e. for each subset of $ob(I)$ of cardinality 2. Besides the morphisms of I and the identity morphisms K has the morphisms $k_{i,j} \to i$ and $k_{i,j} \to j$ for $i \neq j \in I$. Then $\mathcal{D}$ induces a diagram $\mathcal{D}' \colon K \to \mathcal{B}$ and again, U is the colimit of the diagram obtained by composing $\mathcal{D}$ with the inclusion functor $\mathcal{B} \to$ **Set**. For $k \in K$ we use the notation $U_k := \mathcal{D}'(k)$.

Let A be an object of $\mathfrak{C}$ together with morphisms $\phi_i \colon A \to \mathcal{F}(U_i)$ such that $\rho_{U_j}^{U_i} \circ \phi_i = \phi_j$ holds whenever there exists an I-morphism $i \to j$. We show that this defines morphisms $\phi_k \colon A \to \mathcal{F}(U_k)$ such that $\rho_{U_l}^{U_k} \circ \phi_k = \phi_l$ whenever there exists a K-morphism $k \to l$. For $i, j \in I$ consider $k = k_{i,j}$ and $x \in U_k$. Then for every open $V \subseteq U_{i,j}^x$ we have

$$\rho_V^{U_{i_1^x}} \circ \phi_{i_1^x} = \ldots = \rho_V^{U_{i_{n_x}^x}} \circ \phi_{i_{n_x}}.$$

Indeed, assuming that $U_{i_l^x} \subseteq U_{i_{l+1}^x}$ we have

$$\rho_V^{U_{i_{l+1}^x}} \circ \phi_{i_{l+1}^x} = \rho_V^{U_{i_l^x}} \circ \rho_{U_{i_l^x}}^{U_{i_{l+1}^x}} \circ \phi_{i_{l+1}^x} = \rho_V^{U_{i_l^x}} \circ \phi_{i_l^x}.$$

Now, for $x \in U_k$ we set $\phi_x := \rho^{U_i}_{U^x_{i,j}} \circ \phi_i = \rho^{U_j}_{U^x_{i,j}} \circ \phi_j$. For $x, y \in U_k$ restricting to $U^x_{i,j} \cap U^y_{i,j}$ then gives an equality $\rho^{U^x_{i,j}}_{U^x_{i,j} \cap U^y_{i,j}} \circ \phi_x = \rho^{U^y_{i,j}}_{U^x_{i,j} \cap U^y_{i,j}} \circ \phi_y$ and by the sheaf property on $\mathcal{B}$ there exists a unique morphism $\phi_k \colon A \to \mathcal{F}(U_k)$ such that $\phi_x = \rho^{U_k}_{U^x_{i,j}} \circ \phi_k$ for each x. Now, uniqueness implies $\rho^{U_i}_{U_k} \circ \phi_i = \phi_k = \rho^{U_j}_{U_k} \circ \phi_j$.

Again by the sheaf property, there exists a unique morphism $\phi \colon A \to \mathcal{F}(U)$ such that $\phi_k = \rho^U_{U_k} j \circ \phi$ for all $k \in K$. For $\phi' \colon A \to \mathcal{F}(U)$ with $\phi_i = \rho^U_{U_i} \circ \phi'$ for all $i \in I$ we then also have

$$\rho^U_{U_{k_{i,j}}} \circ \phi = \rho^{U_i}_{U_{k_{i,j}}} \circ \phi_i = \phi_{k_{i,j}} = \rho^{U_i}_{U_{k_{i,j}}} \circ \phi_i = \rho^U_{U_{k_{i,j}}} \circ \phi'$$

for all $i, j \in I$ and by uniqueness of ϕ we conclude $\phi' = \phi$, which establishes (ii).

Suppose that (iii) holds and let $U = \bigcup_{i \in I} U_i \in \mathcal{B}$ with $U_i \in \mathcal{B}$. Let J be the opposite of the (upward-)directed category of finite subsets of I with inclusions as morphisms. Then U is also the **Set**-colimit of the diagram $\mathcal{E}$ sending $j \in J$ to $U_j := \bigcap_{i \in j} U_i$. By assumption, $\mathcal{F}(U)$ is the limit of $\mathcal{F} \circ \mathcal{E}$. Let $\phi_i \colon A \to \mathcal{F}(U_i)$ be morphisms for $i \in I$ such that we always have $\rho^{U_{i'}}_{U_i \cap U_{i'}} \circ \phi_{i'} = \rho^{U_i}_{U_i \cap U_{i'}} \circ \phi_i$. Then composing with the appropriate restricition maps gives a system of morphisms $\phi_j \colon A \to \mathcal{F}(U_j)$ satisfying the necessary compatibility conditions. Thus, there exists a unique morphism $\phi \colon A \to \mathcal{F}(U)$ with $\phi_j = \rho^U_{U_j} \circ \phi$. For any morphism $\phi' \colon A \to \mathcal{F}(U)$ with $\phi_i = \rho^U_{U_i} \circ \phi'$ for $i \in I$ we obtain $\phi_j = \rho^U_{U_j} \circ \phi'$ by composing with the appropriate restricition maps. Now, uniqueness of ϕ implies $\phi = \phi'$, showing that $\mathcal{F}(U)$ is the limit over all $\rho^{U_i}_{U_i \cap U_{i'}}$. □

PROPOSITION III.1.0.10. *Let $\mathcal{F}$ be a $\mathfrak{C}$-presheaf on X and let $\mathcal{B}$ be any basis of Ω_X. Then $\mathcal{F}$ is a $\mathfrak{C}$-sheaf if and only if the folllowing conditions are satisfied:*

(i) *$\mathcal{F}$ is a $\mathfrak{C}$-sheaf with respect to $\mathcal{B}$.*
(ii) *for $W \in \Omega_X$ we have $\mathcal{F}(W) = \lim_{U \in \mathcal{B} \cap \mathcal{P}(W)} \mathcal{F}(U)$, i.e. $\mathcal{F}(W)$ is the limit over the diagram defined by all morphisms $\rho^U_V \colon \mathcal{F}(U) \to \mathcal{F}(V)$ where $U, V \in \mathcal{B}$ with $V \subseteq U \subseteq W$.*

PROOF. If $\mathcal{F}$ is a $\mathfrak{C}$-sheaf then Corollary III.1.0.8 yields (i) and (ii). For the converse suppose that conditions (i) and (ii) hold. For an open set $V \in \Omega_X$ denote by $\mathcal{B}_V$ the intersection of $\mathcal{B}$ with the powerset of V. Let $V = \bigcup_{j \in J} V_j$ be an open cover. Let $\phi_j \colon A \to \mathcal{F}(V_j)$ be morphisms with $\rho^{V_j}_{V_j \cap V_k} \circ \phi_j = \rho^{V_k}_{V_j \cap V_k} \circ \phi_k$. Let $U \in \mathcal{B}_V$. For every $W \in \mathcal{B}_{U \cap V_j}$ set $\phi_{W,j} := \rho^{V_j}_W \circ \phi_j$. Then for each two $j, k \in J$ with $W \in V_j \cap V_k$ we have $\phi_{W,j} = \phi_{W,k}$ and therefore set $\phi_W := \phi_{W,j}$. Let $\mathcal{U}$ be the set of those $W \in \mathcal{B}_U$ which are contained in some V_j. Then by (i) there exists a unique morphism $\phi_U \colon A \to \mathcal{F}(U)$ with $\phi_W = \rho^U_W \circ \phi_U$ for every $W \in \mathcal{U}$. Moreover, for $U' \in \mathcal{B}_V$ with $U' \subseteq U$ we have $\phi_{U'} = \rho^U_{U'} \circ \phi_U$. Now, (ii) provides a unique morphism $\phi \colon A \to \mathcal{F}(V)$ with $\phi_U = \rho^V_U \circ \phi$ for every $U \in \mathcal{B}_V$. Since

$$\rho^{V_j}_W \circ \phi_j = \phi_W = \rho^V_W \circ \phi = \rho^{V_j}_W \circ (\rho^V_{V_j} \circ \phi)$$

holds for every $W \in \mathcal{B}_{V_j}$ condition (ii) implies $\phi_j = \rho^V_{V_j} \circ \phi$. For uniqueness, consider $\phi' \colon A \to \mathcal{F}(V)$ with $\phi_j = \rho^V_{V_j} \circ \phi'$. For $U \in \mathcal{B}_V$ and $W \in \mathcal{B}_{U \cap V_j}$ we then have $\phi_W = \rho^V_W \circ \phi' = \rho^U_W \circ \rho^V_U \circ \phi'$. Now (i) implies $\phi_U = \rho^V_U \circ \phi'$ and (ii) in turn gives $\phi = \phi'$. □

III.2. Sheaves of graded algebras and modules

Given any category $\mathfrak{C}$, $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(X)$ and $\mathcal{S}h_{\mathfrak{C}}(X)$ denote the categories of $\mathfrak{C}$-presheaves and -sheaves on X, respectively. For the remainder of this chapter let A be a graded monoid/$\mathbb{F}_1$-algebra/ring and let $\mathfrak{C}$ denote one of the categories $\mathbf{GrAlg}_A$

or $\mathbf{GrMod}_A$. Let γ be the structure map of a $gr(A)$-algebra/-module K. By $\mathfrak{C}^\gamma$ we then denote $\mathbf{GrAlg}_A^\gamma$ resp. $\mathbf{GrMod}_A^\gamma$.

In this section, we define several categories of $\mathfrak{C}$- resp. $\mathfrak{C}^\gamma$-(pre-)sheaves and basic notions derived from such (pre-)sheaves. Afterwards, we treat the sheaf property and sheafification before turning to the construction of adjoining one presheaf to another.

Definition III.2.0.1. A *graded (pre-)sheaf* of modules/algebras over a monoid, $\mathbb{F}_1$-algebra or ring B is a (pre-)sheaf $\mathcal{R}$ of B-modules/-algebras together with a decomposition of presheaves $\mathcal{R} = \coprod_{w\in gr(\mathcal{R})(X)} \mathcal{R}_w$ into subpresheaves of B-modules with $gr(\mathcal{R})(X)$ being a set resp. an abelian group set such that we have $\mathcal{R}_w\mathcal{R}_{w'} \subseteq \mathcal{R}_{w+w'}$ for all $w, w' \in gr(\mathcal{R})$.

Remark III.2.0.2. Graded (pre-)sheaves of B-algebras/-modules canonically form a subcategory of (pre-)sheaves of graded algebras/modules with fixed accompanying object over the 0-graded ring B. In the presheaf case, this is an equality.

The above notion is useful in the description of invariant structure sheaves of quasi-torus actions and their Cox sheaves, as well as structure sheaves and Cox sheaves of graded schemes of Krull type. However, it is too strong for the general case of structure sheaves of graded schemes because due to absence of noetherianity of the respective topological spaces the latter may fail to be **Ring**-sheaves, instead only being $\mathbf{GrRing}^K$-sheaves.

Definition III.2.0.3. For a $\mathfrak{C}$-presheaf $\mathcal{R}$ fix the following notations.

(i) If A is a graded ring and $\mathcal{R}$ is a $\mathfrak{C}$-presheaf then $\mathcal{R}^{\mathrm{hom}}$ denotes the composition of $\mathcal{R}$ with the functor $(-)^{\mathrm{hom}}$ from graded A-algebras resp. -modules to graded A^{hom}-algebras/-modules. If A is a graded monoid or $\mathbb{F}_1$-algebra then we set $\mathcal{R}^{\mathrm{hom}} := \mathcal{R}$ and if A is a graded monoid than we take $\mathcal{R}^{\mathrm{hom}} \setminus 0$ to mean just $\mathcal{R}$.
(ii) $gr(\mathcal{R})$ denotes the composition of $\mathcal{R}$ with the grading object functor gr.
(iii) If A is a graded monoid/$\mathbb{F}_1$-algebra and $\mathfrak{C}= \mathbf{GrAlg}_A$, then $\mathcal{R}^*$ denotes the composition of $\mathcal{R}$ with the units functor $(-)^*$ from $\mathfrak{C}$ to simple graded monoids.

Construction III.2.0.4. Let $\mathcal{R}$ be a $\mathfrak{C}$-presheaf on X. If for open subsets $U \subseteq V$ of X the $gr(\mathcal{R})$-restricition from V to U maps $\mathrm{degsupp}(\mathcal{R}(V))$ into $\mathrm{degsupp}(\mathcal{R}(U))$ then the resulting presheaf $\mathrm{degsupp}(\mathcal{R})$ of sets is the *degree support presheaf* associated to $\mathcal{R}$, and we say that $\mathrm{degsupp}(\mathcal{R})$ *exists*.

A sufficient condition for the existence of $\mathrm{degsupp}(\mathcal{R})$ would be that $\mathcal{R}^{\mathrm{hom}} \setminus 0$ defines a presheaf, i.e. $\rho_U^V(\mathcal{R}(V)^{\mathrm{hom}} \setminus 0) \subseteq \mathcal{R}(U)^{\mathrm{hom}} \setminus 0$ holds for all open subsets $U \subseteq V$ of X. In this case, $\mathrm{degsupp}(\mathcal{R})$ is the image presheaf of the homomorphism $\deg\colon \mathcal{R}^{\mathrm{hom}} \setminus 0 \to gr(\mathcal{R})$ of presheaves of sets.

Remark III.2.0.5. Let $\mathcal{R}$ be a $\mathfrak{C}$-presheaf on X and let $B \subseteq X$ be closed and irreducible.

(i) If A is a graded ring then $(\mathcal{R}_B)^{\mathrm{hom}} = (\mathcal{R}^{\mathrm{hom}})_B$.
(ii) If A is a graded monoid or $\mathbb{F}_1$-algebra and $\mathfrak{C}= \mathbf{GrAlg}_A$ then we have $(\mathcal{R}_B)^* = (\mathcal{R}^*)_B$.
(iii) If $\mathrm{degsupp}(\mathcal{R})$ exists then we have $\mathrm{degsupp}(\mathcal{R}_B) = \mathrm{degsupp}(\mathcal{R})_B$ in $gr(\mathcal{R}_B) = gr(\mathcal{R})_B$.
(iv) If A is an $\mathbb{F}_1$-algebra without zero divisors and $\mathcal{R} \setminus 0$ is a presheaf of graded $(A \setminus 0)$-algebras/-modules then we have $(\mathcal{R} \setminus 0)_B = \mathcal{R}_B \setminus 0$.

Not to be confused with the degree support presheaf defined above is the following:

Definition III.2.0.6. Let $\mathcal{R}$ be a $\mathfrak{C}^\gamma$-presheaf, where γ is a $gr(A)$-algebra/-module structure on K. The *degree support set* of $\mathcal{R}$ is the set $\mathrm{degsupp}^{set}(\mathcal{R})$ of all $w \in K$ for which $\mathcal{R}_w$ is not the zero-sheaf, i.e. the union over all $\mathrm{degsupp}(\mathcal{R}(U))$ for $U \in \Omega_X$.

Remark III.2.0.7. Let X be irreducible and let $\mathcal{R}$ be a $\mathfrak{C}^\gamma$-presheaf on X. Then $\mathrm{degsupp}^{set}(\mathcal{R})$ is equal to the degree support of the stalk $\mathcal{R}_X$ of $\mathcal{R}$ at X.

Definition III.2.0.8. The category of *(pre-)sheaves of graded A-algebras/-modules with fixed accompaniment*, denoted $(\mathcal{P}r)\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$, is the full subcategory of $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(X)$ defined by all $(\mathcal{P}r)\mathcal{S}h_{\mathfrak{C}^\delta}(X)$ where δ runs through the structure maps of all algebras/modules over $gr(A)$. Its objects are called $\mathfrak{C}^{\mathrm{fix}}$-(pre-)sheaves.

Recall that a presheaf $\mathcal{G}$ is constant if the restriction maps between sections of arbitrary open sets are identity maps.

Remark III.2.0.9. $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$ is just the subcategory of those presheaves $\mathcal{R}$ of graded A-algebras/-modules for which the presheaf $gr(\mathcal{R})$ is constant. In particular, if (ϕ, ψ) is a morphism in $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$ then ψ is a morphism of constant presheaves of $gr(A)$-algebras/-modules, i.e. $\psi_X = \psi_U$ holds for all open $U \subseteq X$, and we treat ψ as a homomorphism of $gr(A)$-algebras/-modules. $\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$ is the full subcategory of those $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$-objects $\mathcal{R}$ which are $\mathfrak{C}^{gr(\mathcal{R})(X)}$-sheaves. Its intersection with the category of sheaves of graded A-algebras/-modules is the category of sheaves of 0-graded A-algebras/-modules.

Construction III.2.0.10. Let $\mathcal{R}$ be an object of $(\mathcal{P}re-)\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$. For each $w \in gr(\mathcal{R}(X))$ the (pre-)sheaf $\mathcal{R}_w(U) := \mathcal{R}(U)_w$ of abelian groups is called the *w-th homogeneous component* of $\mathcal{R}$. This defines the structure of a graded presheaf of A_0-algebras resp. -modules depending on whether $\mathfrak{C}$ was the category of graded A-algebras or -modules. Moreover, for a subgroup $G \subseteq gr(\mathcal{R}(X))$ the (pre-)sheaf $\mathcal{R}_G := \coprod_{w \in G} \mathcal{R}_w$ is the corresponding *Veronese subalgebra.*

Definition III.2.0.11. A homomorphism ϕ of $\mathfrak{C}$- or $\mathfrak{C}^\gamma$-presheaves on X is called *Veronesean* if ϕ_U is Veronesean in the sense of Definition II.1.2.4 for each open $U \subseteq X$.

Remark III.2.0.12. Let $\mathcal{R}$ be an object of $(\mathcal{P}re-)\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$ and let $B \subseteq X$ be closed and irreducible. Then for each $w \in gr(\mathcal{R})$ the inclusion $(\mathcal{R}_w)_B \subseteq (\mathcal{R}_B)$ is surjective onto $(\mathcal{R}_B)_w$.

We now turn to statements on the sheaf property and sheafification.

Remark III.2.0.13. If A is a graded ring then a $\mathfrak{C}^{\mathrm{fix}}$-sheaf $\mathcal{F}$ on a noetherian topological space X is automatically a sheaf of non-graded A-algebras/-modules due to Proposition II.1.3.8.

Remark III.2.0.14. If A is a graded ring then a $\mathfrak{C}$- resp. $\mathfrak{C}^\gamma$-presheaf $\mathcal{R}$ is a sheaf if and only if $\mathcal{R}^{\mathrm{hom}}$ is a sheaf of graded A^{hom}-algebras/-modules.

Remark III.2.0.15. Depending on whether A is a monoid, $\mathbb{F}_1$-algebra or ring let $\mathfrak{D}$ denote the category of sets, pointed sets or abelian groups. Let γ be a fixed $gr(A)$-algebra/-module structure on K. Then a $\mathfrak{C}^\gamma$-presheaf $\mathcal{R}$ is a $\mathfrak{C}^\gamma$-sheaf if and only if each $\mathcal{R}_w, w \in K$ is a $\mathfrak{D}$-sheaf. i.e. a **Set**-sheaf.

Remark III.2.0.16. Let $\mathcal{R}$ be an object of $\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$. Then for each family $\{U_i\}_{i \in I}$ of open sets, where I is non-empty, $gr(\mathcal{R}(\bigcup_i U_i))$ is the limit of the diagram defined by all morphisms $gr(\mathcal{R}(U_i)) \to gr(\mathcal{R}(U_i \cap U_j)), i, j \in I$. Due to Proposition II.1.3.8 this means that $\mathcal{R}(\bigcup_i U_i)$ is the $\mathfrak{C}$-limit of the diagram defined by all morphisms $\mathcal{R}(U_i) \to \mathcal{R}(U_i \cap U_j)$. Thus, $\mathcal{R}$ is a $\mathfrak{C}$-sheaf if and only if $gr(\mathcal{R}(\emptyset)) \cong \{0\}$.

CONSTRUCTION III.2.0.17. Let $\mathcal{R}$ be an object of $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\text{fix}}}(X)$ resp. $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(X)$. Let $\mathcal{G}$ denote $gr(\mathcal{R})$ resp. its sheafification. For $U \in \Omega_X$ and $w \in \mathcal{G}(U)$ let $\mathcal{R}^\sharp(U)_w$ be the set/pointed set/group of all $(f_x)_{x\in U} \in \prod_{x\in U} (\mathcal{R}_x)_{w_x}$ such that for every $x \in U$ there exist $V \in \Omega_{U,x}$ and $g \in \mathcal{R}(V)^{\text{hom}}$ with $g_y = f_y$ for every $y \in V$. Then

$$\mathcal{R}^\sharp(U) := \coprod_{w\in gr(\mathcal{R})^\sharp(U)} \mathcal{R}^\sharp(U)_w.$$

defines a $\mathfrak{C}$-sheaf on X. The *sheafification functor* $(-)^\sharp$ thus defined is left adjoint to the inclusion of $\mathfrak{C}$-sheaves (with fixed accompanying $gr(A)$-algebra/-module) on X into $\mathfrak{C}$-presheaves (with fixed accompanying $gr(A)$-algebra/-module) on X. By restricition to subcategories, we obtain a left adjoint to the inclusion of $\mathcal{S}h_{\mathfrak{C}^\gamma}(X)$ into $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^\gamma}(X)$.

REMARK III.2.0.18. Let $\mathcal{R}$ be an object of $\mathcal{S}h_{\mathfrak{C}^{\text{fix}}}(X)$ resp. $\mathcal{S}h_{\mathfrak{C}}(X)$ and let $\mathcal{S} \subseteq \mathcal{R}$ be a subpresheaf. For each $x \in X$ denote by $\imath_x : \{x\} \to X$ the canonical inclusion. Let $\mathcal{S}^{(x)}$ be the preimage of $(\imath_x)_*\imath_x^{-1}\mathcal{S}$ under the canonical homomorphism $\mathcal{R} \to (\imath_x)_*\imath_x^{-1}\mathcal{R}$. Then the sheafification of $\mathcal{S}$ is canonically isomorphic to the intersection over all $\mathcal{S}^{(x)}$ for $x \in X$.

Recall that a sheaf is *constant* if it is isomorphic to the sheafification of a constant presheaf.

EXAMPLE III.2.0.19. Let $\mathcal{S}$ be a constant $\mathfrak{C}^\gamma$-sheaf on an irreducible space X and let $\mathcal{R}$ be a subpresheaf. Then the following hold:

(i) All restricition maps to sections over non-empty sets are monomorphisms, as are canonical maps to and between stalks. In particular, two sections on U and V which define the same stalk at some point agree on $U \cap V$. Moreover, we have $\mathcal{R}_Y = \bigcup_{U\in\Omega_{X,Y}} \mathcal{R}(U)$ in $\mathcal{S}(X)$.

(ii) $\mathcal{R}$ is a $\mathfrak{C}^\gamma$-sheaf if and only if it is a **Set**-sheaf, i.e. if and only if $\mathcal{R}$ sends unions to intersections.

III.3. Algebras and modules over sheaves and spaces with structure sheaves

We discuss operations such as tensor products of $\mathcal{O}$-(pre-)modules, radicals of $\mathcal{O}$-ideals and adjunction of one sheaf to another, as well as gluing of spaces with structure sheaves. $\gamma : gr(A) \to K$ will denote a fixed $gr(A)$-algebra structure map. First, we treat algebras and modules over (pre-)sheaves, in particular, ideals of a (pre-)sheaf.

DEFINITION III.3.0.1. For a $\mathfrak{C}$-/$\mathfrak{C}^\gamma$-/$\mathfrak{C}^{\text{fix}}$-(pre-)sheaf $\mathcal{R}$ the category of objects under $\mathcal{R}$ is denoted $(\mathbf{Pre}-)\mathbf{Alg}_{\mathcal{R}}$. Its objects are called *(pre-)algebras over* $\mathcal{R}$ or $\mathcal{R}$-(pre-)algebras. If $\mathcal{R}$ is a $\mathfrak{C}$- or $\mathfrak{C}^{\text{fix}}$-(pre-)sheaf then for a fixed $gr(\mathcal{R})$-algebra δ we denote by $(\mathbf{Pre}-)\mathbf{Alg}^\delta_{\mathcal{R}}$ the subcategory of $\mathcal{R}$-algebras with accompanyment δ.

DEFINITION III.3.0.2. Let $\mathfrak{C}$ and $\mathfrak{D}$ denote the categories of graded A-algebras and -modules, respectively. Let $\mathcal{O}$ be a $\mathfrak{C}$-/$\mathfrak{C}^\gamma$ or $\mathfrak{C}^{\text{fix}}$-(pre-)sheaf and correspondingly, let $\mathcal{M}$ be a $\mathfrak{D}$-/$\mathfrak{D}^\delta$- or $\mathfrak{D}^{\text{fix}}$-(pre-)sheaf on X, where in the second case, δ denotes the structure map of a module over the $gr(A)$-algebra $\gamma : gr(A) \to K$.

An $\mathcal{O}$*-(pre-)module structure on* $\mathcal{M}$ consists of homomorphisms $\mu : \mathcal{O}\times\mathcal{M} \to \mathcal{M}$ and $\lambda : gr(\mathcal{O}) \times gr(\mathcal{M}) \to gr(\mathcal{M})$ of presheaves of sets such that μ_U and λ_U define a graded $\mathcal{O}(U)$-module structure on $\mathcal{M}(U)$ for every open $U \subseteq X$. We then say that $\mathcal{M}$ is an $\mathcal{O}$*-(pre-)module* with μ understood.

A morphism of $\mathcal{O}$-(pre-)modules is a homomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ of presheaves of A-modules together with a homomorphism $\psi : gr(\mathcal{M}) \to gr(\mathcal{M}')$ of presheaves of $gr(A)$-modules such that each (ϕ_U, ψ_U) is a morphism of $\mathcal{O}(U)$-modules. The category thus defined is denoted $\mathbf{Mod}_{\mathcal{O}}$ resp. $\mathbf{PreMod}_{\mathcal{O}}$.

In case $\mathcal{O}$ is a $\mathfrak{C}$-(pre-)sheaf consider a $gr(\mathcal{O})$-(pre-)module $\mathcal{G}$ with structure homomorphism of presheaves λ. Then $(\mathbf{Pre}-)\mathbf{Mod}_{\mathcal{O}}^{\lambda}$ denotes the subcategory of $(\mathbf{Pre}-)\mathbf{Mod}_{\mathcal{O}}$ whose objects all have λ as their accompanying $gr(\mathcal{O})$-module structure, and whose morphisms are all accompanied by $\mathrm{id}_{\mathcal{G}}$.

In case $\mathcal{O}$ is a $\mathfrak{C}^{\mathrm{fix}}$-(pres-)sheaf consider a $gr(\mathcal{O}(X))$-module G with structure map λ. Then $(\mathbf{Pre}-)\mathbf{Mod}_{\mathcal{O}}^{\lambda}$ denotes the subcategory of $(\mathbf{Pre}-)\mathbf{Mod}_{\mathcal{O}}$ whose objects all have λ as their accompanying $gr(\mathcal{O}(X))$-module structure, and whose morphisms are all accompanied by id_G.

Remark III.3.0.3. Let $\mathcal{O}$ be a $\mathfrak{C}$-/$\mathfrak{C}^{\mathrm{fix}}$-presheaf and let $\mathcal{M}$ be an object of $\mathbf{PreMod}_{\mathcal{O}}$ resp. $\mathbf{PreMod}_{\mathcal{O}}^{\lambda}$. Then applying sheafification yields an object $\mathcal{M}^{\sharp}$ of $\mathbf{Mod}_{\mathcal{O}^{\sharp}}$, $\mathbf{Mod}_{\mathcal{O}^{\sharp}}^{\lambda^{\sharp}}$ or $\mathbf{Mod}_{\mathcal{O}^{\sharp}}^{\lambda}$, the last two cases depending on whether λ defines a $gr(\mathcal{O})$- or $gr(\mathcal{O}(X))$-module structure.

Construction III.3.0.4. Let $\mathcal{O}$ be a $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$- or $\mathfrak{C}^{\mathrm{fix}}$-(pre-)sheaf and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}$-(pre-)modules. Then the *tensor product of* $\mathcal{M}$ *and* $\mathcal{N}$ is the $\mathcal{O}$-premodule $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ which sends U to the tensor product $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$, resp. the $\mathcal{O}$-module obtained via sheafification of the former.

If λ is the structure homomorphism of a $gr(\mathcal{O})$-(pre-)algebra $\mathcal{G}$ or a $gr(\mathcal{O}(X))$-algebra G, depending on whether $\mathcal{O}$ is a $\mathfrak{C}$- or a $\mathfrak{C}^{\mathrm{fix}}$-(pre-)sheaf, then the tensor product of $(\mathbf{Pre}-)\mathbf{Mod}_{\mathcal{O}}^{\lambda}$-objects $\mathcal{M}$ and $\mathcal{N}$ is the $\mathbf{PreMod}_{\mathcal{O}}^{\lambda}$-object $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$ which sends U to the $\mathbf{GrMod}_{\mathcal{O}(U)}^{\lambda_U}$- or $\mathbf{GrMod}_{\mathcal{O}(U)}^{\lambda}$-object $\mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$, resp. the $\mathbf{Mod}_{\mathcal{O}}^{\lambda}$-object obtained via sheafification of the former.

Definition III.3.0.5. Let $\mathcal{R}$ be a $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-(pre-)sheaf of and let $\mathcal{I}$ be an $\mathcal{R}$-sub(pre-)module of $\mathcal{R}$, i.e. an $\mathcal{R}$-(pre-)ideal. The *graded radical* of $\mathcal{I}$ is the $\mathcal{R}$-(pre-)ideal $\sqrt{\mathcal{I}}^{\mathrm{gr}}$ assigning $\sqrt{\mathcal{I}(U)}^{\mathrm{gr}}$ to U. $\mathcal{I}$ is *homogeneously radical* if $\mathcal{I} = \sqrt{\mathcal{I}}^{\mathrm{gr}}$.

Remark III.3.0.6. For a preideal $\mathcal{I}$ of the $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-sheaf $\mathcal{R}$ the following hold:

(i) If $\mathcal{I}$ is homogeneously radical then all its stalks are homogeneously radical. In case $\mathcal{I}$ is an $\mathcal{R}$-ideal the converse also holds.
(ii) If $\mathcal{I}$ is an $\mathcal{R}$-ideal and $\mathcal{B}$ is a basis of X then $(\sqrt{\mathcal{I}}^{\mathrm{gr}})^{\sharp}(U)$ is generated by those (homogeneous) $f \in \mathcal{R}(U)$ which restrict to elements of $\sqrt{\mathcal{I}}^{\mathrm{gr}}(U_i)$ for some family $\{U_i\}_{i \in I} \subseteq \mathcal{B}$ covering U.
(iii) $(\sqrt{\mathcal{I}}^{\mathrm{gr}})^{\sharp}(U) = \sqrt{\mathcal{I}}^{\mathrm{gr}}(U)$ holds for each quasi-compact U, which means that $(\sqrt{\mathcal{I}}^{\mathrm{gr}})^{\sharp} = \sqrt{\mathcal{I}}^{\mathrm{gr}}$ holds if X is noetherian.
(iv) If $\mathcal{I}$ is an $\mathcal{O}_X$-ideal and X has a basis $\mathcal{B}$ of quasi-compact open subsets then $\mathcal{I}$ is homogeneously radical if and only if $\mathcal{I}(U)$ is homogeneously radical for each $U \in \mathcal{B}$.

Next, we treat the concept of adjoining one presheaf to another.

Construction III.3.0.7. Let A be a graded monoid/$\mathbb{F}_1$-algebra and let B be a graded $\mathbb{F}_1$-algebra/ring. Let $\mathfrak{C}$ denote $\mathbf{GrAlg}_A$ or $\mathbf{GrMod}_A$ and correspondingly, let $\mathfrak{D}$ denote $\mathbf{GrAlg}_B$ or $\mathbf{GrMod}_B$. Fix a $gr(A)$-algebra/-module structure map γ and a $gr(B)$-algebra/-module δ structure map. Likewise, let $\mathfrak{E}$ denote $\mathbf{GrAlg}_{A[B]}$ or $\mathbf{GrMod}_{A[B]}$, and denote by $\delta\gamma$ the induced $gr(A)[gr(B)]$-algebra/-module structure.

For a $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-/$\mathfrak{C}^{\mathrm{fix}}$-presheaf $\mathcal{C}$ and a $\mathfrak{D}$-/$\mathfrak{D}^{\delta}$-/$\mathfrak{D}^{\mathrm{fix}}$-presheaf $\mathcal{R}$ on X let $\mathcal{R}[\mathcal{C}]$ be the $\mathfrak{E}$-/$\mathfrak{E}^{\gamma}$-/$\mathfrak{E}^{\mathrm{fix}}$-presheaf assigning $\mathcal{R}(U)[\mathcal{C}(U)]$ to U. This defines a functor from the category of $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-/$\mathfrak{C}^{\mathrm{fix}}$-presheaves to the category of $\mathfrak{E}$-/$\mathfrak{E}^{\delta\gamma}$-/$\mathfrak{E}^{\mathrm{fix}}$-prealgebras resp. premodules over $\mathcal{R}$ which by Lemma A.0.0.2 is left-adjoint to the forgetful functor.

Remark III.3.0.8. For a presheaf $\mathcal{R}[\mathcal{C}]$ of the type constructed above the stalk at a closed irreducible $Y \subseteq X$ is canonically isomorphic to $\mathcal{R}_Y[\mathcal{C}_Y]$ due to Proposition II.1.5.7.

Remark III.3.0.9. Let X be irreducible and let $\mathcal{R}$ be a presheaf of (constantly) graded $\mathbb{F}_1$-algebras/rings and let $\mathcal{C}$ be a presheaf of (constantly) graded algebras or modules over $\{1\}$ resp. $\mathbb{F}_1$. Suppose further that preimages of zero are zero under the restricition maps of $\mathcal{R}$ (and $\mathcal{C}$ if applicable). Then by Proposition II.1.5.7 $\mathcal{R}[\mathcal{C}]$ is a sheaf if and only if $\mathcal{R}[\mathcal{C}](\emptyset)$ is terminal and for each family of open sets $\{U_i\}_{i\in I}$ with $I \neq \emptyset$, $\mathcal{R}(\bigcup_i U_i)$ is the limit of the diagram defined by all the restricitions $\mathcal{R}(U_i) \to \mathcal{R}(U_i \cap U_j)$ and $\mathcal{C}(\bigcup_i U_i)$ is the limit of the diagram defined by all the restricitions $\mathcal{C}(U_i) \to \mathcal{C}(U_i \cap U_j)$.

Example III.3.0.10. For a graded $\mathbb{F}_1$-algebra/ring A denote by $A[\mathcal{C}]$ the presheaf $\mathcal{A}[\mathcal{C}]$ constructed in Construction III.3.0.7 with $\mathcal{A}$ being the constant presheaf assigning A.

Construction III.3.0.11. Let $\mathcal{F}$ be a $A[\mathfrak{C}]$-presheaf on X. Then for each open $U \subseteq X$ there exists a unique $\mathfrak{C}$-object $\mathcal{G}(U)$ with $\mathcal{F}(U) = A[\mathcal{G}(U)]$. This defines a $\mathfrak{C}$-presheaf with the restrictions being obtained from $\mathcal{G}(U) \to \mathcal{F}(U) \to \mathcal{F}(V)$ by taking preimages under $\mathcal{G}(V) \to \mathcal{F}(V)$.

Remark III.3.0.12. The functor $A[-]\colon \mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(X) \to \mathcal{P}r\mathcal{S}h_{A[\mathfrak{C}]}(X)$ is inverse to the functor $\mathfrak{f}$ sending a $A[\mathfrak{C}]$-presheaf to its induced $\mathfrak{C}$-presheaf. If A is a 0-graded field then the latter functor is isomorphic to $(-)/A^*$. If additionally, the objects of $\mathfrak{C}$ are all *canonically graded* in the sense that $\mathbb{F}_1[\operatorname{degsupp}(-)]$ is isomorphic to $\mathrm{id}_{\mathfrak{C}}$, then $\mathfrak{f}$ is also isomorphic to the functor obtained by composing a $\mathcal{P}r\mathcal{S}h_{A[\mathfrak{C}]}(X)$-object with $\mathbb{F}_1[\operatorname{degsupp}(-)]$. All of the above statements repect the sheaf-property.

Lastly, we discuss spaces with structure (pre-)sheaves.

Definition III.3.0.13. The categories of *spaces with $\mathfrak{C}$-/$\mathfrak{C}^\gamma$- resp. $\mathfrak{C}^{\mathrm{fix}}$-structure (pre-)sheaves* have as its objects triples $(X, \Omega_X, \mathcal{O}_X)$ where (X, Ω_X) is a topological space and $\mathcal{O}_X$ is a $\mathfrak{C}$-/$\mathfrak{C}^\gamma$- resp. $\mathfrak{C}^{\mathrm{fix}}$-(pre-)sheaf on Ω_X, the latter being called the *structure (pre-)sheaf* of X.

Morphisms are pairs $\phi\colon X \to Y, \phi^*\colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ of continuous maps and homomorphisms of $\mathfrak{C}$-/$\mathfrak{C}^\gamma$- resp. $\mathfrak{C}^{\mathrm{fix}}$-presheaves on Ω_Y. Usually the morphism (ϕ, ϕ^*) will just be denoted ϕ with ϕ^* understood. The composition of $\phi\colon X \to Y$ and $\psi\colon Y \to Z$ is the pair $(\psi\circ\phi, \psi_*\phi^*\circ\psi^*)$. The categories thus defined are denoted $\mathbf{Sp}^{\mathcal{P}r\mathcal{S}h_{\mathfrak{C}}}$,$\mathbf{Sp}^{\mathcal{S}h_{\mathfrak{C}}}$, $\mathbf{Sp}^{\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^\gamma}}$, $\mathbf{Sp}^{\mathcal{S}h_{\mathfrak{C}^\gamma}}$, $\mathbf{Sp}^{\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}}$ and $\mathbf{Sp}^{\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}}$, respectively.

Definition III.3.0.14. The categories of *spaces with stalkwise homogeneously local $\mathfrak{C}$-/$\mathfrak{C}^\gamma$ resp. $\mathfrak{C}^{\mathrm{fix}}$-structure (pre-)sheaves* are the subcategories of spaces with $\mathfrak{C}$-/$\mathfrak{C}^\gamma$ resp. $\mathfrak{C}^{\mathrm{fix}}$-structure (pre-)sheaves whose objects satisfy that the stalks at all points are homogeneously local.

A morphism $\phi\colon X \to Y, \phi^*\colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ in one of these subcategories must satisfy that for each $x \in X$ the canonical map

$$\phi^*_x\colon \mathcal{O}_{Y,\phi(x)} \xrightarrow{\phi^*_{\phi(x)}} (\phi_*\mathcal{O}_X)_{\phi(x)} = \operatorname{colim} \mathcal{O}_{X|\phi^{-1}(\Omega_{Y,\phi(x)})} \longrightarrow \mathcal{O}_{X,x}$$

is homogeneously local. The subcategories thus defined are denoted with an index *loc*, e.g. $\mathbf{Sp}^{\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}}_{\mathrm{loc}}$.

Construction III.3.0.15. Let $(X, \Omega_X, \mathcal{O}_X)$ be a space with a (stalkwise homogeneously local) $\mathfrak{C}$-/$\mathfrak{C}^\gamma$-/$\mathfrak{C}^{\mathrm{fix}}$-structure (pre-)sheaf. Then each open subset $U \subseteq X$ defines an *open subobject* $(U, \Omega_{X|U}, \mathcal{O}_{X|U})$ of $(X, \Omega_X, \mathcal{O}_X)$ where $\Omega_{X|U}$ is the collection of Ω_X-open subsets of U and $\mathcal{O}_{X|U}$ is the restriction of $\mathcal{O}_X$ to $\Omega_{X|U}$.

Definition III.3.0.16. A morphism of spaces with (stalkwise homogeneously local) $\mathfrak{C}$-/$\mathfrak{C}^\gamma$-/$\mathfrak{C}^{\mathrm{fix}}$-structure (pre-)sheaves is an *open embedding* if it has an open image and defines an isomorphism onto the open subobject given by its image.

Next, we consider colimits of diagrams of open embeddings, i.e. gluing.

Construction III.3.0.17. Let D be a small I-diagram of spaces with (stalkwise homogeneously local) $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-/$\mathfrak{C}^{\mathrm{fix}}$-structure sheaves such that all occuring morphisms are open embeddings. Set $D(i) = (X_i, \Omega_{X_i}, \mathcal{O}_{X_i})$ for $i \in I$. Let (X, Ω_X) be the **Top**-colimit, i.e. the **Set**-colimit X endowed with the final topology of the canonical maps $\phi_i \colon X_i \to X$. Then each ϕ_i defines a homeomorphism onto an Ω_X-open subset of X which is also denoted X_i. For each $U \in \Omega_X$ we define $\mathcal{O}_X(U)$ as the limit of the induced diagram given by all $\mathcal{O}_{X_i}(U \cap X_i)$. The triple $(X, \Omega_X, \mathcal{O}_X)$ is then a colimit of D and the canonical maps ϕ_i are open embeddings.

Remark III.3.0.18. Let $(X, \Omega_X, \mathcal{O}_X)$ be a space with (stalkwise homogeneously local) $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-/$\mathfrak{C}^{\mathrm{fix}}$-structure sheaf and let $\mathcal{U}$ be any cover of X. Then $(X, \Omega_X, \mathcal{O}_X)$ is the colimit of the diagram given by all the canonical inclusions

$$(U \cap V, \Omega_{U\cap V}, \mathcal{O}_{X|U\cap V}) \longrightarrow (U, \Omega_U, \mathcal{O}_{X|U}).$$

Recall that if $\mathfrak{K}$ is a category with all directed colimits then a continuous map $\phi \colon X \to Y$ gives rise to an *inverse image functor* ϕ^{-1} from $\mathcal{P}r\mathcal{S}h_{\mathfrak{K}}(Y)$ to $\mathcal{P}r\mathcal{S}h_{\mathfrak{K}}(X)$ by sending a presheaf $\mathcal{G}$ on Y to the assignment

$$\phi^{-1}\mathcal{G} \colon U \mapsto \operatorname*{colim}_{\substack{V \in \Omega_Y \\ \phi(U) \subseteq V}} \mathcal{G}(V).$$

If $\mathfrak{K}$ has all limits then composing the above with sheafification constitutes the *inverse image functor* ϕ^{-1} from $\mathcal{S}h_{\mathfrak{K}}(Y)$ to $\mathcal{S}h_{\mathfrak{K}}(X)$. The canonical isomorphisms $\mathrm{id} \to \phi_* \circ \phi^{-1}$ and $\phi^{-1} \circ \phi_* \to \mathrm{id}$ induce an adjunction realising (ϕ^{-1}, ϕ_*) as an adjoint pair.

Construction III.3.0.19. Let $\phi \colon X \to Y, \phi^* \colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ be a morphism of spaces with (stalkwise homogeneously local) $\mathfrak{C}$-/$\mathfrak{C}^{\gamma}$-/$\mathfrak{C}^{\mathrm{fix}}$-structure (pre-)sheaves. For an $\mathcal{O}_Y$-(pre-)algebra/-module $\mathcal{F}$ we write $\phi^*\mathcal{F}$ for the presheaf $\mathcal{O}_X \otimes_{\phi^{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{F}$ resp. for its sheafification. For a morphism $\alpha \colon \mathcal{F} \to \mathcal{G}$ we write $\phi^*\alpha$ for $\mathrm{id}_{\mathcal{O}_X} \otimes \phi^{-1}\alpha$ resp. for its image under sheafification.

This defines a functor called the *inverse image* by ϕ from $\mathcal{O}_Y$-(pre-)algebras/-modules to $\mathcal{O}_X$-(pre-)algebras/-modules which by abuse of notation will be denoted ϕ^*. Moreover, the canonical isomorphisms $\mathrm{id} \to \phi_* \circ \phi^*$ and $\phi^* \circ \phi_* \to \mathrm{id}$ induce an adjunction realising (ϕ^*, ϕ_*) as an adjoint pair.

Remark III.3.0.20. Let $\phi \colon X \to Y$ be a morphism of spaces with $\mathfrak{C}$-structure (pre-)sheaves. Then the restriction of the inverse image functor ϕ^* to $\mathcal{O}_Y$-(pre-)submodules is isomorphic to the functor sending $\mathcal{I}$ to the $\mathcal{O}_X$-(pre-)submodule generated by the image of $\mathcal{I}$ under the morphism $\phi^* \colon \mathcal{O}_Y \to \phi_*\mathcal{O}_X$.

III.4. Sheaves of Krull type

Here, we define (pre-)sheaves of Krull type - the sheaf-theoretic analogon of K-Krull rings. This property occurs in the structure sheaves of (graded) schemes of Krull type (in particular in those of normal prevarieties) as well as divisorial algebras and Cox sheaves on such spaces. Throughout, let $\mathfrak{D}$ denote one of the categories **GrMon**, $\mathbf{GrAlg}_{\mathbb{F}_1}$ and **GrRing** and let K be an abelian group.

Definition III.4.0.1. A *discrete value (pre-)sheaf* on X is a (pre-)sheaf $\mathcal{Z}$ of partially ordered abelian groups with values in $\{0, \mathbb{Z}\}$ and identity or zero-maps as restricition maps.

Remark III.4.0.2. The stalk of a discrete value presheaf $\mathcal{Z}$ at a closed irreducible subset $B \subseteq X$ is $\mathbb{Z}$ if $\mathcal{Z}(U) = \mathbb{Z}$ holds for each neighbourhood U of B, and it is 0 otherwise.

Example III.4.0.3. Let B be an irreducible closed subset of the irreducible topological space X. Then the *skyscraper sheaf* $\mathbb{Z}^{(B)}$ with value $\mathbb{Z}$ at B defined by

$$\mathbb{Z}^{(B)}(U) := \begin{cases} \mathbb{Z} & \text{if } B \cap U \neq \emptyset \\ 0 & \text{if } B \cap U = \emptyset \end{cases}$$

is a discrete value sheaf with restriction maps $\varrho_V^U := \mathrm{id}_{\mathbb{Z}}$ for $V \subseteq U$ with $B \cap V \neq \emptyset$ and $\varrho_V^U = 0$ otherwise. The sheaf axioms follow from the fact that $U = \bigcup_{i \in I} U_i$ intersects B if and only if some U_i intersects B. The stalk $\mathbb{Z}_C^{(B)}$ at an irreducible closed subset $C \subseteq X$ is $\mathbb{Z}$ if $C \subseteq B$ and 0 otherwise.

Example III.4.0.4. If $\mathcal{Z}$ and $\mathcal{Z}'$ are discrete value presheaves on X then so is the presheaf $\mathcal{Z} + \mathcal{Z}'$ assigning $\mathcal{Z}(U) + \mathcal{Z}'(U) \subseteq \mathbb{Z}$ to $U \in \Omega_X$.

Remark III.4.0.5. A discrete value (pre-)sheaf $\mathcal{Z}$ defines a (pre-)sheaf $\mathcal{Z}_{\geq 0}$ via $\mathcal{Z}_{\geq 0}(U) := \mathcal{Z}(U)_{\geq 0}$. For a closed irreducible $A \subseteq X$ we have $(\mathcal{Z}_{\geq 0})_A = (\mathcal{Z}_A)_{\geq 0}$.

Definition III.4.0.6. Let $\mathcal{S}$ be a $\mathfrak{D}$- resp. $\mathfrak{D}^K$-presheaf such that each $\mathcal{S}(U)$ is simple/simply graded and denote $\mathcal{S}^*$ resp. $(\mathcal{S}^{\hom})^*$ by $\mathcal{S}'$. A *discrete graded valuation* on $\mathcal{S}$ is a morphism $\nu \colon \mathcal{S}' \to \mathcal{Z}$ to a discrete value presheaf such that each ν_U is surjective and either a discrete graded valuation or zero. The associated *graded valuation presheaf* is the subpresheaf $\mathcal{S}_\nu \subseteq \mathcal{S}$ generated by $\nu^{-1}(\mathcal{Z}_{\geq 0})$.

In case $gr(\mathcal{S})$ is constantly zero we speak of discrete valuations and their discrete valuation (pre-)sheaves.

Remark III.4.0.7. For a discrete graded valuation ν on $\mathcal{S}$ and an irreducible closed $B \subseteq X$ we have canonical isomorphisms $\nu^{-1}(\mathcal{Z}_{\geq 0})_B \cong \nu_B^{-1}(\mathcal{Z}_{\geq 0, B})$ and $(\mathcal{S}_\nu)_B \cong (\mathcal{S}_B)_{\nu_B}$.

Definition III.4.0.8. For a family $\{\nu_j\}_{j \in J}$ of graded valuations on a $\mathfrak{D}$- resp. $\mathfrak{D}^K$-presheaf $\mathcal{S}$ with simple/simply graded sections $\mathcal{R} := \bigcap_{j \in J} \mathcal{S}_{\nu_j}$ is called

(i) *locally of Krull type* if if for each $x \in X$ there exists $U \in \Omega_{X,x}$ such that for each (homogeneous, non-zero) $f \in \mathcal{R}(U)$ only finitely many $\nu_{j,U}(f)$ are non-zero.
(ii) *of Krull type with respect to a basis* $\mathcal{B}$ of Ω_X if for each $U \in \mathcal{B}$ and each (homogeneous, non-zero) $f \in \mathcal{R}(U)$ only finitely many $\nu_{j,U}(f)$ are non-zero. If $\mathcal{B} = \Omega_X$ then $\mathcal{R}$ is *globally of Krull type* or just *of Krull type*.

In all cases, $\{\nu_j\}_{j \in J}$ is said to define $\mathcal{R}$ in $\mathcal{S}$.

Construction III.4.0.9. Let $\mathcal{R}$ be of Krull type in $\mathcal{S}$ with defining family $\{\nu_j\}_{j \in J}$ and denote $\mathcal{S}^*$ resp. $(\mathcal{S}^{\hom})^*$ by $\mathcal{S}'$. Then

$$\mathrm{div}_{\mathcal{R}} := \prod_{j \in J} \nu_j \colon \mathcal{S}' \longrightarrow \prod_{j \in J} \mathcal{Z}_j$$

is a homomorphism of presheaves of abelian groups and $\mathrm{div}_{\mathcal{R}}^{-1}(\prod_{j \in J} \mathcal{Z}_{j, \geq 0})$ equals $\mathcal{R}, \mathcal{R} \setminus 0$ or $\mathcal{R}^{\hom} \setminus 0$ respectively. Moreover, setting

$$\mathcal{J}(U) := \{j \in J | \mathcal{Z}_j(U) = \mathbb{Z}\} = \{j \in J | \nu_j \neq 0\}$$

defines a $\mathbf{Set}^{op}$-presheaf, called the *index presheaf* of the family $\{\nu_j\}_{j \in J}$.

Proposition III.4.0.10. *Let $\mathcal{R}$ be of Krull type in a $\mathfrak{D}$- or $\mathfrak{D}^K$-presheaf $\mathcal{S}$ with defining family $\{\nu_j\}_{j \in J}$. Then the following hold:*

(i) *If all $\mathcal{Z}_j$ are sheaves then so is $\mathcal{J}$, meaning it respects arbitrary unions.*
(ii) *If all $\mathcal{Z}_j$ are sheaves and $\mathcal{S}$ is a $\mathfrak{D}$-, $\mathfrak{D}^K$-sheaf or K-graded and a $\mathbf{Set}$-sheaf then so is $\mathcal{R}$.*

Proof. Set $U = \bigcup_i U_i$. For (i) to see that $\bigcup_i \mathcal{J}(U_i) = \mathcal{J}(U)$ note that $\mathcal{Z}_j(U)$, being a limit of all morphisms $\mathcal{Z}_j(U_i \cap U_h) \to \mathcal{Z}_j(U_i)$, is zero if and only if all $\mathcal{Z}_j(U_i)$ are zero.

In (ii) we consider the cases of monoids, $\mathbb{F}_1$-algebras and rings in this order. Firstly, $\mathrm{div}_{\mathcal{R}}^{-1}(\prod_j \mathcal{Z}_{j,\geq 0}) \subseteq \mathcal{S}'$ is a subsheaf of (K-)graded monoids because $\prod_j \mathcal{Z}_{j,\geq 0} \subseteq \prod_j \mathcal{Z}_j$ is a subsheaf of trivially (K-)graded monoids. Secondly, since all restricition maps of $\mathcal{S}$ map non-zero elements to non-zero ones, $\mathrm{div}_{\mathcal{R}}^{-1}(\prod_j \mathcal{Z}_{j,\geq 0}) \sqcup \{0\}$ is then a (graded) subsheaf of graded $\mathbb{F}_1$-algebras. This implies that in the third case $\mathcal{S}_\nu$ is a (graded) subsheaf of graded rings.

Lastly, suppose that $\mathcal{S}$ is (K-graded and) a **Set**-sheaf and consider elements $f^{(i)} \in \mathcal{S}_{\nu_j}(U_i)$ with $f^{(i)}_{|U_i \cap U_h} = f^{(h)}_{|U_i \cap U_h}$. Then there exists $f \in \mathcal{S}(U)$ with $f_{|U_i} = f^{(i)}$. For each $w \in K$ we then have $\nu_{j,U}(f_w)_{|U_i} = \nu_{j,U_i}(f^{(i)}_w) \geq 0$ for all i and hence $\nu_{j,U}(f_w) \geq 0$ because $\mathcal{Z}_{j,\geq 0}$ is a sheaf, which means $f \in \mathcal{S}_{\nu_j}(U)$. □

Proposition III.4.0.11. *Let $\mathcal{R}$ be of Krull type in $\mathcal{S}$ with defining family $\{\nu_j\}_{j \in J}$. Then the following hold:*

(i) *The stalk $\mathcal{J}_x$ at $x \in X$ is*

$$\mathcal{J}_x = \bigcap_{U \in \Omega_{X,x}} \mathcal{J}(U) = \{j \in J | \mathcal{Z}_{j,x} = \mathbb{Z}\} = \{j \in J | \nu_{j,x} \neq 0\}.$$

(ii) *Then $\mathcal{R}_x$ is of Krull type in $\mathcal{S}_x$ with defining family $\{\nu_{j,x}\}_{j \in J_x}$.*

Proof. In (i) note that $j \in J_x$ if and only if $\mathcal{Z}_j(U) \neq 0$ for all $U \in \Omega_{X,x}$, i.e. $\mathcal{Z}_{j,x} \neq 0$. For (ii) set $\mathcal{H} := \prod_{j \in J} \mathcal{Z}_{j,\geq 0}$. In $\mathcal{S}'_x$ we then have

$$\mathrm{div}_{\mathcal{R}}^{-1}(\mathcal{H})_x = \mathrm{div}_{\mathcal{R},x}^{-1}(\mathcal{H}_x) = \mathrm{div}_{\mathcal{R},x}^{-1}(\prod_{j \in J_x} \mathbb{Z}_{\geq 0})$$

and hence $\mathcal{R}_x = \bigcap_{j \in J_x} (\mathcal{S}_x)_{\nu_{j,x}} \subseteq \mathcal{S}_x$. □

Definition III.4.0.12. Let $\mathcal{B}$ be a basis of Ω_X. A defining family $\{\nu_j\}_{j \in J}$ for a presheaf of Krull type $\mathcal{R}$ in $\mathcal{S}$ is the family of *essential graded valuations with respect to $\mathcal{B}$* if for each $U \in \mathcal{B}$ the localization of $\mathcal{R}(U)$ by all (non-zero/homogeneous non-zero) elements is canonically isomorphic to $\mathcal{S}(U)$ and $\{\nu_{j,U}\}_{j \in \mathcal{J}(U)}$ form the essential graded valuations of $\mathcal{R}(U)$.

Example III.4.0.13. Let $\mathcal{S}$ be a presheaf of simply graded $\mathbb{F}_1$-algebras/rings and let $\mathcal{N}$ be a presheaf of simply graded monoids/$\mathbb{F}_1$-algebras. Let $\{\mu_i\}_{i \in I}$ and $\{\nu_j\}_{j \in J}$ be defining families for subpresheaves $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{M} \subseteq \mathcal{N}$ of Krull type. Then the folllowing hold:

(i) Each μ_i and ν_j extend trivially to graded valuations $\overline{\mu_i}$ and $\overline{\nu_j}$ on $\mathcal{S}[N]$, and $\{\overline{\mu_i}\}_i \cup \{\overline{\nu_j}\}_j$ is a family defining $\mathcal{R}[M]$ as a subpresheaf of Krull type in $\mathcal{S}[N]$. If $\{\mu_i\}_i$ and $\{\nu_j\}_j$ are the essential graded valuations with respect to $\mathcal{B}$ then so are $\{\overline{\mu_i}\}_i \cup \{\overline{\nu_j}\}_j$.

(ii) For each i and j we obtain a graded valuation $\mu_i + \nu_j$ on $\mathcal{S}[\mathcal{N}]$ with range $\mathcal{Z}_i + \mathcal{Z}_j$, where $\mathcal{Z}_i$ and $\mathcal{Z}_j$ are the respective range presheaves of μ_i and ν_j, via $(\mu_i + \nu_j)_U(f\chi^n) := \mu_{i,U}(f) + \nu_{j,U}(n)$ for (homogeneous) units $f \in \mathcal{S}(U), n \in \mathcal{N}(U)$. Moreover, $\{\mu_i + \nu_j\}_{i,j}$ defines a subpresheaf of Krull type in $\mathcal{S}[N]$, and if $I = J$ then so does the subfamily $\{\mu_i + \nu_i\}_i$. In this case, if $\{\mu_i\}_i$ or $\{\nu_i\}_i$ are the essential graded valuations with respect to $\mathcal{B}$ then so are $\{\mu_i + \nu_i\}_i$ because they restrict to the essential graded valuations on $\mathcal{S}$ resp. $\mathcal{N}$.

III.5. Component-wise bijective epimorphisms

In this section we list general properties of CBEs of $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(X)$- or $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$-objects. Of particular use will be the result on the behaviour of the Krull property under CBEs, see Proposition III.5.0.7.

Definition III.5.0.1. A morphism $\pi\colon \mathcal{F} \to \mathcal{G}, \psi\colon gr(\mathcal{F}) \to gr(\mathcal{G})$ of $\mathfrak{C}$-presheaves is a *component-wise bijective (epi-)morphism* (CB(E)) if each (π_U, ψ_U) is a CB(E).

Remark III.5.0.2. Let $\phi\colon \mathcal{F} \to \mathcal{G}, \psi\colon gr(\mathcal{F}) \to gr(\mathcal{G})$ be a CBE of $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$-objects. Then due to Remark III.2.0.15 $\mathcal{F}$ is a $\mathfrak{C}^{\mathrm{fix}}$-sheaf if and only if $\mathcal{G}$ is one, because $\mathcal{F}_w$ is a sheaf if and only if $\mathcal{G}_{\psi(w)}$ is one.

The following is a consequence of Proposition II.1.2.13.

Proposition III.5.0.3. *For a $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$-morphism $\pi\colon \mathcal{F} \to \mathcal{G}$ accompanied by a group homomorphism ψ denote by $\mathcal{N}$ the constant subpresheaf assigning the monoid of (homogeneous) elements in the preimage of $\pi_U^{-1}(1_{\mathcal{G}(U)})$.*

Then (π, ψ) is a CBE if and only if there exists a subpresheaf $\mathcal{N}' \subseteq \mathcal{N}$ of groups with bijective restricition maps such that $\deg\colon \mathcal{N}'(X) \to \ker(\psi)$ *is bijective, $\sim_{\mathcal{N}'(U)}$ equals the kernel relation of π_U and we have* $\mathrm{im}(\pi) = \mathcal{G}_{\mathrm{im}(\psi)}$.

Proposition III.5.0.4. *Suppose that A is a graded rings and let $\pi\colon \mathcal{F} \to \mathcal{G}$ be a CBE of $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$. Then $\mathcal{F}$ is a sheaf of sets if and only if $\mathcal{G}$ and* $\ker(\pi)$ *are.*

Proof. Let $c\colon gr(\mathcal{G}) \to gr(\mathcal{F})$ be a map of sets such that $\psi \circ c = id_{gr(\mathcal{G})}$. For an open cover $U = \bigcup_{i \in I} U_i$ set $U_{i,j} := U_i \cap U_j$ for $i, j \in I$.

First, suppose that $\mathcal{F}$ is a sheaf. If $g = \sum_{w \in gr(\mathcal{G})} g_w \in \mathcal{G}(U)$ restricts to $0_{\mathcal{G}(U_i)}$ for each $i \in I$ then in particular $g_{w|U_i} = (g_{|U_i})_w = 0_{\mathcal{G}(U_i)}$. For $w \in gr(\mathcal{G})$ we then have $(\pi_{|\mathcal{F}_{c(w)}})_U^{-1}(g_w)_{|U_i} = (\pi_{|\mathcal{F}_{c(w)}})_{U_i}^{-1}(g_{w|U_i}) = 0_{\mathcal{F}(U_i)}$ for ever $i \in I$ which gives $g_w = 0_{\mathcal{G}(U)}$ and $g = 0_{\mathcal{G}(U)}$.

If $g^{(i)} = \sum_{w \in gr(\mathcal{G})} g_w^{(i)} \in \mathcal{G}(U_i)$ satisfy ${g^{(i)}}_{|U_{i,j}} = {g^{(j)}}_{|U_{i,j}}$ then

$$\sum_{w \in gr(\mathcal{G})} {g_w^{(i)}}_{|U_{i,j}} = {g^{(i)}}_{|U_{i,j}} = {g^{(j)}}_{|U_{i,j}} = \sum_{w \in gr(\mathcal{G})} {g_w^{(j)}}_{|U_{i,j}}$$

and since $\mathcal{G}(U_{i,j})$ is $gr(\mathcal{G})$-graded we obtain ${g_w^{(i)}}_{|U_{i,j}} = {g_w^{(j)}}_{|U_{i,j}}$ for every $w \in gr(\mathcal{G})$. Now set $f_{c(w)}^{(i)} := (\pi_{U_i})_{|\mathcal{F}(U_i)_{c(w)}}^{-1}(g_w^{(i)}) \in \mathcal{F}(U_i)_{c(w)}$ for $i \in I$ and $w \in gr(\mathcal{G})$. Then

$$f_{c(w)|U_{i,j}}^{(i)} = (\pi_{U_{i,j}})_{|\mathcal{F}(U_{i,j})_{c(w)}}^{-1}({g_w^{(i)}}_{|U_{i,j}}) = (\pi_{U_{i,j}})_{|\mathcal{F}(U_{i,j})_{c(w)}}^{-1}({g_w^{(j)}}_{|U_{i,j}}) = f_{c(w)|U_{i,j}}^{(j)}$$

For $i \in I$ set $f^{(i)} := \sum_{w \in gr(\mathcal{G})} f_{c(w)}^{(i)} \in \mathcal{F}(U_i)$. Note that this is a finite sum because $g^{(i)}$ is a finite sum and it is a decomposition into $gr(\mathcal{F})$-homogeneous parts because c is a section. Then

$$f_{|U_{i,j}}^{(i)} = \sum_{w \in gr(\mathcal{G})} f_{c(w)|U_{i,j}}^{(i)} = \sum_{w \in gr(\mathcal{G})} f_{c(w)|U_{i,j}}^{(j)} = f_{|U_{i,j}}^{(j)}$$

and gluing gives $f \in \mathcal{F}(U)$ with $f_{|U_i} = f^{(i)}$. In particular, $f_{c(w)|U_i} = f_{c(w)}^{(i)}$ for every $w \in gr(\mathcal{G})$ and $i \in I$. Moreover, for every $v \in gr(\mathcal{F}) \setminus \mathrm{im}(c)$ we have $f_{v|U_i} = f_v^{(i)} = 0$ for every i and thus $f_v = 0$. We claim that $g := \pi_U(f)$ restricts to $g^{(i)}$ on each U_i. Indeed, we have $g_w = \pi_U(f_{c(w)})$ and thus

$$g_{|U_i} = \sum_{w \in gr(\mathcal{G})} \pi_U(f_{c(w)})_{|U_i} = \sum_{w \in gr(\mathcal{G})} \pi_{U_i}(f_{c(w)}^{(i)}) = g^{(i)}.$$

Therefore, $\mathcal{G}$ is a sheaf and thus $\ker(\pi)$ is also a sheaf.

For the converse, suppose that $\mathcal{G}$ and $\ker(\pi)$ are sheaves. If $f \in \mathcal{F}(U)$ restricts to $0_{\mathcal{F}(U_i)}$ on every U_i then the same holds for each f_w. Thus, $\phi_U(f_w)_{|U_i} = \phi_{U_i}(f_{w|U_i}) = 0_{\mathcal{G}(U_i)}$ for every i which means that $\phi_U(f_w) = 0_{\mathcal{G}(U)}$. But then $f_w = 0_{\mathcal{F}(U)}$ and $f = 0_{\mathcal{F}(U)}$.

Now, let $f_i \in \mathcal{F}(U_i)$ with $f_{i|U_{i,j}} = f_{j|U_{i,j}}$ for all $i, j \in I$. Set $g_i := \pi_{U_i}(f_i)$ for all $i \in I$. Then $g_{i|U_{i,j}} = g_{j|U_{i,j}}$ and since $\mathcal{G}$ is a sheaf there exists $g \in \mathcal{G}(U)$ with $g_{|U_i} = g_i$. Let $f' \in \mathcal{F}(U)$ with $\pi_U(f) = g$. Then the elements $f_i - f'_{|U_i}$ lie in $\ker(\pi)(U_i)$ and satisfy

$$(f_i - f'_{|U_i})_{|U_{i,j}} = (f_j - f'_{|U_j})_{|U_{i,j}}$$

and since $\ker(\pi)$ is a sheaf there exists $h \in \ker(\pi)(U)$ with $h_{|U_i} = f_i - f'_{|U_i}$. Now, $f := h + f' \in \mathcal{F}(U)$ is the desired element which restricts to f_i on U_i. □

Remark III.5.0.5. If $\pi\colon \mathcal{F} \to \mathcal{G}$ is a CBE between sheaves of graded rings then $\pi(\mathcal{F}) = \mathrm{im}(\mathcal{F}) = \mathcal{G}$ is the image of π considered as a morphism of sheaves as well as a morphism of presheaves.

Proposition III.5.0.6. *A $\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$-morphism $\pi\colon \mathcal{F} \to \mathcal{G}, \psi$ is a CBE if and only if every $\pi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is a CBE of graded A-algebras.*

Proof. First, note that because direct sums commute with colimits and hence direct sums of presheaves commute with stalks we have $(\mathcal{F}_w)_x = (\mathcal{F}_x)_w$ for every $w \in gr(\mathcal{F})$ and $x \in X$. Now, $\pi_{|\mathcal{F}_w}\colon \mathcal{F}_w \to \mathcal{G}_{\psi(w)}$ is an isomorphism of sheaves if and only if $(\pi_{|\mathcal{F}_w})_x\colon (\mathcal{F}_w)_x \to (\mathcal{G}_{\psi(w)})_x$ is an isomorphism of abelian groups, i.e. $\pi_{x|(\mathcal{F}_x)_w}\colon (\mathcal{F}_x)_w \to (\mathcal{G}_x)_{\psi(w)}$ is an isomorphism. □

Lastly, we consider the behaviour of (graded) valuations and graded presheaves of Krull type under CBEs.

Proposition III.5.0.7. *Let $\pi\colon \mathcal{F} \to \mathcal{G}, \psi$ be a CBE of $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}(X)$-objects. Suppose that each $\mathcal{F}(U)$ is $gr(\mathcal{F})$-simple, i.e. each $\mathcal{R}(U)$ is $gr(\mathcal{G})$-simple. Let $\mathcal{R} \subseteq \mathcal{F}$ and $\mathcal{S} \subseteq \mathcal{G}$ be subpresheaves such that $\mathcal{S} = \pi(\mathcal{R})$, in other words, we have $\mathcal{R} = \pi^{-1}(\mathcal{S})^{\mathrm{gr}}$. Then the following hold:*

(i) *Every discrete $gr(\mathcal{F})$-valuation ν on $\mathcal{F}$ induces a discrete $gr(\mathcal{G})$-valuation $\overline{\nu}$ on $\mathcal{G}$ via $\overline{\nu}_U(\pi_U(f)) = \nu_U(f)$ for $f \in (\mathcal{F}(U)^{\mathrm{hom}})^*$. Conversely, every discrete $gr(\mathcal{G})$-valuation $\overline{\nu}$ on $\mathcal{G}$ defines a discrete $gr(\mathcal{F})$-valuation on $\mathcal{F}$ via $\nu := \overline{\nu} \circ \pi_{|(\mathcal{F}^{\mathrm{hom}})^*}$. These assignments are mutually inverse.*

(ii) *If ν and $\overline{\nu}$ are corresponding $gr(\mathcal{F})$- resp. $gr(\mathcal{G})$-valuations then π restricts to a CBE $\pi_{|\mathcal{F}_\nu}\colon \mathcal{F}_\nu \to \mathcal{G}_{\overline{\nu}}$ and we have $\mathcal{F}_\nu = \pi^{-1}(\mathcal{G}_{\overline{\nu}})^{\mathrm{gr}}$.*

(iii) *If $\{\nu_i\}_{i\in I}$ and $\{\overline{\nu}_i\}_{i\in I}$ are corresponding families of $gr(\mathcal{F})$- resp. $gr(\mathcal{G})$-valuations then one defines $\mathcal{R}$ in $\mathcal{F}$ as a subpresheaf of Krull type with respect to $\mathcal{B}$ if and only if the other defines $\mathcal{S}$ in $\mathcal{G}$ as a subpresheaf of Krull type with respect to $\mathcal{B}$.*

Proof. Assertions (i) and (ii) are due to Remark II.2.4.9. Assertion (iii) follows from Proposition II.2.5.11. □

CHAPTER IV

Graded schemes over $\mathbb{Z}$ and $\mathbb{F}_1$

The theory of graded schemes over $\mathbb{Z}$ or $\mathbb{F}_1 = \{0,1\}$ is developed analogously to the theory of schemes over $\mathbb{Z}$ or $\mathbb{F}_1$, the last being due to [**15**]. After establishing the (contravariant) equivalence of graded schemes and graded algebras over A in Section IV.1.1 we provide the necessary background for the construction of relative graded spectra of quasi-coherent $\mathcal{O}_X$-algebras in Section IV.1.3.

In Section IV.2.2 we introduce Veronesean good quotients and give their basic properties for the later study of relative graded spectra of Cox sheaves. With view toward the latter we give basic information on homogeneous integrality and reducedness of graded schemes and construct the constant sheaf of graded fraction rings $\mathcal{K}$, see Section IV.2.1). There we also introduce closed subschemes, separatedness and homogeneous noetherianity.

A distinguishing feature of a graded scheme is that the grading of the structure sheaf defines a canonical action by a graded quasi-torus, see Section IV.2.3. Using the canonical functor from graded schemes to (0-graded) schemes from Section IV.1.5 we will later establish an equivalence which sends a (homogeneously) reduced graded scheme of finite type over an algebraically closed field $\mathbb{K}$ to an action of a quasi-torus on a prevariety over $\mathbb{K}$, see Chapter VI.

In Section IV.3.4 we explore the combinatorial nature of $\mathbb{F}_1$-schemes of finite type. As a preparation, Sections IV.3.1 and IV.3.2 conceptualize graded schemes as cofunctors of graded rings which satisfy certain localization conditions. The resulting category of *schematic cofunctors* of graded A-algebras is shown to be equivalent to graded schemes over A via a canonical extension of the $\mathrm{Spec}_{\mathrm{gr}}$-functor on $\mathbf{GrAlg}_A$, see Proposition IV.3.2.11, the essential inverse sending X to the restriction of $\mathcal{O}_X$ to the set of non-empty affine open subsets of X.

Throughout let A denote a fixed $\mathbb{F}_1$-algebra or ring, e.g. A is $\mathbb{F}_1$, $\mathbb{Z}$ or a field $\mathbb{K}$ equipped with the 0-grading, and let $\mathfrak{C}$ denote $\mathbf{GrAlg}_A$.

IV.1. The category of graded schemes

There are some publications [**23, 27**] in which graded schemes or sets of homogeneously prime ideals haven been studied, but a standard reference for the basic theory appears not to exist. The seemingly only reference which defines structure sheaves for graded spectra of graded rings,[**8**], only treats noetherian ($\mathbb{Z}$-)graded rings, due to the desire to obtain *$\mathbb{Z}$-graded sheaves of rings*. We develop the theory in full generality which means that the structure sheaf $\mathcal{O}_X$ of $X = \mathrm{Spec}_{\mathrm{gr}}(R)$ is still $gr(R)$-graded presheaf of A-algebras (where $A = \mathbb{Z}$ or $A = \mathbb{F}_1$) and also a $\mathbf{GrAlg}_A^{gr(R)}$-sheaf, but in general not a $\mathbf{Set}$-sheaf, see Example IV.1.1.8. Among the material treated in the following sections are various categorical aspects of the theory, including the (contravariant) equivalence between graded A-algebras and graded schemes over A, the equivalence of graded algebras/modules over R and quasi-coherent $\mathcal{O}_{\mathrm{Spec}_{\mathrm{gr}}(R)}$-algebras/-modules, the equivalence of affine graded schemes over X and quasi-coherent $\mathcal{O}_X$-algebras, the relation between closed subsets of X and quasi-coherent $\mathcal{O}_X$-ideals, as well as functors between different subcategories of graded schemes.

IV.1.1. Affine graded schemes. This section introduces affine graded schemes and establishes the (contravariant) equivalence of affine graded schemes over A and graded algebras over A. Note that structure sheaves are $\mathfrak{C}^{\text{fix}}$-sheaves and not $\mathfrak{C}$-sheaves because $gr(\mathcal{O}(\emptyset))$ is usually non-zero.

CONSTRUCTION IV.1.1.1. Let $\imath\colon A \to R, \psi\colon gr(A) \to gr(R)$ be a graded A-algebra. The set $X = \operatorname{Spec}_{\text{gr}}(R)$ of its $gr(R)$-prime ideals is the *graded spectrum* of R. For $f \in R^{\text{hom}}$ the *principal open set* is the set X_f of all $\mathfrak{p} \in X$ with $f \notin \mathfrak{p}$. The set $\mathcal{B}_X^{\text{pr}}$ of all principal open sets is an $\mathbb{F}_1$-algebra with operation $\cap$, unit elements X and zero element $\emptyset$. For each $U \in \mathcal{B}_X^{\text{pr}}$ let $S_U := \bigcap_{\mathfrak{p}\in U} R^{\text{hom}} \setminus \mathfrak{p}$. The composition of the homomorphisms of $\mathbb{F}_1$-algebras $f \mapsto X_f$ and $U \mapsto S_U$ is then the canonical map sending $f \mapsto \operatorname{face}(f)$.

The topology Ω_X generated by $\mathcal{B}_X^{\text{pr}}$ is the *Zariski topology* Ω_X. The *structure sheaf* $\mathcal{O}_X$ on (X, Ω_X) is the $\mathcal{S}h_{\mathfrak{C}^{\text{fix}}}(X)$-object defined via

$$\mathcal{O}_X(U) := \lim_{W\in\mathcal{B}_X^{\text{pr}}(U)} S_W^{-1} R$$

where the limit is taken in $\mathfrak{C}^{\psi}$ and restriction maps are defined via universal properties of limits.

PROOF. As an intersection of faces containing f, S_{X_f} also contains $\operatorname{face}(f)$. If $g \in S_{X_f}$ then for each $\mathfrak{p} \in X_f$ we have $g \notin \mathfrak{p}$ and hence $\sqrt{\langle f\rangle}^{\text{gr}} \subseteq \sqrt{\langle g\rangle}^{\text{gr}}$. Thus, there exist $n \in \mathbb{N}$ and $h \in R^{\text{hom}}$ with $gh = f^n \in \operatorname{face}(f)$ and we conclude $g \in \operatorname{face}(f)$.

The relation defined by $f \sim g$ if and only if $X_f = X_g$ is a congruence, and as an $\mathbb{F}_1$-algebra $\mathcal{B}_X^{\text{pr}}$ is isomorphic to $R^{\text{hom}}/\sim$. Thus, $U \mapsto S_U$ is a homomorphism because it is the map induced by $f \mapsto \operatorname{face}(f)$. $\square$

REMARK IV.1.1.2. The structure (pre)sheaf $\mathcal{O}_X$ of $X = \operatorname{Spec}_{\text{gr}}(R)$ has the following properties: For $f \in R^{\text{hom}}$ there are canonical isomorphisms of graded A-algebras $\mathcal{O}_X(X_f) \to S_{X_f}^{-1}R \to R_f$. Consequently, for a point $\mathfrak{p} \in X$ there are canonical isomorphisms of graded A-algebras

$$\mathcal{O}_{X,\mathfrak{p}} \xleftarrow{\sim} \operatorname*{colim}_{U\in\mathcal{B}_{X,\mathfrak{p}}^{\text{pr}}} \mathcal{O}_X(U) \xrightarrow{\sim} \operatorname*{colim}_{U\in\mathcal{B}_{X,\mathfrak{p}}^{\text{pr}}} S_U^{-1}R \xrightarrow{\sim} R_{\mathfrak{p}}.$$

because $R^{\text{hom}} \setminus \mathfrak{p}$ is the colimit of the set of all principal faces it contains, partially ordered by inclusion.

DEFINITION IV.1.1.3. An *affine graded scheme* is an object $(X, \mathcal{O}_X)$ of the category of spaces with stalkwise homogeneously local $\mathfrak{C}^{\text{fix}}$-structure sheaves which is isomorphic to the graded spectrum $(\operatorname{Spec}_{\text{gr}}(R), \mathcal{O}_{\operatorname{Spec}_{\text{gr}}(R)})$ associated to some graded A-algebra R. The full subcategory thus defined is denoted $\mathbf{AffGrSch}_A$ resp. $\mathbf{AffGrSch}_{\operatorname{Spec}_{\text{gr}}(A)}$.

DEFINITION IV.1.1.4. Let X be an affine graded scheme.

(i) For $x \in X$ let $\mathfrak{m}_x$ denote the homogeneously maximal ideal of $\mathcal{O}_{X,x}$. Then $\mathcal{I}_x(X) := (\rho_x^X)^{-1}(\mathfrak{m}_x)$ is the *vanishing ideal* of x. Likewise, for a subset $Z \subseteq X$, $\mathcal{I}_Z(X) := \bigcap_{x\in Z} \mathcal{I}_x(X)$ is the *vanishing ideal* of Z.

(ii) For a graded ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ let $V_X(\mathfrak{a})$ be the set of those $x \in X$ with $\mathfrak{a} \subseteq \mathcal{I}_x(X)$. For $f \in \mathcal{O}(X)^{\text{hom}}$ we set $V_X(f) := V_X(\langle f\rangle)$ and define the *principal open subset* associated to f as $X_f := X \setminus V_X(f)$. The set of all principal open subsets is denoted $\mathcal{B}_X^{\text{pr}}$.

The following facts on affine graded schemes are inherited from graded spectra.

REMARK IV.1.1.5. In an affine graded scheme X the topology is generated by $\mathcal{B}_X^{\text{pr}}$. Sending f to X_f defines a homomorphism of $\mathbb{F}_1$-algebras to the set $\mathcal{B}_X^{\text{pr}}$ of all principal open subsets, in which X and $\emptyset$ are unit resp. zero element with respect to the operation $\cap$. Sending $U \in \mathcal{B}_X^{\mathrm{pr}}$ to $S_{X,U} := \bigcap_{x \in U} \mathcal{O}(X)^{\mathrm{hom}} \setminus \mathcal{I}_x(X)$ defines a homomorphism to the $\mathbb{F}_1$-algebra of faces of $\mathcal{O}(X)^{\mathrm{hom}}$. The composition of these homomorphisms sends f to face(f).

Moreover, we have $S_{X,U} = (\rho_U^X)^{-1}_{|\mathcal{O}(X)^{\mathrm{hom}}}((\mathcal{O}(U)^{\mathrm{hom}})^*)$ and the canonical map $S_{X,U}^{-1}\mathcal{O}(X) \to \mathcal{O}(U)$ is an isomorphism. Consequently, applying colimits gives an isomorphism $\mathcal{O}_{X,x} \to \mathcal{O}(X)_{\mathcal{I}_x(X)}$.

Proposition IV.1.1.6. *For an affine graded scheme X consider graded ideals $\mathfrak{a}, \mathfrak{a}_i, i \in I$ and homogeneous elements $f_i, i \in I$ of $\mathcal{O}(X)$ and subsets $Z, Z_i, i \in I$ of X. Then the following hold:*

(i) *If $X = \mathrm{Spec}_{\mathrm{gr}}(R)$ then we have $\mathcal{I}_Z(X) \cong \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$ under the canonical isomorphism $\mathcal{O}(X) \to R$ and $V_X(\mathfrak{a})$ is the set of all $\mathfrak{p} \in X$ containing $\mathfrak{a}$.*
(ii) *We have $V_X(\langle \bigcup_i \mathfrak{a}_i \rangle) = \bigcap_i V_X(\mathfrak{a}_i)$, in particular, $V_X(-)$ reverses inclusions. Thus, we have $\bigcup_i X_{f_i} = X \setminus V_X(\langle f_i | i \in I \rangle)$.*
(iii) *We have $\mathcal{I}_{\bigcup_i Z_i}(X) = \bigcap_i \mathcal{I}_{Z_i}(X)$, and $\mathcal{I}_-(X)$ reverses inclusions;*
(iv) *If I is finite then $V_X(\prod_i \mathfrak{a}_i) = V_X(\bigcap_i \mathfrak{a}_i) = \bigcup_i V_X(\mathfrak{a}_i)$ holds.*
(v) *$V_X(\mathfrak{a}) = V_X(\sqrt{\mathfrak{a}}^{\mathrm{gr}})$ and $\sqrt{\mathfrak{a}}^{\mathrm{gr}} = \mathcal{I}_{V_X(\mathfrak{a})}(X)$.*
(vi) *The closure of Z is $V_X(\mathcal{I}_Z(X))$ and we have $\mathcal{I}_Z(X) = \mathcal{I}_{\overline{Z}}(X)$.*
(vii) *We have $Z \subseteq V_X(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq \mathcal{I}_Z(X)$.*
(viii) *Z is irreducible if and only if $\mathcal{I}_Z(X)$ is homogeneously prime.*

Proof. Throughout, it suffices to consider the case $X = \mathrm{Spec}_{\mathrm{gr}}(R)$. For (ii) note that we have $\langle \bigcup_{i \in I} \mathfrak{a}_i \rangle \subseteq \mathcal{I}_x(X)$ if and only if $\mathfrak{a}_i \subseteq \mathcal{I}_x(X)$ holds for every i. Using (i) we may put assertions (v) and (iv) in terms of R, in which case they follow from Proposition II.1.8.12, and additionally Remark II.1.8.3 in the case of (iv).

In (vi) note that $Z \subseteq V_X(\mathcal{I}_Z(X))$ implies $\overline{Z} \subseteq V_X(\mathcal{I}_Z(X))$. On the other hand, if $Z \subseteq V_X(\mathfrak{a})$ then applying (v) gives $\sqrt{\mathfrak{a}}^{\mathrm{gr}} = \mathcal{I}_{V_X(\mathfrak{a})}(X) \subseteq \mathcal{I}_Z(X)$ as well as $V_X(\mathcal{I}_Z(X)) \subseteq V_X(\sqrt{\mathfrak{a}}^{\mathrm{gr}}) = V_X(\mathfrak{a})$. For the second equation we now calculate $\mathcal{I}_{\overline{Z}}(X) = \mathcal{I}_{V_X(\mathcal{I}_Z(X))}(X) = \sqrt{\mathcal{I}_Z(X)}^{\mathrm{gr}} = \mathcal{I}_Z(X)$.

Assertion (vii) follows from (vi). In (viii) note that if Z is irreducible then $\mathfrak{b}\mathfrak{c} \subseteq \mathcal{I}_Z(X)$ implies $Z \subseteq V_X(\mathfrak{b}) \cup V_X(\mathfrak{c})$ and hence we have $Z \subseteq V_X(\mathfrak{b})$ or $Z \subseteq V_X(\mathfrak{c})$, and consequently $\mathfrak{b} \subseteq \mathcal{I}_Z(X)$ or $\mathfrak{c} \subseteq \mathcal{I}_Z(X)$. If $\mathcal{I}_Z(X)$ is homogeneously prime then $Z \subseteq B \cup C$ implies $\mathcal{I}_B(X)\mathcal{I}_C(X) \subseteq \mathcal{I}_{B \cup C}(X) \subseteq \mathcal{I}_Z(X)$ and we deduce $\mathcal{I}_B(X) \subseteq \mathcal{I}_Z(X)$ or $\mathcal{I}_C(X) \subseteq \mathcal{I}_Z(X)$, which gives $Z \subseteq \overline{Z} \subseteq B$ or $Z \subseteq \overline{Z} \subseteq C$. □

Proposition IV.1.1.7. *The structure presheaf $\mathcal{O}_X$ of $X = \mathrm{Spec}_{\mathrm{gr}}(R)$ is indeed a $\mathfrak{C}^{\mathrm{fix}}$-sheaf. Thus, $(X, \Omega_X, \mathcal{O}_X)$ is a space with stalkwise homogeneously local $\mathfrak{C}^{\mathrm{fix}}$-structure sheaf.*

Proof. Due to Proposition III.1.0.10 it suffices to show that $\mathcal{O}_X$ is a $\mathfrak{C}^{gr(R)}$-sheaf with respect to $\mathcal{B}_X^{\mathrm{pr}}$. For $X_h = \bigcup_{f \in F} X_f$ we have $R_h = \sqrt{\langle h \rangle}_h^{\mathrm{gr}} = \sqrt{\langle F \rangle}_h^{\mathrm{gr}}$ and hence $R_h = \langle F \rangle_h$. Now, Proposition II.1.3.10 shows that R_h is the limit over all the canonical maps $R_{f/1} \to R_{fg/1}$ where $f, g \in F$ and since the isomorphisms $\mathcal{O}_X(X_f) \cong R_{f/1}$ are compatible with the respective diagram structures we obtain that $\mathcal{O}_X(X_h)$ is the limit over all $\rho_{X_{fg}}^{X_f}$ as required. □

Example IV.1.1.8. For $A = \mathbb{Z}$ or $A = \mathbb{F}_1$ consider the $\mathbb{Z}$-graded group algebra $A[\mathbb{Z}]$ and the $\mathbf{GrAlg}_A^{\mathbb{Z}}$-product $R := \prod_{n \in \mathbb{N}} \mathbb{C}[\mathbb{Z}]$ which is a proper A-subalgebra of the $\mathbf{Alg}_A$-product R'. Let $f_n \in R$ be the element whose n-th coordinate is 1 and whose other coordinates are 0. Let U be the union over all the principal subsets X_{f_n} of $X = \mathrm{Spec}_{\mathrm{gr}}(R)$. The sets X_{f_n} are pairwise disjoint. By the $\mathbf{GrAlg}_A^{\mathbb{Z}}$-sheaf property we have $\mathcal{O}_X(U) = R \neq R'$, which means $\mathcal{O}_X$ is no $\mathbf{Alg}_A$-sheaf (i.e. no $\mathbf{Set}$-sheaf).

Proposition IV.1.1.9. *An affine graded scheme X is quasi-compact and sober. Moreover, a subset $Z \subseteq X$ is irreducible if and only if the set $\{\mathcal{I}_x(X) | x \in Z\}$ has a unique minimal element, which in that case equals $\mathcal{I}_Z(X)$.*

Proof. It suffices to consider a graded spectrum $X = \mathrm{Spec}_{\mathrm{gr}}(R)$. Firstly, if $X = \bigcup_{f\in F} X_f$ with some $F \subseteq \mathcal{O}(X)^{\mathrm{hom}}$ then $V_X(\langle F\rangle) = \emptyset$, i.e. $\langle F\rangle = \mathcal{O}(X)$. Thus, there exists a finite subset $F' \subseteq F$ such that $1 \in \langle F'\rangle$, which means $X = \bigcup_{f\in F'} X_f$.

Secondly, let $Y \subseteq X$ be closed and irreducible. Then $\mathcal{I}_Y(X)$ is homogeneously prime and the corresponding point $\mathfrak{p} \in X$ satisfies $\overline{\{\mathfrak{p}\}} = V_X(\mathcal{I}_Y(X)) = Y$. For uniqueness note that if $\overline{\{\mathfrak{q}\}} = Y$ then we have $\mathcal{I}_{\mathfrak{q}}(X) = \mathcal{I}_Y(X) = \mathcal{I}_{\mathfrak{p}}(X)$ and applying the canonical isomorphism $\mathcal{O}(X) \to R$ gives $\mathfrak{q} = \mathfrak{p}$.

Thirdly, if Z is irreducible then the generic point z of $\overline{Z}$ lies in Z and we have $\mathcal{I}_z(X) = \mathcal{I}_Z(X)$. In the converse case, $\mathcal{I}_Z(X)$ is homogeneously prime and thus Z is irreducible. $\square$

Construction IV.1.1.10. For graded A-algebras $\imath\colon A \to R, \lambda\colon gr(A) \to gr(R)$ and $\imath'\colon A \to R', \lambda'\colon gr(A) \to gr(R')$ and let $\alpha\colon R' \to R, \psi\colon gr(R') \to gr(R)$ be a morphism. Then the map

$$\phi := \mathrm{Spec}_{\mathrm{gr}}(\alpha)\colon X := \mathrm{Spec}_{\mathrm{gr}}(R) \longrightarrow X' := \mathrm{Spec}_{\mathrm{gr}}(R'), \quad \mathfrak{p} \longmapsto \alpha^{-1}(\mathfrak{p})^{\mathrm{gr}}.$$

is Zariski-continuous, because $\phi^{-1}(V(\mathfrak{a}')) = V(\langle\alpha(\mathfrak{a}')\rangle)$ holds for each graded ideal $\mathfrak{a}'$ of R' and hence $\phi^{-1}(X'_g) = X_{\alpha(g)}$ holds for each $g \in R'^{\mathrm{hom}}$. Moreover, for a graded ideal $\mathfrak{a}$ of R we have $\overline{\phi(V(\mathfrak{a}))} = V(\alpha^{-1}(\mathfrak{a}))$. For each $U' \in \mathcal{B}^{\mathrm{pr}}_{X'}$ we have $\mathrm{face}(\alpha(S_{U'})) = S_{\phi^{-1}(U')}$. The homomorphism $\mathrm{Spec}_{\mathrm{gr}}(\alpha)^* := \phi^*\colon \mathcal{O}_{X'} \to \phi_*\mathcal{O}_X$ with accompanying map ψ is given as

$$\mathcal{O}_{X'}(U) = \lim_{V'\in\mathcal{B}^{\mathrm{pr}}_X(U)} S_{V'}^{-1}R' \longrightarrow \lim_{V'\in\mathcal{B}^{\mathrm{pr}}_X(U)} S_{\phi^{-1}(V')}^{-1}R \to \mathcal{O}_X(\phi^{-1}(U))$$

where we have used the fact that the occuring limits are limits in $\mathbf{GrAlg}_A$ as well as in $\mathbf{GrAlg}_A^{\lambda}$ resp. $\mathbf{GrAlg}_A^{\lambda'}$. Now, (ϕ, ϕ^*) forms a morphism of graded schemes.

Construction IV.1.1.11. Let X be an affine graded scheme. Then the map

$$\jmath_X\colon X \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X)), \quad x \mapsto \mathcal{I}_x(X)$$

is a homeomorphism such that $\jmath(X_f) = \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))_f$ holds for each $f \in \mathcal{O}(X)^{\mathrm{hom}}$. Moreover, the canonical isomorphisms

$$\jmath^*_{X,U}\colon \mathcal{O}(U) \cong S_U^{-1}\mathcal{O}(\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))) \to S^{-1}_{X,\jmath_X^{-1}(U)}\mathcal{O}(X) \cong \mathcal{O}(\jmath_X^{-1}(U))$$

for $U \in \mathcal{B}^{\mathrm{pr}}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))}$ define an isomorphism $\jmath^*_X\colon \mathcal{O}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))} \to (\jmath_X)_*\mathcal{O}_X$ which together with $\jmath_X$ forms an isomorphism of affine graded schemes. For a morphism $\phi\colon X \to Y$ of affine graded schemes we have $\jmath_Y \circ \phi = \mathrm{Spec}_{\mathrm{gr}}(\phi^*_Y) \circ \jmath_X$.

Proof. For surjectivity of $\jmath_X$ consider $\mathfrak{p} \in \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$. Then the generic point x of $V_X(\mathfrak{p})$ satisfies $\mathcal{I}_x(X) = \mathcal{I}_{V_X(\mathfrak{p})}(X) = \sqrt{\mathfrak{p}}^{\mathrm{gr}} = \mathfrak{p}$. For injectivity note that if $\mathcal{I}_x(X) = \mathcal{I}_y(X)$ then $\overline{\{x\}} = V_X(\mathcal{I}_x(X)) = \overline{\{y\}}$ and hence $x = y$.

The equation of morphisms holds set-theoretically because due to locality of $\phi^*_x\colon \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$. Moreover, with respect to the faces defined by principal open subsets we have

$$\mathrm{face}(\phi^*_Y(\jmath^*_{Y,\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(Y))}(S_U))) = \jmath^*_{X,\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))}(\mathrm{face}(\mathrm{Spec}_{\mathrm{gr}}(\phi^*_Y)^*_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(Y))}(S_U))).$$

This gives the equation of homomorphisms of sheaves on $\mathcal{B}^{\mathrm{pr}}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(Y))}$ and hence, on all of $\Omega_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(Y))}$. $\square$

Proposition IV.1.1.12. *The functor* $\mathrm{Spec}_{\mathrm{gr}}\colon \mathbf{GrAlg}_A \to \mathbf{AffGrSch}_A$ *is essentially inverse to the global section functor $\mathcal{O}(-)$ sending X to $\mathcal{O}_X(X)$ and a morphism $\phi\colon X \to Y$ to ϕ^*_Y.*

Proof. The isomorphism from $\mathrm{id}_{\mathbf{AffGrSch}_A}$ to $\mathrm{Spec}_{\mathrm{gr}} \circ \mathcal{O}(-)$ is provided by Construction IV.1.1.11. The isomorphism from $\mathcal{O}(-) \circ \mathrm{Spec}_{\mathrm{gr}}$ to $\mathrm{id}_{\mathbf{GrAlg}_A}$ is provided by Remark IV.1.1.2. $\square$

Proposition IV.1.1.13. *Let $\phi\colon X \to Y$ be a morphism of affine graded schemes. Then $\phi^{-1}(V_Y(\mathfrak{b})) = V_X(\langle \phi_Y^*(\mathfrak{b})\rangle)$ and $\overline{\phi(V_X(\mathfrak{a}))} = V_Y((\phi_Y^*)^{-1}(\mathfrak{a})^{\mathrm{gr}})$ holds for graded ideals $\mathfrak{a} \trianglelefteq \mathcal{O}(X)$ and $\mathfrak{b} \trianglelefteq \mathcal{O}(Y)$. In particular, we have $\phi^{-1}(Y_g) = X_{\phi_Y^*(g)}$ for $g \in \mathcal{O}(Y)^{\mathrm{hom}}$. Likewise, for a closed subset $Z \subseteq Y$ and a subset $W \subseteq X$ we have $\mathcal{I}_{\phi^{-1}(Z)}(X) = \sqrt{\langle \phi_Y^*(\mathcal{I}_Z(Y))\rangle}^{\mathrm{gr}}$ and $\mathcal{I}_{\phi(W)}(Y) = (\phi_Y^*)^{-1}(\mathcal{I}_W(X))^{\mathrm{gr}}$.*

Proof. For $x \in X$ we have $\mathcal{I}_{\phi(x)}(Y) = (\phi_Y^*)^{-1}(\mathcal{I}_x(X))^{\mathrm{gr}}$ due to graded locality of the map ϕ_x^* of stalks. Consequently, we have $\phi^{-1}(V_Y(\mathfrak{b})) = V_X(\langle \phi_Y^*(\mathfrak{b})\rangle)$. as well as $\mathcal{I}_{\phi(W)}(W) = (\phi_Y^*)^{-1}(\mathcal{I}_Z(X))^{\mathrm{gr}}$. From this, we obtain

$$\mathcal{I}_{\phi^{-1}(Z)}(X) = \mathcal{I}_{\phi^{-1}(V_Y(\mathcal{I}_Z(Y)))}(X) = \mathcal{I}_{V_X(\langle \phi_Y^*(\mathcal{I}_Z(Y))\rangle)}(X) = \sqrt{\langle \phi_Y^*(\mathcal{I}_Z(Y))\rangle}^{\mathrm{gr}}$$

and

$$\begin{aligned}\overline{\phi(V_X(\mathfrak{a}))} &= V_Y(\mathcal{I}_{\phi(V_X(\mathfrak{a}))}(Y)) = V_Y((\phi_Y^*)^{-1}(\mathcal{I}_{V_X(\mathfrak{a})}(X))^{\mathrm{gr}}) = V_Y((\phi_Y^*)^{-1}(\sqrt{\mathfrak{a}}^{\mathrm{gr}})^{\mathrm{gr}}) \\ &= V_Y(\sqrt{(\phi_Y^*)^{-1}(\mathfrak{a})^{\mathrm{gr}}}^{\mathrm{gr}}) = V_Y((\phi_Y^*)^{-1}(\mathfrak{a})^{\mathrm{gr}}).\end{aligned}$$

$\square$

Proposition IV.1.1.14. *Each principal open subset X_f of an affine graded scheme X defines an affine graded subscheme $(X_f, \Omega_{X|X_f}, \mathcal{O}_{X|X_f})$. Moreover, each $Z \subseteq X$ satisfies $\mathcal{I}_{Z\cap X_f}(X_f) = \langle \rho^X_{X_f}(\mathcal{I}_Z(X))\rangle$.*

Proof. For a graded A-algebra R the localization map $\imath_f\colon R \to R_f$ induces a morphism of affine graded schemes $\mathrm{Spec}_{\mathrm{gr}}(\imath_f)$ under which $\mathrm{Spec}_{\mathrm{gr}}(R_f)_{g/f^n}$ is mapped bijectively onto $\mathrm{Spec}_{\mathrm{gr}}(R)_{fg}$ according to Remark II.1.8.9. Moreover, we have a canonical isomorphism

$$\mathcal{O}(\mathrm{Spec}_{\mathrm{gr}}(R_f)_{g/f^n}) \cong (R_f)_{g/f^n} \cong R_{fg} \cong \mathcal{O}(\mathrm{Spec}_{\mathrm{gr}}(R)_{fg}).$$

Since these isomorphisms are compatible with restrictions between principal open subsets, $\mathrm{Spec}_{\mathrm{gr}}(\imath_f)$ is an $\mathbf{Sp}_{\mathrm{loc}}^{\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}}$-isomorphism onto $\mathrm{Spec}_{\mathrm{gr}}(R)_f$. In general, the isomorphism $\jmath_X$ reduces to an $\mathbf{Sp}_{\mathrm{loc}}^{\mathcal{S}h_{\mathfrak{C}^{\mathrm{fix}}}}$-isomorphism $X_f \cong \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))_f$. Applying the above for $R = \mathcal{O}(X)$ we obtain an isomorphism $X_f \cong \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X)_f)$. The supplement follows from Propositions IV.1.1.13 and II.1.8.13. $\square$

IV.1.2. Graded schemes. This section deals with the topology of graded schemes. Of particular use in the context of quasi-coherent $\mathcal{O}_X$-modules will be Lemma IV.1.2.4, which states that the intersection of two affine open subsets of X is covered by their common principal open subsets.

Definition IV.1.2.1. A *graded scheme* is an object $(X, \mathcal{O}_X)$ of the category of spaces with stalkwise homogeneously local $\mathfrak{C}^{\mathrm{fix}}$-structure sheaves such that X has a cover by open sets U defining affine graded schemes $(U, \mathcal{O}_{X|U})$. Such sets U are called *affine open subsets* of X and their collection is denoted $\mathcal{B}_X$. The full subcategory thus defined is denoted $\mathbf{GrSch}_A$ resp. $\mathbf{GrSch}_{\mathrm{Spec}_{\mathrm{gr}}(A)}$. The category $\mathbf{GrSch}/X$ of graded schemes over X will also be denoted $\mathbf{GrSch}_X$.

First we note that graded schemes inherit sobriety from affine graded schemes due to the following general fact.

Remark IV.1.2.2. A topological space is sober if and only if it is covered by a family of sober open subspaces.

Remark IV.1.2.3. A topological space X is quasi-compact if and only if it has a finite cover by quasi-compact open subspaces.

LEMMA IV.1.2.4. *Let X be a graded scheme and let U, V be affine open subsets. Then any principal open set V_g which is contained in a principal open set $U_f \subseteq V$ is also a principal open subset $U_h = V_g$ of U. Thus, the open subsets that are principal in both U and V form a basis for the topology of $U \cap V$.*

PROOF. Let $f \in \mathcal{O}(U)^{\mathrm{hom}}$ with $U_f \subseteq V$ and let $V_g \subseteq U_f$ with $g \in \mathcal{O}(V)^{\mathrm{hom}}$. Let $g_{|U_f} = h'/f^n \in \mathcal{O}(U_f)$ with some $h' \in \mathcal{O}(U)$ and $n \geq 0$. With $h := h'f$ we calculate

$$\begin{aligned} V_g &= \{x \in U_f;\ g_x \notin \mathfrak{m}_x\} = \{x \in U_f;\ (g_{|U_f})_x = h'_x (f_{|U_f})_x^{-n} \notin \mathfrak{m}_x\} \\ &= \{x \in U_f;\ h'_x \notin \mathfrak{m}_x\} = \{x \in U;\ h'_x f_x \notin \mathfrak{m}_x\} = U_h. \end{aligned}$$

For the supplement, suppose that $W \subseteq U \cap V$ is open. Then $W = \bigcup_{i \in I} U_{f_i}$ is a union of principal subsets of U. Each U_{f_i} is open in V and hence a union $U_{f_i} = \bigcup_{j \in J_i} V_{g_j}$ of principal subsets of V. By the above each set V_{g_j} is also principal in U. □

REMARK IV.1.2.5. For a point x of a graded scheme X any $U \in \mathcal{B}_{X,x}$ defines a canonical morphism

$$\jmath\colon \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,x}) \xrightarrow{\mathrm{Spec}_{\mathrm{gr}}(\rho_x^U)} \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_X(U)) \cong U \to X.$$

$\jmath$ does not depend on the choice of U and maps bijectively onto the set of all points which specialize to x. Moreover, for each $\mathfrak{p} \in \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,x})$ the induced map of stalks $\jmath_{\mathfrak{p}}^*\colon \mathcal{O}_{X,\jmath(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,x}),\mathfrak{p}}$ is an isomorphism.

REMARK IV.1.2.6. Let X be a sober topological space and let A be a closed non-empty subset resp. a point closure. If A is minimal among closed non-empty subsets resp. point closures then $A = \{x\}$ consists of a closed point.

Affine graded $\mathbb{F}_1$-schemes have a stronger property than quasi-compactness which is discussed below.

PROPOSITION IV.1.2.7. *Let X be a topological space. Then the following are equivalent:*

(i) *The set of non-empty closed sets has a (unique) minimal element, i.e. the intersection of all non-empty closed sets is non-empty.*
(ii) *The set of point closures has a unique minimal element.*
(iii) *X is an element of every family of open sets which covers X, i.e. the union of all proper open subsets is proper.*

PROOF. If (i) holds then uniqueness is a consequence of finite intersections of closed sets being closed. Furthermore, the minimal closed set must be a closure of a point due to minimality. Suppose that (ii) holds. Then each point x in the minimal point closure P satisfies $\overline{\{x\}} = P$ by minimality. Each further non-empty closed set B then satisfies $P \subseteq \overline{\{y\}}$ for each of its points y, which establishes (i). If $X_i, i \in I$ form an open cover of X then there exists $i \in I$ with $P \cap X_i \neq \emptyset$. Since $X \setminus X_i$ is closed, but does not contain P, it must be empty by (i). □

REMARK IV.1.2.8. The point corresponding to the ideal of non-units in a graded $\mathbb{F}_1$-algebra R is the unique closed point of X and is contained in all closures of points of X, as was observed in [**15**].

PROPOSITION IV.1.2.9. *A graded $\mathbb{F}_1$-scheme X is affine if and only if it satisfies one of the conditions of Proposition IV.1.2.7.*

PROOF. Suppose that X contains a unique point closure $\overline{\{x\}}$ which lies in the closures of all points of X. Let V be an open affine neighbourhood of x. Then the closure of x contains the closed point x' of V, which means $\overline{\{x\}} = \overline{\{x'\}}$ and hence $x = x'$ by sobriety. Consequently, a point $y \in X$ cannot lie in $X \setminus V$ because then so would x'. □

IV.1.3. Quasi-coherent $\mathcal{O}_X$-modules and affine morphisms. Throughout, we consider (pre)sheaves on a fixed graded scheme $(X, \mathcal{O}_X)$. The goal of this section is to introduce the relative graded spectrum of quasi-coherent $\mathcal{O}_X$-algebras and relate this construction to the concept of affine graded schemes over X, i.e. graded schemes over X with affine structure morphisms.

DEFINITION IV.1.3.1. An $\mathcal{O}_X$-prealgebra/-premodule on X is *quasi-coherent* if for all $U \in \mathcal{B}_X$ and $f \in \mathcal{O}(U)^{\text{hom}}$ the canonical homomorphism $\mathcal{M}(U)_f \to \mathcal{M}(U_f)$ of $\mathcal{O}(U_f)$-algebras/-modules is an isomorphism.

PROPOSITION IV.1.3.2. *An $\mathcal{O}_X$-prealgebra/-premodule $\mathcal{M}$ is quasi-coherent if and only if for each $V \in \mathcal{B}_X$ and every $U \in \mathcal{B}_V$ the canonical homomorphism $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{N}(V) \to \mathcal{N}(U)$ of $\mathcal{O}(U)$-algebras/-modules is an isomorphism.*

PROOF. Suppose that $\mathcal{M}$ is quasi-coherent. Let $\mathcal{W}$ denote the set of common principal open subsets of U and V. For each $W \in \mathcal{W}$ the canonical homomorphism $\mathcal{O}(W) \otimes_{\mathcal{O}(V)} \mathcal{M}(V) \to \mathcal{M}(W)$ due to principality. Using quasi-compactness and Proposition II.1.3.10 we see firstly, that $\mathcal{M}(U)$ is the limit over all $\mathcal{M}(W)$ with $W \in \mathcal{W}$ and secondly, that $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} \mathcal{M}(V)$ is a limit of the diagram given by all $\mathcal{O}(W) \otimes_{\mathcal{O}(V)} \mathcal{M}(V)$ with $W \in \mathcal{W}$. This gives the assertion. $\square$

REMARK IV.1.3.3. The sum $\sum_i \mathcal{N}_i$ of quasi-coherent $\mathcal{O}_X$-subpremodules $\mathcal{N}_i$ of a given quasi-coherent $\mathcal{O}_X$-premodule $\mathcal{N}$ is quasi-coherent due to additivity of localization.

EXAMPLE IV.1.3.4. Let $\mathcal{N}$ be a quasi-coherent $\mathcal{O}_X$-premodule. Then every $f \in \mathcal{N}(X)^{\text{hom}}$ defines a quasi-coherent $\mathcal{O}_X$-subpremodule $\mathcal{O}_X f$ which assigns $\mathcal{O}_X(U) f_{|U}$ to $U \in \Omega_X$. More generally, if $M \leq \mathcal{N}(X)$ is a graded $\mathcal{O}_X(X)$-submodule then $\mathcal{O}_X M := \sum_{f \in M^{\text{hom}}} \mathcal{O}_X f$ is a quasi-coherent $\mathcal{O}_X$-subpremodule of $\mathcal{N}$.

REMARK IV.1.3.5. Let $\mathcal{N}$ be a quasi-coherent $\mathcal{O}_X$-premodule/-prealgebra. For $U \in \mathcal{B}_X$ and $x \in U$ we then have a canonical isomorphism $\mathcal{N}_x \cong \mathcal{N}(U)_{I_U(x)}$ due to Example II.1.5.4.

PROPOSITION IV.1.3.6. *A quasi-coherent $\mathcal{O}_X$-prealgebra/-premodule $\mathcal{N}$ is a sheaf with respect to $\mathcal{B}_X$ in the sense of Definition III.1.0.6.*

PROOF. Let $V \in \mathcal{B}_X$ and let $\mathcal{U} \subseteq \mathcal{B}_X$ be a cover of V such that V is the **Set**-colimit of the diagram given by $\mathcal{U}$. Then due to Proposition II.1.3.10 $\mathcal{N}(U)$ is the limit of the diagram given by all $\mathcal{N}(W_U)$ where $W_U \in \mathcal{W}_U := \mathcal{B}_U^{\text{pr}} \cap \mathcal{B}_V^{\text{pr}}$ and $\mathcal{N}(V)$ is the limit of the diagram given by all $\mathcal{N}(W)$ where $W \in \bigcup_{U \in \mathcal{U}} \mathcal{W}_U$. Note that $\bigcup_{U \in \mathcal{U}} \mathcal{W}_U$ is a $\cap$-stable subcategory of $\mathcal{B}_V^{\text{pr}}$ whose collection of morphisms is the union over all $\text{Mor}(\mathcal{W}_U)$. Thus, $\mathcal{N}(V)$ is the limit of the diagram given by all $\mathcal{N}(U)$ where $U \in \mathcal{U}$. $\square$

CONSTRUCTION IV.1.3.7. Suppose that $(X, \mathcal{O}_X)$ is the graded spectrum of a graded A-algebra R and let $\mathfrak{D}$ denote $\mathbf{GrAlg}_R^\lambda$ or $\mathbf{GrMod}_R^\lambda$ where λ denotes a fixed $gr(R)$-algebra/-module (resp. its structure map). The $\mathcal{O}_X$-algebra/-module $\mathcal{N} := N^\sim$ associated to a $\mathfrak{D}$-object N is defined via $\mathcal{N}(U) := \lim_{W \in \mathcal{B}_U^{\text{pr}}} S_W^{-1} N$ for $U \in \Omega_X$ where restriction maps are induced by the universal property. If N was an R-algebra then the maps $S_W^{-1} R \to S_W^{-1} N$ induce maps $\mathcal{O}_X(U) \to \mathcal{N}(U)$ which form a homomorphism $\mathcal{O}_X \to \mathcal{N}$ of presheaves of constantly graded A-algebras with accompaniment λ.

In the case of an R-module let $w \in gr(R)$ and $v \in gr(N)$ and consider for all $W \in \mathcal{B}_U^{\text{pr}}$ the maps

$$\mathcal{O}_X(U)_w \times \mathcal{N}(U)_v \longrightarrow (S_W^{-1} R)_w \times (S_W^{-1} N)_v \longrightarrow (S_W^{-1} N)_{\lambda(w,v)}.$$

The universal property of $\mathcal{N}(U)_{\lambda(w,v)} = \lim_{W\in\mathcal{B}_U^{\mathrm{pr}}}(S_W^{-1}N)_{\lambda(w,v)}$ induces a map $\mathcal{O}_X(U)_w \times \mathcal{N}(U)_v \to \mathcal{N}(U)_{\lambda(w,v)}$. These maps fit together to a scalar multiplication giving $\mathcal{N}(U)$ an $\mathcal{O}_X(U)$-structure with accompaniment λ.

PROPOSITION IV.1.3.8. *For a graded A-algebra R with accompanying map ψ let $X = \mathrm{Spec}_{\mathrm{gr}}(R)$. Then sending a $\mathfrak{D}$-object N to $N^\sim$ constitutes a functor to the category of quasi-coherent $\mathcal{O}_X$-modules/-algebras which is essentially inverse to the functor sending $\mathcal{N}$ to $\mathcal{N}(X)$.*

PROOF. By construction, $N^\sim$ is quasi-coherent and thus a sheaf with respect to $\mathcal{B}_X^{\mathrm{pr}}$ due to Proposition IV.1.3.6. By Proposition III.1.0.10 $N^\sim$ is now a sheaf of graded A-algebras/-modules with fixed accompaniment and hence an $\mathcal{O}_X$-algebra/-module. For a morphism $N \to N'$ the morphism $N^\sim \to N'^\sim$ is defined via universal properties of limits. Due to quasi-coherence, the canonical map $N \to N^\sim(X)$ is an isomorphism and so is the morphism $\mathcal{N}(X)^\sim \to \mathcal{N}$ induced by the universal property of limits. Using these isomorphisms one constructs the required isomorphisms of functors. □

PROPOSITION IV.1.3.9. *Let $\mathcal{N}$ be a $\mathcal{O}_X$-prealgebra/-premodule and let $\mathcal{U} \subseteq \mathcal{B}_X$ be a cover of X. Then $\mathcal{N}$ is quasi-coherent if and only if for each $U \in \mathcal{U}$ and $W \in \mathcal{B}_U^{\mathrm{pr}}$ the canonical map $S_{U,W}^{-1}\mathcal{N}(U) \to \mathcal{N}(W)$ is an isomorphism and for each $V \in \mathcal{B}_X$, $\mathcal{N}(V)$ is the limit over all $\mathcal{N}(W)$ where $W \in \bigcup_{U\in\mathcal{U}} \mathcal{B}_U^{\mathrm{pr}} \cap \mathcal{B}_V^{\mathrm{pr}}$.*

PROOF. If $\mathcal{N}$ is quasi-coherent then Proposition IV.1.3.2 verifies the first condition and Proposition IV.1.3.6 the second. In the converse situation, consider $V \in \mathcal{B}_X$ and $V_f \in \mathcal{B}_V^{\mathrm{pr}}$. Then $\mathcal{N}(V)$ is the limit over all $\mathcal{N}(W)$ where $W \in \bigcup_{U\in\mathcal{U}} \mathcal{B}_V^{\mathrm{pr}} \cap \mathcal{B}_U^{\mathrm{pr}}$. Since localization is exact, it commutes with finite limits and hence $\mathcal{N}(V)_f$ is the limit over all $\mathcal{N}(W)_{f_{|W}} = \mathcal{N}(W_{f_{|W}})$. By assumption this limit is $\mathcal{N}(V_f)$ because $\mathcal{B}_{V_f}^{\mathrm{pr}} \cap \mathcal{B}_U^{\mathrm{pr}}$ is the set of all $W_{f_{|W}}$ where $W \in \mathcal{B}_V^{\mathrm{pr}} \cap \mathcal{B}_U^{\mathrm{pr}}$. □

CONSTRUCTION IV.1.3.10. Let $\mathcal{M}$ be an $\mathcal{O}_X$-algebra/-module and let $U \in \mathcal{B}_X$. Then the canonical maps

$$\lim_{W\in\mathcal{B}_U^{\mathrm{pr}}(V)} S_{U,W}^{-1}\mathcal{M}(U) \longrightarrow \lim_{W\in\mathcal{B}_U^{\mathrm{pr}}(V)} \mathcal{M}(W) \cong \mathcal{M}(V)$$

for $V \in \Omega_U$ define a homomorphism $\mathcal{M}(U)^\sim \to \mathcal{M}_{|U}$.

COROLLARY IV.1.3.11. *An $\mathcal{O}_X$-algebra/-module $\mathcal{M}$ is quasi-coherent if and only if there exists a cover $\mathcal{U} \subseteq \mathcal{B}_X$ of X such that the canonical map $\mathcal{M}(U)^\sim \to \mathcal{M}_{|U}$ is an isomorphism for each $U \in \mathcal{U}$. In this case, the statement also holds for $\mathcal{U} = \mathcal{B}_X$.*

PROPOSITION IV.1.3.12. *For a quasi-coherent $\mathcal{O}_X$-prealgebra/-premodule $\mathcal{N}$ the canonical homomorphism $\mathcal{N}(U) \to \mathcal{N}^\sharp(U)$ is an isomorphism for each $U \in \mathcal{B}_X$. In particular, $\mathcal{N}^\sharp$ is also quasi-coherent.*

PROOF. Injectivity follows from Lemma II.1.8.7. For surjecitivity, consider $(f^{(x)})_{x\in U} \in \mathcal{N}^\sharp(U)_w$. Then there exist $h_i \in \mathcal{O}(U)_{v_i}, i = 1, \dots, d$ with $U = \bigcup_i U_{h_i}$ as well as $n_i \in \mathbb{N}_0$ and $g_i \in \mathcal{O}(U)_{w-n_iv_i}$ such that $f^{(x)} = (h_i)_x^{-n_i}(g_i)_x$ holds for all $x \in U_{h_i}$. Since we have $(g_ih_j^{n_i}/(h_ih_j)^{n_i})_x = (g_jh_i^{n_j}/(h_ih_j)^{n_j})_x$ for all $x \in U_{h_ih_j}$, Lemma II.1.8.7 implies $g_ih_j^{n_i}/(h_ih_j)^{n_i} = g_jh_i^{n_j}/(h_ih_j)^{n_j}$ and thus Proposition II.1.3.10 yields a $g \in \mathcal{N}(U)_w$ with $g_{|U_{h_i}} = g_i/h_i^{n_i}$ for each i, which means g is mapped to $(f^{(x)})_{x\in U}$. □

PROPOSITION IV.1.3.13. *Let X be a graded scheme and $f \in \mathcal{O}_X(X)^{\mathrm{hom}}$. Then $f_{|X_f}$ is a unit in $\mathcal{O}_X(X_f)$. Moreover, for a quasi-coherent $\mathcal{O}_X$-algebra/-module $\mathcal{N}$*

the induced map $(\varrho^X_{X_f})_f \colon \mathcal{N}(X)_f \to \mathcal{N}(X_f)$ *of* $\mathcal{O}(X)_f$*-algebras/-modules is injective if* X *has a finite cover* $\mathcal{U} \subseteq \mathcal{B}_X$*, i.e.* X *is quasi-compact, and surjective if additionally,* $U \cap V$ *is quasi-compact for all* $U, V \in \mathcal{U}$*.*

Proof. Note that X_f is the **Set**-colimit of the sets $U_{f_{|U}}$ where $U \in \mathcal{B}_X$. For every $U \in \mathcal{B}_X$ set $g_U := 1/(f_{|U}) \in \mathcal{O}(U_{f|U}) = \mathcal{O}(U)_f$. Then for $W \in \mathcal{B}_U$ we have $g_{U|W_{f_{|W}}} = g_W$ and thus there exists an element $g \in \mathcal{O}(X_f)$ with $g_{|U_{f_{|U}}} = g_U$ and it satisfies $(fg)_{|U} = f_{|U} g_U = 1$ for all U which means $fg = 1$ in $\mathcal{O}(X_f)$.

Let $g/f^n \in \mathcal{N}(X)_f$ be an (homogeneous) element. If $g/f^n \in \ker((\varrho^X_{X_f})_f)$ then $f^n_{|X_f} g_{|X_f} = 0$ and hence $g_{|X_f} = 0$. For each $U \in \mathcal{B}_X$ we then have $g_{|U}/f^n_{|U} = 0$ which means there exists $m_U \geq 0$ with $f^{m_U}_{|U} g_{|U} = 0$. If X has a finite cover by $\mathcal{U} \in \mathcal{B}_X$ then we find $m \geq 0$ with $(f^m g)_{|U} = 0$ for $U \in \mathcal{U}$ and hence $f^m g = 0$, i.e. $g/f^n = 0$.

Lastly, let $h \in \mathcal{N}(X_f)$ be an (homogeneous) element. Then there exist (homogeneous) $g_U \in \mathcal{N}(U)$ and $m_U \in \mathbb{N}_0$ such that $h_{|U_f} = f^{-m_U}_{|U_f}(g_U)_{|U_f}$. For $U, V \in \mathcal{U}$ we then have $f^{-m_V}_{|(U\cap V)_f}(g_V)_{(U\cap V)_f} = h_{|(U\cap V)_f} = f^{-m_U}_{|(U\cap V)_f}(g_U)_{(U\cap V)_f}$ and hence $(f^{m_U}_{|V} g_V)_{|(U\cap V)_f} = (f^{m_V}_{|U} g_U)_{|(U\cap V)_f}$. If each $U \cap V$ is quasi-compact injectivity gives $(f^{m_U}_{|V} g_V)_{|U\cap V} = (f^{m_V}_{|U} g_U)_{|U\cap V}$. Let $m := \max_U m_U$ and set $g'_U := f^{m-m_U}_{|U} g_U$. Then $(g'_U)_{|U\cap V} = (g'_V)_{|U\cap V}$ and thus there exists $g' \in \mathcal{O}(X)$ with $g'_{|U} = g'_U$ and by construction we have $f^{-m}_{|U_f} g'_{|U_f} = h_{|U_f}$ and hence $f^{-m}_{|X_f} g'_{|X_f} = h$. □

Remark IV.1.3.14. Let X be a graded scheme over A. Proposition II.1.3.10 implies that for a quasi-coherent $\mathcal{O}_X$-algebra/-module $\mathcal{N}$ being of finite type over $\mathcal{O}_X$ resp. A need be checked only on an affine cover of X.

Definition IV.1.3.15. A morphism $\phi : X \to Y$ of graded schemes is *affine* if for every affine open $V \subseteq Y$ the preimage $\phi^{-1}(V)$ is affine.

Proposition IV.1.3.16. *A morphism* $\phi\colon X \to Y$ *of graded schemes is affine if for some cover* $Y = \bigcup_{i\in I} V_i$ *by affine* V_i *the preimages* $\phi^{-1}(V_i)$ *are affine.*

Proof. Each $U \in \mathcal{B}_Y$ is a union of finitely many common principal subsets $U_{f_j} = (V_i)_{g_j^{(i)}}$ of U and the V_i, so $\phi^{-1}(U)$ is the union of the (affine) principal sets $\phi^{-1}(U)_{\phi^*_U(f_j)} = \phi^{-1}(V_i)_{\phi^*_{V_i}(g_j^{(i)})}$. Moreover, we have $1 \in \langle f_j | j\rangle$ and hence $1 \in \langle \phi^*_U(f_j) | j\rangle$. Thus, U is affine by Proposition IV.1.5.2. □

Construction IV.1.3.17. Let X be a graded scheme and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-prealgebra. Then for each $U, V \in \mathcal{B}_X$ with $U \subseteq V$ we have an open embedding $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{A}(U)) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{A}(V))$ obtained by covering U with all principal common subsets of U and V, and using Propositions IV.1.3.6 and II.1.3.10. The resulting colimit

$$\mathrm{Spec}_{gr,X}(\mathcal{A}) = \operatorname*{colim}_{U\in\mathcal{B}_X} \mathrm{Spec}_{\mathrm{gr}}(\mathcal{A}(U))$$

together with the affine morphism $\phi\colon Y := \mathrm{Spec}_{gr,X}(\mathcal{A}) \to X$ induced by the morphisms $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{A}(U)) \to U$ is called the *relative graded spectrum* of $\mathcal{A}$. Note that $\mathcal{A}$ is a $\mathfrak{C}^{\mathrm{fix}}$-*sheaf* if and only if the canonical homomorphism $\mathcal{A} \to \phi_*\mathcal{O}_Y$ is an isomorphism.

We will later consider graded relative spectra of Cox sheaves. Moreover, we will apply the construction in the general case of $\mathcal{O}_X$-*pre*algebras when we define the canonical functor from graded schemes to schemes.

Proposition IV.1.3.18. *For a graded scheme* X *let* $\mathbf{AffGrSch}_X$ *be the subcategory of* $\mathbf{GrSch}_X$ *whose objects are affine morphisms to* X*. Then the following* *are mutually essentially inverse equivalences*

$$\begin{aligned} \{\textit{quasi-coherent } \mathcal{O}_X\textit{-algebras}\} &\longleftrightarrow \mathbf{AffGrSch}_X \\ \mathrm{Spec}_{gr,X}\colon [\mathcal{O}_X \to \mathcal{A}] &\longmapsto [\mathrm{Spec}_{gr,X}(\mathcal{A}) \to X] \\ [\mathcal{A} \to \mathcal{B}] &\longmapsto [\mathrm{Spec}_{gr,X}(\mathcal{B}) \to \mathrm{Spec}_{gr,X}(\mathcal{A})] \\ [\mathcal{O}_X \xrightarrow{\imath^*} \imath * \mathcal{O}_Y] &\longleftarrow [X \xrightarrow{\imath} Y] \\ [\mathcal{O}_Y \xrightarrow{\phi^*} \phi_*\mathcal{O}_Z] &\longleftarrow [Y \xrightarrow{\phi} Z] \end{aligned}$$

PROOF. Suppose that ϕ is affine. For $U \in \mathcal{B}_Y$ and $f \in \mathcal{O}_Y(U_f)^{\mathrm{hom}}$ we have

$$\phi_*\mathcal{O}_X(U_f) = \mathcal{O}_X(\phi^{-1}(U)_{\phi_U^*(f)}) \cong \mathcal{O}_X(\phi^{-1}(U))_{\phi_U^*(f)} = \phi_*\mathcal{O}_X(U)_f$$

which shows well-definedness. □

IV.1.4. Vanishing sets and an affineness criterion. Following the discussion of basic properties of vanishing sets of quasi-coherent $\mathcal{O}_X$-(pre-)ideals we give an affineness criterion, see Propositions IV.1.4.8, which we apply in the discussion of affine morphisms whose comorphisms are CBEs, see Proposition IV.1.4.13.

CONSTRUCTION IV.1.4.1. Let $\mathcal{I}$ be a quasi-coherent $\mathcal{O}_X$-preideal. Then

$$V_X(\mathcal{I}) := \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}) := \{x \in X \mid \mathcal{I}_x \neq \mathcal{O}_{X,x}\} = \{x \in X \mid \mathcal{I}_x \subseteq \mathfrak{m}_{X,x}\}$$

is the *vanishing set* of $\mathcal{I}$. For $f \in \mathcal{O}(X)^{\mathrm{hom}}$ we set $V_X(f) := V_X(\mathcal{O}_X f)$ and call $X_f := X \setminus V_X(f)$ the *principal open subset* associated to f.

CONSTRUCTION IV.1.4.2. Let $A \subseteq X$ be a subset of a graded scheme. Then setting

$$\mathcal{I}_A(U) := \bigcap_{x \in U \cap A} (\rho_x^U)^{-1}(\mathfrak{m}_{X,x})$$

for $U \in \Omega_X$ defines an $\mathcal{O}_X$-ideal called the *sheaf of vanishing ideals* associated to A.

REMARK IV.1.4.3. For a subset $A \subseteq X = \mathrm{Spec}_{\mathrm{gr}}(R)$ the canonical isomorphism $R \cong \mathcal{O}_X(X)$ restricts to an isomorphism $I(A) \cong \mathcal{I}_A(X)$. Consequently, we have $V_X(\mathcal{I}_A) = V(I(A)) = \overline{A}$.

PROPOSITION IV.1.4.4. *Let A be a subset of X and let $\mathcal{J}, \mathcal{J}_i, i \in I$ be a quasi-coherent graded $\mathcal{O}_X$-preideals.*

(i) *For each $U \in \mathcal{B}_X$ we have $\mathcal{I}_A(U) = \mathcal{I}_{A\cap U}(U)$ and $V_U(\mathcal{J}(U)) = V_U(\mathcal{J}_{|U})$ with respect to the notions from Definition IV.1.1.4.*
(ii) *$\mathcal{I}_A$ equals $\sqrt{\mathcal{I}_A}^{\mathrm{gr}}$ and is quasi-coherent since we have $(\mathcal{I}_A)_{|U} = \mathcal{I}_{A\cap U}$ for each $U \in \Omega_X$,*
(iii) *$V_X(\mathcal{J}) = V_X(\mathcal{J}^\sharp)$ is closed because we have $V_U(\mathcal{J}_{|U}) = V_X(\mathcal{J}) \cap U$ for each $U \in \Omega_X$,*
(iv) *we have $V_X(\mathcal{I}_A) = \overline{A}^X$,*
(v) *we have $\mathcal{I}_{V_X(\mathcal{J})} = (\sqrt{\mathcal{J}}^{\mathrm{gr}})^\sharp$,*
(vi) *if I is finite then $V_X(\prod_i \mathcal{J}_i) = V_X(\bigcap_i \mathcal{J}_i) = \bigcup_i V_X(\mathcal{J}_i)$.*
(vii) *we have $V_X(\sum_i \mathcal{J}_i) = \bigcap_i V_X(\mathcal{J}_i)$.*
(viii) *$\mathcal{I}_-$ and $V_X(-)$ reverse inclusions of sets resp. quasi-coherent graded $\mathcal{O}_X$-preideals.*
(ix) *A is irreducible if and only if $\mathcal{J}\mathcal{I} \subseteq \mathcal{I}_A$ implies $\mathcal{J} \subseteq \mathcal{I}_A$ or $\mathcal{I} \subseteq \mathcal{I}_A$ for each two quasi-coherent graded $\mathcal{O}_X$-preideals. In these cases, each $\mathcal{I}_A(U)$ is homogeneously prime.*

PROOF. Assertion (i) follows from the respective definitions. In (ii) quasi-coherence follows from Proposition IV.1.1.14 and each $\mathcal{I}_A(U)$ is homogeneously radical as an intersection of homogeneously prime ideals. Assertion (iii) follows from Remark IV.1.4.3.

Assertion (iv) follows from the fact that due to Proposition IV.1.1.6 we have $V_U(\mathcal{I}_{A|U}) \cap V = \overline{A \cap V}^V$ for each $V \in \mathcal{B}_U$. For (v) we use (ii), (iii), Proposition IV.1.1.6 and Remark III.3.0.6 to argue that

$$\mathcal{I}_{V_X(\mathcal{J})}(U) = \mathcal{I}_{V_U(\mathcal{J}_{|U})}(U) = \sqrt{\mathcal{J}(U)}^{\mathrm{gr}} = (\sqrt{\mathcal{J}}^{\mathrm{gr}})^\sharp(U)$$

holds for $U \in \mathcal{B}_X$.

For (vi) and (vii) note that due to Proposition IV.1.1.6 for each $U \in \mathcal{B}_X$ we have

$$\begin{aligned} V_X(\prod_i \mathcal{J}_i) \cap U = V_U(\prod_i (\mathcal{J}_i)_{|U}) = V(\prod_i \mathcal{J}_i(U)) = V_X(\bigcap_i \mathcal{J}_i) \cap U = V_U(\bigcap_i (\mathcal{J}_i)_{|U}) \\ = V(\bigcap_i \mathcal{J}_i(U)) = \bigcup_i V_X(\mathcal{J}_i) \cap U = \bigcup_i V_U((\mathcal{J}_i)_{|U}) = \bigcup_i V(\mathcal{J}(U)) \end{aligned}$$

and

$$\begin{aligned} V_X(\sum_i \mathcal{J}_i) \cap U = V_U(\sum_i (\mathcal{J}_i)_{|U}) = V(\sum_i \mathcal{J}_i(U)) = \bigcap_i V(\mathcal{J}_i(U)) \\ = \bigcap_i V_U((\mathcal{J}_i)_{|U}) = \bigcap_i V_X(\mathcal{J}_i) \cap U. \end{aligned}$$

In (ix) first suppose that A is irreducible. If $\mathcal{I}\mathcal{J} \subseteq \mathcal{I}_A$ holds then we have $A \subseteq V_X(\mathcal{I}_A) \subseteq V_X(\mathcal{I}\mathcal{J}) = V_X(\mathcal{I}) \cup V_X(\mathcal{J})$ and we deduce that $A \subseteq V_X(\mathcal{I})$ or $A \subseteq V_X(\mathcal{J})$ holds. Thus, we have $\mathcal{I} \subseteq \mathcal{I}_{V_X(\mathcal{I})} \subseteq \mathcal{I}_A$ or $\mathcal{J} \subseteq \mathcal{I}_{V_X(\mathcal{J})} \subseteq \mathcal{I}_A$. For the converse suppose that $A \subseteq B \cup C$ holds with closed subsets $B, C \subseteq X$. Then $\mathcal{I}_B\mathcal{I}_C \subseteq \mathcal{I}_B \cap \mathcal{I}_C = \mathcal{I}_{B\cup C} \subseteq \mathcal{I}_A$ holds and by assumption we have $\mathcal{I}_B \subseteq \mathcal{I}_A$ or $\mathcal{I}_C \subseteq \mathcal{I}_A$, which gives $A \subseteq V_X(\mathcal{I}_A) \subseteq V_X(\mathcal{I}_B) = B$ or $A \subseteq V_X(\mathcal{I}_C) = C$. Finally, consider $f, g \in \mathcal{O}(U)^{\mathrm{hom}}$ with $fg \in \mathcal{I}_A(U)$. Then $\mathcal{O}_U f \mathcal{O}_U g \subseteq (\mathcal{I}_A)_{|U} = \mathcal{I}_{A\cap U}$ and by irreducibility of $A \cap U$ we obtain $\mathcal{O}_U f \subseteq \mathcal{I}_{A\cap U}$ or $\mathcal{O}_U g \subseteq \mathcal{I}_{A\cap U}$ and hence $f \in \mathcal{I}_A(U)$ or $g \in \mathcal{I}_A(U)$. □

PROPOSITION IV.1.4.5. *Consider a subset Y of a graded scheme X and a graded ideal $\mathfrak{a}$ of $\mathcal{O}(X)$. Then with respect to the topology Ω'_X generated by all X_f for $f \in \mathcal{O}(X)^{\mathrm{hom}}$ the following hold:*

(i) *We have $Y \subseteq V_X(\mathfrak{a})$ if and only if $\mathfrak{a} \subseteq I_X(Y)$. Consequently, $V_X(I_X(Y))$ is the Ω'_X-closure of Y and we have $\mathfrak{a} \subseteq I_X(V_X(\mathfrak{a}))$.*

(ii) *Y is Ω'_X-irreducible if and only if $I_X(Y)$ is homogeneously prime. In particular, Ω_X-irreducibility of Y implies homogeneous primality of $I_X(Y)$.*

PROOF. Regarding (i) note that $Y \subseteq V_X(\mathfrak{a})$ means $Y \subseteq V_X(f)$ for each $f \in \mathfrak{a}^{\mathrm{hom}}$, i.e. $f_x \in \mathfrak{m}_x$ for each $x \in Y$ and each $f \in \mathfrak{a}^{\mathrm{hom}}$. In other words, for each $f \in \mathfrak{a}^{\mathrm{hom}}$ we have $f \in I_X(Y)$. This means $\mathfrak{a} \subseteq I_X(Y)$.

In (ii) note that if Y is Ω'_X-irreducible and $\mathfrak{a}, \mathfrak{b}$ are graded ideals of $\mathcal{O}(X)$ with $\mathfrak{a}\mathfrak{b}$ then $Y \subseteq V_X(\mathfrak{a}) \cup V_X(\mathfrak{b})$ implies $Y \subseteq V_X(\mathfrak{a})$ or $Y \subseteq V_X(\mathfrak{b})$, i.e. $\mathfrak{a} \subseteq I_X(Y)$ or $\mathfrak{b} \subseteq I_X(Y)$. Conversely, consider Ω'_X-closed sets A, B with $Y \subseteq A \cup B$, where $A = V_X(\mathfrak{a})$ and $B = V_X(\mathfrak{b})$ holds with graded ideals $\mathfrak{a}, \mathfrak{b}$ of $\mathcal{O}(X)$. Then $\mathfrak{a}\mathfrak{b} \subseteq I_X(Y)$ gives $\mathfrak{a} \subseteq I_X(Y)$ or $\mathfrak{b} \subseteq I_X(Y)$ which means $Y \subseteq A$ or $Y \subseteq B$. □

REMARK IV.1.4.6. If $\phi\colon X \to Y$ is a morphism of graded schemes, $V \subseteq Y$ is an open set and $g \in \mathcal{O}_Y(V)^{\mathrm{hom}}$ then $\phi^{-1}(V_g) = \phi^{-1}(V)_{\phi_V^*(g)}$. Indeed, for an element $x \in \phi^{-1}(V)$ we have $g_{\phi(x)} \in (\mathcal{O}_{Y,\phi(x)}^{\mathrm{hom}})^*$ if and only if $\phi_x^*(g_{\phi(x)}) = \phi_V^*(g)_x \in (\mathcal{O}_{X,x}^{\mathrm{hom}})^*$ by locality of ϕ_x^*.

Remark IV.1.4.7. A morphism $\phi\colon U \to X$ of affine graded schemes is an open embedding if and only if there exist (homogeneous) $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $U = U_{\phi_X^*(f_1)} \cup \ldots \cup U_{\phi_X^*(f_n)}$ and the restrictions $U_{\phi_X^*(f_i)} \to X_{f_i}$ are isomorphisms, i.e. $\mathcal{O}(U) = \langle \phi_X^*(f_1), \ldots, \phi_X^*(f_n) \rangle$ and the natural maps $\mathcal{O}(X)_{f_i} \to \mathcal{O}(U)_{\phi_X^*(f_i)}$ are isomorphisms.

Indeed, if ϕ is an open embedding then $\phi(U) = X_{f_1} \cup \ldots \cup X_{f_n}$ holds with $f_i \in \mathcal{O}(X)^{\mathrm{hom}}$ and we have $U = \phi^{-1}(\phi(U)) = U_{\phi_X^*(f_1)} \cup \ldots \cup U_{\phi_X^*(f_n)}$, in particular $\mathcal{O}(U) = \langle \phi_X^*(f_1), \ldots, \phi_X^*(f_n) \rangle$.

Proposition IV.1.4.8. *Let X be a graded scheme. Then an open set $U \subseteq X$ is affine if and only if there exist $U_1, \ldots, U_n \in \mathcal{B}_U$ which cover U such that each face $\tau_i := (\rho^U_{U_i})^{-1}((\mathcal{O}(U_i)^{\mathrm{hom}})^*)$ is principal, $\tau_i^{-1}\tau_k^{-1}\rho^U_{U_i \cap U_k}$ is an isomorphism, $U_i \cap U_k$ is affine and we have $\mathcal{O}(U) = \langle \tau_1^\circ, \ldots, \tau_n^\circ \rangle$. If X has an affine cover $X = \bigcup_{j \in J} X_j$ then the U_i may be chosen as principal open subsets of the X_j.*

Proof. If the second set of conditions is satisfied then we have open principal open embeddings $U_i \cap U_k \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(U)) =: U'$ which are compatible and whose images cover U' due to Proposition II.1.3.10. Thus, they fit together to an isomorphism $U \to U'$.

For the supplement, suppose that U is affine. Then $U \cap X_i$ is covered by common principal subsets of U and X_i. By quasi-compactness, U is covered by finitely many such sets. $\square$

Remark IV.1.4.9. If $X = \bigcup_{i \in I} X_i$ is an open affine cover of a graded $\mathbb{F}_1$-scheme then $\mathcal{B}_X = \bigcup_i \mathcal{B}^{\mathrm{pr}}_{X_i}$, because each $U \in \mathcal{B}_X$ is a union of principal subsets $(X_i)_{f_{i,j}}$ and by Proposition IV.1.2.7 (iii) it must equal one of these.

Remark IV.1.4.10. Let $\phi\colon X \to Y$ be a morphism of affine graded schemes. Then the composition of the inverse image functor ϕ^*, restricted to quasi-coherent $\mathcal{O}_Y$-algebras/-modules, with the global sections functor to $\mathcal{O}(X)$-algebras/-modules is isomorphic to the composition of the global sections functor with the functor sending an $\mathcal{O}(Y)$-algebra/-module R to $\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} R$.

Denote by ϕ^+ the functor from $\mathcal{O}_Y$-ideals to $\mathcal{O}_X$-ideals from Remark III.3.0.20. Then the composition of ϕ^+, restricted to quasi-coherent $\mathcal{O}_Y$-ideals, with the global sections functor is isomorphic to the composition of the global sections functor and the functor sending an $\mathcal{O}(Y)$-ideal $\mathfrak{a}$ to $\langle \phi_X^*(\mathfrak{a}) \rangle_{\mathcal{O}(X)}$.

Remark IV.1.4.11. Let $\phi\colon X \to Z$ be an affine morphism of graded schemes. Then the direct resp. inverse image functors ϕ_* and ϕ^* both respect quasi-coherence.

Remark IV.1.4.12. Let $\phi\colon X \to Z$ be an affine morphism of graded schemes. For each quasi-coherent $\mathcal{O}_X$-preideal $\mathcal{I}$ we have $V_Z(\phi_*\mathcal{I}) = \overline{\phi(V_X(\mathcal{I}))}$. Moreover, for each quasi-coherent $\mathcal{O}_Z$-preideal $\mathcal{J}$ we have $V_X(\langle \phi^*(\mathcal{J}) \rangle) = \phi^{-1}(V_Z(\mathcal{J}))$.

Proposition IV.1.4.13. *Let $\phi\colon X \to Z$ be an affine morphism of graded schemes such that $\phi^*\colon \mathcal{O}_Z \to \phi_*\mathcal{O}_X$ is a CBE. Then the following hold:*

(i) *ϕ is a homeomorphism.*

(ii) *The canonical map $\mathcal{B}_Z \to \mathcal{B}_X$ is bijective. Moreover, for each $U \in \mathcal{B}_Z$ the canonical map $\mathcal{B}^{\mathrm{pr}}_U \to \mathcal{B}^{\mathrm{pr}}_{\phi^{-1}(U)}$ is bijective.*

(iii) *The canonical functors between quasi-coherent $\mathcal{O}_Z$-ideals and quasi-coherent $\mathcal{O}_X$-ideals are mutually essentially inverse equivalences.*

(iv) *For $x \in X$ the taking image resp. preimage under ϕ defines mutually inverse bijections between the (affine) neighbourhoods of x and $\phi(x)$. Moreover, the canonical map $\phi_x^*\colon \mathcal{O}_{Z,\phi(x)} \to (\phi_*\mathcal{O}_X)_{\phi(x)} = \mathcal{O}_{X,x}$ is a CBE.*

Proof. We assume that X is affine. By Proposition II.1.2.17 taking graded preimages resp. images under ϕ_Z^* defines a bijection between the graded ideals of

$\mathcal{O}(X)$ and $\mathcal{O}(Z)$ which preserves graded principality and primality. Therefore, ϕ is bijective and we have $\phi(X_{\phi_Z^*(f)}) = Z_f$ for $f \in \mathcal{O}(Z)^{\mathrm{hom}}$. This gives (i) and bijectivity of the map $\mathcal{B}_Z^{\mathrm{pr}} \to \mathcal{B}_X^{\mathrm{pr}}$. Moreover, if $\phi^{-1}(U)$ is affine then U is affine by Proposition IV.1.4.8. For (iii) we use the setting of Remark IV.1.4.10 and apply Proposition II.1.2.17. Assertion (iv) follows from Remark II.1.3.5. $\square$

IV.1.5. Adjoint pairs and limits. In this section we consider functors between various categories of graded schemes and consider constructions such as finite limits.

Proposition IV.1.5.1. *The following defines a covariant functor*

$$\begin{aligned} \mathfrak{aff}\colon \mathbf{GrSch}_A &\longrightarrow \mathbf{AffGrSch}_A \\ (X, \mathcal{O}_X) &\longmapsto (\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_X(X)), \mathcal{O}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_X(X))}) \\ (X \xrightarrow{\phi} Y, \mathcal{O}_Y \xrightarrow{\phi^*} \phi_*\mathcal{O}_X) &\longmapsto (\mathrm{Spec}_{\mathrm{gr}}(\phi_Y^*), \mathrm{Spec}_{\mathrm{gr}}(\phi_Y^*)^*) \end{aligned}$$

with the following properties:

(i) *For a graded scheme $(X, \mathcal{O}_X)$ we have a morphism*
$$\alpha_X\colon X \longrightarrow \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_X(X)), \quad x \longmapsto I_X(\{x\})$$
which satisfies $\alpha_X^{-1}(\mathfrak{aff}(X)_f) = X_f$ for $f \in \mathcal{O}(X)^{\mathrm{hom}}$.

(ii) *A graded scheme $(X, \mathcal{O}_X)$ is affine if and only if the canonical morphism α_X is an isomorphism.*

(iii) $\mathfrak{aff}$ *is canonically left adjoint to the inclusion* $\mathbf{AffGrSch} \to \mathbf{GrSch}$.

Proof. Let $(\phi, \phi^*) : (X, \mathcal{O}_X) \to (\mathrm{Spec}_{\mathrm{gr}}(R), \mathcal{O}_{\mathrm{Spec}_{\mathrm{gr}}(R)}) =: (Y, \mathcal{O}_Y)$ be an isomorphism. For every $x \in X$ and $\mathfrak{p} := \phi(x)$ we have, in the notation of Section IV.1.1, a commutative diagram

$$\begin{tikzcd} \mathcal{O}_X(X) \arrow[d, "\rho_x^X"'] & \mathcal{O}_Y(Y) \arrow[l, "\phi_Y^*"'] \arrow[d, "\rho_{\phi(x)}^Y"] & R \arrow[l, "\jmath_R"'] \arrow[d, "\imath_{\mathfrak{p}}"] \\ \mathcal{O}_{X,x} & \mathcal{O}_{Y,\phi(x)} \arrow[l, "\phi_{\phi(x)}^*"'] & R_{\mathfrak{p}} \arrow[l, "\jmath_{\mathfrak{p}}"'] \end{tikzcd}$$

whose rows are isomorphisms. We show that α_X equals the morphism

$$\alpha' := \mathrm{Spec}_{\mathrm{gr}}((\phi_Y^*)^{-1}) \circ \mathrm{Spec}_{\mathrm{gr}}((\jmath_R)^{-1}) \circ \phi : X \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_X(X)).$$

For $\mathfrak{p}$ we know that $\mathfrak{m}_{\mathfrak{p}} = \jmath_{\mathfrak{p}}(S_{\mathfrak{p}}^{-1}\mathfrak{p})$ and $\imath_{\mathfrak{p}}^{-1}(S_{\mathfrak{p}}^{-1}\mathfrak{p}) = \mathfrak{p}$ thus we may calculate

$$\begin{aligned} \alpha'(x) &= \phi_Y^*(\jmath_R(\phi(x))) = \phi_Y^*(\jmath_R(\imath_{\phi(x)}(S_{\phi(x)}^{-1}\phi(x)))) \\ &= \phi_Y^*((\rho_{\phi(x)}^Y)^{-1}(\mathfrak{m}_{\phi(x)})) = (\rho_x^X)^{-1}(\mathfrak{m}_x) = \alpha_X(x) \end{aligned}$$

In (i) continuity of α_X follows from the supplement. Moreover, Proposition IV.1.3.13 provides canonical maps

$$\mathcal{O}_{\mathfrak{aff}(X)}(\mathfrak{aff}(X)_f) = \mathcal{O}_X(X)_f \to \mathcal{O}_X(X_f) = \mathcal{O}_X(\alpha_X^{-1}(\mathfrak{aff}(X)_f))$$

and applying limits defines a morphism of graded sheaves $\alpha_X^*\colon \mathcal{O}_{\mathfrak{aff}(X)} \to (\alpha_X)_*\mathcal{O}_X$. Assertion (iii) is an application of Lemma A.0.0.2. $\square$

Proposition IV.1.5.2. *A graded scheme X over A is affine if and only if there exist (homogeneous) $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $\mathcal{O}(X) = \langle f_1, \ldots, f_n \rangle$ and each X_{f_i} is affine.*

Proof. Suppose the second condition holds. By Proposition IV.1.4.5 we have $X = \bigcup_i X_{f_i}$. Moreover, each $X_{f_i} \cap X_{f_j} = (X_{f_i})_{f_{j|X_{f_i}}}$ is affine and hence the canonical morphism $\mathcal{O}(X)_{f_i} \to \mathcal{O}(X_{f_i})$ is an isomorphism by Proposition IV.1.3.13. For $f = f_i$ and $f = f_i f_j$ we have canonical isomorphisms $X_f \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X)_f)$ by

Proposition IV.1.5.1 which define compatible open embeddings $X_f \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$. Their images cover $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$ because the f_i generate $\mathcal{O}(X)$. Thus, the induced morphism $X \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$ is an isomorphism. □

Proposition IV.1.5.3. *Let S be a graded scheme and consider $gr(\mathcal{O}_S)$-algebras $\kappa\colon gr(\mathcal{O}_S) \to K$ and $\lambda\colon gr(\mathcal{O}_S) \to L$ and a homomorphism $\psi\colon K \to L$ of $gr(\mathcal{O}_S)$-algebras. Denoting the coarsening and augmentation functors from Section II.1.2 by co^ψ resp. aug^ψ we have the following:*

(i) *Sending a K-graded scheme X over S to $\mathrm{Spec}_X(\mathrm{co}^\psi \circ \mathcal{O}_X)$ defines a functor $\mathbf{GrSch}_S^K \to \mathbf{GrSch}_S^L$.*
(ii) *If ψ is surjective then sending an L-graded scheme $(Y, \mathcal{O}_Y)$ over S to $(Y, \mathrm{aug}^\psi \circ \mathcal{O}_Y)$ defines a functor $\mathbf{GrSch}_S^L \to \mathbf{GrSch}_S^K$ which is left adjoint to the functor from (i).*

Proof. Using the canonical functors sending X to $\mathrm{Spec}_X(\mathrm{co}^\psi \circ \mathcal{O}_X) \to X$ and Y to $(Y, \mathcal{O}_Y) \to (Y, \mathrm{aug}^\psi \circ \mathcal{O}_Y)$ we apply Lemma A.0.0.4. □

Proposition IV.1.5.4. *The inclusion functor from schemes to graded schemes is left adjoint to the functor $\mathfrak{s}^0$ which sends X to $\mathrm{Spec}_{gr,X}(\mathrm{co}^0 \circ \mathcal{O}_X)$, where co^0 denotes the forgetful functor form graded rings to rings. Moreover, the following hold:*

(i) *The canonical affine morphism $\mathfrak{s}^0(X) \to X$ is surjective.*
(ii) *The inclusion of affine schemes into affine graded schemes is left adjoint to the restriction of $\mathfrak{s}^0$ to affine graded schemes. Moreover, both these functors commute with taking principal subsets.*
(iii) *If X is a colimit of a diagram $D\colon I \to \mathbf{GrSch}$ of open embeddings then $\mathfrak{s}^0(X)$ is a colimit of $\mathfrak{s}^0 \circ D$. Thus, $\mathfrak{s}^0$ preserves open embeddings.*

Proof. The main assertion follows from Lemma A.0.0.4, assertion (i) from Proposition II.1.8.15.

□

Proposition IV.1.5.5. *The above functor commutes with fibre products and maps affine graded schemes to affine schemes. Moreover, it respects localization.*

Remark IV.1.5.6. The restriction of the structure sheaf of $\mathfrak{s}^0(X)$ to the initial topology of the canonical map $\jmath_X \colon \mathfrak{s}^0(X) \to X$ is naturally $gr(X)$-graded since we have $\mathcal{O}_{\mathfrak{s}^0(X)}(\jmath_X^{-1}(U)) = \mathcal{O}_X(U)$ for every open $U \subseteq X$.

Construction IV.1.5.7. Let I be a category with finitely many objects and morphisms. Let $D\colon I \to \mathbf{GrSch}_S$ be a diagram assigning i to $\pi_i\colon X_i \to S$. Let J be the set of all pairs $(U, (V_i)_{i\in I})$ where $U \in \mathcal{B}_S$ and $V_i \in \mathcal{B}_{\pi_i^{-1}(U)}$ are such that for each I-morphism $\alpha\colon i \to i'$ we have $D(\alpha)(V_i) \subseteq V_{i'}$. Then D restricts to a $\mathbf{GrSch}_U$-diagram D_j for each $j = (U, (V_i)_i) \in J$. The limit $\pi_j\colon Y_j \to U$ of D_j together with the morphisms $\phi_{j,i}\colon Y_j \to V_i$ is defined by taking the image of the limit of $\mathcal{O} \circ D_j$ under Spec and employing the isomorphisms $\mathrm{Spec}(\mathcal{O}(V_i)) \cong V_i$ and $\mathrm{Spec}(\mathcal{O}(U)) \cong U$. Note that this $\mathbf{GrSch}_U$-limit is also a $\mathbf{GrSch}_S$-limit.

Due to compatibility of colimits of graded A-algebras with localization, see Remark II.1.5.3, we have canonical open embeddings $Y_j \to Y_{j'}$ of schemes over S for all $j \le j' \in J$. Let Y be the $\mathbf{GrSch}$-colimit of the diagram $J \to \mathbf{GrSch}, j \mapsto Y_j$. Then each morphism π_j induces a morphism $\pi\colon Y \to S$ and each morphism $\phi_{j,i}$ induces a morphism $\phi_i\colon Y \to X_i$ of graded schemes over S. Then π and $\{\phi_i\}_i$ together form the limit of D.

Remark IV.1.5.8. Let $\phi\colon X \to Y$ be a morphism of graded schemes over $\mathbb{F}_1$ resp. $\mathbb{Z}$. Then due to Example A.0.0.3, the associated *base change* functor

$\mathbf{GrSch}_Y \to \mathbf{GrSch}_X, [Z \to Y] \mapsto [Z \times_\phi X \to X]$ is right adjoint to the functor sending a morphism to X to its composition with ϕ.

Construction IV.1.5.9. Fix a graded scheme X over $\mathbb{Z}$ and let Z be a graded scheme over $\mathbb{F}_1$. For $U \in \mathcal{B}_Z$ and $f \in \mathcal{O}(U)$ we have an open embedding $\mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(U_f)]) \to \mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(U)])$ of graded schemes over X. Thus, all $V \in \mathcal{B}_U$ yield open embeddings $\mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(V)]) \to \mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(U)])$ of graded schemes over X. Taking the colimit over $\mathcal{B}_Z$ we obtain a graded scheme $X[Z]$ over X, which we call the *free graded scheme over X in Z* and also denote by $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_X[\mathcal{O}_Z])$.

$X[Z]$ is covered by all $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(W)[\mathcal{O}(U)])$ where $W \in \mathcal{B}_X$ and $U \in \mathcal{B}_Z$. For such W and U the canonical map $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(W)[\mathcal{O}(U)]) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(U))$ which sends a homogeneously prime ideal to its intersection with $\mathcal{O}(U)$ is continuous. Note that in a generalized setting of *graded sesquiad schemes*, i.e. spaces with structure sheaves which are locally sets of homogeneously prime ideals of graded sesquiads, this map should be a morphism. For each $f \in \mathcal{O}(W)^{\mathrm{hom}}$ the canonical map $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(W_f)[\mathcal{O}(U)]) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(U))$ is the composition of the aforementioned map and the open embedding $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(W_f)[\mathcal{O}(U)]) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(W)[\mathcal{O}(U)])$. Thus, we have obtain a well-defined continuous map $\mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(U)]) \to U$. For $g \in \mathcal{O}(U)$ the respective maps, together with the principal open inclusions $U_g \to U$ and $\mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(U_g)]) \to \mathrm{Spec}_{\mathrm{gr},X}(\mathcal{O}_X[\mathcal{O}(U)])$ form a commutative diagram. Therefore, these maps fit together to a continuous map $X[Z] \to Z$, which in the more general setting should turn out to be a morphism of graded sesquiad schemes.

For a morphism $Z \to Z'$ we obtain an induced morphism $X[Z] \to X[Z']$ of graded schemes over X. This turns $X[-]$ into a functor from graded schemes over $\mathbb{F}_1$ to graded schemes over X.

Example IV.1.5.10. Let M be an abelian monoid with cancellation and let k be a simply graded ring, e.g. a 0-graded field. Let $R := k[M]$ be the canonically $gr(k) \oplus Q(M)$-graded monoid algebra of M over k. Then there mutually inverse order reversing bijections

$$\begin{aligned} \mathrm{faces}(M) &\longleftrightarrow \mathrm{Spec}_{Q(M)}(R) \\ \tau &\longmapsto \langle \chi^w \mid w \in M \setminus \tau \rangle \\ \deg(R^{\mathrm{hom}} \setminus \mathfrak{p}) &\longleftarrow\!\shortmid\ \mathfrak{p} \end{aligned}$$

with both sets being ordered by inclusion. In this setting, $\mathfrak{s}_k^{\mathrm{gr}}$ is isomorphic to the functor $k[-]$ which replaces each structure sheaf $\mathcal{O}_X$ with $k[\mathcal{O}_X]$.

Remark IV.1.5.11. Let A be a graded ring and let S be a graded A-algebra. Take the forgetful functor from graded S-algebras to graded $\mathbb{F}_1$-algebras and compose with $\mathrm{Spec}_{\mathrm{gr}}$ on one side and with the global sections functor on the other. The resulting functor $\mathfrak{i}$ is left adjoint to $\mathrm{Spec}_{\mathrm{gr}}(S)[-]$ due to Proposition II.1.4.8. Moreover, for each graded S-algebra R we have a canonical injection

$$\mathrm{Spec}_{\mathrm{gr}}(R) \longrightarrow \mathrm{Spec}(R^{\mathrm{hom}}), \quad \mathfrak{p} \longmapsto \mathfrak{p} \cap R^{\mathrm{hom}}$$

which is continuous. Indeed, the preimage of $V(\mathfrak{a})$ is $V(\langle \mathfrak{a} \rangle_R)$. Consequently, $\mathfrak{i}$ preserves inclusions of principal open subsets.

IV.2. Properties of graded schemes and their morphisms

IV.2.1. Homogeneous integrality, regularity and closed graded subschemes. In this section we assemble the preliminaries for Chapter V and some further basic properties of graded schemes, including homogeneous integrality, reducedness and noetherianity, as well as closed subschemes and separatedness.

Remark IV.2.1.1. Let $\mathcal{I}$ be a quasi-coherent $\mathcal{A}$-ideal of a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$. By Remark III.3.0.6 and Proposition II.1.8.13 $\mathcal{I}$ is homogeneously radical if and only if $\mathcal{I}(U) = \sqrt{\mathcal{I}(U)}^{\mathrm{gr}}$ holds for each member U of an affine cover of X. This is because for each principal open subset U_f of such a U we have

$$\mathcal{I}(U_f) = \mathcal{I}(U)_f = \sqrt{\mathcal{I}(U)}^{\mathrm{gr}}_f = \sqrt{\mathcal{I}(U)_f}^{\mathrm{gr}} = \sqrt{\mathcal{I}(U_f)}^{\mathrm{gr}}.$$

Definition IV.2.1.2. A graded scheme X is *homogeneously integral/reduced* if $\mathcal{O}_X(U)$ is homogeneously integral resp. $\{0_{\mathcal{O}_X(U)}\}$ is homogeneously radical for each $U \in \Omega_X$.

Proposition IV.2.1.3. *Let X be an irreducible graded scheme which admits an affine cover $\mathcal{U}$ such that $\mathcal{O}(U)$ is homogeneously integral for each $U \in \mathcal{U}$. Then X is homogeneously integral.*

Proof. First note that for each non-empty $W \in \mathcal{B}_U^{\mathrm{pr}}$ where $U \in \mathcal{U}$ the sections $\mathcal{O}(W)$ are again homogeneously integral. Here, if W' is principal in W then the restriction $\mathcal{O}(W) \to \mathcal{O}(W')$ is an injective localization map. Secondly, $V \in \Omega_X$ is covered by all $W \in \mathcal{B}_U^{\mathrm{pr}} \cap \Omega_V$ where $U \in \mathcal{U}$. Thus, $\mathcal{O}(V)$ is the limit over all these $\mathcal{O}(W)$, and is consequently homogeneously integral by Proposition II.1.5.8. $\square$

Proposition IV.2.1.4. *A graded scheme X is irreducible and homogeneously reduced if and only if no $\mathcal{O}(U)$ has homogeneous zero divisors.*

Proof. If no $\mathcal{O}(U)$ has homogeneous zero divisors then X is homogeneously reduced. Suppose there exist $U, V \in \Omega_X$ with $U \cap V = \emptyset$. Then $\mathcal{O}(U \cup V) \cong \mathcal{O}(U) \times \mathcal{O}(V)$ has homogeneous zero divisors - a contradiction.

Conversely, suppose that X is irreducible and homogeneously reduced. Let $U \in \Omega_X$ and let $fg = 0$ with homogeneous non-zero $f, g \in \mathcal{O}(U)$. Then $U = V_U(f) \cup V_U(g)$ means we may assume $U = V_U(f)$. For each $W \in \mathcal{B}_U$ we then have $W = V_W(f_{|W})$, i.e. $f_{|W} \in \sqrt{\{0_{\mathcal{O}(W)}\}}^{\mathrm{gr}}$ which by homogenous reducedness means $f_{|W} = 0$. Thus, we obtain $f = 0$. $\square$

Definition IV.2.1.5. For a homogeneously integral graded scheme X the constant sheaf of *rational fractions* $\mathcal{K}$ is defined as the sheaf assigning the stalk at the generic point ξ of X to each non-empty open set of X.

Proposition IV.2.1.6. *For a homogeneously integral graded scheme X, $\mathcal{O}_X$ is a subsheaf of $\mathcal{K}$.*

Proof. Due to quasi-coherence the canonical (localization) map $\mathcal{O}(U) \to \mathcal{K}(U)$ is a monomorphism for affine open sets U and by Proposition II.1.5.8 for all open sets. $\square$

Construction IV.2.1.7. Let X be a graded scheme and let $\mathcal{I}$ be a quasi-coherent $\mathcal{O}_X$-ideal. Then the morphism $Z := \mathrm{Spec}_{gr,X}(\mathcal{O}_X/\mathcal{I}) \to X$ defines a homeomorphism $\jmath$ onto its image Y, and $(Y, \jmath_*\mathcal{O}_Z)$ is called the *closed graded subscheme* of X associated to $\mathcal{I}$. $\jmath$ is then an isomorphism of graded schemes and the inclusion $\imath\colon Y \to X$ naturally becomes a morphism of graded schemes. Specifically, we have canonical isomorphisms between $(U \cap V(\mathcal{I}), \mathcal{O}_Y(U \cap V(\mathcal{I})))$ and the graded spectrum of $\mathcal{O}(U)/\mathcal{I}(U)$ for all $U \in \mathcal{B}_X$.

Definition IV.2.1.8. A morphism $\phi\colon Y \to X$ of graded schemes is a *closed embedding* if it factors into a product $Y \xrightarrow{\sim} \phi(Y) \xrightarrow{\imath} X$ of an isomorphism and an inclusion of a closed graded subscheme into X.

EXAMPLE IV.2.1.9. Let R be a graded ring and $\mathfrak{a}$ a graded ideal of R. Then the canonical map $\operatorname{Spec}_{\mathrm{gr}}(R \to R/\mathfrak{a})$ is a closed embedding. In the general case, a morphism $\phi\colon X \to Z$ of affine graded schemes is a closed embedding if and only if $\phi^*_Z\colon \mathcal{O}(Z) \to \mathcal{O}(X)$ is a surjection with bijective accompanying map.

REMARK IV.2.1.10. A morphism $\phi\colon Z \to X$ of graded schemes is a closed embedding if and only if ϕ is affine and $\phi_*\mathcal{O}_Z$ is the image $\mathfrak{C}^{\mathrm{fix}}$-*sheaf* of ϕ^*, and in that case ϕ factors into an isomorphism onto $V_X(\ker(\phi^*))$ and an inclusion morphism.

DEFINITION IV.2.1.11. A graded scheme X over Y with structure morphism $\phi\colon X \to Y$ is of *finite type* if there exists an affine cover $Y = U_1 \cup \ldots \cup U_m$ such that each of the preimages has a finite affine cover $\phi^{-1}(U_i) = V_{i,1} \cup \ldots \cup V_{i,n_i}$ and each of the homomorphisms $\mathcal{O}_Y(U_i) \to \mathcal{O}_X(V_{i,j})$ defines a finitely generated $\mathcal{O}_Y(U_i)$-algebra.

DEFINITION IV.2.1.12. Let X be a graded scheme over Y with structure morphism $\phi\colon X \to Y$ and denote by $(X \times_Y X)^{gr(\mathcal{O}_X)}$ the product of ϕ with itself in $\mathbf{GrSch}_Y^{gr(\mathcal{O}_X)}$. Then the diagonal morphism $\Delta_\phi\colon X \to X \times_Y X$ factors in to the diagonal morphism $\Delta_\phi^{gr(\mathcal{O}_X)}\colon X \to (X \times_Y X)^{gr(\mathcal{O}_X)}$ and the canonical (affine) morphism $(X \times_Y X)^{gr(\mathcal{O}_X)} \to X \times_Y X$. ϕ is separated if $\Delta_\phi^{gr(\mathcal{O}_X)}$ is a closed embedding.

DEFINITION IV.2.1.13. A graded scheme X is *of affine intersection* if the intersection of each two affine open subsets is again affine.

DEFINITION IV.2.1.14. A graded scheme X is *locally homogeneously noetherian* if $\mathcal{O}(U)$ is homogeneously noetherian for each $U \in \mathcal{B}_X$. If X is also quasi-compact then it is called *homogeneously noetherian*.

REMARK IV.2.1.15. If X is homogeneously noetherian then Ω_X is noetherian.

REMARK IV.2.1.16. Due to Proposition II.1.3.10 a graded scheme X is locally homogeneously noetherian if it is covered by affine open U with homogeneously noetherian sections. In particular, if X is affine then it is homogeneously noetherian if and only if $\mathcal{O}(X)$ is homogeneously noetherian.

EXAMPLE IV.2.1.17. A graded scheme of finite type over a homogeneously noetherian A is homogeneously noetherian. If A is a graded ring this is due to the homogeneous version of Hilbert's basis theorem which is given in Theorem II.1.7.5. If A is a graded monoid/$\mathbb{F}_1$-algebra the claim follows from Proposition I.1.2.4.

DEFINITION IV.2.1.18. A point p of a homogeneously noetherian graded scheme X is called *regular* if $\mathcal{O}_{X,p}$ is regularly graded. X_{reg} denotes the set of regular points of X. If $X = X_{\mathrm{reg}}$ then X is called *regular*.

The theory may now be developped in the same way as for schemes. For graded schemes over $\mathbb{F}_1$ we have the following:

REMARK IV.2.1.19. Due to Proposition I.2.7.6 a point p of a homogeneously noetherian graded scheme X over $\mathbb{F}_1$ is regular if and only if its stalk is factorial. Moreover, if $p \in X$ is regular and $p \in \overline{\{p'\}}$ then p' is also regular. Thus, the set X_{reg} of regular points of X is open.

IV.2.2. Veronesean good quotients. To distinguish the good quotients defined below from quotients by actions we term them *Veronesean*, since they are defined in terms of Veronese subalgebras.

DEFINITION IV.2.2.1. A morphisms $q\colon \widehat{X} \to X$ is a *(Veronesean) good quotient* if q is affine, $\psi\colon gr(\mathcal{O}_X) \to gr(\mathcal{O}_{\widehat{X}})$ is injective and $q^*\colon \mathcal{O}_X \to (q_*\mathcal{O}_{\widehat{X}})_{\psi(gr(\mathcal{O}_X))}$ is an isomorphism. X is then called the *quotient space* of q. If $gr(\mathcal{O}_X) = 0$, i.e. X is a scheme, then q is said to be a good quotient *by* $gr(\mathcal{O}_X)$ or by the $gr(\mathcal{O}_X)$-grading.

EXAMPLE IV.2.2.2. For a Veronese subalgebra $\imath\colon R_G \to R$ the morphism $\mathrm{Spec}_{\mathrm{gr}}(\imath)$ is a good quotient.

PROPOSITION IV.2.2.3. *For a good quotient $q\colon \widehat{X} \to X$ the following hold:*

(i) *For $x \in X$ the preimage $q^{-1}(x)$ contains a unique point $\widehat{x}$, called the* special point over x, *which is contained in all closed set B with $x \in q(B)$, in particular in all closures of points in $q^{-1}(x)$. Moreover, we have $\mathcal{O}_{X,x} = (\mathcal{O}_{\widehat{X},\widehat{x}})_{gr(\mathcal{O}_X)}$. Furthermore, x is closed in $U \in \mathcal{B}_{X,x}$ if and only if $\widehat{x}$ is closed in $q^{-1}(U)$, and $I_{q^{-1}(U)}(\widehat{x})$ is the special ideal over $I_U(x)$ from Proposition II.1.8.14.*
(ii) *q is surjective.*
(iii) *q is closed.*
(iv) *$q(\bigcap_i \widehat{X}_i) = \bigcap_i q(\widehat{X}_i)$ holds for all closed $\widehat{X}_i \subseteq \widehat{X}$.*

PROOF. In (i) consider x and $U \in \mathcal{B}_{X,x}$. The special point over x is then defined as the point $\widehat{x}$ of $q^{-1}(U)$ corresponding to the special ideal over $I_U(x)$ from Proposition II.1.8.14. Since the formation of special ideals over homogeneously prime ideals commutes with localization $\widehat{x}$ is independent of the choice of U. The desired properties of $\widehat{x}$ may be checked in the affine neighbourhoods $q^{-1}(U)$ and thus follow from Proposition II.1.8.14. The existence of special points in particular implies surjectivity of q.

For (iii) consider $U \in \mathcal{B}_X$ and $x \in U$. For a closed set $Z \subseteq \widehat{X}$ with $x \in \overline{q(Z)}$ we have $I_{q^{-1}(U)}(Z \cap q^{-1}(U)) \cap \mathcal{O}_X(U) \subseteq I_U(x)$ and hence $I_{q^{-1}(U)}(Z \cap q^{-1}(U))$ contains the special ideal over $I_U(x)$ which means the special point $\widehat{x}$ over x lies in Z. In (iv) let $U \in \mathcal{B}_X$. Then Proposition II.1.2.17 gives

$$\mathcal{O}_X(U) \cap \sum_i I_{q^{-1}(U)}(\widehat{X}_i \cap q^{-1}(U)) = \sum_i \mathcal{O}_X(U) \cap I_{q^{-1}(U)}(\widehat{X}_i \cap q^{-1}(U))$$

which means $q(\bigcap_i \widehat{X}_i) \cap U = \bigcap_i q(\widehat{X}_i) \cap U$. □

PROPOSITION IV.2.2.4. *For a good quotient $q\colon \widehat{X} \to X$ the following hold:*

(i) *For each $U \in \Omega_X$ the restriction $q_{|q^{-1}(U)}\colon q^{-1}(U) \to U$ is a good quotient.*
(ii) *For a quasi-coherent $\mathcal{O}_{\widehat{X}}$-preideal $\mathcal{I}$ denote $\mathrm{Spec}_{gr,X}(\mathcal{O}_X/(q^*)^{-1}(q_*\mathcal{I}))$ and $\mathrm{Spec}_{gr,\widehat{X}}(\mathcal{O}_{\widehat{X}}/\mathcal{I})$ by Z resp. $\widehat{Z}$. Then q induces a good quotient $q_Z\colon \widehat{Z} \to Z$ which together with the closed embeddings $\widehat{Z} \to \widehat{X}$ and $Z \to X$ forms a commutative diagram.*

PROOF. For (ii) first note that the composition of canonical homomorphisms $\mathcal{O}_X \to q_*\mathcal{O}_{\widehat{X}} \to q_*\mathcal{O}_{\widehat{X}}/q_*\mathcal{I}$ is equal to the composition of the canonical homomorphisms $\mathcal{O}_X \to \mathcal{O}_X/(q^*)^{-1}(q_*\mathcal{I}) \to q_*\mathcal{O}_{\widehat{X}}/q_*\mathcal{I}$, with the last factor being Veronesean. Applying $\mathrm{Spec}_{gr,X}$ gives the desired diagram because we have canonical isomorphisms $\widehat{Z} \to \mathrm{Spec}_{gr,X}(q_*\mathcal{O}_{\widehat{X}}/q_*\mathcal{I})$, $\widehat{X} \to \mathrm{Spec}_{gr,X}(q_*\mathcal{O}_{\widehat{X}})$ and $X \to \mathrm{Spec}_{gr,X}(\mathcal{O}_X)$. □

DEFINITION IV.2.2.5. A bijective good quotient is called *geometric*.

EXAMPLE IV.2.2.6. Let R be a $\mathbb{Z}$-graded ring with a unit $f \in R^{\mathrm{hom}}$ of non-zero degree. Then $\mathrm{Spec}_{\mathrm{gr}}(R_0 \subseteq R)$ is a geometric good quotient. For injectivity, let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}_{\mathrm{gr}}(R)$ with $\mathfrak{p} \cap R_0 = \mathfrak{q} \cap R_0$. Then for $a \in \mathfrak{p}^{\mathrm{hom}} \setminus 0$ there exist $m, n \in \mathbb{Z}$ with $m \deg_{\mathbb{Z}}(a) = n \deg_{\mathbb{Z}}(f)$. Then $a^m f^{-n} \in \mathfrak{p} \cap R_0 = \mathfrak{q} \cap R_0$ and hence $a^m \in \mathfrak{q}$, i.e. $a \in \mathfrak{q}$.

REMARK IV.2.2.7. If $q\colon \widehat{X} \to X$ is geometric then q is a homeomorphism and each $\widehat{x} \in \widehat{X}$ is special over $q(\widehat{x})$. Indeed, the single point in $q^{-1}(q(\widehat{x}))$ is by definition special over $q(\widehat{x})$.

Definition IV.2.2.8. Let $q\colon \widehat{X} \to X$ be a Veronesean quotient and let $A \subseteq X$ be a closed set. The *special set over* A is the closure of the set of all $\widehat{x} \in \widehat{X}$ such that $q(\widehat{x}) \in A$ and $\widehat{x}$ is the special point over x.

Proposition IV.2.2.9. *For a good quotient $q\colon \widehat{X} \to X$ and the closed subset $Z \subseteq X$ the following hold:*

(i) *$\widehat{Z}$ is the unique closed subset of $\widehat{X}$ which satisfies $q(\widehat{Z}) = Z$ and is contained in all closed sets whose image under q contains Z. Moreover, $\widehat{Z}$ is the intersection over all closed sets whose image under q contains Z.*
(ii) *for $U \in \mathcal{B}_X$, $\mathcal{I}_{\widehat{Z}}(q^{-1}(U))$ is the special ideal over $\mathcal{I}_Z(U)$.*
(iii) *$\widehat{Z}$ is irreducible if and only if Z is so. In particular, the special set over $\overline{\{x\}}$ is the closure $\overline{\{\widehat{x}\}}$ of the special point $\widehat{x}$ over x.*
(iv) *If $\widehat{X}$ is homogeneously integral/reduced/quasi-compact then so is X.*

Proof. In (i) first note that the set $\widetilde{Z}$ of all special points over points of Z has image Z, and continuity gives $\widehat{Z} = \overline{\widetilde{Z}} \subseteq q^{-1}(Z)$. Next, we show that a closed subset $\widehat{Y} \subseteq \widehat{X}$ contains $\widetilde{Z}$ (equivalently, $\widehat{Z}$) if and only if $Z \subseteq q(\widehat{Y})$. If $\widehat{Y}$ is closed with $Z \subseteq q(\widehat{Y})$ then the special point $\widehat{x}$ over a $x \in Z$ is contained in the closure of some point of $\widehat{Y}$ with image x and hence $\widehat{x} \in \widehat{Y}$. Conversely, if $\widehat{Z} \subseteq \widehat{Y}$ then $Z \subseteq q(\widehat{Y})$. For uniqueness, consider a minimal closed set $\widehat{Y}'$ with $Z \subseteq q(\widehat{Y}')$. Then $\widehat{Z} \subseteq \widehat{Y}'$ implies $\widehat{Z} = \widehat{Y}'$. In (ii) we calculate

$$\mathcal{I}_{\widehat{Z}}(q^{-1}(U)) = \mathcal{I}_{\widetilde{Z}}(q^{-1}(U)) = \bigcap_{x \in Z} \mathcal{I}_{\widehat{x}}(q^{-1}(U)) = \bigcap_{x \in Z} \widehat{\mathcal{I}_x(U)} = \widehat{\mathcal{I}_Z(U)}.$$

In (iii) suppose that $Z = \overline{\{x\}}$ is irreducible. Then $\overline{\{\widehat{x}\}}$ is the minimal closed set with image Z which by (i) means $\overline{\{\widehat{x}\}} = \widehat{Z}$. In assertion (iv) the statement concerning homogeneous reducedness follows from Remark IV.2.1.1 applied to the cover $\bigcup_{U \in \mathcal{B}_X} U$. The statement on homogeneous integrality follows from the fact Veronese subalgebras inherit homogeneous integrality from their containing algebras. The statement on quasi-compactness follows from surjectivity of q. ☐

IV.2.3. The canonical quasi-torus action on a graded scheme. We briefly introduce the concept of graded group schemes and show that each graded scheme X over A comes with a natural action by the quasi-torus $\mathrm{Spec}_{\mathrm{gr}}(A[gr(X)])$. This constitutes a functor from graded schemes to actions of graded group schemes. Moreover, we relate quasi-tori over $\mathbb{F}_1$ and $\mathbb{K}$ to one another.

In the above definition, group object and action refer to the concepts defined in terms of $\mathbf{GrSch}_S$-products and commutative diagrams of $\mathbf{GrSch}_S$-morphisms.

Below, we continue to make the assumption that all occuring algebras are commutative.

Definition IV.2.3.1. A *graded bialgebra* over A consists of a graded A-algebra $u\colon A \to R, gr(u)\colon gr(A) \to gr(R)$ together with an A-algebra homomorphism $\mu\colon R \to R \otimes_A R, gr(\mu)\colon gr(R) \to gr(R) \otimes_{gr(A)} gr(R)$ called the *comultiplicaltion*, which satisfies the coassociativity condition

$$(\mu \otimes \mathrm{id}_R) \circ \mu = (\mathrm{id}_R \otimes \mu) \circ \mu,$$
$$(gr(\mu) \otimes \mathrm{id}_{gr(R)}) \circ gr(\mu) = (\mathrm{id}_{gr(R)} \otimes gr(\mu)) \circ gr(\mu),$$

and an A-algebra homomorphism $cu\colon R \to A, gr(cu)\colon gr(R) \to gr(A)$, called the *counit*, and satisfy the equations

$$((u \circ cu) \cdot \mathrm{id}_R) \circ \mu = \mathrm{id}_R = (\mathrm{id}_R \cdot (u \circ cu)) \circ \mu,$$
$$((gr(u) \circ gr(cu)) + \mathrm{id}_{gr(R)}) \circ gr(\mu) = \mathrm{id}_{gr(R)} = (\mathrm{id}_{gr(R)} + (gr(u) \circ gr(cu))) \circ gr(\mu).$$

A *morphism* of graded bialgebras over A is a morphism of graded algebras that is also compatible with the respective comultiplications and counits.

Construction IV.2.3.2. For a graded bialgebra R over A the set $M(R)$ of *homogeneous monoid-like elements* is the set of those $f \in R^{\mathrm{hom}}$ whose image under comultiplicaltion and under the counit is $f \otimes f$ resp. 1_A. $M(R)$ is a submonoid of R^{hom} which is graded by the kernel $\ker(gr(cu))$ of the map accompanying the counit.

For a morphism $\phi\colon R \to R', \psi\colon gr(R) \to gr(R')$ of graded bialgebras over A the restrictions $M(\phi)\colon M(R) \to M(R'), \ker(gr(cu)) \to \ker(gr(cu'))$ form a morphism of graded monoids.

Remark IV.2.3.3. Each graded monoid R is canonically a graded bialgebra over $\{1\}$, with the diagonal map $\Delta\colon R \to R \times R = R \otimes_{\{1\}} R, r \mapsto r \otimes r$ together with $gr(\Delta)\colon gr(R) \to gr(R) \times gr(R), w \mapsto (w, w)$ serving as comultiplication. Moreover, each homomorphism of graded monoids is automatically a morphism of graded bialgebras over $\{1\}$. By definition, we have $M(R) = R$. Conversely, consider a bialgebra R over $\{1\}$ with comultiplication $cm\colon R \to R \otimes_{\{1\}} R$. The counit cu is the unique map $R \to \{1\}$. By the counit axiom, each coordinate of $cm(f)$ equals f, i.e. we have $cm = \Delta$. Thus, endowing a graded monoid with this canonical bialgebra structure is inverse to the forgetful functor from graded bialgebras over $\{1\}$ to graded monoids.

Remark IV.2.3.4. Due to Lemma A.0.0.4 the functor $M(-)$ from Construction IV.2.3.2 is right adjoint to the functor $A[-]$ which sends a graded monoid N to $A[N]$. The adjunction is defined using the canonical morphisms $A[M(R)] \to R$ and $N \cong M(A[N])$.

Lemma IV.2.3.5. *If A^{hom} is simple then the set of homogeneous monoid-like elements of a graded bialgebra R over A is A-linearly independent.*

Proof. Assume that $n \in \mathbb{N}_0$ is minimal with the property that there exists a linear combination $f = \sum_{i=1}^{n} a_i f_i$ with $a_i, \in A^{\mathrm{hom}}$ and distinct $f_i, f \in M(R)$ such that $\deg(a_i f_i) = \deg(f)$. Minimality of n gives $a_i \neq 0$ and linear independence of $f_1, \ldots, f_n$. By Corollary II.1.6.13 $(f_i \otimes f_j)$ are linearly independent and hence

$$\sum_{i,j} a_i a_j (f_i \otimes f_j) = f \otimes f = \sum_i a_i (f_i \otimes f_i)$$

implies $a_i = 1$, which gives a contradiction if $n = 1$, and $a_i a_j = 0$ for $i \neq j$, which gives a contradiction if $n > 1$. □

Definition IV.2.3.6. A *graded Hopf algebra R over A* is a graded A-bialgebra together with an A-algebra endomorphism α on R, called the *antipode*. A *morphism* of graded Hopf algebras over A is a homomorphism of the underlying graded A-bialgebras.

Remark IV.2.3.7. Morphisms of graded Hopf algebras commute with antipodes.

Remark IV.2.3.8. The functor sending simple graded monoid G to its to its associated graded $\{1\}$-bialgebra together with the antipode $g \mapsto g^{-1}$ is inverse to the forgetful functor from graded Hopf algebras over $\{1\}$ to simple graded monoids.

Remark IV.2.3.9. The adjunction from Remark IV.2.3.4 restricts to an adjunction of the functor sending a simple graded monoid G to the Hopf algebra $A[G]$, where the antipode is $A[g \mapsto g^{-1}]$, and the functor sending a Hopf algebra R to its homogeneous monoid-like elements $M[R]$, which are then called its homogeneous *group-like* elements. In the latter case, note that the antipode sends a monoid-like element to its multiplicative inverse.

DEFINITION IV.2.3.10. Let Z be a graded scheme.

(i) The group objects of $\mathbf{GrSch}_Z$ are called *graded group schemes over* Z.
(ii) An *action* of a graded group scheme over Z on a graded scheme over Z is an action of a group object in $\mathbf{GrSch}_Z$ on an object of $\mathbf{GrSch}_Z$.

PROPOSITION IV.2.3.11. *The anti-equivalence of affine graded schemes over A and graded algebras over A induces an anti-equivalence of affine graded group schemes over A and graded Hopf algebras over A.*

CONSTRUCTION IV.2.3.12. Let H be a graded group scheme over Z with structure morphism ϕ. Let $\mathcal{M}(\mathcal{O}_H)(U)$ be the set of those $f \in \phi_*\mathcal{O}_H(U)^{\mathrm{hom}}$ such that $e^*_{\phi^{-1}(U)}(f) = 1_{\mathcal{O}_Z(U)}$ and we have $m^*_{\phi^{-1}(U)}(f) = (pr_1)^*_{\phi^{-1}(U)}(f)(pr_2)^*_{\phi^{-1}(U)}(f)$ with respect to the projection morphisms $pr_1, pr_2 \colon H \times_Z H \to H$. This defines a sheaf $\mathcal{M}(\mathcal{O}_H)$ of constantly $\ker(gr(e^*_Z))$-graded monoids, called the sheaf of *homogeneous group-like elements* of H. Moreover, if $f \in \mathcal{M}(\mathcal{O}_H)(U)$ is invertible in $\phi_*\mathcal{O}_H(U)$ then it is invertible in $\mathcal{M}(\mathcal{O}_H)(U)$.

For a morphism $\theta \colon H \to H'$ of graded group schemes over Z with structure morphisms ϕ and ϕ' the homomorphism $\phi'_*\mathcal{O}_{H'} \to \phi_*\mathcal{O}_H$ restricts to a homomorphism $\mathcal{M}(\mathcal{O}_{H'}) \to \mathcal{M}(\mathcal{O}_H)$ of constantly graded sheaves of graded monoids. This turns $\mathcal{M}$ into a functor from graded group schemes over Z to sheaves of constantly graded monoids on Z.

PROOF. Let $U = \bigcup_i U_i$ be an open cover and consider $f_i \in \mathcal{M}(\mathcal{O}_H)(U_i)_w$ with $f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}$ for all i,j. Let $f \in \mathcal{O}_H(\phi^{-1}(U))_w$ be the unique element with $f_{|U_i} = f_i$. Then $m^*_{\phi^{-1}(U)}(f)$ and $(pr_1)^*_{\phi^{-1}(U)}(f)(pr_2)^*_{\phi^{-1}(U)}(f)$ both restrict to $m^*_{\phi^{-1}(U_i)}(f_i)$ on each $\phi^{-1}(U_i)$ and therefore coincide.

Concerning inverse elements note that if there exists $g \in \mathcal{O}_H(\phi^{-1}(U))$ with $fg = 1$ then $m^*_{\phi^{-1}(U)}(g)$ equals $(pr_1)^*_{\phi^{-1}(U)}(g)(pr_2)^*_{\phi^{-1}(U)}(g)$ because the latter is inverse to $m^*_{\phi^{-1}(U)}(f) = (pr_1)^*_{\phi^{-1}(U)}(f)(pr_2)^*_{\phi^{-1}(U)}(f)$. $\square$

REMARK IV.2.3.13. For a graded group scheme H over Z with affine structure morphism ϕ the multiplication m, unit e and inverse i are also affine, due to Proposition IV.1.3.16. Moreover, for each $U \in \mathcal{B}_Z$ the set of group-like elements of the graded Hopf algebra $\mathcal{O}_H(\phi^{-1}(U))$ over $\mathcal{O}(U)$ is then $\mathcal{M}(\mathcal{O}_H)(U)$.

DEFINITION IV.2.3.14. A graded group scheme H over Z is a *graded quasi-torus over* Z if $\mathcal{M}(\mathcal{O}_H)(Z)$ has a bijective degree map and the canonical morphism $H \to \mathrm{Spec}_{\mathrm{gr},Z}(\mathcal{O}_Z[\mathcal{M}(\mathcal{O}_H)(Z)])$ is an isomorphism. A *graded torus over* Z additionally satisfies that $\mathcal{M}(\mathcal{O}_H(Z))$ is free. A *morphism* of graded quasi-tori is a morphism of graded group schemes over Z.

PROPOSITION IV.2.3.15. *Consider the faithful contravariant functor $\mathfrak{i}$ sending an abelian group L to $\mathrm{Spec}_{\mathrm{gr},Z}(\mathcal{O}_Z[L])$ and the contravariant functor $\mathfrak{f}$ sending a graded quasi-torus H over Z to the underlying group of $\mathcal{M}(\mathcal{O}_H)(Z)$. These are mutually essentially inverse. Moreover, if Z is 0-graded then $\mathfrak{f}$ is isomorphic to the contravariant functor sending H to $gr(\mathcal{O}_H)$.*

PROOF. Consider a graded quasi-torus H over Z with structure morphism ϕ. Then the canonical maps $\mathfrak{f}(H) \to \mathcal{O}(H) \to \mathcal{O}(\phi^{-1}(U))$ induce an isomorphism $\mathcal{O}_Z[\mathfrak{f}(H)] \to \phi_*\mathcal{O}_H$ which gives rise to the defining isomorphism $\eta_H \colon H \to \mathfrak{i}(\mathfrak{f}(H))$. For a morphism $\theta \colon H \to H'$ the induced morphism $\eta_\theta \colon \mathfrak{i}(\mathfrak{f}(H)) \to \mathfrak{i}(\mathfrak{f}(H'))$ together with η_H and $\eta_{H'}$ form a commutative diagram. This constitutes an isomorphism η from the identity functor to $\mathfrak{i} \circ \mathfrak{f}$.

Let L be an abelian group. Then for each $U \in \mathcal{B}_Z$ the canonical homomorphism $L \to M(\mathcal{O}(U)[L]) = \mathcal{M}(\mathcal{O}_{\mathfrak{i}(L)})(U)$ is bijective, as are all restriction maps of $\mathcal{M}(\mathcal{O}_{\mathfrak{i}(L)})$ to principal open subsets and hence so is the induced homomorphism

$\tau_L\colon L \to \mathfrak{f}(\mathfrak{i}(L))$. Again, this is compatible with morphisms, and we obtain an isomorphism τ from the identity to $\mathfrak{f} \circ \mathfrak{i}$. □

PROPOSITION IV.2.3.16. *Over a (0-graded) base field $\mathbb{K}$ the following hold:*

(i) *Let K be an abelian group. Then an element of $\mathbb{K}[K]$ is K-homogeneous if and only if its image under comultiplication is a pure tensor. Thus, a morphism $\mathbb{K}[K] \to \mathbb{K}[K']$ of Hopf algebras over $\mathbb{K}$ maps $\{\chi^w\}_{w\in K}$ to $\{\chi^{w'}\}_{w'\in K'}$ and hence respects the canonical grading structures.*

(ii) *Let R be a $\mathbb{K}$-algebra, let K be an abelian group and let $\zeta\colon R \to \mathbb{K}[K] \otimes_{\mathbb{K}} R$ be a coaction. Then denoting by R_w the set of $f \in R$ with $\zeta(f) = \chi^w \otimes f$ for $w \in K$ we obtain a K-grading of R.*

(iii) *The functor $\mathfrak{f}$ sending a graded quasi-torus Q over $\mathbb{F}_1$ to $(Q, \mathbb{K}[\mathcal{O}_Q])$ is essentially inverse to the functor $\mathfrak{g}$ sending a graded quasi-torus H over $\mathbb{K}$ to $(H, (\mathcal{O}_H^{\mathrm{hom}})/\mathbb{K}^*)$. Moreover, $\mathfrak{g}$ is isomorphic to the functor $\mathfrak{k}$ sending H to $(H, \mathbb{F}_1[\mathrm{degsupp}(\mathcal{O}_H)])$, equipped with the canonical $gr(\mathcal{O}_H)$-grading.*

PROOF. In (i) note that the comultiplication of $\sum_{i\in I} a_i\chi^{w_i}$ is $\sum_i a_i(\chi^{w_i} \otimes \chi^{w_i})$, which can only be a pure tensor if $|I| \leq 1$ since $\{\chi^w \otimes \chi^v\}_{w,v}$ is a $\mathbb{K}$-basis of $\mathbb{K}[K] \otimes_{\mathbb{K}} \mathbb{K}[K]$. Consequently, $\{\chi^w\}_{w\in K}$ is the set of group-like elements of $\mathbb{K}[K]$. The supplement follows from the fact that morphisms of Hopf algebras send group-like elements to group-like elements.

In (ii) consider a graded quasi-torus Q over $\mathbb{F}_1$. Since $\mathcal{O}(Q)$ is simple Q is a singleton. Thus, due to Example IV.1.5.10 $(Q, \mathbb{K}[\mathcal{O}_Q])$ is an affine graded scheme over $\mathbb{K}$. Due to Remark IV.1.5.11 $\mathfrak{f}$ preserves products and thus also the group object structure. Conversely, for a graded quasi-torus H over $\mathbb{K}$, the global sections are simply graded and hence H is a singleton. By Remark II.1.4.16 we have a canonical isomorphism $\eta_H\colon \mathbb{K}[\mathcal{O}(H)^{\mathrm{hom}}/\mathbb{K}^*] \to \mathcal{O}(H)$ and thus Example IV.1.5.10 implies that $(H, \mathcal{O}_H^{\mathrm{hom}}/\mathbb{K}^*)$ is an affine graded scheme over $\mathbb{F}_1$. Since we have a canonical isomorphism

$$(\mathcal{O}(H) \otimes_{\mathbb{K}} \mathcal{O}(H))^{\mathrm{hom}}/\mathbb{K}^* \cong (\mathcal{O}(H)^{\mathrm{hom}}/\mathbb{K}^*) \otimes_{\mathbb{F}_1} (\mathcal{O}(H)^{\mathrm{hom}}/\mathbb{K}^*),$$

$(Q, \mathcal{O}_Q) := (H, \mathcal{O}_H^{\mathrm{hom}}/\mathbb{K}^*)$ is a graded group scheme. Moreover, we have a canonical isomorphism

$$M(\mathcal{O}(H)^{\mathrm{hom}}/\mathbb{K}^*) \cong M(\mathcal{O}(H)) \cong (\mathcal{O}(H)^{\mathrm{hom}}/\mathbb{K}^*) \setminus \{0\}$$

which means Q is a graded quasi-torus. η_H induces an isomorphism $\mathfrak{f}(\mathfrak{g}(H)) \to H$. These define an isomorphism between $\mathfrak{f}\circ\mathfrak{g}$ and the identity functor. For an arbitrary graded quasi-torus Q over $\mathbb{F}_1$ the isomorphism $\mathcal{O}(Q) \to \mathbb{K}[\mathcal{O}(Q)]^{\mathrm{hom}}/\mathbb{K}^*$ induces an isomorphism $Q \to \mathfrak{g}(\mathfrak{f}(Q))$. These define an isomorphism between $\mathfrak{g} \circ \mathfrak{f}$ and the identity functor.

Concerning the supplement, note that for each graded quasi-torus H over $\mathbb{K}$ the degree map induces an isomorphism $\mathcal{O}_H^{\mathrm{hom}}/\mathbb{K}^* \to \mathbb{K}[\mathrm{degsupp}(\mathcal{O}_H)]$ and hence an isomorphism $\mathfrak{k}(H) \to \mathfrak{g}(H)$. These constitute an isomorphism from $\mathfrak{k}$ to $\mathfrak{g}$. □

CONSTRUCTION IV.2.3.17. Let R be a graded algebra over A. Then we obtain a *coaction*

$$R \longrightarrow A[gr(R)] \otimes_A R, \quad R_w \ni f \longmapsto \chi^w \otimes f$$

of the Hopf A-algebra $A[gr(R)]$ on R. This defines a functor $\mathfrak{ca}$ from graded algebras over A to coactions of Hopf A-algebras. $\mathfrak{ca}$ is compatible with graded localizations.

CONSTRUCTION IV.2.3.18. Let Z be a graded scheme and let $\phi\colon X \to Z$ be a graded scheme over Z. Then $G := \mathrm{Spec}_Z(\mathcal{O}_Z[gr(\mathcal{O}_X)])$ acts on X as follows. Then the actions

$$\mathrm{Spec}_U([\mathcal{O}(V) \longrightarrow \mathcal{O}(U)[gr(\mathcal{O}_X)] \otimes_{\mathcal{O}(U)} \mathcal{O}(V), \quad \mathcal{O}(V)_w \ni f \longmapsto \chi^w \otimes f])$$

for $U \in \mathcal{B}_Z$ and $V \in \mathcal{B}_{\phi^{-1}(U)}$ fit together to the *canonical action* $\lambda_X : G \times_Z X \to X$ on X. Sending a graded scheme X over Z to its canonical action λ_X constitutes a functor from $\mathbf{GrSch}_Z$ to actions of group objects on graded schemes.

Proof. Let A be a graded ring and let $\phi\colon R \to Z, \psi\colon gr(R) \to gr(S)$ be a morphism of graded rings over A. Then the canonical maps form a commutative diagram of graded algebras over A:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ A[gr(R)] \otimes_A R \arrow[r, dashed] & A[gr(S)] \otimes_A S \end{tikzcd}$$

□

Construction IV.2.3.19. Let $\mathbb{K}$ be a field. For an abelian group K and a coaction $\zeta\colon R \to \mathbb{K}[K] \otimes_{\mathbb{K}} R$ we obtain a K-grading on R by defining R_w as the set of all $f \in R$ with $\zeta(f) = \chi^w \otimes f$.

Proof. First observe that since $M(\mathbb{K}[K])$ form an $\mathbb{K}$-basis of $\mathbb{K}[K]$ the canonical map $\mathbb{K} \otimes_{\mathbb{K}} R \to \mathbb{K}[K] \otimes_{\mathbb{K}} R$ is a monomorphism. Consequently, if $\zeta(f) = \chi^w \otimes g$ then coaction axioms on counits give $1 \otimes f = 1 \otimes g$ and hence $f = g$. Another consequence is that $R_w = \zeta^{-1}(\mathbb{K}\chi^w \otimes R)$.

Now, consider $f \in R$. Then $\zeta(f) = \sum_{i=1}^n \chi^{w_i} \otimes g_i$ holds with pairwise different w_i and certain $g_i \in R$. Likewise, we have $\zeta(g_i) = \sum_{j=1}^{n_i} \chi^{w_{i,j}} \otimes g_{i,j}$ with pairwise different $w_{i,j}$ and certain $g_{i,j} \in R$. Applying the axioms of coactions we obtain

$$\sum_{i=1}^{n} \chi^{w_i} \otimes \chi^{w_i} \otimes g_i = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \chi^{w_i} \otimes \chi^{w_{i,j}} \otimes g_{i,j}$$

and linear indendence gives $\zeta(g_i) = \chi^{w_i} \otimes g_i$. Applying our first observation to g_1 and $f - \sum_{i=2}^n g_i$ we obtain $f = \sum_{i=1}^n g_i$. Lastly, consider pairwise different $w_1, \dots, w_m \in K$ and $f_j \in R_{w_j}$ with $\sum_{j=1}^m f_j = 0$. Then by Corollary II.1.6.14 $\sum_{j=1}^m \chi^{w_j} \otimes f_j = 0$ gives $f_j = 0$ for all j. □

Proposition IV.2.3.20. *The above construction defines a functor from coactions of group algebras over $\mathbb{K}$ to graded algebras over $\mathbb{K}$ which is inverse to the functor from Construction IV.2.3.17.*

Proof. Consider coactions $\zeta\colon R \to \mathbb{K}[K] \otimes_{\mathbb{K}} R$ and $\zeta'\colon R' \to \mathbb{K}[K'] \otimes_{\mathbb{K}} R'$ and a pair of morphisms $\theta\colon \mathbb{K}[K] \to \mathbb{K}[K']$ and $\phi\colon R \to R'$ forming a morphism of coactions. By Proposition IV.2.3.16 we have $\theta = \mathbb{K}[\psi]$ with a unique group homomorphism $\psi\colon K \to K'$. For $f \in R_w$ we then have

$$\zeta'(\phi(f)) = (\theta \otimes \phi)(\zeta(f)) = (\mathbb{K}[\psi] \otimes \phi)(\chi^w \otimes f) = \chi^{\psi(w)} \otimes \phi(f),$$

i.e. $\phi(f) \in R'_{\psi(w)}$ as required. □

Example IV.2.3.21. Let X be a graded scheme over a zero graded ring/$\mathbb{F}_1$-algebra A and let K be a group. Then the canonical (projection) map $\operatorname{Spec}_{\mathrm{gr}}(A[K]) \times_A X \to X$ is a Veronesean good quotient.

IV.3. Graded schemes via schematic cofunctors

Throughout, we will consider a partially ordered set I as a category where a morphism $i \to j$ is a pair (i, j) such that $i \leq j$. For a graded scheme $(X, \mathcal{O}_X)$ over A the restriction $\mathcal{O}_{X|\mathcal{B}_X \setminus \{\emptyset\}}$ to the set of non-empty affine open sets, ordered by inclusion, is a cofunctor (i.e. a contravariant functor) of graded A-algebras with the property that the colimit of $\mathrm{Spec}_{\mathrm{gr}} \circ \mathcal{O}_{X|\mathcal{B}_X \setminus \{\emptyset\}}$ in the category of locally A-algebraed spaces is $(X, \mathcal{O}_X)$.

We show that assigning this cofunctor to a given graded scheme over A constitutes an anti-equivalence of graded schemes with the category $\mathfrak{J}$ of schematic cofunctors of graded A-algebras, which are cofunctors subject to certain localization requirements. This may be considered an extension of the anti-equivalence of graded A-algebras and affine graded schemes over A. In the next two sections we treat $\mathfrak{J}$-objects and -morphisms, respectively, before turning to the description of properties of (morphisms of) graded schemes in terms of schematic cofunctors. The case of $\mathbb{F}_1$-schemes of finite type allows further descriptions because points are then in canonical bijection with affine open subsets, see Section IV.3.4. The present topic will be concluded in Section V.3.4 of the next chapter with the treatment of combinatorial schematic functors which may be used for the description of Krull schemes of finite type over $\mathbb{F}_1$.

IV.3.1. Schematic cofunctors of graded rings and $\mathbb{F}_1$-algebras. This section is dedicated to the definition of the objects of the category $\mathfrak{J}$ of schematic cofunctors. Again, we denote by $\mathfrak{C}$ the category $\mathbf{GrAlg}_A$ for a fixed graded $\mathbb{F}_1$-algebra/ring A, e.g. $A = \mathbb{F}_1$ or $A = \mathbb{Z}$.

DEFINITION IV.3.1.1. A *schematic cofunctor* or simply, a $\mathfrak{J}$-*object* is a contravariant functor $\mathcal{O}\colon J \to \mathfrak{C}$ from a partially ordered set J, with ρ_i^j denoting the morphism which $\mathcal{O}$ assigns to $i \leq j$, such that the following hold:

(i) $gr \circ J$ is constant.
(ii) The set $J_{\leq j}^{\mathrm{pr}}$ of *principal elements below* j, which are $i \in J_{\leq j}$ such that $\tau_i := (\rho_i^j)^{-1}_{|\mathcal{O}(j)^{\mathrm{hom}}}((\mathcal{O}(i)^{\mathrm{hom}})^*)$ is principal and $\tau_i^{-1}\rho_i^j$ is bijective, is isomorphic to the reversely inclusion ordered set of proper principal faces of $\mathcal{O}(j)^{\mathrm{hom}}$ via the map $i \mapsto \tau_i$ whose inverse is denoted $\tau \mapsto i_\tau$;
(iii) For each $j, k \in J$ and $i \in J_{\leq j} \cap J_{\leq k}$ there exist $i_1, \dots, i_n \in J_{\leq i}^{\mathrm{pr}} \cap J_{\leq j}^{\mathrm{pr}} \cap J_{\leq k}^{\mathrm{pr}}$ such that $\mathcal{O}(i) = \langle (\rho_{i_l}^i)^{-1}_{|\mathcal{O}(i)^{\mathrm{hom}}}((\mathcal{O}(i_l)^{\mathrm{hom}})^*)^\circ | l = 1, \dots, n\rangle$ holds.
(iv) If $i_{\tau_1}, \dots, i_{\tau_n} \in J_{\leq j}^{\mathrm{pr}}$ satisfy $\mathcal{O}(j) = \langle \tau_1^\circ, \dots, \tau_n^\circ\rangle$ then $j = \sup_{l=1}^n i_{\tau_l}$.
(v) If for $i_1, \dots, i_n$ each $i_{k,l} := \max J_{\leq i_k} \cap J_{\leq i_l}$ exists and for the limit R over all $\rho_{i_{k,l}}^{i_{k,k}}$ with projections $pr_{i_{k,l}}$ each $\tau_k := (pr_{i_{k,k}})^{-1}_{|R^{\mathrm{hom}}}((\mathcal{O}(i_k)^{\mathrm{hom}})^*)$ is principal, each $(\tau_k\tau_l)^{-1}pr_{i_{k,l}}$ is bijective and $R = \langle \tau_1^\circ, \dots, \tau_n^\circ\rangle$, then $j := \sup_{l=1}^n i_l$ exists, and $\mathcal{O}(j)$ and $\rho_{i_{k,l}}^j$ form a limit over all $\rho_{i_{k,l}}^{i_{k,k}}$.

REMARK IV.3.1.2. If A is an $\mathbb{F}_1$-algebra then (iii) and (iv) amount to the condition $J_{\leq j}^{\mathrm{pr}} = J_{\leq j}$ for all $j \in J$. Axiom (v) is then vacuous.

CONSTRUCTION IV.3.1.3. Let $\mathcal{O}\colon J \to \mathfrak{C}$ be a $\mathfrak{J}$-object. For $i \leq j \in J$ the morphism $\mathrm{Spec}_{\mathrm{gr}}(\rho_i^j)\colon \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(i)) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(j))$ is an open embedding of graded schemes over A. Consequently, $\mathrm{Spec}_{\mathrm{gr}} \circ \mathcal{O}$ is a diagram of open embeddings of graded schemes over A, whose colimit is denoted $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$. For $j \in J$ let U_j denote the image of the morphism $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(j)) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$.

PROOF. $\mathrm{Spec}_{\mathrm{gr}}(\rho_i^j)$ is an open embedding due to axiom (iii) of $\mathfrak{J}$-objects. □

PROPOSITION IV.3.1.4. *For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ the morphism of partially ordered sets $\alpha\colon J \to \mathcal{B}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})} \setminus \{\emptyset\}, j \mapsto U_j$ is an isomorphism and we have a natural isomorphism $\mathcal{O}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})} \circ \alpha \to \mathcal{O}$. Moreover, α restricts to isomorphisms $J_{\leq j} \to \mathcal{B}_{U_j} \setminus \{\emptyset\}$ and $J_{\leq j}^{\mathrm{pr}} \to \mathcal{B}_{U_j}^{\mathrm{pr}} \setminus \{\emptyset\}$ for each $j \in J$.*

PROOF. The second supplement is a direct consequence of axiom (ii). For injectivity of α and order preservation of the inverse, let $a, b \in J$ with $U_a \subseteq U_b$. Then there exist $i_1, \dots, i_n \in J_{\leq a} \cap J_{\leq b}$ with $U_a = \bigcup_l U_{i_l}$. Due to (iii) for each i_l we find finitely many $j_{l,k} \in J^{\mathrm{pr}}_{\leq a} \cap J^{\mathrm{pr}}_{\leq b} \cap J^{\mathrm{pr}}_{\leq i_l}$ with $U_{i_l} = \bigcup_k U_{j_{l,k}}$. Since $U_a = \bigcup_{k,l} U_{j_{l,k}}$ the relative interiors of the corresponding faces $\tau_{l,k}$ of $\mathcal{O}(a)$ then generate $\mathcal{O}(a)$ as an ideal and using (iv) we deduce $a = \sup_{l,k} j_{l,k} \leq b$.

For $U \in \mathcal{B}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})} \setminus \{\emptyset\}$ Proposition IV.1.4.8 gives $i_1, \dots, i_n \in J$ such that $U = \bigcup_l U_{i_l}$, each $U_{i_k} \cap U_{i_l}$ is affine and for $\tau_k := (\rho^U_{U_{i_k}})^{-1}_{|\mathcal{O}(U)^{\mathrm{hom}}}((\mathcal{O}(U_{i_k})^{\mathrm{hom}})^*)$ each $(\tau_k \tau_l)^{-1} \rho^U_{U_k \cap U_l}$ is an isomorphism and $R := \mathcal{O}(U)$ as an ideal is generated by all τ_k°. Moreover, $U_{i_k} \cap U_{i_l}$ is principal in U_{i_k} and U_{i_l} with the defining face being generated by $\rho^U_{U_{i_k}}(\tau_l)$ resp. $\rho^U_{U_{i_l}}(\tau_k)$. Correspondingly, there are $i_{k,l} \in J^{\mathrm{pr}}_{\leq i_k}$ and $i'_{k,l} \in J^{\mathrm{pr}}_{\leq i_l}$ with $U_{i_{k,l}} = U_{i_k} \cap U_{i_l} = U_{i'_{k,l}}$ and hence $i_{k,l} = i'_{k,l}$. R is the limit of all $\rho^{i_k}_{i_{k,l}}$. By condition (v) $j = \sup_l i_l$ exists and we have $U \subseteq U_j$ and the canonical map $\mathcal{O}(U) \to \mathcal{O}(j) \to \mathcal{O}(U_j)$ is a composition of isomorphisms, which gives $U = U_j$. □

Definition IV.3.1.5. Let $\mathcal{O}\colon J \to \mathfrak{C}$ be an $\mathfrak{J}$-object. A $\mathfrak{J}$-subobject of $\mathcal{O}$ is a $\mathfrak{J}$-object $\mathcal{O}'\colon J' \to \mathfrak{C}$ such that $J' \subseteq J$ is a partially ordered subset and $\mathcal{O}' = \mathcal{O}_{|J'}$. More loosely, we also call J' a subobject if $\mathcal{O}_{|J'}\colon J' \to \mathfrak{C}$ is a subobject.

Remark IV.3.1.6. For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ a subset $J' \subseteq J$ carries a subobject structure if and only if J' contains all elements j allowing $i_{\tau_1}, \dots, i_{\tau_n} \in J^{\mathrm{pr}}_{\leq j} \cap J'$ with $\mathcal{O}(j) = \langle \tau_1^\circ, \dots, \tau_n^\circ \rangle$ and hence all elements below its elements, the last condition being sufficient if A is an $\mathbb{F}_1$-algebra. Consequently, intersections of subobjects are subobjects, and if A is a $\mathbb{F}_1$-algebra then arbitrary unions of subobjects are subobjects.

Construction IV.3.1.7. For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ the subobject *generated by* a subset $L \subseteq J$ is defined on the set J' of those $j \in J$ allowing $i_{\tau_1}, \dots, i_{\tau_n} \in J^{\mathrm{pr}}_{\leq j}$ which are also principal below some $l \in L$ such that $\mathcal{O}(j) = \langle \tau_1^\circ, \dots, \tau_n^\circ \rangle$. L is then called a generating subset of J'.

Proof. For $i_{\tau_1}, \dots, i_{\tau_n} \in J^{\mathrm{pr}}_{\leq j} \cap J'$ such that $\mathcal{O}(j) = \langle \tau_1^\circ, \dots, \tau_n^\circ \rangle$ there exist $k_{\tau_{a,1}}, \dots, k_{\tau_{a,m_a}} \in J^{\mathrm{pr}}_{\leq i_{\tau_a}}$ which are principal below some elements of L such that $\mathcal{O}(i_{\tau_a}) = \langle \tau_{a,1}^\circ, \dots, \tau_{a,m_a}^\circ \rangle$. Then each $k_{\tau_{a,t}}$ is principal below j and some element of L, and we have

$$\mathcal{O}(j) = \langle (\rho^j_{k_{\tau_{a,t}}})^{-1}_{|\mathcal{O}(j)^{\mathrm{hom}}}((\mathcal{O}(k_{\tau_{a,t}})^{\mathrm{hom}})^*)^\circ \mid a = 1, \dots, n, t = 1, \dots, m_a \rangle,$$

because the respective open subsets of $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$ satisfy $U_j = \bigcup_{a,t} U_{k_{\tau_{a,t}}}$. □

Remark IV.3.1.8. The subobject generated by $L \subseteq J$ is the minimal subobject of $\mathcal{O}$ defined on a superset of L and we have $\bigcup_{j \in \langle L \rangle} U_j = \bigcup_{l \in L} U_l$ in $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$. Moreover, the subobject generated by a union of subsets $L_i \subseteq J, i \in I$ is the same as the subobject generated by the union over the subobjects generated by each L_i.

IV.3.2. Morphisms of schematic cofunctors. In this section, various classes of morphisms of the category $\mathfrak{J}$ of schematic cofunctors are defined and in Proposition IV.3.2.11 the equivalence of schematic cofunctors over A with graded algebras over A is established.

Definition IV.3.2.1. A $\mathfrak{J}$-morphism from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$ is a functor $\mathfrak{f}\colon J \to \mathbf{SubCat}(J')$, together with a family $\mathfrak{f}^* = \{\mathfrak{f}^*_j\}_{j \in J}$ of cones $\mathfrak{f}^*_j\colon \mathcal{O}(j) \to \mathcal{O}'_{|\mathfrak{f}(j)}$ for $j \in J$ such that the following hold:

(i) $\mathfrak{f}^*_i \circ \rho^j_i$ is a subcone of $\mathfrak{f}^*_j$ for $i \leq j$, i.e. $\mathfrak{f}^*_{j,i'} = \mathfrak{f}^*_{i,i'} \circ \rho^j_i$ holds for $i' \in \mathfrak{f}(i)$;

(ii) for $i_\tau \in J^{\mathrm{pr}}_{\leq j}$, $\mathfrak{f}(i_\tau)$ consists of those $k' \in J'$ for which there exists $j' \in \mathfrak{f}(j)$ such that k' is the element of $J'^{\mathrm{pr}}_{\leq j'}$ defined by the face generated by $\mathfrak{f}^*_{j,j'}(\tau)$;

(iii) $\mathfrak{f}(k) \cap \mathfrak{f}(l)$ is the subobject of J' generated by $\bigcup_{j \in J_{\leq k} \cap J_{\leq l}} \mathfrak{f}(j)$;

(iv) J' is generated by $\bigcup_{j \in J} \mathfrak{f}(j)$.

The composition of a morphism $(\mathfrak{g},\mathfrak{g}^*)$ from $\mathcal{O}'$ to $\mathcal{O}''\colon J''\to\mathfrak{C}$ with $(\mathfrak{f},\mathfrak{f}^*)$ is the assignment sending $j\in J$ to the subobject generated by $\bigcup_{j'\in\mathfrak{f}(j)}\mathfrak{g}(j')$ together with the family of cones $(\mathfrak{g}^*\circ\mathfrak{f}^*)_j$ where $(\mathfrak{g}^*\circ\mathfrak{f}^*)_{j,j''}$ is the map induced by the homomorphisms $\mathfrak{g}^*_{j,'i''}\circ\mathfrak{f}^*_{j,j'}$ for $j'\in\mathfrak{f}(j)$ and $i''\in\mathfrak{g}(j')\cap J''^{\mathrm{pr}}_{\leq j''}$. $\mathrm{id}_{\mathcal{O}}$ is the assignment $j\mapsto J_{\leq j}$ together with the family of cones $(\rho^j_i)_{i\in J_{\leq j}}$ for $j\in J$. This constitutes a category $\mathfrak{J}$, also denoted as $\mathfrak{J}_A$.

LEMMA IV.3.2.2. *Compositions of $\mathfrak{J}$-morphisms are indeed $\mathfrak{J}$-morphisms.*

PROOF. Let $(\mathfrak{f},\mathfrak{f}^*)$ and $(\mathfrak{f}',\mathfrak{f}'^*)$ be morphisms from $\mathcal{O}\colon J\to\mathfrak{C}$ to $\mathcal{O}'\colon J'\to\mathfrak{C}$ resp. from $\mathcal{O}'$ to $\mathcal{O}''\colon J''\to\mathfrak{C}$. For well-definedness of their composition $(\mathfrak{g},\mathfrak{g}^*)$ first consider $j\in J$, $j',k'\in\mathfrak{f}(j)$ and $i''\in\mathfrak{f}'(j')\cap\mathfrak{f}'(k')$. By axiom (iii) there exist $i''_{\tau_1},\dots,i''_{\tau_n}\in J''^{\mathrm{pr}}_{\leq i''}$ which also belong to $\bigcup_{l'\in J'_{\leq j'}\cap J'_{\leq k'}}\mathfrak{f}'(l')$ such that $\mathcal{O}(i'')$ is generated as an ideal by all τ_a°. Fix $l'_a\in J'_{\leq j'}\cap J'_{\leq k'}$ with $i''_{\tau_a}\in\mathfrak{f}'(l'_a)$ and denote by $\tau_{a,b}$ the face generated by $\tau_a\tau_b$. For all a,b we then have

$$\begin{aligned}\rho^{i''}_{i''_{\tau_{a,b}}}\circ f'^*_{j',i''}\circ f^*_{j,j'} &= f'^*_{l'_a,i''_{\tau_{a,b}}}\circ\rho^{j'}_{l'_a}\circ f^*_{j,j'} = f'^*_{l'_a,i''_{\tau_{a,b}}}\circ f^*_{j,l'_a}\\ &= f'^*_{l'_a,i''_{\tau_{a,b}}}\circ\rho^{k'}_{l'_a}\circ f^*_{j,k'} = \rho^{i''}_{i''_{\tau_{a,b}}}\circ f'^*_{k',i''}\circ f^*_{j,k'}\end{aligned}$$

and since $\mathcal{O}(i'')$ is the limit of the diagram given by all $\rho^{i''_{\tau_a}}_{i''_{\tau_{a,b}}}$ the universal property gives $f'^*_{j',i''}\circ f^*_{j,j'}=f'^*_{k',i''}\circ f^*_{j,k'}$ as desired. This defines a cone from $\mathcal{O}(j)$ to the restriction of $\mathcal{O}''$ to $J''^{\mathrm{pr}}_{\leq j''}\cap\bigcup_{j'\in\mathfrak{f}(j)}\mathfrak{f}'(j')$, for which $\mathcal{O}''(j'')$ together with the maps $\rho^{j''}_{i''}$ is a limit, and hence we obtain an induced map $(\mathfrak{f}'^*\circ\mathfrak{f}^*)_{j,j''}\colon\mathcal{O}(j)\to\mathcal{O}''(j'')$.

For axiom (ii) let $i_\tau\in J^{\mathrm{pr}}_{\leq j}$ and consider an element j'' of the subobject $\mathfrak{g}(j)$ generated by $\bigcup_{j'\in\mathfrak{f}(j)}\mathfrak{f}'(j')$. Let $i''_{\tau''}\in J''^{\mathrm{pr}}_{\leq j''}\cap\mathfrak{g}(j)$ be the element corresponding to $\tau'':=\mathrm{face}(\mathfrak{g}^*_{j,j''}(\tau))$ and denote $P:=J''^{\mathrm{pr}}_{\leq j''}\cap\bigcup_{j'\in\mathfrak{f}(j)}\mathfrak{f}'(j')$. For $i''\in P\cap\mathfrak{f}'(j')$ with defining face $\eta''_{i''}$ we have

$$\sigma''_{i''}:=\mathrm{face}(\rho^{j''}_{i''}(\tau''))=\mathrm{face}(\mathfrak{g}^*_{j,i''}(\tau))=\mathrm{face}(\mathfrak{f}'^*_{j',i''}(\mathrm{face}(\mathfrak{f}^*_{j,j'}(\tau)))$$

and hence axiom (ii) gives $j''_{\sigma''_{i''}}\in\mathfrak{f}'(i'_{\mathrm{face}(\mathfrak{f}^*_{j,j'}(\tau))})$ and $i'_{\mathrm{face}(\mathfrak{f}^*_{j,j'}(\tau))}\in\mathfrak{f}(i_\tau)$ which implies $j''_{\sigma''_{i''}}\in\mathfrak{g}(i_\tau)$. Since $\mathcal{O}(i''_{\tau''})=\langle\sigma''^\circ_{i''}\mid i''\in P\rangle$ we obtain $i''_{\tau''}\in\mathfrak{g}(i_\tau)$.

For axiom (iii) consider $j''\in J''$ which allows $k''_{\tau''_1},\dots,k''_{\tau''_n},l''_{\eta''_1},\dots,l''_{\eta''_m}\in J''^{\mathrm{pr}}_{\leq j''}$ such that $k''_{\tau''_a}\in\mathfrak{f}'(k'_a)$ and $l''_{\eta''_b}\in\mathfrak{f}'(l'_b)$ hold with certain $k'_a\in\mathfrak{f}(k)$ and $l'_b\in\mathfrak{f}(l)$ and we have

$$\mathcal{O}''(j'')=\langle\tau''^\circ_1,\dots,\tau''^\circ_n\rangle=\langle\eta''^\circ_1,\dots,\eta''^\circ_m\rangle.$$

Then the relative interiors of all $\mathrm{face}(\tau''_a\eta''_b)$ generate $\mathcal{O}''(j'')$ and each of the corresponding principal elements $i''_{a,b}$ lies in $\mathfrak{f}'(k'_a)\cap\mathfrak{f}'(l'_b)$. Axiom (iii) for $\mathfrak{f}'$ now yields $t''_{a,b,1},\dots,t''_{a,b,p_{a,b}}\in J''^{\mathrm{pr}}_{\leq i''_{\sigma''_{a,b}}}$ such that the relative interiors of their defining faces generate $\mathcal{O}''(i''_{\sigma''_{a,b}})$ and $t''_{a,b,c}\in\mathfrak{f}'(t'_{a,b,c})$ holds for a certain $t'_{a,b,c}\in J'_{\leq k'_a}\cap J'_{\leq l'_b}$. Axiom (ii) for $\mathfrak{f}$ gives $j'_{a,b,c,1},\dots,j'_{a,b,c,q_{a,b,c}}\in J'^{\mathrm{pr}}_{\leq t'_{a,b,c}}$ such that the relative interiors of the corresponding faces generate $\mathcal{O}'(t'_{a,b,c})$ and we have $j'_{a,b,c,d}\in\mathfrak{f}(j_{a,b,c,d})$ with a certain $j_{a,b,c,d}\in J_{\leq k}\cap J_{\leq l}$. The image of the defining face of $j'_{a,b,c,d}$ defines an element $j''_{a,b,c,d}\in J''^{\mathrm{pr}}_{\leq t''_{\sigma''_{a,b,c}}}\cap\mathfrak{f}'(j'_{a,b,c,d})$ by (ii). The relative interiors of the defining faces of $j''_{a,b,c,d}$ then generate $\mathcal{O}''(j'')$ and axiom (iii) is verified for the composition.

To show that J'' is generated by $\bigcup_j\bigcup_{j'\in\mathfrak{f}(j)}\mathfrak{f}'(j')$ let $j''\in J''$ and consider $i''_{\tau''_1},\dots,i''_{\tau_n}\in J''^{\mathrm{pr}}_{\leq j''}$ where $i''_{\tau''_a}\in\mathfrak{f}(j'_a)$ such that $\mathcal{O}''(j'')=\langle\tau''^\circ_1,\dots,\tau''^\circ_n\rangle$. Let $i'_{\tau'_{a,1}},\dots,i'_{\tau'_{a,m_a}}\in J'^{\mathrm{pr}}_{\leq j'_a}$ with $i'_{\tau'_{a,k}}\in\mathfrak{f}(j_{a,k})$ satisfy $\mathcal{O}'(j'_a)=\langle\tau'^\circ_{a,1},\dots,\tau'^\circ_{a,m_a}\rangle$. Then the element $j''_{\eta''_{a,k}}\in J''^{\mathrm{pr}}_{i''_{\tau_a}}$ corresponding to $\eta''_{a,k}:=\mathrm{face}(\mathfrak{f}'^*_{j'_a,i''_{\tau''_a}}(\tau'_{a,k}))$ lies in $\mathfrak{f}'(i'_{\tau'_{a,k}})$

and we have $\mathcal{O}''(i''_{\tau''_a}) = \langle \eta''^{\circ}_{a,1}, \dots, \eta''^{\circ}_{a,m_a} \rangle$. Consequently, $\mathcal{O}''(j'')$ is generated as an ideal by the relative interiors of the faces defined by $(\mathcal{O}(j''_{\eta''_{a,k}})^{\mathrm{hom}})^*$. $\square$

Construction IV.3.2.3. Consider a morphism $(\mathfrak{f}, \mathfrak{f}^*)$ from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$. Then each $j \in J$ defines a morphism

$$\phi_j \colon \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}'_{|\mathfrak{f}(j)}) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(j)) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$$

and these induce a morphism $(\phi, \phi^*) := \mathrm{Spec}_{\mathrm{gr}}(\mathfrak{f}, \mathfrak{f}^*)\colon \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}') \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$. This constitutes a contravariant functor $\mathrm{Spec}_{\mathrm{gr}}$ from $\mathfrak{J}$ to $\mathbf{GrSch}_A$. Moreover, the canonical isomorphism $\alpha'\colon J' \to \mathcal{B}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}')}$ then restricts to an isomorphism $\mathfrak{f}(j) \to \mathcal{B}_{\phi^{-1}(U_j)} \setminus \{\emptyset\}$ for each $j \in J$.

Proof. Proposition II.1.3.10 together with $\mathfrak{J}$-morphism axioms (iii) resp. (iv) of implies that firstly, for $i \leq j$, ϕ_i is the composition of ϕ_j and the canonical embedding $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}'_{|\mathfrak{f}(i)}) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}'_{|\mathfrak{f}(j)})$ and secondly, the canonical morphism $\mathrm{colim}_{j \in J} \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}'_{|\mathfrak{f}(j)}) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}')$ is an isomorphism. $\square$

Remark IV.3.2.4. For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ each subobject $J' \subseteq J$ defines an open embedding $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{|J'}) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$ onto $\bigcup_{j \in J'} U_j$.

Remark IV.3.2.5. Let $(\mathfrak{f}, \mathfrak{f}^*)$ be a morphism from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$. Then for $i_\tau \in J^{\mathrm{pr}}_{\leq j}$ and $j' \in \mathfrak{f}(j)$ the element $i'_{\tau'} \in J'^{\mathrm{pr}}_{\leq j'}$ corresponding to the face τ' generated by $\mathfrak{f}^*_{j,j'}(\tau)$ the canonical diagram

$$\begin{tikzcd}
\tau^{-1}\mathcal{O}(j) \arrow[r, "\tau^{-1}(\imath_{\tau'} \circ \mathfrak{f}^*_{j,j'})"] \arrow[d, "\tau^{-1}\rho^j_{i_\tau}"', "\cong"] & \tau'^{-1}\mathcal{O}'(j') \arrow[d, "\tau'^{-1}\rho^{j'}_{i'_{\tau'}}"', "\cong"] \\
\mathcal{O}(i_\tau) \arrow[r, "\mathfrak{f}^*_{i_\tau, i'_{\tau'}}"] & \mathcal{O}'(i'_{\tau'})
\end{tikzcd}$$

commutes due to the universal property of localizations.

Construction IV.3.2.6. For a graded scheme $(X, \mathcal{O}_X)$, the inclusion-order turns $(\mathcal{O}_X)_{|\mathcal{B}_X \setminus \{\emptyset\}}$ into a $\mathfrak{J}$-object. A morphism $(\phi, \phi^*)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces a $\mathfrak{J}$-morphism consisting of the map $[U \mapsto \mathcal{B}_{\phi^{-1}(U)} \setminus \{\emptyset\}]$ and the family of cones $(\rho^{\phi^{-1}(U)}_V \circ \phi^*_U)_{V \in \mathcal{B}_{\phi^{-1}(U)} \setminus \{\emptyset\}}$, where U runs through all of $\mathcal{B}_Y \setminus \{\emptyset\}$. This constitutes a contravariant functor $\mathcal{O}_{|\mathcal{B}}$ from $\mathbf{GrSch}_A$ to $\mathfrak{J}$.

Proof. For well-definedness of $\mathcal{O}_{|\mathcal{B}}$ note that $(\mathcal{O}_X)_{|\mathcal{B}_X \setminus \{\emptyset\}}$ satisfies the defining axiom (vi) due to Proposition IV.1.4.8, and axiom (iv) is due to Lemma IV.1.2.4. Axiom (v) holds because $f_1, \dots, f_n \in R^{\mathrm{hom}}$ generate R as an ideal if and only if $\mathrm{Spec}_{\mathrm{gr}}(R)$ is covered by their principal subsets. $\square$

Remark IV.3.2.7. For a graded scheme $(X, \mathcal{O}_X)$ the subobject generated by a subset $\mathcal{U} \subseteq \mathcal{B}_X \setminus \{\emptyset\}$ is $\mathcal{B}_{\bigcup \mathcal{U}} \setminus \{\emptyset\}$.

Definition IV.3.2.8. Let $\mathfrak{J}'$ be the category with object class $ob(\mathfrak{J})$ whose morphisms from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}\colon J' \to \mathfrak{C}$ are those $\mathfrak{J}$-morphisms $(\mathfrak{f}, \mathfrak{f}^*)$ such that for each $j' \in J'$ there exists a uniquely minimal $j \in J$ with $j' \in \mathfrak{f}(j)$.

Definition IV.3.2.9. Let $\mathfrak{J}^{\mathrm{covar}}$ be the category with object class $ob(\mathfrak{J})$ whose morphisms from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}\colon J' \to \mathfrak{C}$ are pairs (α, α^*) consisting of morphisms $\alpha\colon J \to J'$ of partially ordered sets and natural transformations $\alpha^*\colon \mathcal{O}' \circ \alpha \to \mathcal{O}$ of functors such that for $i'_{\tau'} \in J'^{\mathrm{pr}}_{\leq j'}$ the set $\alpha^{-1}(J_{\leq j'})$ consists of those $k \in J$ for which there exists $j \in \alpha^{-1}(J'_{\leq j'})$ such that k is the element of $J^{\mathrm{pr}}_{\leq j}$ corresponding to the face generated by $\alpha^*_j(\rho^{j'}_{\alpha(j)}(\tau'))$.

Again, when there is need to emphasize the basis A we use the notations $\mathfrak{J}'_A$ and $\mathfrak{J}^{\mathrm{covar}}_A$ for the categories defined above.

PROPOSITION IV.3.2.10. *Then the following hold:*

(i) *$\mathfrak{J}'$ has all $\mathfrak{J}$-isomorphisms.*

(ii) *We obtain mutually inverse functors between $\mathfrak{J}'$ and $\mathfrak{J}^{\mathrm{covar}}$ by sending a $\mathfrak{J}'$-morphism $(\mathfrak{f},\mathfrak{f}^*)$ from $\mathcal{O}$ to $\mathcal{O}'$ to the map $\alpha\colon j'\mapsto \min_{j'\in\mathfrak{f}(j)} j$ together with the natural transformation α^* defined at j' by $\mathfrak{f}^*_{\alpha(j'),j'}$, and sending a $\mathfrak{J}^{\mathrm{covar}}$-morphism (α,α^*) from $\mathcal{O}'$ to $\mathcal{O}$ to the assignment $\mathfrak{f}\colon j\mapsto \alpha^{-1}(J_{\leq j})$ and the family $(\alpha^*_{j'}\circ\rho^{j}_{\alpha(j')})_{j'\in\mathfrak{f}(j)}$ where j runs through all of J.*

(iii) *For a $\mathfrak{J}^{\mathrm{covar}}$-morphism (α,α^*) from $\mathcal{O}'$ to $\mathcal{O}$ each of the homomorphisms $\alpha^*_{j'}$ is (homogeneously) local.*

PROOF. In (i) we show that for mutually inverse $\mathfrak{J}$-isomorphisms $(\mathfrak{f},\mathfrak{f}^*)$ and $(\mathfrak{g},\mathfrak{g}^*)$ from $\mathcal{O}\colon J\to\mathfrak{C}$ to $\mathcal{O}'\colon J'\to\mathfrak{C}$ and vice versa, each $\mathfrak{f}(j)$ and $\mathfrak{g}(j')$ has a unique maximal element, and that $\max\mathfrak{g}(-)$ and $\max\mathfrak{f}(-)$ define mutually inverse isomorphisms between J and J'. Indeed, we have $J'_{\leq j'}=\bigcup_{i\in\mathfrak{g}(j')}\mathfrak{f}(i)$ which means there exists $j\in\mathfrak{g}(j')$ with $j'\in\mathfrak{f}(j)$, and hence $\mathfrak{f}(j)=J'_{\leq j'}$. Moreover,

$$J_{\leq j}\subseteq\mathfrak{g}(j')\subseteq\bigcup_{i'\in\mathfrak{f}(j)}\mathfrak{g}(i')=J_{\leq j}$$

which shows $j'=\max\mathfrak{f}(j)=\max\mathfrak{f}(\max\mathfrak{g}(j'))$. Furthermore, $\max\mathfrak{f}(-)$ is order-preserving because $\mathfrak{f}(i)\subseteq\mathfrak{f}(k)$ holds for $i\leq k$.

In (ii) consider a $\mathfrak{J}'$-morphism $(\mathfrak{f},\mathfrak{f}^*)$ from $\mathcal{O}$ to $\mathcal{O}'$ and set $\alpha(j'):=\min_{j'\in\mathfrak{f}(j)} j$. Then we have $j'\in\alpha^{-1}(J_{\leq j})=\mathfrak{f}(j)$ because $\alpha(j')\leq j$ implies $j'\in\mathfrak{f}(\alpha(j'))\subseteq\mathfrak{f}(j)$ and conversely, $j'\in\mathfrak{f}(j)$ implies $\alpha(j')\leq j$. If a $\mathfrak{O}^{\mathrm{covar}}$-morphism (α,α^*) from $\mathcal{O}'$ to $\mathcal{O}$ is given then $\alpha(j')$ is the minimum over all $j\in J$ with $j'\in\alpha^{-1}(J_{\leq j})$ because firstly, $j'\in\alpha^{-1}(J_{\leq\alpha(j')})$ and secondly, for each $j\in J$ with $j'\in\alpha^{-1}(J_{\leq j})$ we have $\alpha(j')\leq j$.

For (iii) consider $f\in\mathcal{O}(\alpha(j'))^{\mathrm{hom}}$ with $\alpha^*_{j'}(f)\in\mathcal{O}'(j)^*$ and set $\tau:=\mathrm{face}(f)$. Then we have $j'=i'_{\mathrm{face}(\alpha^*_{j'}(\tau))}\in\alpha^{-1}(J_{\leq i_\tau})$ and minimality implies $\alpha(j')=i_\tau$ and hence $f\in\mathcal{O}(\alpha(j'))^*$. $\square$

PROPOSITION IV.3.2.11. *The functors $\mathrm{Spec}_{\mathrm{gr}}\colon\mathfrak{J}\to\mathbf{GrSch}_A$ and $\mathcal{O}_{|\mathcal{B}}$ constructed above are mutually essentially inverse contravariant equivalences of categories.*

PROOF. For functoriality of $\mathrm{Spec}_{\mathrm{gr}}$ consider morphisms $(\mathfrak{f},\mathfrak{f}^*)$ and $(\mathfrak{f}',\mathfrak{f}'^*)$ from $\mathcal{O}\colon J\to\mathfrak{C}$ to $\mathcal{O}'\colon J'\to\mathfrak{C}$ resp. from $\mathcal{O}'$ to $\mathcal{O}''\colon J''\to\mathfrak{C}$ and denote their composition by $(\mathfrak{g},\mathfrak{g}^*)$. Denoting for $j\in J$, $j'\in\mathfrak{f}(j)$ and $j''\in\mathfrak{f}'(j')$ the induced morphisms by $\phi_{j',j}\colon U'_{j'}\to U_j$, $\phi'_{j'',j'}\colon U''_{j''}\to U'_{j'}$ and $\psi_{j'',j}\colon U''_{j''}\to U_j$ and the canonical embedding by $\imath_j\colon U_j\to\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$ we calculate

$$(\mathrm{Spec}_{\mathrm{gr}}(\mathfrak{f},\mathfrak{f}^*)\circ\mathrm{Spec}_{\mathrm{gr}}(\mathfrak{f}',\mathfrak{f}'^*))_{|U''_{j''}}=\imath_j\circ\phi_{j',j}\colon\phi'_{j'',j'}=\imath_j\circ\psi_{j'',j}=\mathrm{Spec}_{\mathrm{gr}}(\mathfrak{g},\mathfrak{g}^*)_{|U''_{j''}}$$

which gives $\mathrm{Spec}_{\mathrm{gr}}(\mathfrak{f},\mathfrak{f}^*)\circ\mathrm{Spec}_{\mathrm{gr}}(\mathfrak{f}',\mathfrak{f}'^*)=\mathrm{Spec}_{\mathrm{gr}}(\mathfrak{g},\mathfrak{g}^*)$ because the sets $U''_{j''}$ of the above type cover $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}'')$.

For functoriality of $\mathcal{O}_{|\mathcal{B}}$ consider morphisms $\phi\colon(X,\mathcal{O}_X)\to(X',\mathcal{O}_{X'})$ and $\phi'\colon(X',\mathcal{O}_{X'})\to(X'',\mathcal{O}_{X''})$. For $U''\in\mathcal{B}_{X''}\setminus\{\emptyset\}$ the union over all $\mathcal{B}_{\phi^{-1}(U')}\setminus\{\emptyset\}$ where $U'\in\mathcal{B}_{\phi'^{-1}(U'')}\setminus\{\emptyset\}$ covers $(\phi'\circ\phi)^{-1}(U'')$ and hence generates $\mathcal{B}_{(\phi'\circ\phi)^{-1}(U'')}\setminus\{\emptyset\}$. Moreover, for non-empty $U'\in\mathcal{B}_{\phi'^{-1}(U'')}$ and $U\in\mathcal{B}_{\phi^{-1}(U')}$ we have

$$\rho_U^{(\phi'\circ\phi)^{-1}(U'')}\circ(\phi'\circ\phi)^*_{U''}=\rho_U^{\phi^{-1}(U')}\circ\phi^*_{U'}\circ\rho_{U'}^{(\phi'^{-1}(U''))}\circ\phi'^*_{U''}$$

which shows functoriality.

Now, the canonical isomorphisms $\mathrm{Spec}_{\mathrm{gr}}((\mathcal{O}_X)_{|\mathcal{B}_X\setminus\{\emptyset\}}) \to X$ form the desired natural isomorphism between $\mathrm{id}_{\mathbf{GrSch}_A}$ and $\mathrm{Spec}_{\mathrm{gr}} \circ \mathcal{O}_{|\mathcal{B}}$. For an $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ the isomorphism $\alpha\colon J \to \mathcal{B}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})} \setminus \{\emptyset\}$ from Construction IV.3.1.3 together with the natural isomorphism $\alpha^*\colon \mathcal{O} \to \mathcal{O}_{|\mathcal{B}}(\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})) \circ \alpha$ defined at j by the canonical isomorphism $\alpha_j^*\colon \mathcal{O}(j) \to \mathcal{O}(U_j)$ forms an $\mathfrak{J}^{\mathrm{covar}}$-isomorphism. Proposition IV.3.2.10 yields a corresponding $\mathfrak{J}$-isomorphism and since the above is compatible with $\mathfrak{J}$-morphisms we obtain a natural isomorphism between $\mathrm{id}_{\mathfrak{J}}$ and $\mathcal{O}_{|\mathcal{B}} \circ \mathrm{Spec}_{\mathrm{gr}}$. □

IV.3.3. Describing properties and neighbourhood bases. Here, we give first descriptions of algebro-geometric properties in terms of schematic cofunctors. The study of neighbourhood bases leads to another functor which is isomorphic to $\mathrm{Spec}_{\mathrm{gr}}$.

Remark IV.3.3.1. A morphism (ϕ, ϕ^*) of graded schemes corresponding to a $\mathfrak{J}$-morphism $(\mathfrak{f}, \mathfrak{f}^*)$ from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$ is

(i) dominant if and only if each $\mathfrak{f}(j)$ is non-empty;
(ii) affine if and only if for some (and hence each) generating subset L of J and each $l \in L$ the set $\mathfrak{f}(l)$ has a greatest element if it is non-empty; note that all these together then generate J';
(iii) a good quotient if and only if for some (and hence each) generating subset L of J and each $l \in L$ the set $\mathfrak{f}(l)$ has a greatest element l' and $\mathfrak{f}^*_{l,l'}$ is Veronesean.

Remark IV.3.3.2. A morphism (ϕ, ϕ^*) of graded schemes corresponding to a $\mathfrak{J}^{\mathrm{covar}}$-morphism (α, α^*) is a good quotient if and only if each $\alpha^{-1}(i')$ has a greatest element i and α_i^* is Veronesean.

Definition IV.3.3.3. A $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ such that J allows a finite generating subset is *finitely generated*.

Remark IV.3.3.4. A $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ is finitely generated if and only if the corresponding graded scheme X over A is quasi-compact. This is because $L \subseteq J$ generates J if and only if $\{U_l\}_{l\in L}$ covers X.

Definition IV.3.3.5. Let (X, Ω_X) be a topological space and let $\mathcal{B}$ be a basis of Ω_X.

(i) For $A \subseteq X$ the induced basis of the set $\Omega_{X,A}$ of neighbourhoods of A, which are $U \in \Omega_X$ intersecting A non-trivially, is denoted $\mathcal{B}_A := \mathcal{B} \cap \Omega_{X,A}$.
(ii) A $\mathcal{B}$-neighbourhood basis is a subset $\mathcal{W} \subseteq \mathcal{B} \setminus \{\emptyset\}$ such that if $V \in \mathcal{W}$ lies in the union of $\mathcal{U} \subseteq \mathcal{B}$ then $\mathcal{U} \cap \mathcal{W} \neq \emptyset$. $\mathcal{W}$ is called *irreducible* if it is non-empty and $\mathcal{W} \subseteq \mathcal{W}' \cup \mathcal{W}''$ implies $\mathcal{W} \subseteq \mathcal{W}'$ or $\mathcal{W} \subseteq \mathcal{W}''$ for all $\mathcal{B}$-neighbourhood bases $\mathcal{W}'$ and $\mathcal{W}''$.

Remark IV.3.3.6. $\Omega_{X,A}$ is determined by $\mathcal{B}_A$ as the set of those $U \in \Omega_X$ which contain some $V \in \mathcal{B}_A$. Moreover, we have $\mathcal{B}_A = \mathcal{B}_{\overline{A}}$.

Example IV.3.3.7. Each open $U \subseteq X$ determines the $\mathcal{B}$-neighbourhood basis $\mathcal{B}_{X\setminus U} = \mathcal{B} \setminus \mathcal{B}_U$.

Proposition IV.3.3.8. *Let (X, Ω_X) be a topological space and let $\mathcal{B}$ be a basis of Ω_X. Then the following hold:*

(i) *All unions of $\mathcal{B}$-neighbourhood bases are again $\mathcal{B}$-neighbourhood bases.*
(ii) *The map $A \mapsto \mathcal{B}_A$ commutes with arbitrary unions of subsets.*
(iii) *$A \mapsto \mathcal{B}_A$ defines an inclusion-preserving bijection between the closed subsets of X and the set of $\mathcal{B}$-neighbourhood bases, with the inverse sending $\mathcal{W}$ to $A_{\mathcal{W}} := X \setminus \bigcup(\mathcal{B} \setminus \mathcal{W})$, and both maps preserve irreducibility.*

(iv) *A $\mathcal{B}$-neighbourhood basis $\mathcal{W}$ is irreducible if for $U, V \in \mathcal{W}$ there exists $W \in \mathcal{W}$ with $W \subseteq U$ and $W \subseteq V$.*

Proof. In (iii) we first note that A is the intersection of the complements $X \setminus U$ of all $U \in \mathcal{B}_A$, i.e. the closure of A. For a $\mathcal{B}$-neighbourhood basis $\mathcal{W}$, consider $U \in \mathcal{B}$. If U intersects $A_{\mathcal{W}}$ trivially then it is contained in $\bigcup \mathcal{B} \setminus \mathcal{W}$ which gives $U \notin \mathcal{W}$. Conversely, if $U \notin \mathcal{W}$ then $U \cap A_{\mathcal{W}} \subseteq U \cap (X \setminus U) = \emptyset$ and we conclude $\mathcal{B}_{A_{\mathcal{W}}} = \mathcal{W}$. Now, (i) and (ii) give preservation of irreducibility under the first map, which gives the same for the second. In (iv) note that for $A \subseteq X$ with $\mathcal{W} = \mathcal{B}_A$ the given condition characterizes irreducibility of A and thus irreducibility of $\mathcal{W}$. □

Remark IV.3.3.9. Let $\phi\colon X \to X'$ be a morphism and let $A \subseteq X$. Then $\mathcal{B}_{\phi(A)}$ is the set of those $U \in \mathcal{B}_{X'}$ with $\mathcal{B}_{\phi^{-1}(U)} \cap \mathcal{B}_A \neq \emptyset$.

Definition IV.3.3.10. For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$, $J' \subseteq J$ is an *irreducible neighbourhood basis* if and only if for $k, l \in J'$ there exists $m \in J'$ with $m \leq k$ and $m \leq l$, and whenever $k \in J'$ allows $i_{1,l}, \ldots, i_{n_l,l} \in J^{\mathrm{pr}}_{\leq k} \cap J^{\mathrm{pr}}_{\leq j_l}$ for certain $j_1, \ldots, j_d \in J$ such that $\mathcal{O}(k)$ is generated as an ideal by all $\rho^k_{i_{m,l}}((\mathcal{O}(i_{m,l})^{\mathrm{hom}})^*)^\circ$ then some j_l belongs to J'.

Remark IV.3.3.11. For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ the canonical isomorphism $\alpha\colon J \to \mathcal{B}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})}$ induces bijections between the set of (irreducible) neighbourhood bases p of $\mathcal{O}$ and the set of (irreducible) $\mathcal{B}_{\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})}$-neighbourhood bases $\mathcal{W}$. For each $U \in \mathcal{W}$ we recover $\mathcal{W}$ as the set of $V \in \mathcal{B}_X$ with $\mathcal{B}_V \cap \mathcal{W} \cap \mathcal{B}^{\mathrm{pr}}_U \neq \emptyset$. Consequently, each p is recovered for $j \in p$ as the set of those $i \in J$ with $J_{\leq i} \cap p \cap J^{\mathrm{pr}}_{\leq j} \neq \emptyset$.

Construction IV.3.3.12. For a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ let $\mathfrak{n}(\mathcal{O})$ be the set of irreducible neighbourhood bases. The topology generated by all the sets V_j of those $p \in \mathfrak{n}(\mathcal{O})$ which contain j is denoted $\Omega_{\mathfrak{n}(\mathcal{O})}$. Note that we have $U_i \subseteq U_j$ for $i \leq j$. For $U \in \Omega_{\mathfrak{n}(\mathcal{O})}$ let $\mathcal{O}_{\mathfrak{n}(\mathcal{O})}(U)$ be the limit over all $\mathcal{O}(j)$ with $V_j \subseteq U$. The maps induced by universal properties of limits turn $\mathcal{O}_{\mathfrak{n}(\mathcal{O})}$ into a presheaf on $\Omega_{\mathfrak{n}(\mathcal{O})}$. For $p \in \mathfrak{n}(\mathcal{O})$ we then have a canonical isomorphism $\mathcal{O}_{\mathfrak{n}(\mathcal{O}),p} \cong \operatorname{colim} \mathcal{O}_{|p}$. This turns $(\mathfrak{n}(\mathcal{O}), \Omega_{\mathfrak{n}(\mathcal{O})}, \mathcal{O}_{\mathfrak{n}(\mathcal{O})})$ into a graded scheme over A.

Proposition IV.3.3.13. *In the above notation, $(\mathfrak{n}(\mathcal{O}), \Omega_{\mathfrak{n}(\mathcal{O})}, \mathcal{O}_{\mathfrak{n}(\mathcal{O})})$ is canonically isomorphic to $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$.*

Proof. The canonical isomorphism $\alpha\colon J \to \mathcal{B}_X$, where $X := \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O})$, defines a bijection $\beta\colon \mathfrak{n}(\mathcal{O}) \to X$ sending p to the generic point of $X \setminus \bigcup(\mathcal{B}_X \setminus \alpha(p))$, and $x \in X$ to $\alpha^{-1}(\mathcal{B}_{X,x})$. We have $p \in V_j$ if and only if $J^{\mathrm{pr}}_{\leq j} \cap p$ generates p, which holds if and only if $\alpha(J^{\mathrm{pr}}_{\leq j} \cap p) = \mathcal{B}^{\mathrm{pr}}_{U_j} \cap \alpha(p)$ generates $\alpha(p) = \mathcal{B}_{\beta(p)}$, i.e. if and only if $\beta(p) \in U_j$. In particular, this shows continuity of β and $\gamma := \beta^{-1}$. Moreover, the canonical isomorphisms $\mathcal{O}(j) \to \mathcal{O}_X(U_j)$ extend to isomorphisms $\mathcal{O}_{\mathfrak{n}(\mathcal{O})}(V) \to \mathcal{O}_X(\beta(V))$ and we obtain homomorphisms γ^* and β^* turning (β, β^*) and (γ, γ^*) into mutually inverse morphisms of graded schemes over A. □

Construction IV.3.3.14. Consider a $\mathfrak{J}$-morphism $(\mathfrak{f}, \mathfrak{f}^*)$ from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$. The map $\phi\colon \mathfrak{n}(\mathcal{O}') \to \mathfrak{n}(\mathcal{O})$ sending $p' \in \mathfrak{n}(\mathcal{O}')$ to the set of $j \in J$ with $p' \cap \mathfrak{f}(j) \neq \emptyset$ then satisfies $\phi^{-1}(V_j) = \bigcup_{j' \in \mathfrak{f}(j)} V'_{j'}$, in particular, ϕ is continuous. For $U \in \Omega_{\mathfrak{n}(\mathcal{O})}$ we obtain a homomorphism $\phi^*_U\colon \mathcal{O}_{\mathfrak{n}(\mathcal{O})} \to \mathcal{O}_{\mathfrak{n}(\mathcal{O}')}(\phi^{-1}(U))$ as the map induced by all $\mathfrak{f}^*_{j,j'}$ for $V_j \subseteq U$ and $j' \in \mathfrak{f}(j)$. The pair $\mathfrak{n}(\mathfrak{f}, \mathfrak{f}^*) := (\phi, \phi^*)$ then forms a morphism of graded schemes over A.

Proposition IV.3.3.15. *The functor $\mathfrak{n}(-)$ is naturally isomorphic to* $\mathrm{Spec}_{\mathrm{gr}}$.

Proof. For functoriality, consider morphisms $(\mathfrak{f}, \mathfrak{f}^*)$ $(\mathfrak{f}', \mathfrak{f}'^*)$ from $\mathcal{O}\colon J \to \mathfrak{C}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$ resp. from $\mathcal{O}'$ to $\mathcal{O}''\colon J'' \to \mathfrak{C}$ and denote their composition by $(\mathfrak{g}, \mathfrak{g}^*)$. Let $(\phi, \phi^*) := \mathfrak{n}(\mathfrak{f}, \mathfrak{f}^*)$, $(\phi', \phi'^*) := \mathfrak{n}(\mathfrak{f}', \mathfrak{f}'^*)$ and $(\psi, \psi^*) := \mathfrak{n}(\mathfrak{g}, \mathfrak{g}^*)$ be the corresponding morphisms of graded schemes. To see that $\psi = \phi \circ \phi'$ holds note that for $j \in J$ there exists $j' \in \mathfrak{f}(j) \cap \mathfrak{n}(\mathfrak{f}', \mathfrak{f}'^*)(p'')$ if and only if there exists $j'' \in p'' \cap \bigcup_{j' \in \mathfrak{f}(j)} \mathfrak{f}'(j')$, i.e. if and only if there exists $j'' \in p'' \cap \mathfrak{g}(j)$. For each $j \in J$ as well as all $j' \in \mathfrak{f}(j)$ and $j'' \in \mathfrak{f}'(j')$ we have $\mathfrak{g}^*_{j,j''} = \mathfrak{f}'^*_{j',j''} \circ \mathfrak{f}^*_{j,j'}$ and hence

$$\rho^{\psi^{-1}(V_j)}_{V''_{j''}} \circ \psi^*_{V_j} = \rho^{\phi'^{-1}(V'_{j'})}_{V''_{j''}} \circ \phi'^*_{V'_{j'}} \circ \rho^{\phi^{-1}(V_j)}_{V'_{j'}} \circ \phi^*_{V_j} = \rho^{\psi^{-1}(V_j)}_{V''_{j''}} \circ \phi'^*_{\phi^{-1}(V_j)} \circ \phi^*_{V_j}.$$

Since such $V''_{j''}$ form a basis of $\psi^{-1}(V_j)$ the sheaf properties give $\psi^*_{V_j} = \phi'^*_{\phi^{-1}(V_j)} \circ \phi^*_{V_j}$, and since V_j, $j \in J$ form a basis of $\Omega_{\mathfrak{n}(\mathcal{O})}$ we have $\psi^* = (\phi_* \phi'^*) \circ \phi^*$ as desired.

Next we show that the morphisms constructed in the proof of Proposition IV.3.3.13 satisfy $\operatorname{Spec}_{\mathrm{gr}}(\mathfrak{f}, \mathfrak{f}^*) \circ \beta_{\mathcal{O}'} = \beta_{\mathcal{O}} \circ \mathfrak{n}(\mathfrak{f}, \mathfrak{f}^*)$. For $j' \in \mathfrak{f}(j)$ and $p' \in V_j$ we have The neighbourhood basis of $\operatorname{Spec}_{\mathrm{gr}}(\mathfrak{f}, \mathfrak{f}^*)(\beta_{\mathcal{O}'}(p'))$ is the set of those U_j for which there exists $U'_{j'} \in \mathcal{B}_{\operatorname{Spec}_{\mathrm{gr}}(\mathfrak{f},\mathfrak{f}^*)^{-1}(U_j)} \cap \alpha_{\mathcal{O}'}(p')$, i.e. there exists $j' \in \mathfrak{f}(j) \cap p'$. On the other hand the neighbourhood basis of $\beta_{\mathcal{O}}(\phi(p'))$ is $\alpha_{\mathcal{O}}(\phi(p'))$, i.e. the set of those U_j such that $j \in \phi(p')$, i.e. $\mathfrak{f}(j) \cap p' \neq \emptyset$. Thus, the diagram of continuous maps commutes. Since the homomorphisms of structure sheaves are all defined in terms of the maps ρ^j_i the diagram of morphisms of graded schemes commutes. $\square$

Remark IV.3.3.16. Let X be the graded scheme corresponding to a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$. Then X is irreducible if and only if J defines a collection of neighbourhoods of an irreducible closed subset. Moreover, X is integrally graded if and only if J defines a collection of neighbourhoods of an irreducible closed subset and each $\mathcal{O}(j)$ is integrally graded, i.e. if and only if for all $j, k \in J$ we have $J_{\leq j} \cap J_{\leq k} \neq \emptyset$.

Definition IV.3.3.17. For each of the categories $\mathfrak{J}, \mathfrak{J}'$ and $\mathfrak{J}^{\mathrm{covar}}$ we obtain a category denoted $\mathfrak{J}_{\mathrm{int}}$, $\mathfrak{J}'_{\mathrm{int}}$ resp. $\mathfrak{J}^{\mathrm{covar}}_{\mathrm{int}}$ as follows: An object is a pair of a $\mathfrak{J}$-object $\mathcal{O}\colon J \to \mathfrak{C}$ and a simply graded $R \in \mathfrak{C}$, such that $\mathcal{O}$ maps into $\mathbf{GrSubAlg}_A(R)$, in particular all restrictions ρ^j_i are inclusions, and each of the canonical maps $Q_{\mathrm{gr}}(\mathcal{O}(j)) \to R$ is an isomorphism.

A morphism from $(\mathcal{O}, R)$ to $(\mathcal{O}', R')$ is a morphism $(\mathfrak{f}, \mathfrak{f}^*)$ resp. (α, α^*) in the given category such that (with respect to the morphism $(\mathfrak{f}, \mathfrak{f}^*)$ derived from (α, α^*)) each $\mathfrak{f}(j)$ is non-empty together with a $\mathfrak{C}$-morphism $\phi\colon R \to R'$, $\psi\colon gr(R) \to gr(R')$ such that $\mathfrak{f}^*_{j,j'}$ equals $\phi_{|\mathcal{O}(j)}\colon \mathcal{O}(j) \to \mathcal{O}(j')$ for each $j' \in \mathfrak{f}(j)$. The composition of morphisms is defined in terms of the composition of the constituent parts.

Construction IV.3.3.18. Let $\mathfrak{K}$ be the subcategory of $\mathfrak{J}$, $\mathfrak{J}'$ resp. $\mathfrak{J}^{\mathrm{covar}}$ whose objects $\mathcal{O}\colon J \to \mathfrak{C}$ satisfy that for all $j, k \in J$ we have $J_{\leq j} \cap J_{\leq k} \neq \emptyset$ and each $\mathcal{O}(j)$ is integrally graded, and whose morphisms $(\mathfrak{f}, \mathfrak{f}^*)$ (resp. (α, α^*)) satisfy $\mathfrak{f}(j) \neq \emptyset$ for each $j \in J$ (for the morphism $(\mathfrak{f}, \mathfrak{f}^*)$ derived from (α, α^*)).

For a $\mathfrak{K}$-object $\mathcal{O}$ the canonical injections $\mathcal{O}(j) \to R := \mathcal{K}(X)$ define a $\mathfrak{J}_{\mathrm{int}}$-object $(\widetilde{\mathcal{O}}, R)$. For a $\mathfrak{K}$-morphism $(\mathfrak{f}, \mathfrak{f}^*)$ from $\mathcal{O}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$ the induced homomorphism $\phi\colon \mathcal{K}(X) \to \mathcal{K}(X')$ together with $\mathfrak{f}$ and the induced homomorphisms $\widetilde{\mathfrak{f}}^*_{j,j'}$ for $j' \in \mathfrak{f}(j)$ form a $\mathfrak{J}_{\mathrm{int}}$-morphisms. This constitutes a functor from $\mathfrak{K}$ to $\mathfrak{J}_{\mathrm{int}}$, $\mathfrak{J}'_{\mathrm{int}}$ resp. $\mathfrak{J}^{\mathrm{covar}}_{\mathrm{int}}$.

Proposition IV.3.3.19. *The functor defined above is essentially inverse to the forgetful functor.*

IV.3.4. Schematic cofunctors of graded $\mathbb{F}_1$-schemes of finite type. The key statement of this section is that for $\mathbb{F}_1$-scheme of finite type, points are in bijection to affine open sets, see Proposition IV.3.4.3. From now on let $A = \mathbb{F}_1$ and $\mathfrak{C} = \mathbf{GrAlg}_{\mathbb{F}_1}$ Below, we consider a graded scheme X with the partial specialization order where $x \leq y$ if and only if x specializes to y, with anti-symmetry following from uniqueness of generic points.

Definition IV.3.4.1. Let $\mathfrak{O}$ resp. $\mathfrak{O}_{\mathbb{F}_1}^{\mathrm{covar}}$ be the full subcategory of $\mathfrak{J}_{\mathbb{F}_1}$ resp. $\mathfrak{J}_{\mathbb{F}_1}^{\mathrm{covar}}$ constituted by all $\mathfrak{J}_{\mathbb{F}_1}$-objects $\mathcal{O}\colon J \to \mathbf{GrAlg}_{\mathbb{F}_1}$ such that each $\mathcal{O}(j)$ is of finite type over $\mathbb{F}_1$.

Remark IV.3.4.2. For each $\mathfrak{O}$-object $\mathcal{O}\colon J \to \mathfrak{C}$, J is finite and hence generated by $J^{\max}$.

Proposition IV.3.4.3. *For a graded scheme $(X, \mathcal{O}_X)$ of finite type over $\mathbb{F}_1$ the following hold:*

(i) *Sending $U \in \mathcal{B}_X$ to the unique closed point x_U in U and $x \in X$ to its minimal affine neighbourhood U_x constitutes mutually inverse isomorphisms of partially ordered sets with X partially ordered by specialization.*
(ii) *U_x is the set of points specializing to x and $\rho_x^{U_x}$ is an isomorphism.*

Proof. In (i) let $V \in \mathcal{B}_x$. Then $\mathcal{O}(V)$ is generated as a monoid by certain elements $f_1, \ldots, f_n$. Let f be the product of those f_i which are lie in $(\rho_x^V)^{-1}(\mathcal{O}_{X,x}^*)$. Then f is not contained in any proper face and $U_x := V_f$ is the minimal element of $\mathcal{B}_V$ which contains x. For any further $W \in \mathcal{B}_x$, $W \cap V$ is covered by common principal subsets of V and W, meaning that x lies in one of these. By minimality in $\mathcal{B}_V$ we conclude $V_f \subseteq W$.

For (ii) first note that if y specializes to x then $y \in U_x$ and conversely, no proper principal subset of U_x contains x, which means that x is the point corresponding to the maximal ideal of $\mathcal{O}(U_x)$. In particular, $\rho_x^{U_x}$ is bijective and $x = x_{U_x}$.

Now, functoriality of $x \mapsto U_x$ is a consequence of (ii). For $U = U_{x_U}$ note that minimality of U_{x_U} implies $U_{x_U} \subseteq U$. The converse holds because x_U belongs to all Ω_U-closed subsets and hence lies in no proper principal subset of U. □

Construction IV.3.4.4. For a graded scheme $(X, \mathcal{O}_X)$ of finite type over $\mathbb{F}_1$ the functor (with respect to specialization) $\omega_X : X \to \mathfrak{C}$ sending $x \in X$ to $\mathcal{O}_{X,x}$ is a $\mathfrak{O}$-object. For a morphism $(\phi, \phi^*)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of graded schemes of finite type over $\mathbb{F}_1$ the (specialization preserving) map ϕ together with the natural maps $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ forms a $\mathfrak{O}_{\mathbb{F}_1}^{\mathrm{covar}}$-morphism. This constitutes a (covariant) functor to $\mathfrak{O}_{\mathbb{F}_1}^{\mathrm{covar}}$.

Proposition IV.3.4.5. *The following hold:*

(i) *$\mathfrak{O}' = \mathfrak{O}$, i.e. $\mathfrak{O}$ is a subcategory of $\mathfrak{J}'_{\mathbb{F}_1}$.*
(ii) *The functor ω sending X to ω_X is naturally isomorphic to the composition of the restriction of $\mathcal{O}_{|\mathcal{B}}$ to graded schemes of finite type over $\mathbb{F}_1$ with the anti-equivalence $\mathfrak{O}' \to \mathfrak{O}^{\mathrm{covar}}$.*

Proof. Let $(\phi, \phi^*)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism. In (i) observe that for $U \in \mathcal{B}_X$ the set $U_{\phi(x_U)} \in \mathcal{B}_Y$ is uniquely minimal among all $V \in \mathcal{B}_Y$ with $\phi(U) \subseteq V$, i.e. $U \subseteq \mathcal{B}_{\phi^{-1}(V)}$.

In (ii) the canonical isomorphism $X \to \mathcal{B}_X \setminus \{\emptyset\}$ from Proposition IV.3.4.3 together with the family of isomorphisms $\rho_x^{U_x}$ ensures that ω_X is an $\mathfrak{O}^{\mathrm{covar}}$-object and the pair constitutes the required $\mathfrak{O}^{\mathrm{covar}}$-isomorphism from ω_X to $(\mathcal{O}_X)_{|\mathcal{B}_X \setminus \{\emptyset\}}$. This constitues a natural isomorphism due to the proof of (i) and the fact that $\phi_x^* \circ \rho_{\phi(x)}^{U_{\phi(x)}} = \rho_x^{U_x} \circ \rho_{U_x}^{\phi^{-1}(U_{\phi(x)})} \circ \phi_{U_{\phi(x)}}^*$ holds for each $x \in X$. □

CHAPTER V

Cox sheaves on graded schemes of Krull type

In this chapter we discuss algebraic and geometric properties of Cox sheaves on graded schemes of Krull type as well as their global sections, which are called *Cox algebras*. The latter were introduced for toric varieties as *homogeneous coordinate rings* in [**10**] and their finite generation was shown in [**19**] to guarantee a normal variety's good behaviour under Mori's Minimal Model Program. Known properties of Cox algebras (of normal prevarieties) include integrality, normality and graded factoriality [**3, 4, 7**].

We study algebraic properties of Cox sheaves as a whole and show that they naturally are graded sheaves of Krull type whose grading group and defining graded valuations satisfy several extra conditions. These conditions characterize Cox sheaves and may be formulated entirely in terms of the sheaf itself, i.e. they provide an *intrinsic* algebraic characterization, see Section V.2.2.

Our basic geometric objects are graded schemes over A which are (locally) of Krull type, the base A being a graded algebra over $\mathbb{Z}$ or $\mathbb{F}_1$. In the non-graded case some aspects of Krull schemes were discussed [**21**]. We stress a sheaf-theoretic point of view on Weil divisors on a graded scheme X of Krull type so that the graded valuations ν_Y to the skyscraper sheaves $\mathbb{Z}^{(Y)}$ sum up to a homomorphism of presheaves $\mathrm{div}\colon (\mathcal{K}^{\mathrm{hom}})^* \to \mathrm{WDiv}$ in terms of which one defines PDiv and Cl, as well as the $\mathcal{O}_X$-modules $\mathcal{O}_X(D)$, see Section V.1.

Leading to our results on Cox sheaves Section V.2.1 discusses the more general notion of natural divisorial $\mathcal{O}_X$-algebras of Krull type. of which the divisorial algebra $\mathcal{O}(\mathrm{WDiv}(X))$ is an example. For graded schemes which are locally of Krull type we distinguish finite and arbitrary Weil divisors, the latter being only locally finite. This distinction leads to the concepts of quasi-Cox sheaves and Cox sheaves, which coincide in the presence of quasi-compactness.

The study of graded (quasi-)characteristic spaces, i.e. of relative graded spectra is the subject of Section V.3. After giving a set of characterizing conditions in the general case, we study the influence of finite generation conditions on the level of grading groups in Section V.3.2. Graded schemes of finite and Krull type over $\mathbb{F}_1$ as well as their class groups and graded characteristic spaces are treated in Section V.3.3. Lastly, we modify the setting of Section IV.2.3 to establish a covariant equivalence between $\mathbb{F}_1$-schemes of finite and Krull type and the category of *schematic combinatorial functors*, allowing us to translate graded characteristic spaces of these $\mathbb{F}_1$-schemes into more combinatorial terms. The results presented in this Chapter were in part published by the author in [**6**].

V.1. Divisors and their $\mathcal{O}_X$-modules on graded schemes of Krull type

V.1.1. Graded schemes of Krull type and their structure sheaves. We define prime divisors on graded schemes which are (locally) of Krull type and show that they define graded valuations which realize the structure sheaf as a sheaf of Krull type (with respect to the basis affine open sets).

Definition V.1.1.1. An integral graded scheme X is *locally of Krull type* if for some open affine cover $X = \bigcup_{i\in I} X_i$ every $\mathcal{O}(X_i)$ is a $gr(X)$-Krull ring. X is of *Krull type* or a *graded Krull scheme* if it is quasi-compact and locally of Krull type.

Definition V.1.1.2. Let X be a graded scheme which is locally of Krull type. A *prime divisor* on X is an irreducible closed subset Y of codimension one. For each open $U \subseteq X$ denote by $\mathcal{Y}_X(U)$ (or just $\mathcal{Y}(U)$) the set of prime divisors Y on X which intersect U non-trivially.

Remark V.1.1.3. By Section III.4 the assignment $\mathcal{Y}\colon U \mapsto \mathcal{Y}(U)$ defines a sheaf to the opposite category $\mathrm{Set}^{\mathrm{op}}$ of the category of sets. Moreover, $\mathcal{Y}$ commutes with finite intersections due to irreducibility of prime divisors. The stalk at an irreducible closed subset $A \subseteq X$ is $\mathcal{Y}_A = \{Y \in \mathcal{Y}(X) \mid A \subseteq Y\}$. Consequently, we have $\mathcal{Y}(U) = \bigcup_{x \in U} \mathcal{Y}_x$.

Remark V.1.1.4. Let $X = \mathrm{Spec}_{gr}(R)$ be the graded spectrum of a K-graded ring of Krull type. Then $\mathfrak{p} \mapsto \overline{\{\mathfrak{p}\}}$ defines a bijection $\mathfrak{P}(R) \to \mathcal{Y}(X)$ whose inverse assigns the genereric point. Moreover, we have $V(\mathfrak{q}) = \bigcap_{\mathfrak{q} \supseteq \mathfrak{p} \in \mathfrak{P}(R)} V(\mathfrak{p})$ by Remark II.2.5.6 for $\mathfrak{q} \in X$.

Construction V.1.1.5. Let X be an integral graded scheme which is locally of Krull type. By Remark V.1.1.4 each stalk $\mathcal{O}_{X,Y}$ at a prime divisor Y is a discrete graded valuation ring defined by a normed discrete graded valuation $\nu_{Y,X}$ on $\mathcal{K}(X) \cong Q_{\mathrm{gr}}(\mathcal{O}_{X,Y})$. We obtain a discrete graded valuation $\nu_Y\colon (\mathcal{K}^{\mathrm{hom}})^* \to \mathbb{Z}^{(Y)}$ to the skyscraper sheaf corresponding to Y by defining $\nu_{Y,U}(f)$ as $\nu_{Y,X}(f)$ if $Y \in \mathcal{Y}(U)$ and as 0 otherwise. Thus, $\mathcal{K}_{\nu_Y}(U)$ equals $\mathcal{O}_{X,Y}$ if $Y \in \mathcal{Y}(U)$ and $\mathcal{K}$ otherwise.

Proposition V.1.1.6. *For a graded scheme X which is locally of Krull type the following hold:*

(i) $\mathcal{O}_X = \bigcap_{Y \in \mathcal{Y}(X)} \mathcal{K}_{\nu_Y}$,
(ii) *for every open $U \subseteq X$ contained in a quasi-compact open subset of X and every $f \in (\mathcal{K}(X)^{\mathrm{hom}})^*$ only finitely many of the values $\nu_{Y,U}(f)$ are non-zero.*
(iii) *for every affine open $U \subseteq X$ the graded ring $\mathcal{O}_X(U)$ is of Krull type with essential graded valuations $\{\nu_{Y,U}\}_{Y \in \mathcal{Y}(U)}$.*

Proof. Let $X = \bigcup_{i \in I} X_i$ be an affine open cover such that each $\mathcal{O}(X_i)$ is of Krull type. To show (i) we use firstly that the assertions holds for localizations $U = (X_i)_f$, secondly that we always have $\mathcal{O}(\bigcup_j U_j) = \bigcap_j \mathcal{O}(U_j)$ and thirdly that $\mathcal{Y}$ commutes with unions. Assertion (ii) holds for localizations $(X_i)_f$ and hence also for finite unions of such.

In (iii) first note that by (ii) the sections over an affine open U are of Krull type. Thus, $\mathcal{Y}(U)$ is in natural bijection with $\mathfrak{P}(\mathcal{O}(U))$, and for corresponding Y and $\mathfrak{p}$ the graded valuations $\nu_{Y,U}$ and $\nu_{\mathfrak{p}}$ both are normed and define $\mathcal{O}_{X,Y} = \mathcal{O}(U)_{\mathfrak{p}}$ in $\mathcal{K}(X) = Q_{\mathrm{gr}}(\mathcal{O}(U))$ which means that they coincide. □

Remark V.1.1.7. Let X be a graded scheme of Krull type and let $A \subseteq X$ be closed and irreducible. Denote by $\jmath\colon \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,A}) \to X$ the inclusion morphism of all points which specialize to the generic point of A. Then the assignments $Y \mapsto (\rho_Y^A)^{-1}(\mathfrak{m}_{X,Y})$ and $\mathfrak{p} \mapsto \overline{\{\jmath(\mathfrak{p})\}}$ define mutually inverse natural bijections between $\mathcal{Y}_A$ and $\mathfrak{P}(\mathcal{O}_{X,A})$.

Corollary V.1.1.8. *Let X be an integral graded scheme. Then X is of Krull type if and only if X is quasi-compact and $\mathcal{O}_X$ is of Krull type, and in this case $\{\nu_Y\}_{Y \in \mathcal{Y}(X)}$ are the essential graded valuations of $\mathcal{O}_X$.*

Remark V.1.1.9. Let X be a graded scheme of Krull type and let $U \in \mathcal{B}_X$. By construction, the canonical isomorphism $\jmath\colon \mathcal{K}(U) \to Q_{\mathrm{gr}}(\mathcal{O}(U))$ respects valuations, i.e. for a prime divisor $Y \in \mathcal{Y}(U)$ and the corresponding homogeneously prime divisor $\mathfrak{p} := I(Y \cap U) \in \mathfrak{P}(\mathcal{O}(U))$ we have $\nu_{Y,X} = \nu_{\mathfrak{p}} \circ \jmath_{|(\mathcal{K}(U)^{\mathrm{hom}})^*}$.

REMARK V.1.1.10. Let X be a graded scheme of Krull type and let $A \subseteq X$ be closed and irreducible. Then the locally graded ring $\mathcal{O}_{X,A}$ is of Krull type with essential graded valuations $\{\nu_{Y,X}\}_{Y\in\mathcal{Y}_A}$, because for an affine neighbourhood U of A we have $\mathcal{O}_{X,A} \cong \mathcal{O}(U)_{I(A\cap U)}$. In particular, we have

$$\mathfrak{m}_A \cap (\mathcal{O}_{X,A}^{\mathrm{hom}} \setminus \{0\}) = \{f \in \mathcal{O}_{X,A}^{\mathrm{hom}} \setminus \{0\} \mid \text{ there is } Y \in \mathcal{Y}_A \text{ with } \nu_{Y,X}(f) > 0\}.$$

REMARK V.1.1.11. Let X be locally of Krull type and let $Y \in \mathcal{Y}(X)$. Then due to Remark II.2.5.7 there exists $U \in \mathcal{B}_{X,Y}$ such that $I(Y\cap U) = \langle f\rangle$ holds for some $f \in \mathcal{O}(U)^{\mathrm{hom}}$.

PROPOSITION V.1.1.12. *Let $\phi\colon X \to X'$ be a dominant morphism between graded schemes which are locally of Krull type such that $\phi(X)$ intersects each prime divisor of X'. Then the preimage of a prime divisor contains a prime divisor.*

PROOF. For $Y' \in \mathcal{Y}(X')$ let $U' \in \mathcal{B}_{X',Y'}$ be such that $I(Y'\cap U') = \langle f\rangle$ holds for some $f \in \mathcal{O}(U')^{\mathrm{hom}}$. Let $x \in X$ be such that $\phi(x)$ is the generic point of Y'. Then there exists $U \in \mathcal{B}_{\phi^{-1}(U')} \cap \mathcal{B}_{X,x}$ and $\phi_{|U}^{-1}(Y'\cap U') = V(\phi_{U'}^*(f)_{|U})$ contains x and is hence purely one-codimensional and so is its closure. $\square$

V.1.2. Weil divisors and class groups. Here, we give a sheaf-theoretic definition of Weil divisors, class groups and pullbacks thereof.

CONSTRUCTION V.1.2.1. Let X be a graded scheme which is locally of Krull type. Then $\prod_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)}$ is a sheaf of partially ordered abelian groups with each $Y \in \mathcal{Y}(X)$ defining a homomorphism $pr_Y\colon \prod_{Y'\in\mathcal{Y}(X)} \mathbb{Z}^{(Y')} \to \mathbb{Z}^{(Y)}$. The direct sum of presheaves $\mathrm{WDiv}_X^{\mathrm{fin}} := \bigoplus_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)} \subseteq \prod_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)}$ is the presheaf of *finite Weil divisors*. The sheaf of *Weil divisors* on X is the sheafification WDiv_X of $\mathrm{WDiv}_X^{\mathrm{fin}}$ in $\prod_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)}$. In other words, $\mathrm{WDiv}_X(U)$ is the group of those $D = (d_Y)_Y$ in $\prod_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)}(U)$ for which there exists a cover by $V \in \Omega_U$ such that each $D_{|V}$ belongs to $\bigoplus_{Y\in\mathcal{Y}(V)} \mathbb{Z}$. Elements of $\mathrm{WDiv}_X(U)$ are also written as *locally finite formal sums* $D = \sum_{Y\in\mathcal{Y}(U)} d_Y Y$. The subsheaves $\mathrm{WDiv}_{X,\geq 0}$ and $\mathrm{WDiv}_{X,\geq 0}^{\mathrm{fin}}$ of *effective* (finite) Weil divisors assign those $D = \sum_Y d_Y Y$ in $\mathrm{WDiv}_X(U)$ resp. $\mathrm{WDiv}_X^{\mathrm{fin}}(U)$ with $d_Y \geq 0$ for all $Y \in \mathcal{Y}_X(U)$.

PROOF. To show that WDiv is isomorphic to the sheafification of $\mathrm{WDiv}_X^{\mathrm{fin}}$ we observe note that the canonical map $D \mapsto (D_x)_{x\in U}$ is inverse to the map sending $(E_x^{(x)})_{x\in U}$, where $E^{(x)} \in \bigoplus_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)}(V^{(x)})$ with some $V^{(x)} \in \Omega_{X,x}$, to $(pr_{\overline{\{\eta\}},\overline{\{\eta\}}}(E_\eta^{(\eta)}))_{\eta\in H(U)}$ where η runs through the set $H(U)$ of points of U with one-codimensional closure. $\square$

When we are concerned with only a single graded scheme X which is locally of Krull type we will omit the index and use the notations WDiv, $\mathrm{WDiv}_{\geq 0}$, $\mathrm{WDiv}^{\mathrm{fin}}$ and $\mathrm{WDiv}_{\geq 0}^{\mathrm{fin}}$.

REMARK V.1.2.2. In the above situation, we have $\mathrm{WDiv}_X(U) = \mathrm{WDiv}_X^{\mathrm{fin}}(U)$ for each $U \in \Omega_X$ which is contained in some quasi-compact open subset of X. Moreover, an element $D \in \prod_{Y\in\mathcal{Y}(X)} \mathbb{Z}^{(Y)}(V)$ belongs to $\mathrm{WDiv}_X(V)$ if and only if $D_{|U} \in \mathrm{WDiv}_X^{\mathrm{fin}}(U)$ holds for each member U of a cover $\mathcal{U} \subseteq \mathcal{B}_V$ of V (resp. for $\mathcal{U} = \mathcal{B}_V$).

REMARK V.1.2.3. If X is locally of Krull type then $\mathcal{Y}(X)$ is in canonical bijection to the set of minimal positive elements of $\mathrm{WDiv}(X)$ (and indeed, of $\prod_{Y\in\mathcal{Y}(X)} \mathbb{Z}$ and $\mathrm{WDiv}^{\mathrm{fin}}(X)$), which is also the set of prime elements of $\mathrm{WDiv}(X)_{\geq 0}$ (and of $\prod_{Y\in\mathcal{Y}(X)} \mathbb{N}_0$ and $\mathrm{WDiv}_{\geq 0}^{\mathrm{fin}}(X)$).

REMARK V.1.2.4. For an open subset U of a graded scheme X which is locally of Krull type the bijection $\mathcal{Y}_U(U) \to \mathcal{Y}_X(U)$ induces an isomorphism between WDiv_U and $(\mathrm{WDiv}_X)_U$ which restricts to an isomorphism between $\mathrm{WDiv}_U^{\mathrm{fin}}$ and $(\mathrm{WDiv}_X^{\mathrm{fin}})_{|U}$.

CONSTRUCTION V.1.2.5. Let X be locally of Krull type. Then the *divisor homomorphism* is $\mathrm{div} := \sum_Y \nu_Y : (\mathcal{K}^{\mathrm{hom}})^* \to \mathrm{WDiv}$. By PDiv we denote the image presheaf of *principal divisors*. Its sheafification CaDiv in WDiv is the sheaf of *Cartier divisors*. Specifically, $\mathrm{CaDiv}(U)$ is the set of all $D \in \mathrm{WDiv}(U)$ such that for each $x \in U$ we have $D_x \in \mathrm{PDiv}_x$. We then have $\mathcal{O}_X^{\mathrm{hom}} \setminus \{0\} = \mathrm{div}^{-1}(\mathrm{WDiv}_{\geq 0})$ and $(\mathcal{O}_X^{\mathrm{hom}})^* = \ker(\mathrm{div})$. For the presheaf $\mathrm{Cl} := \mathrm{coker}\,(\mathrm{div})$ of *class groups* the canonical homomorphism is denoted $c\colon \mathrm{WDiv} \to \mathrm{Cl}$. The quotient $\mathrm{CaDiv} \,/\, \mathrm{PDiv}$ is the presheaf Pic of *Picard groups*.

CONSTRUCTION V.1.2.6. Let X be locally of Krull type and let $x \in X$ be a point, with the canonical inclusion morphism denoted $\imath_x \colon \{x\} \to X$. Then the preimage of the constant sheaf $(\imath_x)_*\imath_x^{-1}\,\mathrm{PDiv}_X$ under the canonical homomorphism $\mathrm{WDiv}_X \to (\imath_x)_*\imath_x^{-1}\,\mathrm{WDiv}_X$ is the sheaf $\mathrm{PDiv}_X^{(x)}$ of Weil divisors which are *principal near x*. Applying $c\colon \mathrm{WDiv}_X \to \mathrm{Cl}_X$ to $\mathrm{PDiv}_X^{(x)}$ gives the kernel $\mathrm{Cl}_X^{(x)}$ of the canonical homomorphism $\mathrm{Cl}_X \to (\imath_x)_*\imath_x^{-1}\,\mathrm{Cl}_X$. In this notation we have

$$\mathrm{CaDiv}_X = \bigcap_{x\in X} \mathrm{PDiv}_X^{(x)} = \bigcap_{\substack{x\in X\\ \overline{\{x\}}=\{x\}}} \mathrm{PDiv}_X^{(x)}, \quad \mathrm{Pic}_X = \bigcap_{x\in X} \mathrm{Cl}_X^{(x)} = \bigcap_{\substack{x\in X\\ \overline{\{x\}}=\{x\}}} \mathrm{Cl}_X^{(x)}.$$

REMARK V.1.2.7. Let X be a graded scheme of Krull type and let $U \subseteq X$ be open and affine. Then the bijection $\mathcal{Y}(U) \to \mathfrak{P}(\mathcal{O}(U))$ extends to an isomorphism $\phi\colon \mathrm{WDiv}(U) \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U))$ of partially ordered groups and with the isomorphism $\theta\colon \mathcal{K}(U) \to Q_{\mathrm{gr}}(\mathcal{O}(U))$ we have $\mathrm{div}^{\mathrm{gr}}_{\mathcal{O}(U)} \circ \theta_{|(\mathcal{K}(X)^{\mathrm{hom}})^*} = \phi \circ \mathrm{div}_U$. Consequently, we have

$$\phi(D) = \Big\{ \sum_{w\in K} f_w \in Q_{\mathrm{gr}}(\mathcal{O}(U)) \;\Big|\; f_w = 0 \text{ or } \mathrm{div}^{\mathrm{gr}}_{\mathcal{O}(U)}(f_w) \geq \phi(D) \Big\} \leq Q_{\mathrm{gr}}(\mathcal{O}(U)),$$

see Section II.2.5. ϕ restricts to an isomorphism $\mathrm{PDiv}(U) \to \mathrm{PDiv}_{\mathrm{gr}}(\mathcal{O}(U))$ and hence induces an isomorphism $\mathrm{Cl}(U) \to \mathrm{Cl}_{\mathrm{gr}}(\mathcal{O}(U))$.

REMARK V.1.2.8. Let X be a graded scheme of Krull type and let $A \subseteq X$ be a closed irreducible subset. Then the bijection $\mathcal{Y}_A \to \mathfrak{P}(\mathcal{O}_{X,A})$ extends to an isomorphism of partially ordered groups

$$\phi_A\colon \mathrm{WDiv}_A = \bigoplus_{Y\in\mathcal{Y}(X)} \mathbb{Z}_A^{(Y)} = \bigoplus_{Y\in\mathcal{Y}_A} \mathbb{Z} \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}_{X,A})$$

which together with the canonical isomorphism $\theta_A\colon \mathcal{K}_A = \mathcal{K}(X) \to Q_{\mathrm{gr}}(\mathcal{O}_{X,A})$ satisfies $\mathrm{div}^{\mathrm{gr}}_{\mathcal{O}_{X,A}} \circ (\theta_A)_{|(\mathcal{K}_A^{\mathrm{hom}})^*} = \phi_A \circ \mathrm{div}_A$. Consequently, ϕ_A restricts to an isomorphism $\mathrm{PDiv}_A \to \mathrm{PDiv}_{\mathrm{gr}}(\mathcal{O}_{X,A})$ and induces an isomorphism $\mathrm{Cl}_A \to \mathrm{Cl}_{\mathrm{gr}}(\mathcal{O}_{X,A})$.

CONSTRUCTION V.1.2.9. Let X be a graded scheme which is locally of Krull type and satisfies $\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}^{\mathrm{fin}}(X)$, i.e. $\mathrm{PDiv} \subseteq \mathrm{WDiv}^{\mathrm{fin}}$. Then the quotient $\mathrm{Cl}^{\mathrm{fin}} := \mathrm{WDiv}^{\mathrm{fin}} \,/\, \mathrm{PDiv}$ is the presheaf of *finite divisor classes* on X.

REMARK V.1.2.10. Let X be locally of Krull type with $\mathrm{PDiv} \subseteq \mathrm{WDiv}^{\mathrm{fin}}$ and let $V \subseteq U$ be open subsets. Then we have an exact sequence

$$\textstyle\sum_{Y\in\mathcal{Y}(U)\setminus\mathcal{Y}(V)} \mathbb{Z}Y \longrightarrow \mathrm{Cl}^{\mathrm{fin}}(U) \longrightarrow \mathrm{Cl}^{\mathrm{fin}}(V) \longrightarrow 0.$$

CONSTRUCTION V.1.2.11. Let $\phi\colon X \to X'$ be a dominant morphism between graded schemes which are locally of Krull type. Then we have a homomorphism $\mathrm{PDiv}_{X'} \to \phi_*\,\mathrm{PDiv}_X$ which sends $\mathrm{div}_{U'}(f')$ to $\mathrm{div}_{\phi^{-1}(U')}(\phi^*_{U'}(f'))$.

For each $Y \in \mathcal{Y}(X)$ we obtain a homomorphism $\mathrm{WDiv}_{X'} \to \phi_* \mathbb{Z}^{(Y)}$ as follows: If $\mathrm{Cl}_{\overline{\phi(Y)}} \neq 0$ or $Y \notin \mathcal{Y}(\phi^{-1}(U'))$ then $\mathrm{WDiv}_{X'}(U') \to \mathbb{Z}^{(Y)}(\phi^{-1}(U'))$ is the zero map. Otherwise, it is the canonical homomorphism

$$\mathrm{WDiv}_{X'}(U') \longrightarrow \mathrm{WDiv}_{X',\overline{\phi(Y)}} = \mathrm{PDiv}_{X',\overline{\phi(Y)}} \longrightarrow \mathrm{PDiv}_{X,Y} \cong \mathbb{Z}^{(Y)}(\phi^{-1}(U')).$$

The induced homomorphism $\mathrm{WDiv}_{X'} \to \phi_* \prod_{Y \in \mathcal{Y}(X)} \mathbb{Z}^{(Y)}$ has image in $\phi_* \mathrm{WDiv}_X$ and thereby defines the *pullback homomorphism* $\phi^*\colon \mathrm{WDiv}_{X'} \to \phi_* \mathrm{WDiv}_X$.

Proof. Consider $U' \in \mathcal{B}_X$ and $U \in \mathcal{B}_{\phi^{-1}(U')}$ and denote the canonical homomorphism by $\alpha := \rho_U^{\phi^{-1}(U')} \circ \phi_{U'}^*\colon \mathcal{O}(U') \to \mathcal{O}(U)$. Then for each $Y' \in \mathcal{Y}_{X'}(U')$ there exist only finitely many $Y \in \mathcal{Y}_X(U)$ with $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ and $\phi(Y) \subseteq Y'$ because by Proposition I.2.6.8 for each $\mathfrak{p}' \in \mathfrak{P}_{\mathrm{gr}}(\mathcal{O}(U'))$ there exist only finitely many $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(\mathcal{O}(U))$ with $\mathrm{Cl}(\mathcal{O}(U')_{\alpha^{-1}(\mathfrak{p})}) = 0$ and $\mathfrak{p}' \subseteq \alpha^{-1}(\mathfrak{p})$. $\square$

Remark V.1.2.12. For a dominant morphism $\phi\colon X \to X'$ between graded schemes of Krull type the following hold:

(i) We have $\mathrm{pr}_{Y,X}(\phi_{X'}^*(\mathrm{div}_{X'}(f))) = \nu_{Y,X}(\phi_{X'}^*(f))$ for each $f \in (\mathcal{K}(X')^{\mathrm{hom}})^*$ and $Y \in \mathcal{Y}(X)$ with $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$. Thus, if $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ holds for all $Y \in \mathcal{Y}(X)$ then we also have a pullback homomorphism $\phi^*\colon \mathrm{Cl}_{X'} \to \mathrm{Cl}_X$ of presheaves.

(ii) If X and X' are affine then the canonical maps form a commutative diagram:

$$\begin{array}{ccc} \mathrm{WDiv}(X') & \longrightarrow & \mathrm{WDiv}(X) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(X')) & \longrightarrow & \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(X)) \end{array}$$

(iii) For a closed irreducible $A \subseteq X$ (e.g. $A = \overline{\{x\}}$) we have a canonical commutative diagram

$$\begin{array}{ccc} \mathrm{WDiv}_{X',\overline{\phi(A)}} & \xrightarrow{\phi_A^*} & \mathrm{WDiv}_{X,A} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}_{X',\overline{\phi(A)}}) & \longrightarrow & \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}_{X,A}). \end{array}$$

(iv) If ϕ is an open embedding then $\phi_{X'}^*\colon \mathrm{WDiv}_{X'}(X') \to \mathrm{WDiv}_X(X)$ equals the composition of the restriction map $\rho_{\phi(X')}^{X'}$ with the canonical isomorphism $\mathrm{WDiv}_X(\phi(X')) \cong \mathrm{WDiv}_{X'}(X')$.

Corollary V.1.2.13. *Let $\phi\colon X \to X'$ be a dominant affine morphism between graded schemes which are locally of Krull type. Then the following are equivalent:*

(i) *ϕ induces a bijection $\mathcal{Y}(X) \to \mathcal{Y}(X'), Y \mapsto \overline{\phi(Y)}$ and for each $Y \in \mathcal{Y}(X)$ we have $\nu_Y \circ \phi_{|(\mathcal{K}_{X'}^{\mathrm{hom}})^*} = \nu_{\overline{\phi(Y)}}$.*

(ii) *The pullback $\phi_{X'}^*\colon \mathrm{WDiv}(X') \to \mathrm{WDiv}(X)$ restricts to a bijection of the sets of prime divisors.*

(iii) *$\phi_{X'}^*\colon \mathrm{WDiv}(X') \to \mathrm{WDiv}(X)$ is an isomorphism of partially ordered groups.*

(iv) *$\phi^*\colon \mathrm{WDiv}_{X'} \to \phi_* \mathrm{WDiv}_X$ is an isomorphism (of sheaves of partially ordered groups).*

(v) *For each $U' \in \mathcal{B}_{X'}$ the canonical map $\mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U')) \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U))$ is an isomorphism.*

PROOF. The equivalence of the last to conditions is due to the fact that $\mathcal{B}_{X'}$ is a basis and the above remark. Condition (iii) implies (ii) because $\mathcal{Y}(X)$ is the set of minimal positive elements of $\mathrm{WDiv}(X)$. Finally, (i) implies (v) because $\mathcal{Y}(U)$ generates $\mathrm{WDiv}(U)$. $\square$

PROPOSITION V.1.2.14. *Let $\phi\colon X \to X'$ and $\phi'\colon X' \to X''$ be dominant morphisms between graded schemes which are locally of Krull type. Let $Y \in \mathcal{Y}(X)$ satisfy $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ and $\mathrm{Cl}_{\overline{\phi'(\phi(Y))}} = 0$. Then the respective pullbacks of Weil divisors satisfy $\phi'_*\phi_* pr_Y \circ (\phi' \circ \phi)^* = \phi'_*\phi_* pr_Y \circ \phi'_*\phi^* \circ \phi'^*$.*

PROOF. By assumption the restriction $(\phi' \circ \phi)^*_Y\colon \mathrm{PDiv}_{\overline{\phi'(\phi(Y))}} \to \mathrm{PDiv}_Y$ is the composition of the restricitions $\phi'^*_{\overline{\phi(Y)}}\colon \mathrm{PDiv}_{\overline{\phi'(\phi(Y))}} \to \mathrm{PDiv}_{\overline{\phi(Y)}}$ and $\phi^*_Y\colon \mathrm{PDiv}_{\overline{\phi(Y)}} \to \mathrm{PDiv}_Y$. For $U \in \Omega_{X'',\overline{\phi'(\phi(Y))}}$ and $D \in \mathrm{WDiv}_{X''}(U)$ we then calculate

$$
\begin{aligned}
pr_{Y,\phi^{-1}(\phi'^{-1}(U))}((\phi' \circ \phi)^*_U D) &= ((\phi' \circ \phi)^*_U D)_Y = (\phi' \circ \phi)^*_Y D_{\overline{\phi'(\phi(Y))}} \\
&= \phi^*_Y(\phi'^*_{\overline{\phi(Y)}} D_{\overline{\phi'(\phi(Y))}}) = \phi^*_Y((\phi'^*_U D)_{\overline{\phi(Y)}}) \\
&= (\phi^*_{\phi'^{-1}(U)}(\phi'^*_U D))_Y = ((\phi'_*\phi^*)_U(\phi'^*_U D))_Y \\
&= ((\phi'_*\phi^* \circ \phi'^*)_U D)_Y \\
&= pr_{Y,\phi^{-1}(\phi'^{-1}(U))}((\phi'_*\phi^* \circ \phi'^*)_U D).
\end{aligned}
$$

$\square$

EXAMPLE V.1.2.15. Let X be locally of Krull type such that $gr(\mathcal{O}_X) = K \oplus F$ holds for some free abelian group F and an abelian group K. The projection map $pr_K\colon gr(\mathcal{O}_X) \to K$ then defines an $\mathcal{O}_X$-algebra $\mathcal{O}_X \to \mathcal{A}$ and a morphism $\pi\colon X_K := \mathrm{Spec}_X(\mathcal{A}) \to X$. By Theorem II.2.5.15 X_K is locally of Krull type, by Lemma II.2.5.14 the pullback of Weil divisors is injective and commutes with the divisor homomorphism, and the induced pullback $\mathrm{Cl}(X) \to \mathrm{Cl}(X_K)$ is injective.

For the last statement, consider $D \in \mathrm{WDiv}(X)$ with $\pi^*_X D = \mathrm{div}_{X_K}(f)$ for some $f \in (\mathcal{K}(X_K)^{\mathrm{hom}})^*$. By Theorem II.2.5.15 $R := \mathcal{K}(X)$ is a K-Krull ring, and it satisfies $Q_K(R) = \mathcal{K}(X_K)$. For each $U \in \mathcal{B}_X$ the element of $\mathrm{Div}(\mathcal{O}(\pi^{-1}(U)))$ corresponding to $\pi^*_U(D_{|U})$ is supported solely on K-prime divisors which contain a non-zero $K \oplus F$-homogeneous element. Lemma II.2.5.14 now gives $\mathrm{div}_{R,K}(f) = 0$ and hence f is a K-homogeneous unit of R and thereby $K \oplus F$-homogeneous. We thus have $\pi^*_X D = \pi^*_X \mathrm{div}_X(f)$ and conclude $D = \mathrm{div}_X(f)$.

V.1.3. Support and cones of divisors. In this section we treat the notions of support of divisors and stable base loci, as well as various related monoids resp. cones of divisor classes.

DEFINITION V.1.3.1. Let X be locally of Krull type and let $U \in \Omega_X$. The *support* of $D \in \mathrm{WDiv}(U)$ is the (closed) intersection $|D|$ of U with the union over the prime divisors occuring with non-zero coefficient in D.

REMARK V.1.3.2. $X \setminus |D|$ is the set of $x \in X$ with $D_x = 0_x \in \mathrm{WDiv}_x$.

REMARK V.1.3.3. Let X be a graded scheme of Krull type, let $U \subseteq X$ be an open set and $f \in \mathcal{O}(U)^{\mathrm{hom}} \setminus \{0\}$. Then

$$U_f = \{x \in U \mid f_x \in \ker(\mathrm{div}_x)\} = U \setminus |\mathrm{div}_U(f)|$$

In particular, if U is affine then $V_U(f) = |\mathrm{div}_U(f)|$.

PROPOSITION V.1.3.4. *Let $\phi\colon X \to X'$ be a dominant morphism between graded schemes which are locally of Krull type. Then the inclusion $|\phi^*_{X'}D'| \subseteq \phi^{-1}(|D'|)$ is an equality for each $D' \in \mathrm{WDiv}(X')_{\geq 0}$ such that $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ holds for all $Y \in \mathcal{Y}(X)$ with $\phi(Y) \subseteq |D'|$.*

Proof. Consider $x \in X$ and assume that $D' = Y'$ is prime. If $\phi(x) \in |Y'|$ then $Y'_{\phi(x)} > 0$ and the assumption gives $(\phi^*_{X'} Y')_x = \phi^*_x(Y'_{\phi(x)}) > 0$, i.e. $x \in |\phi^*_{X'} Y'|$. $\square$

Proposition V.1.3.5. *Let X be locally of Krull type such that the intersection of affine subsets is affine. Then $X \setminus U = \bigcup_{Y \in \mathcal{Y}(X) \setminus \mathcal{Y}(U)} |Y|$ holds for each non-empty $U \in \mathcal{B}_X$. If X is quasi-compact then $\mathcal{Y}(X) \setminus \mathcal{Y}(U)$ is finite.*

Proof. Let $U, W \in \mathcal{B}_X \setminus \{\emptyset\}$. Then there exists $0 \neq f \in I(W \setminus U)^{\mathrm{hom}}$ and we have $W_f \subseteq U \cap W$, in particular, $\mathcal{Y}(W) \setminus \mathcal{Y}(U \cap W)$ is finite. Note that if X has a finite cover $\mathcal{W}$ by such W then $\mathcal{Y}(X) \setminus \mathcal{Y}(U)$ is the finite union over the finite sets $\mathcal{Y}(W) \setminus \mathcal{Y}(U \cap W)$.

Consequently, the complement U' in W of all prime divisors contained in $X \setminus U$. Then we have $U \cap W \subseteq U'$ and $\mathcal{Y}(U \cap W) = \mathcal{Y}(U')$. Since X is of Krull type this implies $\mathcal{O}(U \cap W) = \mathcal{O}(U')$. Now, if W_g is a principal subset contained in U', then the restricition monomorphism $\mathcal{O}(W_g) \to \mathcal{O}((U \cap W)_{g_{|U \cap W}})$ is surjective because $\mathcal{Y}(W_g) = \mathcal{Y}((U \cap W)_{g_{|U \cap W}})$. Then $(U \cap W)_{g_{|U \cap W}} \subseteq W_g$ is an isomorphism because both are affine, which means $W_g \subseteq U \cap W$ and we conclude $U \cap W = U'$. $\square$

In the following we use the notation

$$\mathrm{Cl}(X)_{\mathbb{Q}} := \mathbb{N}^{-1}\,\mathrm{WDiv}(X)/\mathbb{N}^{-1}\,\mathrm{PDiv}(X) \cong \mathbb{N}^{-1}\,\mathrm{Cl}(X)$$

for the vector space of *rational divisor classes*. The following invariants are used to introduce (semi-)ample cones later on.

Definition V.1.3.6. Let X be a graded scheme which is locally of Krull type and let $B \subseteq X$ be closed and irreducible.

(i) $S_{\mathrm{WDiv}(X),B}$ resp. $\omega_{\mathrm{WDiv}(X),B}$ is the functor on $\Omega^{op}_{X,B}$ which assigns to U the submonoid of those D in $\mathrm{WDiv}(X)$ resp. $\mathbb{N}^{-1}\,\mathrm{WDiv}(X)$ with $D_{|U} \geq 0$ and $D_B = 0_B$. Furthermore, we set $S_{\mathrm{Cl}(X),B} := c_X \circ S_{\mathrm{WDiv}(X),B}$ and $\omega_{\mathrm{Cl}(X),B} := \mathbb{N}^{-1} c_X \circ \omega_{\mathrm{WDiv}(X),B}$.

(ii) For $U \in \Omega_{X,B}$ let $S^{\mathrm{aff}}_{\mathrm{WDiv}(X),B}(U)$ resp. $\omega^{\mathrm{aff}}_{\mathrm{WDiv}(X),B}(U)$ be the set of those D in $S_{\mathrm{WDiv}(X),B}(U)$ resp. $\omega_{\mathrm{WDiv}(X),B}(U)$ with $U \setminus |D| \in \mathcal{B}_X$. Again, the image under c_X resp. $\mathbb{N}^{-1} c_X$ is denoted $S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U)$ resp. $\omega^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U)$.

Remark V.1.3.7. If X is of Krull type then for $D \in S_{\mathrm{WDiv}(X),B}(U)$ we have $S_{\mathrm{WDiv}(X),B}(U \setminus |D|) = S_{\mathrm{WDiv}(X),B}(U)_D$. Moreover, for $U, V \in \Omega_{X,B}$ with $U \subseteq V$ we have $S_{\mathrm{WDiv}(X),B}(V)^\circ \subseteq S_{\mathrm{WDiv}(X),B}(U)^\circ$. In the same way, we obtain functors $\omega^\circ_{\mathrm{WDiv}(X),x}$, $S^\circ_{\mathrm{Cl}(X),x}$ and $\omega^\circ_{\mathrm{Cl}(X),x}$.

Proposition V.1.3.8. *Let X be locally of Krull type. Then for each B the following hold:*

(i) *Each $S^{\mathrm{aff}}_{\mathrm{WDiv}(X),B}(U)$ is $\mathbb{N}$-invariant and the saturation $\mathrm{sat}(S^{\mathrm{aff}}_{\mathrm{Cl}(X),B})(U)$ of $S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U)$ is a semigroup. Moreover, we have*

$$\omega^{\mathrm{aff}}_{\mathrm{WDiv}(X),B}(U) = \mathbb{N}^{-1} S^{\mathrm{aff}}_{\mathrm{WDiv}(X),B}(U), \quad \omega^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U) = \mathbb{N}^{-1} S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U).$$

(ii) *For $U, V \in \Omega_{X,B}$ with $U \subseteq V$ we have*

$$\mathrm{sat}(S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(V)) \subseteq \mathrm{sat}(S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U)), \quad \omega^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(V)) \subseteq \omega^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U)).$$

Proof. For (i) consider $v, w \in \mathrm{Cl}(X)$ such that there exist $m, n \in \mathbb{N}$ and $D \in mv$, $E \in nw$ with $D_{|U}, E_{|U} \geq 0$ and $U \setminus |D|, U \setminus |E| \in \mathcal{B}_{X,B}$. Then there exists a principal $(U \setminus |D|)_f$ which is contained in $U \setminus |D+E|$ and intersects B non-trivially. For $k \in \mathbb{N}$ large enough we then have $(k(nD+mE) + \mathrm{div}_X(f))_{|U} \geq 0$ and $U \setminus |k(nD+mE) + \mathrm{div}_X(f)| = (U \setminus |D|)_f$. Thus, $kmn(v+w) \in \mathrm{sat}(S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U))$.

In (ii) let $U \subseteq V$ with $U \in \Omega_{X,B}$ and consider $w \in \mathrm{Cl}(X)$ and $n \in \mathbb{N}$ as well as $D \in w$ with $D_{|V} \geq 0$ and $V \setminus |D| \in \mathcal{B}_{X,B}$. Let $(V \setminus |D|)_f \in \mathcal{B}^{\mathrm{pr}}_{V \setminus |D|,B}$ with

$(V \setminus |D|)_f \subseteq U \setminus |D|$. Then $(nD + \mathrm{div}(f))_{|U} \geq 0$ holds with some large $n \in \mathbb{N}$ such that $(V \setminus |D|)_f = V \setminus |nD + \mathrm{div}(f)| = U \setminus |nD + \mathrm{div}(f)|$. Thus, $nw \in S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}(U)$. $\square$

Definition V.1.3.9. Let X be of Krull type and let w be an element of $\mathrm{Cl}(X)$ resp. $\mathrm{Cl}(X)_{\mathbb{Q}}$. The $\mathcal{P}(X)^{op}$-presheaf $\mathrm{Bas}(w)$ resp. $\mathrm{StBas}(w)$ of *base loci* resp. *stable base loci* assigns to $U \in \Omega_X$ the (closed) set of those $x \in U$ with $w \notin S_{\mathrm{Cl}(X),\overline{\{x\}}}(U)$ resp. $w \notin \omega_{\mathrm{Cl}(X),\overline{\{x\}}}(U)$.

Remark V.1.3.10. Let X be of Krull type, let $B \subseteq X$ be closed and irreducible, and consider $U \in \Omega_{X,B}$. Then $B \cap U$ is contained in $\mathrm{Bas}(w)(U)$ resp. $\mathrm{StBas}(w)(U)$ if and only if w does not lie in $S_{\mathrm{Cl}(X),B}(U)$ resp. $\omega_{\mathrm{Cl}(X),B}(U)$.

Remark V.1.3.11. The preimage of $\mathrm{Bas}(v)_x$ resp. $\mathrm{StBas}(w)_x$ under the morphism $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,x}) \to X$ is $\mathrm{Bas}(v_x)(\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,x}))$ resp. $\mathrm{StBas}(w_x)(\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}_{X,x}))$.

Remark V.1.3.12. Let X be locally of Krull type. Then with suitable elements of $\mathrm{Cl}(X)$, $\mathrm{Cl}(X)_{\mathbb{Q}}$ and $\mathbb{Q}_{>0}$ we have

$$\begin{aligned} \mathrm{Bas}(w+w') &\subseteq \mathrm{Bas}(w) \cup \mathrm{Bas}(w'), & \mathrm{StBas}(v+v') &\subseteq \mathrm{StBas}(v) \cup \mathrm{StBas}(v'), \\ \mathrm{StBas}(\lambda v) &= \mathrm{StBas}(v), & \mathrm{StBas}(w/1) &= \textstyle\bigcap_{n\in\mathbb{N}} \mathrm{Bas}(nw). \end{aligned}$$

Remark V.1.3.13. Let X be locally of Krull type and let $B \subseteq X$ be closed and irreducible. Then the colimit over all $S_{\mathrm{WDiv}(X),B}(U)$ where $U \in \Omega_{X,B}$ is $\ker(\rho^X_B) = \langle S_{\mathrm{WDiv}(X),B}(X) \rangle$. Consequently, the colimit over all $\omega_{\mathrm{Cl}(X),B}(U)$ where $U \in \Omega_{X,B}$ is the group $\langle S_{\mathrm{Cl}(X),B}(X) \rangle$ of all classes w which are principal near B.

Definition V.1.3.14. Let X be locally of Krull type. The presheaves SAmple_X and Ample_X of *semiample* resp. *ample* rational divisor classes are defined via

$$\mathrm{SAmple}_X(U) := \bigcap_{x\in U} \omega_{\mathrm{Cl}(X),\overline{\{x\}}}(U), \quad \mathrm{Ample}_X(U) := \bigcap_{x\in U} \omega^{\mathrm{aff}}_{\mathrm{Cl}(X),\overline{\{x\}}}(U).$$

for $U \in \Omega_X$. The presheaf Mov_X of *moving* rational divisor classes is defined via

$$\mathrm{Mov}_X(U) := \bigcap_{Y\in\mathcal{Y}_X(U)} \omega_{\mathrm{Cl}(X),Y}(U).$$

Proposition V.1.3.15. *Let $\phi\colon X \to X'$ be a dominant morphism between graded schemes of Krull type. Then for each closed irreducible $B \subseteq X$ the following hold:*

(i) *We have $\phi^*_{X'} \circ S_{\mathrm{WDiv}(X'),\overline{\phi(B)}} \subseteq \phi_* S_{\mathrm{WDiv}(X),B}$. If each $Y \in \mathcal{Y}(X)$ satisfies $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ then we have $\phi^*_{X'} \circ S_{\mathrm{Cl}(X'),\overline{\phi(B)}} \subseteq \phi_* S_{\mathrm{Cl}(X),B}$ and $\phi_*\mathrm{Bas}(\phi^*_{X'}w) \subseteq \phi^{-1}(\mathrm{Bas}(w))$.*

(ii) *If each $Y \in \mathcal{Y}(X)$ satisfies $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ and $\phi^*_{X'}$ is injective and maps prime divisors to positive multiples of prime divisors then we have*
$$(\phi^*_{X'})^{-1}\phi_* S_{\mathrm{WDiv}(X),B} = S_{\mathrm{WDiv}(X'),\overline{\phi(B)}}.$$

(iii) *If $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ holds for each $Y \in \mathcal{Y}(\phi^{-1}(U))$ and the canonical homomorphism $\phi^*_U\colon \mathcal{O}_{X'}(D')(U) \to \mathcal{O}_X(\phi^*_{X'}D')(\phi^{-1}(U))$ is surjective then $[\phi^*_{X'}D'] \in S_{\mathrm{Cl}(X),B}(\phi^{-1}(U))$ implies $[D'] \in S_{\mathrm{Cl}(X'),\overline{\phi(B)}}(U)$. Consequently, we have $\phi_*\mathrm{Bas}(\phi^*_{X'}[D'])(U) = \phi^{-1}(\mathrm{Bas}([D']))(U)$.*

(iv) *If ϕ is affine then $\phi^*_{X'} \circ S^{\mathrm{aff}}_{\mathrm{WDiv}(X'),\overline{\phi(B)}} \subseteq \phi_* S^{\mathrm{aff}}_{\mathrm{WDiv}(X),B}$. If each $Y \in \mathcal{Y}(X)$ satisfies $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ then we have $\phi^*_{X'} \circ S^{\mathrm{aff}}_{\mathrm{Cl}(X'),\overline{\phi(B)}} \subseteq \phi_* S^{\mathrm{aff}}_{\mathrm{Cl}(X),B}$.*

Proof. In (i) note that for $D' \in S_{\mathrm{WDiv}(X'),\overline{\phi(B)}}(U)$ we have $\phi^*_{X'}D'_{|\phi^{-1}(U)} \geq 0$ and $0_B = \phi^*_B(D'_{\overline{\phi(B)}}) = \phi^*_{X'}(D')_B$. In (ii) let $D' \in \mathrm{WDiv}(X')$ with $\phi^*_{X'}D'_{|\phi^{-1}(U)} \geq 0$ and $\phi^*_{X'}D'_B = 0_B$. Consider $Y' \in \mathcal{Y}(U)$ and the unique $Y \in \mathcal{Y}(\phi^{-1}(U))$ in $|\phi^*_{X'}Y'|$.

Then $0 \le pr_Y(\phi_{X'}^* D') = pr_{Y'}(D') pr_Y(\phi_{X'}^* Y')$ implies $pr_{Y'}(D') \ge 0$. If $Y' \in \mathcal{Y}_{\overline{\phi(B)}}$ then we have $B \subseteq \phi^{-1}(Y') = Y$. Now, $0 = pr_Y(\phi_{X'}^* D')$ implies $pr_{Y'}(D') = 0$.

For (iii) suppose that $X \setminus |\phi_{X'}^* D' + \mathrm{div}_X(f)| \in \Omega_{X,B}$ holds with a non-zero $f \in \mathcal{O}_X(\phi_{X'}^* D')(\phi^{-1}(U))^{\mathrm{hom}}$. Then we have $f = \sum_{i=1}^n \phi_{X'}^*(f_i)$ with certain non-zero $f_i \in \mathcal{O}_{X'}(D')(U)^{\mathrm{hom}}$. Thus, B is not contained in

$$\bigcap_{i=1}^n |\phi_{X'}^*(D' + \mathrm{div}_{X'}(f_i))| \subseteq |\phi_{X'}^* D' + \mathrm{div}_X(f)|$$

and hence some $X' \setminus |\mathrm{div}_{X'}(f_i) + D'|$ lies in $\Omega_{X,B}$. Assertion (iv) follows from (i). $\square$

V.1.4. Divisorial $\mathcal{O}_X$-modules. Here, we show that all *divisorial* $\mathcal{O}_X$-modules are precisely the modules $\mathcal{O}_X(D)$ and that they are in natural bijection with the group $\mathrm{WDiv}(X)$, see Proposition V.1.4.7. Furthermore, we prove quasi-coherence of divisorial $\mathcal{O}_X$-modules, see Proposition V.1.4.8.

DEFINITION V.1.4.1. Let X be locally of Krull type. A graded $\mathcal{O}_X$-submodule $\mathcal{G} \le_{\mathcal{O}_X} \mathcal{K}$ is *divisorial* if

(i) $\mathcal{G}(U) = \bigcap_{Y \in \mathcal{Y}(U)} \mathcal{G}_Y$ holds for each open $U \subseteq X$,
(ii) the stalk $\mathcal{G}_Y$ at each $Y \in \mathcal{Y}(X)$ is a principal $\mathcal{O}_{X,Y}$-module with homogeneus generator,
(iii) and for each $U \in \mathcal{B}_X$ we have $\mathcal{G}_Y = \mathcal{O}_{X,Y}$ for all but finitely many $Y \in \mathcal{Y}(U)$.

REMARK V.1.4.2. Note that due to Example III.2.0.19 divisorial $\mathcal{O}_X$-modules are sheaves of sets.

CONSTRUCTION V.1.4.3. For a graded scheme X which is locally of Krull type the set $\mathrm{Div}(\mathcal{O}_X)$ of divisorial $\mathcal{O}_X$-submodules of $\mathcal{K}$ is a monoid with neutral element $\mathcal{O}_X$, where we define the operation via $\mathcal{G} * \mathcal{H}(U) := \bigcap_{Y \in \mathcal{Y}(U)} \mathcal{G}_Y \mathcal{H}_Y$. Note that $(\mathcal{G} * \mathcal{H})_Y = \mathcal{G}_Y \mathcal{H}_Y$, and with respect to inclusion $\mathcal{G} * \mathcal{H}$ is the smallest element of $\mathrm{Div}(\mathcal{O}_X)$ which contains $\mathcal{G}\mathcal{H}$. We have a canonical homomorphism

$$\mathrm{div}_{\mathcal{O}_X} \colon (\mathcal{K}(X)^{\mathrm{hom}})^* \longrightarrow \mathrm{Div}(\mathcal{O}_X), \quad f \longmapsto \mathcal{O}_X f$$

whose image and cokernel are denoted $\mathrm{PDiv}(\mathcal{O}_X)$ and $\mathrm{Cl}(\mathcal{O}_X)$ respectively.

PROOF. Let $f_Y \in (\mathcal{G} * \mathcal{H})_Y^{\mathrm{hom}} \setminus \{0\}$ where $f \in \mathcal{G} * \mathcal{H}(U)^{\mathrm{hom}} \setminus \{0\}$ with a neighbourhood U of Y. Then $f_Y \in \mathcal{G}_Y \mathcal{H}_Y$ because $Y \in \mathcal{Y}(U)$.

Conversely, let $\mathcal{G}_Y = \mathcal{O}_{X,Y} g$ and $\mathcal{H}_Y = \mathcal{O}_{X,Y} h$ and $a \in \mathcal{O}_{X,Y}^{\mathrm{hom}} \setminus \{0\}$. By axiom (iii) the complement $U \subseteq X$ of all prime divisors $Y' \ne Y$ such that $\mathcal{G}_{Y'}$ and $\mathcal{H}_{Y'}$ do not both equal $\mathcal{O}_{X,Y'}$. Let $V \subseteq U$ be a neighbourhood of Y with $\mathrm{div}_V(agh) \in \mathbb{Z}Y$. Then $agh \in \mathcal{G} * \mathcal{H}(V)^{\mathrm{hom}} \setminus \{0\}$ and hence $(agh)_Y \in (\mathcal{G} * \mathcal{H})_Y^{\mathrm{hom}} \setminus \{0\}$.

Let $\mathcal{F} \in \mathrm{Div}(\mathcal{O}_X)$ contain $\mathcal{G}\mathcal{H}$. Then $\mathcal{G}_Y \mathcal{H}_Y = (\mathcal{G}\mathcal{H})_Y \subseteq \mathcal{F}_Y$ holds for each $Y \in \mathcal{Y}(X)$, and taking the intersection over these stalks we see that $\mathcal{G} * \mathcal{H}(U) \subseteq \mathcal{F}(U)$ holds for each open $U \subseteq X$. $\square$

DEFINITION V.1.4.4. Let X be locally of Krull type. The $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ associated to a divisor $D \in \mathrm{WDiv}(X)$ is defined by

$$\mathcal{O}_X(D)(U) := \left\{ \sum_{w \in K} f_w \in \mathcal{K}(U) \;\middle|\; f_w = 0 \text{ or } \mathrm{div}_U(f_w) \ge -D_{|U} \right\}$$

We will also use the notation $\mathcal{O}(D)$ for $\mathcal{O}_X(D)$ when no confusion can arise.

REMARK V.1.4.5. Let $\phi \colon X \to X'$ be a morphism between graded schemes which are locally of Krull type such that $\mathrm{Cl}_{\overline{\phi(Y)}} = 0$ holds for each $Y \in \mathcal{Y}(X)$. Then the canonical graded homomorphism $\mathcal{K}(X') \to \mathcal{K}(X)$ restricts to a homomorphism $\mathcal{O}_{X'}(D') \to \phi_* \mathcal{O}_X(\phi_{X'}^*(D'))$ for each $D' \in \mathrm{WDiv}(X')$.

EXAMPLE V.1.4.6. Let X be a graded scheme which is locally of Krull type such that $gr(\mathcal{O}_X) = K \oplus F$ with a free group F. Let X' be the induced K-graded scheme, which is the relative spectrum of $\mathcal{O}_X$ equipped with the induced K-grading, and let $\phi\colon X' \to X$ be the canonical morphism. Then $\phi_X^*\colon \mathrm{WDiv}(X) \to \phi_* \mathrm{WDiv}(X')$ is a primality preserving injection, each $Y' \in \mathcal{Y}(X')$ satisfies $\mathrm{Cl}_{\overline{\phi(Y')}} = 0$ and $\mathcal{O}_X(D) \to \phi_*\mathcal{O}_X(\phi_X^*(D))$ is an isomorphism for each $D \in \mathrm{WDiv}(X)$. This follows from the affine case, which is due to Lemma II.2.5.14.

PROPOSITION V.1.4.7. *Let X be locally of Krull type. Then we have mutually inverse isomorphisms*

$$\begin{aligned} \mathrm{WDiv}(X) &\longleftrightarrow \mathrm{Div}(\mathcal{O}_X) \\ D &\longmapsto \mathcal{O}_X(D) \\ -\sum_{Y\in\mathcal{Y}(X)} \min_{f\in\mathcal{G}_Y^{\mathrm{hom}}\setminus\{0\}} \nu_{Y,X}(f)Y &\longleftarrow\!\shortmid \mathcal{G} \end{aligned}$$

which restrict to isomorphisms $\mathrm{PDiv}(X) \cong \mathrm{PDiv}(\mathcal{O}_X)$ and hence induce isomorphisms $\mathrm{Cl}(X) \cong \mathrm{Cl}(\mathcal{O}_X)$. Moreover, if $\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}^{\mathrm{fin}}(X)$ then the above restricts to an isomorphism between $\mathrm{WDiv}^{\mathrm{fin}}(X)$ and those $\mathcal{G} \in \mathrm{Div}(\mathcal{O}_X)$ for which the set of $Y \in \mathcal{Y}(X)$ with $\mathcal{G}_Y \neq \mathcal{O}_{X,Y}$ is finite.

PROOF. For each prime $u_Y \in \mathcal{O}_{X,Y}^{\mathrm{hom}}$ we have $\mathcal{O}_X(D)_Y = \mathcal{O}_{X,Y} u_Y^{-pr_Y(D)}$ which shows (ii). For (iii) note that on $U \in \mathcal{B}_X$, $D_{|U}$ is a finite sum. Since $f \in (\mathcal{K}(X)^{\mathrm{hom}})^*$ satisfies $\mathrm{div}_U(f) \geq -D_{|U}$ if and only if $\mathrm{div}_Y(f) \geq -D_Y$, i.e. $\nu_{Y,U}(f) \geq -pr_Y(D)$ holds for each $Y \in \mathcal{Y}(U)$, $\mathcal{O}_X(D)$ is divisorial. $\square$

PROPOSITION V.1.4.8. *Let X be locally of Krull type and let $D \in \mathrm{WDiv}(X)$ be a Weil divisor on X. Then the following hold:*

(i) *For $U \in \mathcal{B}_X$ denote by $\phi_U\colon \mathrm{WDiv}(U) \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U))$ the canonical isomorphism. Then the canonical isomorphism $Q_{\mathrm{gr}}(\mathcal{O}(U)) \to \mathcal{K}(U)$ restricts to an isomorphism of $\mathcal{O}(U)$-modules $\mathcal{O}_X(D)(U) \longrightarrow \phi_U(-D_{|U})$.*
(ii) *$\mathcal{O}_X(D)$ is quasi-coherent.*

PROOF. Assertion (i) follows directly from Remark V.1.2.7. In (ii) note that for $f \in \mathcal{O}(U)^{\mathrm{hom}} \setminus \{0\}$ we have an isomorphism

$$E_f \longrightarrow \mathcal{O}(D)(U_f), \quad g/f^n \longmapsto f^{-n}g$$

where we have used Remark V.1.2.7. $\square$

REMARK V.1.4.9. For each $U \in \mathcal{B}_X$ we have a canonical isomorphism of graded $\mathcal{O}_U$-modules $\mathcal{O}_X(D)_{|U} \cong \mathcal{O}_U(D_{|U})$. For a closed irreducible $A \subseteq X$ let $\phi_A\colon \mathrm{WDiv}_A \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}_{X,A})$ be the canonical isomorphism. Then the isomorphism $\mathcal{K}(X) \to Q_{\mathrm{gr}}(\mathcal{O}_{X,A})$ restricts to an isomorphism $\mathcal{O}_X(D)_A \to \phi_A(-D_A)$. Indeed, consider an affine open neighbourhood V of A. Since $\mathcal{O}(D)_A \cong \phi_U(-D_{|V})_{I(A)}$ Remark V.1.2.7 gives the assertion.

V.2. Cox sheaves

In this section, we present details on the definition of Cox sheaves and prove their main properties as well as their characterization among graded $\mathcal{O}_X$-algebras of Krull type. A characterizing feature of Cox sheaves is that their Krull structure, i.e. the family of graded valuations $\{\mu_Y\}_{Y\in\mathcal{Y}(X)}$ is compatible with that of $\mathcal{O}_X$, meaning that each μ_Y restricts to the essential graded valuation ν_Y defined by Y. $\mathcal{O}_X$-algebras with this property are called *natural* and studied in Section V.2.1. We characterize general natural $\mathcal{O}_X$-algebras $\mathcal{R}$ of Krull type in terms of their stalks and the Krull structure they define, see Theorem V.2.1.3, and Veronesean ones in terms of existence of a CBE from a *divisorial* $\mathcal{O}_X$-algebra to $\mathcal{R}$, see Theorem V.2.1.9. After these preparations Section V.2.2 sees our main results on the characterizations of Cox sheaves and their global sections in Theorems V.2.2.4 and V.2.2.5, respectively.

V.2.1. Natural and divisorial $\mathcal{O}_X$-algebras of Krull type. The prototype of natural $\mathcal{O}_X$-algebra of Krull type are the algebras $\mathcal{O}_X(K)$ associated to a subgroup $K \leq \mathrm{WDiv}(X)$. In the same way, a homomorphism $K \to \mathrm{WDiv}(X)$ defines an $\mathcal{O}_X$-algebra, see Construction V.2.1.5. General natural $\mathcal{O}_X$-algebras are defined in terms of the Krull property. Next to the characterizations in Theorems V.2.1.3 and V.2.1.9 we study the presheaf homomorphisms defined the Krull structures, localization properties and conditions for graded factoriality of section rings.

Definition V.2.1.1. Let X be locally of Krull type. A *natural $\mathcal{O}_X$-algebra of Krull type* is a graded $\mathcal{O}_X$-algebra $\imath\colon \mathcal{O}_X \to \mathcal{R}$ together with a $gr(\mathcal{R})$-simple constant $\mathcal{K}$-algebra $\bar{\imath}\colon \mathcal{K} \to \mathcal{S}$ wherein $\mathcal{R} = \bigcap_{Y \in \mathcal{Y}(X)} \mathcal{S}_{\mu_Y}$ is of Krull type with respect to $\mathcal{B}_X$ with a defining family $\{\mu_Y\}_{Y \in \mathcal{Y}(X)}$ such that $\bar{\imath}$ restricts to $\imath$ and $\mu_Y \circ \imath_{|(\mathcal{K}^{\mathrm{hom}})^*} = \nu_Y$ holds for $Y \in \mathcal{Y}(X)$.

Remark V.2.1.2. Natural $\mathcal{O}_X$-algebras of Krull type are sheaves of sets due to Example III.2.0.19.

Theorem V.2.1.3. *Let X be locally of Krull type and let $\imath\colon \mathcal{O}_X \to \mathcal{R}$ be a graded $\mathcal{O}_X$-algebra. Then $\mathcal{R}$ is a natural $\mathcal{O}_X$-algebra $\bigcap_{Y \in \mathcal{Y}(X)} \mathcal{S}_{\mu_Y}$ of Krull type in a $\mathcal{K}$-algebra $\bar{\imath}\colon \mathcal{K} \to \mathcal{S}$ with respect to $\mathcal{B}_X$ if and only if $\mathcal{R} \subseteq \mathcal{K}_{\mathcal{R}}$ is a subsheaf and quasi-coherent, we have a canonical isomorphism $\mathcal{O}_{X,A}^{\mathrm{hom}}/(\mathcal{O}_{X,A}^{\mathrm{hom}})^* \cong \mathcal{R}_A^{\mathrm{hom}}/(\mathcal{R}_A^{\mathrm{hom}})^*$ for $A \in \{X\} \cup \mathcal{Y}(X)$, for each open U we have $\mathcal{R}(U) = \bigcap_{Y \in \mathcal{Y}(U)} \mathcal{R}_Y$, and for $V \in \mathcal{B}_X$ each $f \in \mathcal{R}_X^{\mathrm{hom}} \setminus 0$ is a unit in $\mathcal{R}_Y$ for all but finitely many $Y \in \mathcal{Y}_X(V)$.*

Moreover under these conditions the following hold with respect to the notation of Construction III.4.0.9:

(i) *$\{\mu_Y\}_Y$ are the essential graded valuations of $\mathcal{R}$ and we have $\mu_{Y,U} = \mu_{Y,X}$ if $Y \in \mathcal{Y}(U)$ and $\mu_{Y,U} = 0$ otherwise.*

(ii) *$\{\mu_{Y,X}\}_{Y \in \mathcal{Y}_A}$ are the essential graded valuations of $\mathcal{R}_A$ for a closed irreducible $A \subseteq X$. In particular, the homomorphism $\mathcal{K}_{\mathcal{R}} \to \mathcal{S}$ is an isomorphism and we have $\mathcal{S}_{\mu_Y}(X) = \mathcal{R}_Y$ for $Y \in \mathcal{Y}(X)$.*

(iii) *For $U \in \Omega_X$ each $g \in \mathcal{R}(U)^{\mathrm{hom}} \setminus \{0\}$ restricts to a unit on*

$$\begin{aligned} U_g &:= U \setminus |\mathrm{div}_{\mathcal{R},U}(g)| = \{x \in U \mid \mathrm{div}_{\mathcal{R},x}(g_x) = 0\} \\ &= \{x \in U \mid g_x \in (\mathcal{R}_x^{\mathrm{hom}})^*\} \end{aligned}$$

and the canonical map $\mathcal{R}(U)_g \to \mathcal{R}(U_g)$ is an isomorphism.

(iv) *We have a homomorphism*

$$(\mathcal{S}^{\mathrm{hom}})^* \longrightarrow \mathrm{Div}(\mathcal{O}_X), \quad f \longmapsto \bar{\imath}^{-1}(\mathcal{R}f) = \mathcal{O}_X(-\mathrm{div}_{\mathcal{R},X}(f)).$$

Proof. If $\mathcal{R}$ is natural then $\mu_{Y,U}$ equals $\mu_{Y,X}$ if $Y \in \mathcal{Y}(U)$ and 0 otherwise, because $\mathcal{S}$ is constant and the restricition maps of $\mathbb{Z}^{(Y)}$ are identities or zero maps depending on whether or not $Y \in \mathcal{Y}(U)$. Moreover, in the terminology of Section III.4 $\mathcal{Y}$ is the index sheaf $\mathcal{J}$ corresponding to $\{\mu_Y\}_{Y \in \mathcal{Y}(X)}$. Its stalks at X and at $Y \in \mathcal{Y}(X)$ are $\emptyset$ resp. $\{Y\}$, and hence the respective stalks of $\mathcal{R}$ are $\mathcal{S}(X)$ and $\mathcal{S}_{\mu_Y}(X)$. Moreover, the canonical isomorphism $\mathcal{O}_{X,Y}^{\mathrm{hom}}/(\mathcal{O}_{X,Y}^{\mathrm{hom}})^* \to \mathbb{F}_1[\mathbb{N}_0]$ factors into the canonical isomorphism $\mathcal{R}_Y^{\mathrm{hom}}/(\mathcal{R}_Y^{\mathrm{hom}})^* \to \mathbb{F}_1[\mathbb{N}_0]$ and the canonical map $\mathcal{O}_{X,Y}^{\mathrm{hom}}/(\mathcal{O}_{X,Y}^{\mathrm{hom}})^* \to \mathcal{R}_Y^{\mathrm{hom}}/(\mathcal{R}_Y^{\mathrm{hom}})^*$ so that the latter is also an isomorphism. Lastly, for a non-zero $f \in \mathcal{S}(U)^{\mathrm{hom}}$ we have $\mathrm{div}_{\mathcal{R},U}(f) \in \mathrm{WDiv}(U)$ because for each $V \in \mathcal{B}_U$ there exist non-zero $g, h \in \mathcal{R}(V)^{\mathrm{hom}}$ with $f_{|V} = g/h$ and we thus have

$$\mathrm{div}_{\mathcal{R},U}(f)_{|V} = \mathrm{div}_{\mathcal{R},V}(g) - \mathrm{div}_{\mathcal{R},V}(h) \in \mathrm{WDiv}(V).$$

Regarding the further assertions note that by Proposition I.2.6.9, $\{\mu_{Y,X}\}_{Y \in \mathcal{Y}(U)}$ are the essential graded valuations of $\mathcal{R}(U)$ for $U \in \mathcal{B}_X$ because $\{\nu_{Y,X}\}_{Y \in \mathcal{Y}(U)}$

are the essential graded valuations of $\mathcal{O}(U)$. Likewise, $\{\mu_{Y,X}\}_{Y\in\mathcal{Y}_A}$ are the essential graded valuations of the stalk of $\mathcal{R}$ at a closed irreducible $A \subseteq X$ because $\{\nu_{Y,X}\}_{Y\in\mathcal{Y}_A}$ are the essential graded valuations of $\mathcal{O}_{X,A}$. In (iii) note that $\mathcal{R}(U)_g$ is the intersection over all $\mathcal{S}(X)_{\mu_{Y,X}}$ with $Y \in \mathcal{Y}(U)$ and $\mu_{Y,U}(g) = 0$, i.e. $Y \in \mathcal{Y}(U_g)$. Assertion (iv) follows through direct calculation.

Under the second set of conditions the stalk $\mathcal{R}_X$ is simply graded and for each $Y \in \mathcal{Y}(X)$ the stalk $\mathcal{R}_Y$ is a discrete graded valuation ring. Hence the canonical map

$$\mu_{Y,X}\colon (\mathcal{R}_X^{\mathrm{hom}})^* \cong Q(\mathcal{R}_Y^{\mathrm{hom}})^* \to Q(\mathcal{R}_Y^{\mathrm{hom}})^*/(\mathcal{R}_Y^{\mathrm{hom}})^* \cong Q(\mathcal{O}_{X,Y}^{\mathrm{hom}})^*/(\mathcal{O}_{X,Y}^{\mathrm{hom}})^* \cong \mathbb{Z}$$

is a normed discrete graded valuation whose corresponding graded valuation ring is $\mathcal{R}_Y$. Setting $\mu_{Y,U} := \mu_{Y,X}$ if $Y \in \mathcal{Y}(U)$ and $\mu_{Y,U} := 0$ otherwise defines a discrete graded valuation $\nu_Y\colon (\mathcal{K}_{\mathcal{R}}^{\mathrm{hom}})^* \to \mathbb{Z}^{(Y)}$. The family $\{\mu_Y\}_{Y\in\mathcal{Y}(X)}$ realizes as a natural $\mathcal{O}_X$-algebra of Krull type in $\mathcal{K}_{\mathcal{R}}$. □

Proposition V.2.1.4. *Let X be locally of Krull type and let $\mathcal{R}$ be a natural $\mathcal{O}_X$-algebra in $\mathcal{S}$ such that* $\mathrm{im}(\mathrm{div}_{\mathcal{R},U}) = \mathrm{WDiv}^{\mathrm{fin}}(U)$. *Then the following hold:*

(i) *$\mathcal{R}(U)$ is factorially graded,*

(ii) *for any irreducible closed $A \subseteq X$ with $U \in \Omega_{X,A}$ let $\mathfrak{a}(A)$ be the preimage of the maximal graded ideal under the map $\mathcal{R}(U) \to \mathcal{R}_A$. Then $\mathcal{Y}_A$ is the set of all $Y \in \mathcal{Y}(U)$ with $\mathcal{R}(U)^{\mathrm{hom}} \setminus \mathfrak{a}(A) \subseteq \ker(\nu_{Y,X})$. In particular, we have $\mathcal{R}(U)_{\mathfrak{a}(A)} = \mathcal{R}_A$ in $\mathcal{S}(X)$.*

(iii) *The canonical isomorphisms*

$$\mathrm{Div}_{\mathrm{gr}}(\mathcal{R}(U)) \cong (\mathcal{S}(U)^{\mathrm{hom}})^*/(\mathcal{R}(U)^{\mathrm{hom}})^* \cong \mathrm{WDiv}^{\mathrm{fin}}(U)$$

define a bijection $\mathcal{Y}(U) \to \mathfrak{P}(\mathcal{R}(U))$ which sends Y to $\mathfrak{a}(Y)$, and we have $\mu_{Y,U} = \nu_{\mathfrak{a}(Y)}$.

Proof. For (i) note that by assumption the map

$$\mathcal{R}(U)^{\mathrm{hom}} \setminus \{0\}/(\mathcal{R}(U)^{\mathrm{hom}})^* \longrightarrow \mathrm{WDiv}^{\mathrm{fin}}_{\geq 0}(U) \cong \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{N}_0$$

is bijective and thus $\mathcal{R}(U)^{\mathrm{hom}}$ is factorial. For $Y \in \mathcal{Y}(U)$ we now fix $f^Y \in \mathcal{R}(U)^{\mathrm{hom}}$ with $Y = \mathrm{div}_{\mathcal{R},U}(f^Y)$. In (ii) consider $Y \in \mathcal{Y}(U)$ with $\mathcal{R}(U)^{\mathrm{hom}} \setminus \mathfrak{a}(A) \subseteq \ker(\nu_{Y,X})$. Then we have $f^Y \in \mathfrak{a}(A)$ and hence $0 < \mathrm{div}_{\mathcal{R},A}(f^Y) = Y_A$, i.e. $A \subseteq Y$. In (iii) we use that $Q_{\mathrm{gr}}(\mathcal{R}(U)) = \mathcal{S}(X)$ holds and we have $\langle f^Y \rangle = \mathfrak{a}(Y)$ for each $Y \in \mathcal{Y}(U)$. □

Construction V.2.1.5. Let X be locally of Krull type, let M be a graded simple monoid and let $\phi\colon M \to \mathrm{WDiv}(X)$ be a group homomorphism. The associated *divisorial $gr(\mathcal{O}_X) \oplus gr(M)$*-graded natural $\mathcal{O}_X$-algebra is

$$\mathcal{O}_X(M,\phi) := \bigoplus_{w\in M} \mathcal{O}_X(\phi(w))\chi^w \subseteq \mathcal{K}[M].$$

The defining family $\{\mu_Y\}_{Y\in\mathcal{Y}(X)}$ is given via $\mu_{Y,U}(f\chi^w) = \nu_{Y,U}(f) + pr_{Y,U}(\phi(w)_{|U})$ for $U \in \Omega_X$.

Remark V.2.1.6. In the above, we have $\mathrm{div}_{\mathcal{O}_X(M,\phi)}(f\chi^w) = \mathrm{div}_U(f) + \phi(w)_{|U}$ and hence $\mathrm{im}(\mathrm{div}_{\mathcal{O}_X(M,\phi),U}) = \mathrm{PDiv}(U) + \rho_U^X(\mathrm{im}(\phi))$.

Example V.2.1.7. Let X be locally of Krull type and let $L \leq \mathrm{WDiv}(X)$ be a subgroup. The associated $gr(\mathcal{O}_X) \oplus L$-graded divisorial $\mathcal{O}_X$-algebra is

$$\mathcal{O}_X(L) := \mathcal{O}_X(L, L \subseteq \mathrm{WDiv}(X)).$$

Remark V.2.1.8. If $U \subseteq X$ is an open graded subscheme intersecting every prime divisor of X then we have canonical isomorphisms of L-graded $\mathcal{O}_U$-algebras $\mathcal{K}_X[L]_{|U} \cong \mathcal{K}_U[L_{|U}]$ and $\mathcal{O}(L)_{|U} \cong \mathcal{O}(L_{|U})$.

THEOREM V.2.1.9. *Let X be locally of Krull type and let $\mathcal{R}$ be a graded $\mathcal{O}_X$-algebra. Then the following are equivalent:*

(i) *$\mathcal{R}$ is a Veronesean natural $\mathcal{O}_X$-algebra in $\mathcal{S}$,*
(ii) *there exists a divisorial $\mathcal{O}_X$-algebra $\mathcal{A}$ and a CBE $\pi\colon \mathcal{A} \to \mathcal{R}$.*
(iii) *for each surjective homomorphism $\psi\colon gr(\mathcal{O}_X) \oplus F \longrightarrow gr(\mathcal{R})$ of $gr(\mathcal{O}_X)$-algebras such that F is free there exist a divisorial $\mathcal{O}_X$-algebra $\mathcal{A}$ and a CBE $\pi\colon \mathcal{A} \to \mathcal{R}$ accompanied by ψ.*

PROOF. If (iii) holds consider a family $\{v_i\}_{i \in I} \in gr(\mathcal{R})$ whose classes generate $gr(\mathcal{R})/gr(\mathcal{O}_X)$. Then the induced homomorphism $\psi\colon gr(\mathcal{O}_X) \oplus \bigoplus_{i \in I} \mathbb{Z} \to gr(\mathcal{R})$ has the required properties and hence there exists a CBE $\mathcal{A} \to \mathcal{R}$ with accompanying map ψ.

If (ii) holds then by Proposition III.5.0.6 $\pi\colon \mathcal{A} \to \mathcal{R}$ induces a CBE $\mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathcal{R}}$ and by Proposition II.1.2.16 $\mathcal{R}$ is a subsheaf of $\mathcal{K}_{\mathcal{R}}$. By Proposition II.2.5.11 the family which defines $\mathcal{A}$ in $\mathcal{K}_{\mathcal{A}}$ and restricts to $\{\nu_Y\}_{Y \in \mathcal{Y}(X)}$ on $(\mathcal{K}^{\mathrm{hom}})^*$ induces a family $\{\mu_Y\}_{Y \in \mathcal{Y}(X)}$ which defines $\mathcal{R}$ in $\mathcal{K}_{\mathcal{R}}$ and also restricts to $\{\nu_Y\}_Y$ on $(\mathcal{K}^{\mathrm{hom}})^*$.

Suppose that (i) holds and consider a map $\phi\colon F \to gr(\mathcal{R})$ from a free abelian group F such that the map $\psi\colon gr(\mathcal{O}_X) \oplus F \to gr(\mathcal{R}), w + v \mapsto w + \phi(v)$ is surjective. Let $F' := \phi^{-1}(\mathrm{degsupp}(\mathcal{S}(X)))$ and let $\mathcal{S}'$ be the Veronese subalgebra $\mathcal{K}[F]_{gr(\mathcal{O}_X) \oplus F'}$ equipped with the $gr(\mathcal{O}_X) \oplus F$-grading.

Let $\{e_i\}_{i \in I}$ be a basis of F' and choose a non-zero $f_i \in \mathcal{S}(X)_{\phi(e_i)}$ for each $i \in I$. For $v = \sum_i \lambda_i e_i$ set $f^v = \prod_i f_i^{\lambda_i}$. Sending χ^v to f^v then defines a CBE $\pi\colon \mathcal{S}' \to \mathcal{S}$ of $\mathcal{K}$-algebras with accompanying map ψ. Each $\pi_v\colon \mathcal{S}'_{v+gr(\mathcal{O}_X)} \to \mathcal{S}_{\phi(v)+\mathcal{O}_X} = \mathcal{K}f^v$ restricts to an isomorphism $\mathcal{A}_{v+gr(\mathcal{O}_X)} := (\mathcal{R}f^{-v} \cap \mathcal{K})\chi^v \to \mathcal{R}_{\phi(v)+gr(\mathcal{O}_X)}$ because $\mathcal{R}f^{-v} \cap \mathcal{K} = \mathcal{R}_{\phi(v)+gr(\mathcal{O}_X)} f^{-v}$. The $\mathcal{O}_X$-subalgebra $\mathcal{A}$ generated by all $\mathcal{A}_{v+gr(\mathcal{O}_X)}$ is then divisorial and π restricts to a CBE $\pi\colon \mathcal{A} \to \mathcal{R}$. □

REMARK V.2.1.10. Let X be locally of Krull type. For a natural $\mathcal{O}_X$-algebra $\mathcal{R}$ which is of Krull type in $\mathcal{S}$ with respect to $\mathcal{B}_X$ let $pr\colon gr(\mathcal{R}) \to gr(\mathcal{R})/gr(\mathcal{O}_X)$ be the canonical projection. Then in the commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 0 \arrow[d, dashed] & \\
1 \arrow[r] & (\mathcal{O}^{\mathrm{hom}})^* \arrow[r] \arrow[d] & (\mathcal{R}^{\mathrm{hom}})^* \arrow[r, "pr \circ \deg"] \arrow[d] & pr(\deg((\mathcal{R}^{\mathrm{hom}})^*)) \arrow[r] \arrow[d, dashed] & 0 \\
1 \arrow[r] & (\mathcal{K}^{\mathrm{hom}})^* \arrow[r] \arrow[d, "\mathrm{div}"'] & (\mathcal{K}_{\mathcal{R}}^{\mathrm{hom}})^* \arrow[r, "pr \circ \deg"] \arrow[d, "\mathrm{div}_{\mathcal{R}}"'] & pr(\deg((\mathcal{K}_{\mathcal{R}}^{\mathrm{hom}})^*)) \arrow[r] \arrow[ddll, dashed, rounded corners, "\overline{\delta}"] & 0 \\
 & \mathrm{WDiv} \arrow[r, equal] \arrow[d, "c"'] & \mathrm{WDiv} \arrow[d] & & \\
 & \mathrm{Cl} \arrow[r, dashed] \arrow[d] & \mathrm{coker}(\mathrm{div}_{\mathcal{R}}) \arrow[r, dashed] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

of presheaves of abelian groups with exact rows and columns the dashed sequence is also exact. Moreover, if $\mathrm{PDiv}(X) \subseteq \mathrm{WDiv}^{\mathrm{fin}}(X)$ then the same holds if we replace WDiv and Cl with $\mathrm{WDiv}^{\mathrm{fin}}$ resp. $\mathrm{Cl}^{\mathrm{fin}}$.

REMARK V.2.1.11. For each $U \in \mathcal{B}_X$, $\mathcal{O}(U) \subseteq \mathcal{R}(U)$ is natural in the sense of Section II.2.6 due to Proposition II.2.6.5 and Proposition I.2.6.9.

Proposition V.2.1.12. *For a Veronesean natural $\mathcal{O}_X$-algebra $\mathcal{R}$ in $\mathcal{S}$ each stalk $\mathcal{R}_x$ satisfies the following:*

(i) *$\mathcal{O}_{X,x} \subseteq \mathcal{R}_x$ is natural in the sense of Section II.2.6, in particular, $\mathcal{R}_x$ is $gr(\mathcal{R})$-local. Moreover, we have canonical isomorphisms*
$$\mathrm{Cl}_{gr}(\mathcal{R}_x) \cong \mathrm{coker}\,(\mathrm{div}_{\mathcal{R}})_x \cong \mathrm{Cl}_x/\mathrm{im}(c_x \circ \mathrm{div}_{\mathcal{R},x}) = \mathrm{Cl}_x/\mathrm{im}(\overline{\delta}_x).$$

(ii) *We have*
$$\overline{\delta}_X(pr(\deg((\mathcal{R}_x^{\mathrm{hom}})^*))) = \{w \in \mathrm{im}(\overline{\delta}_X) \mid w_x = [0]_x\}.$$

(iii) *For each $U \in \Omega_{X,x}$ we have*
$$\overline{\delta}_X(pr(\deg(\mathcal{R}(U) \cap (\mathcal{R}_x^{\mathrm{hom}})^*)) = S_{\mathrm{Cl}(X),x}(U) \cap \mathrm{im}(\overline{\delta}_X).$$

Proof. Assertion (i) follows from Proposition II.2.6.5. For assertions (ii) and (iii) note that $\mathrm{div}_{\mathcal{R},X}((\mathcal{R}_x^{\mathrm{hom}})^*)$ resp. $\mathrm{div}_{\mathcal{R},X}(\mathcal{R}(U)^{\mathrm{hom}} \cap (\mathcal{R}_x^{\mathrm{hom}})^*)$ is the set of all $D \in \mathrm{im}(\mathrm{div}_{\mathcal{R},X})$ with $D_x \in \mathrm{PDiv}_x$ (and $D_{|U} \geq 0$). $\square$

Remark V.2.1.13. Let $\mathcal{R}$ be a natural $\mathcal{O}_X$-algebra which is locally of Krull type. Let K denote $\mathrm{WDiv}(X)$ resp. $\mathrm{WDiv}^{\mathrm{fin}}$ and correspondingly, let C denote $\mathrm{Cl}(X)$ resp. $\mathrm{Cl}^{\mathrm{fin}}(X)$, in the latter case supposing that $\mathrm{im}(\mathrm{div}_{\mathcal{R},X}) \subseteq \mathrm{WDiv}^{\mathrm{fin}}(X)$. Then the canonical map $(\mathcal{K}_{\mathcal{R}}(X)^{\mathrm{hom}})^* \xrightarrow{\mathrm{div}_{\mathcal{R},X}} K \to C$ induces a homomorphism $\phi\colon \mathcal{K}_{\mathcal{R}}(X)^{\mathrm{hom}})^*/(\mathcal{K}(X)^{\mathrm{hom}})^* \to C$ which is injective/surjective if and only if we have $(\mathcal{R}(X)^{\mathrm{hom}})^* = (\mathcal{O}(X)^{\mathrm{hom}})^*$ resp. $\mathrm{im}(\mathrm{div}_{\mathcal{R},X}) = K$.

Remark V.2.1.14. Let $\mathcal{R}$ be a Veronesean natural $\mathcal{O}_X$-algebra and let $f, f' \in \mathcal{R}(U)_w$ with $f + f' \neq 0$. Then we have
$$|\mathrm{div}_{\mathcal{R},U}(f)| \cap |\mathrm{div}_{\mathcal{R},U}(f')| \cap U \subseteq |\mathrm{div}_{\mathcal{R},U}(f+f')| \cap U.$$
Indeed, if for $x \in U$ the stalks f_x, f'_x are non-units then so is $(f+f')_x$.

Proposition V.2.1.15. *Let X be a graded scheme of Krull type. For each closed point $x \in X$ we then have $S^{\mathrm{aff}}_{\mathrm{WDiv}(X),x} \subseteq S^{\circ}_{\mathrm{WDiv}(X),x}$ and $S^{\mathrm{aff}}_{\mathrm{Cl}(X),x} \subseteq S^{\circ}_{\mathrm{Cl}(X),x}$.*

Proof. Let $q\colon \widehat{X} := \mathrm{Spec}_{\mathrm{gr},\mathrm{X}}(\mathcal{O}_X(\mathrm{WDiv}(X))) \to X$ be the canonical morphism and let $D \in S^{\mathrm{aff}}_{\mathrm{WDiv}(X),x}(U)$. Since the special point $\widehat{x}$ over x is closed $\mathcal{O}(q^{-1}(U \setminus |D|))/\mathcal{I}_{\widehat{x}}(q^{-1}(U \setminus |D|)$ is homogeneously simple and hence its degree support $S_{gr(\mathcal{O}_X)\oplus \mathrm{WDiv}(X),\widehat{x}}(q^{-1}(U \setminus |D|))$ is a group. Therefore,
$$\begin{aligned} S_{\mathrm{WDiv}(X),x}(U)_D &= S_{\mathrm{WDiv}(X),x}(U \setminus |D|) \\ &= pr_{\mathrm{WDiv}(X)}(S_{gr(\mathcal{O}_X)\oplus \mathrm{WDiv}(X),\widehat{x}}(q^{-1}(U \setminus |D|))) \end{aligned}$$
is a group and hence $D \in S_{\mathrm{WDiv}(X),x}(U)^{\circ}$. Likewise,
$$S_{\mathrm{Cl}(X),x}(U)_{[D]} = c_X(pr_{\mathrm{WDiv}(X)}(S_{gr(\mathcal{O}_X)\oplus \mathrm{WDiv}(X),\widehat{x}}(q^{-1}(U \setminus |D|))))$$
is a group and hence $[D] \in S_{\mathrm{Cl}(X),x}(U)^{\circ}$. $\square$

Proposition V.2.1.16. *Let $\phi\colon X \to Z$ be an affine morphism of graded schemes such that $\phi^*\colon \mathcal{O}_Z \to \phi_*\mathcal{O}_X$ is a CBE. Then the following hold:*

(i) *X is of Krull type if and only if Z is so, and in this case the pullback $\mathrm{WDiv}_Z \to \phi_* \mathrm{WDiv}_X$ is an isomorphism of partially ordered groups which commutes with the respective divisor homomorphisms and the induced pullback $\mathrm{Cl}_Z \to \phi_* \mathrm{Cl}_X$ is an isomorphism.*

(ii) *For each $D \in \mathrm{WDiv}(Z)$ the pullback $\mathcal{O}_Z(D) \to \phi_*\mathcal{O}_X(\phi_Z^*(D))$ is a CBE of graded sheaves of graded $\mathbb{K}$-vector spaces. Likewise, for a subgroup $K \leq \mathrm{WDiv}(Z)$ the pullback $\mathcal{O}_Z(K) \to \phi_*\mathcal{O}_X(\phi_Z^*(K))$ is a CBE of graded sheaves of $\mathbb{K}$-algebras.*

Proof. For $U \in \mathcal{B}_Z$ Proposition II.2.5.11 implies that $\mathcal{O}_Z(U)$ is of Krull type if and only if $\mathcal{O}_X(\phi^{-1}(U))$ is so. In particular, X is of Krull type if and only if Z is. Now assume that X and Z are of Krull type. By Proposition IV.1.4.13 (i) the assignment $Y \mapsto \phi(Y)$ constitutes a bijection $\mathcal{Y}(X) \to \mathcal{Y}(Z)$. Moreover, for each $Y \in \mathcal{Y}(X)$ the canonical map $\mathcal{O}_{Z,\phi(Y)} \to \mathcal{O}_{X,Y}$ is a CBE by Proposition IV.1.4.13 (iv). This gives the remaining assertions. □

V.2.2. Cox sheaves and Cox algebras. After defining Cox sheaves and quasi-Cox sheaves for graded schemes which are locally of Krull type we prove the Theorem V.2.2.4 on their characterization, the stated conditions being sufficient if $\mathrm{WDiv}(X)$ is free, e.g. if X is quasi-compact, in which case Cox sheaves and quasi-Cox sheaves are the same. Theorem V.2.2.5 characterizes what it means for an algebra $R_G \subseteq R$ to be a Cox algebra, with the given conditions being sufficient already under the mild assumption that $gr(R)/G$ be finitely generated. Furthermore, we give a constructive approach to (quasi-)Cox sheaves, see Construction V.2.2.6, leading to very mild conditions for their existence. Uniqueness seems unavailable in the general case, but each two Cox sheaves may be connected in some sense via the defining CBEs from $\mathcal{O}(\mathrm{WDiv}(X))$, see Proposition V.2.2.8.

Definition V.2.2.1. Let X be a graded scheme over A. A *CBE of $\mathcal{O}_X$-algebras* is a CBE (component-wise bijective epimorphism) of presheaves of constantly graded A-algebras on X which is also a morphism of (graded) $\mathcal{O}_X$-algebras.

In the following for a graded scheme X which is locally of Krull type we fix the notations $L := \mathrm{WDiv}(X)$ and $L^{\mathrm{fin}} := \mathrm{WDiv}^{\mathrm{fin}}(X)$.

Definition V.2.2.2. Let X be locally of Krull type and write K for L (resp. L^{fin}). A *(quasi-)Cox sheaf* is an $\mathcal{O}_X$-algebra $\mathcal{R}$ allowing a CBE of $\mathcal{O}_X$-algebras $\pi\colon \mathcal{O}_X(K) \to \mathcal{R}, \psi\colon gr(\mathcal{O}_X) \oplus K \to gr(\mathcal{R})$ with $K \cap \psi^{-1}(gr(\mathcal{O}_X)) = \mathrm{PDiv}(X)$. The global sections $\mathcal{R}(X)$ are called a *(quasi-)Cox ring* and the algebra $\mathcal{O}(X) \to \mathcal{R}(X)$ is a *(quasi-)Cox algebra*.

Remark V.2.2.3. In the definition of (quasi-)Cox sheaves, it suffices to require that $\mathcal{R}$ be an $\mathcal{O}_X$-*pre*algebra allowing a CBE as stated. Indeed, by Proposition III.5.0.4 each $\mathcal{R}_{\psi(w,D)}$ is a sheaf because $\mathcal{O}_X(D)_w$ is a sheaf. Moreover, $\mathcal{R}$ is a natural $\mathcal{O}_X$-algebra of Krull type in $\mathcal{K}_{\mathcal{R}}$ with respect to $\mathcal{B}_X$. Thus, $\mathcal{R}$ is even a sheaf of sets because it is a graded subsheaf of the constant sheaf $\mathcal{K}_{\mathcal{R}}$.

Also, it suffices to require that $\mathrm{PDiv}(X)$ be contained in $K \cap \psi^{-1}(gr(\mathcal{O}_X))$ where K denotes L resp. L^{fin} since the converse already holds. Indeed, if an element D of K satisfies $\psi(D) \in gr(\mathcal{O}_X)$ then $\pi_X(\chi^D) \in \mathcal{K}(X)$ and we conclude $D = \mathrm{div}_{\mathcal{O}(K),X}(\chi^D) = \mathrm{div}_{\mathcal{R},X}(\pi_X(\chi^D)) = \mathrm{div}_X(\pi_X(\chi^D))$.

Theorem V.2.2.4. *Let X be a locally of Krull type and let $\mathcal{R}$ be a (graded) $\mathcal{O}_X$-algebra $\mathcal{R}$. If $\mathcal{R}$ is a (quasi-)Cox sheaf then the following hold:*

(i) *$\mathcal{R} = \bigcap_{Y\in\mathcal{Y}(X)} \mathcal{S}_{\mu_Y} \subseteq \mathcal{S}$ is a natural $\mathcal{O}_X$-algebra of Krull type,*

(ii) *the map $\overline{\delta}_X$ from Remark V.2.1.10 is an isomorphism from $gr(\mathcal{R})/gr(\mathcal{O}_X)$ to $\mathrm{Cl}(X)$ (resp. $\mathrm{Cl}^{\mathrm{fin}}(X)$); equivalently, $\mathrm{div}_{\mathcal{R},X}$ has image L (resp. L^{fin}) and kernel $(\mathcal{O}(X)^{\mathrm{hom}})^*$, and we have $\deg((\mathcal{S}(X)^{\mathrm{hom}})^*)+gr(\mathcal{O}_X) = gr(\mathcal{R})$.*

Conversely, if (i) and (ii) hold with respect to L^{fin} then $\mathcal{R}$ is a quasi-Cox sheaf. If (i) and (ii) hold with respect to L and L is free then $\mathcal{R}$ is a Cox sheaf.

Proof. Let K denote L resp. L^{fin}. If $\mathcal{R}$ is a (quasi-)Cox sheaf on X with a CBE $\pi\colon \mathcal{O}(K) \to \mathcal{R}$ as required then it is natural $\mathcal{O}_X$-algebra by Theorem V.2.1.9 $\mathcal{R}$ and we have $\mathrm{im}(\mathrm{div}_{\mathcal{R},X}) = \mathrm{im}(\mathrm{div}_{\mathcal{O}(K),X}) = K$. Since $\mathrm{PDiv}(X) \subseteq K \cap \psi^{-1}(gr(\mathcal{O}_X))$ we have $(\mathcal{R}(X)^{\mathrm{hom}})^* = \pi_X((\mathcal{O}(K)(X)^{\mathrm{hom}})^*) = (\mathcal{O}(X)^{\mathrm{hom}})^*$. Finally, note that $gr(\mathcal{O}(K)) = \deg((\mathcal{O}(K)_\xi^{\mathrm{hom}})^*) + gr(\mathcal{O}_X)$ and hence

$$gr(\mathcal{R}) = \psi(\deg((\mathcal{O}(K)_\xi^{\mathrm{hom}})^*)) + gr(\mathcal{O}_X) = \deg((\mathcal{R}_\xi^{\mathrm{hom}})^*) + gr(\mathcal{O}_X).$$

Conversely, suppose that K is free and that $\mathcal{R}$ satisfies conditions (i) and (ii) with respect to K. Then the map $\overline{\delta}_X$ from Remark V.2.1.10 is an isomorphism from $gr(\mathcal{R})/gr(\mathcal{O}_X)$ to $\mathrm{Cl}^{\mathrm{fin}}(X)$. Let $\overline{\phi}\colon K \to gr(\mathcal{R})/gr(\mathcal{O}_X)$ be the composition with the canonical map. Choosing representatives of $\overline{\phi}$-images for basis elements of K we obtain a surjective homomorphism $\psi\colon gr(\mathcal{O}_X) \oplus K \longrightarrow gr(\mathcal{R})$ of $gr(\mathcal{O}_X)$-algebras. By Theorem V.2.1.9 there exists a CBE of $\mathcal{O}_X$-algebras $\pi\colon \mathcal{O}(L) \to \mathcal{R}$ with accompanying map ψ. Lastly, for $D \in \mathrm{PDiv}(X)$ we have

$$\overline{\delta}_X(\psi(D) + gr(\mathcal{O}_X)) = [\mathrm{div}_{\mathcal{R},X}(\pi_X(\chi^D))] = [\mathrm{div}_{\mathcal{O}(L),X}(\chi^D)] = [D] = [0]$$

i.e. $D \in K \cap \psi^{-1}(gr(\mathcal{O}_X))$. □

Theorem V.2.2.5. *Let $R_G \subseteq R$ be a Veronese A-subalgebra of a graded A-algebra. If there exists a quasi-Cox sheaf $\mathcal{R}$ on a graded scheme X which is locally of Krull type such that we have an isomorphism $\mathcal{R}(X) \cong R$ restricting to an isomorphism $\mathcal{O}(X) \cong R_G$ then the following hold:*

(i) *R^{hom} is factorial,*
(ii) *$(R^{\mathrm{hom}})^* = (R_G^{\mathrm{hom}})^*$,*
(iii) *for every $\mathfrak{p} \in \mathfrak{P}(R)$ we have $\deg(((R_\mathfrak{p})^{\mathrm{hom}})^*) + G = gr(R)$.*

Conversely, if $gr(R)/G$ is finitely generated and conditions (i) - (iii) hold then there exists a graded scheme X of Krull type (and of affine intersection) and a quasi-Cox sheaf $\mathcal{R}$ with an isomorphism $\mathcal{R}(X) \cong R$ which restricts to an isomorphism $\mathcal{O}(X) \cong R_G$.

Part I. If R is a quasi-Cox ring then (ii) follows from Theorem V.2.2.4. Conditions (i) and (iii) are due to Proposition V.2.1.4 and Theorem V.2.1.3. The second part of the proof is found in Section V.3.2. □

Construction V.2.2.6. Let X be locally of Krull type and denote L resp. L^{fin} by K. If $\pi\colon \mathcal{O}(K) \to \mathcal{R}$ is a CBE to a (quasi-)Cox sheaf then sending a principal divisor D to the unique element of $\mathcal{O}(K)(X)^{\mathrm{hom}}_{gr(\mathcal{O}_X)+D} \cap \pi_X^{-1}(1_{\mathcal{R}(X)})$ constitutes a monomomorphism $\kappa\colon \mathrm{PDiv}(X) \to (\mathcal{O}(K)(X)^{\mathrm{hom}})^*$ because the restriction $pr_L\colon \ker(\psi) \to \mathrm{PDiv}(X)$ is an isomorphism. Conversely, for a monomomorphism $\kappa\colon \mathrm{PDiv}(X) \to (\mathcal{O}(K)(X)^{\mathrm{hom}})^*$ with $\kappa(D) \in \mathcal{O}(K)(X)^{\mathrm{hom}}_{gr(\mathcal{O}_X)+D}$ the $\mathcal{O}_X$-module $\mathcal{R}$ which has grading group $(gr(\mathcal{O}_X) \oplus K)/pr_L(\deg(\kappa(\mathrm{PDiv}(X))))$ and assigns

$$\mathcal{O}(K)(U)/\langle \kappa(D)_{|U} - 1_{\mathcal{O}_X(K)(U)} \mid D \in \mathrm{PDiv}(X)\rangle$$

to $U \in \Omega_X$ is a (quasi-)Cox sheaf due to Proposition II.1.2.13.

Proof. If (π, ψ) is a CBE to a (quasi-)Cox sheaf then by Proposition II.1.2.13 deg restricts to an isomorphism of $(\pi_X)^{-1}_{|\mathcal{O}(K)(X)^{\mathrm{hom}}}(1_{\mathcal{R}(X)})$ and $\ker(\psi)$. Injectivity of $pr_L\colon \ker(\psi) \to \mathrm{PDiv}(X)$ follows from $gr(\mathcal{O}_X) \cap \ker(\psi) = \{0\}$. Surjectivity follows from $\mathrm{PDiv}(X) \subseteq K \cap \psi^{-1}(gr(\mathcal{O}_X))$ and component-wise bijectivity.

For the converse note that each π_U as constructed from κ is a CBE of rings by Proposition II.1.2.13 π_U. For bijectivity of the restricition $\psi\colon gr(\mathcal{O}_X) \to gr(\mathcal{O}_X)$ consider $w \in gr(\mathcal{O}_X)$ with $w = \deg(\kappa(D))$ for some $D \in \mathrm{PDiv}(X)$. Then we have $D = pr_L(\deg(\kappa(D))) = 0$ and hence $w = 0$. Thus, π_U is a CBE of $\mathcal{O}(U)$-algebras. □

Corollary V.2.2.7. *Quasi-Cox sheaves on a graded scheme X which is locally of Krull type exist if and only if* $\mathrm{PDiv}(X) \subseteq L^{\mathrm{fin}}$. *A sufficient condition for the existence of Cox sheaves is freeness of* $\mathrm{PDiv}(X)$.

It is well-kown that (quasi-)Cox sheaves are unique up to isomorphism of graded $\mathcal{O}_X$-algebras if $\mathrm{Cl}(X)$ resp. $\mathrm{Cl}^{\mathrm{fin}}(X)$ is free because then each (quasi-)Cox sheaf is isomorphic to $\mathcal{O}_X(K)$ where K is a subgroup of L resp. L^{fin} which maps isomorphically onto $\mathrm{Cl}(X)$ resp. $\mathrm{Cl}^{\mathrm{fin}}(X)$. A further condition enforcing uniqueness up to isomorphism in the case of prevarieties over an algebraically closed field $\mathbb{K}$ is $\mathcal{O}(X)^* = \mathbb{K}^*$ which holds e.g. if X is projective, see [**4**, Sect. I.4.3]. In general, two (quasi-)Cox sheaves are linked via the CBEs from $\mathcal{O}_X(K)$ which has the following consequences.

PROPOSITION V.2.2.8. *Let X be locally of Krull type and let $\mathcal{R}$ and $\mathcal{R}'$ be two (quasi-)Cox sheaves on X. Then for each $U \in \Omega_X$ the following hold:*

(i) *The respective CBEs induce an isomorphism of $\mathbb{F}_1$-algebras*
$$\mathcal{R}(U)^{\mathrm{hom}} \setminus \{0\}/(\mathcal{R}(U)^{\mathrm{hom}})^* \cong \mathcal{R}'(U)^{\mathrm{hom}} \setminus \{0\}/(\mathcal{R}'(U)^{\mathrm{hom}})^*$$
as well as bijections respecting sums, intersections, inclusions, products and ideal quotients between the sets of graded ideals of $\mathcal{R}(U)$ and $\mathcal{R}'(U)$.

(ii) *$\mathcal{R}(U)$ is of finite type over $\mathcal{O}_X(U)$ if and only if $\mathcal{R}'(U)$ is so. If X is a graded scheme over the (0-graded) affine scheme $S = \mathrm{Spec}(B)$, then $\mathcal{R}(U)$ is of finite type over B if and only if $\mathcal{R}'(U)$ is so.*

PROOF. In the first statement is due to Remark II.1.2.14 and the second to Proposition II.1.2.17. Let K denote L resp. L^{fin}. In (ii) let $B \subseteq \mathcal{O}(U)$ be a subalgebra and suppose that $\mathcal{R}(U)$ is of finite type over B. Since $\mathrm{Cl}(X)$ resp. $\mathrm{Cl}^{\mathrm{fin}}(X)$ is isomorphic to $gr(\mathcal{R}(U))/gr(\mathcal{O}(U))$ it is finitely generated and hence there exists a finitely generated subgroup $G \leq K$ which the canonical projection c_X maps onto $\mathrm{Cl}(X)$ resp. $\mathrm{Cl}^{\mathrm{fin}}(X)$. The group $gr(\mathcal{O}_X) \oplus G \cap \ker(\psi)$ is isomorphic to $G \cap \mathrm{PDiv}(X)$ under the isomorphism $pr_L\colon \ker(\psi) \to \mathrm{PDiv}(X)$ from Construction V.2.2.6, in particular, it is finitely generated. By Proposition II.1.7.10 $\mathcal{O}_X(G)(U)$ is of finite type over B and hence, so is $\mathcal{R}'(U)$. □

The above shows that the question of uniqueness is of little practical consequence since all Cox sheaves on a given X behave in the same way.

REMARK V.2.2.9. Let X be locally of Krull type and let $U \in \Omega_X$ satisfy $L = \mathrm{WDiv}_X(U)$ with $\imath\colon U \to X$ denoting the inclusion. Then we have canonical isomorphisms $\mathcal{O}_U(\mathrm{WDiv}_U(U)) \cong \mathcal{O}_X(L)_{|U}$ and $\imath_*\mathcal{O}_U(\mathrm{WDiv}_U(U)) \cong \mathcal{O}_X(L)$, and if $\mathrm{PDiv}(X) \subseteq L^{\mathrm{fin}}$ then the analogous statements hold for the divisorial algebras defined by the respective groups of *finite* Weil divisors. Consequently, the restriction of (quasi-)Cox sheaf on X is a Cox sheaf on U and the direct image of a (quasi-)Cox sheaf on U is a Cox sheaf on X.

PROPOSITION V.2.2.10. *Let $\mathcal{R}$ be a (quasi-)Cox sheaf on X with a defining CBE $\pi\colon \mathcal{O}(K) \to \mathcal{R}$. If $\{1_{\mathcal{O}(X)}\}$ is saturated in $(\mathcal{K}(X)^{\mathrm{hom}})^*$ then the set of all $f \in (\mathcal{K}[K](X)^{\mathrm{hom}})^*$ with $\pi_X(f) = 1_{\mathcal{O}(X)}$ is saturated in $(\mathcal{K}[K](X)^{\mathrm{hom}})^*$.*

PROOF. If $f = g\chi^D$ satisfies $\pi_X(f^n) = 1$ then f^n and hence also f are units of $\mathcal{O}(K)(X)$ which means that D is principal. Thus there exists a unique homogeneous element $h \in \mathcal{K}(X)$ with $\pi_X(h\chi^D) = 1$. We then have $g^n = h^n$ and conclude $g = h$. □

V.3. Graded characteristic spaces

V.3.1. The characterization of graded characteristic spaces. In this section we prove characterization of graded characteristic spaces given in Theorem V.3.1.4 below. Moreover we consider good quotients $q\colon \widehat{X} \to X$ of graded schemes of Krull type and conditions which allow us to relate divisors and class groups of $\widehat{X}$ to those of X.

DEFINITION V.3.1.1. Let X be locally of Krull type. A *graded (quasi-)characteristic space* over X is a graded scheme $\widehat{X}$ over X, given by a morphism $q\colon \widehat{X} \to X$, such that q is affine and $q_*\mathcal{O}_{\widehat{X}}$ is a (quasi-)Cox sheaf on X.

Remark V.3.1.2. $q\colon \widehat{X} \to X$ is a graded (quasi-)characteristic space if and only if it is isomorphic as a graded scheme over X to some $\mathrm{Spec}_{\mathrm{gr,X}}(\mathcal{R}) \to X$ where $\mathcal{R}$ is a (quasi-)Cox sheaf on X.

Proposition V.3.1.3. *Let $q\colon \widehat{X} \to X$ be an affine morphism of graded schemes where X is locally of Krull type. Then q is dominant, $\widehat{X}$ is locally of Krull type and the pullback $q^*\colon \mathrm{WDiv}_X \to q_* \mathrm{WDiv}_{\widehat{X}}$ is an isomorphism (of sheaves of partially ordered abelian groups) if and only if $q^*\colon \mathcal{O}_X \to q_*\mathcal{O}_{\widehat{X}}$ is natural in $q_*\mathcal{K}_{\widehat{X}}$ in the sense of Definition V.2.1.1. In both cases we have a commutative diagram*

$$\begin{tikzcd}
(q_*\mathcal{K}^{\mathrm{hom}}_{\widehat{X}})^* \arrow[rr, "q_*\mathrm{div}^{\widehat{X}}"] \arrow[dr, "\mathrm{div}_{q_*\mathcal{O}_{\widehat{X}}}"'] & & q_* \mathrm{WDiv}_{\widehat{X}} . \\
& \mathrm{WDiv}_X \arrow[ur, "\cong", "q^*"'] &
\end{tikzcd}$$

Proof. If $q_*\mathcal{O}_{\widehat{X}}$ is natural in $q_*\mathcal{K}_{\widehat{X}}$ with defining family $\{\mu_Y\}_{Y\in\mathcal{Y}(X)}$ then $\mathcal{O}(q^{-1}(U))$ is of Krull type for each $U \in \mathcal{B}_X$ and hence $\widehat{X}$ is locally of Krull type. For each $U \in \mathcal{B}_X$ the graded kernel of $q_U^*\colon \mathcal{O}(U) \to \mathcal{O}(q^{-1}(U))$ is trivial and hence $q_{|q^{-1}(U)}\colon q^{-1}(U) \to U$ is dominant. By Theorem V.2.1.3 $\{\mu_{Y,U}\}_{Y\in\mathcal{Y}(U)}$ are the essential graded valuations of $\mathcal{O}(q^{-1}(U))$ and by Proposition I.2.6.9 the canonical map $\mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U)) \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(q^{-1}(U)))$, and hence also the pullback $q_U^*\colon \mathrm{WDiv}_X(U) \to \mathrm{WDiv}_{\widehat{X}}(q^{-1}(U))$, is an isomorphism.

Conversely, suppose that $\widehat{X}$ is locally of Krull type, q is dominant and the homomorphism $q^*\colon \mathrm{WDiv}_X \to q_* \mathrm{WDiv}_{\widehat{X}}$ is an isomorphism. Then for $Y \in \mathcal{Y}$ we have $\widehat{Y} := q_X^*(Y) \in \mathcal{Y}(\widehat{X})$ and set $\mu_Y := q_*\nu_{q_X^*(Y)}$. Then we have $q_*\mathbb{Z}^{(\widehat{Y})} = \mathbb{Z}^{(Y)}$ and $\mu_Y \circ q^*_{|(\mathcal{K}^{\mathrm{hom}})^*} = \nu_Y$ as required. Since q^* restricts to a bijection $\mathcal{Y}(X) \to \mathcal{Y}(\widehat{X})$, $\{\mu_Y\}_{Y\in\mathcal{Y}(X)}$ defines $q_*\mathcal{O}_{\widehat{X}}$ in $q_*\mathcal{K}_{\widehat{X}}$. □

Theorem V.3.1.4. *Let $q\colon \widehat{X} \to X$ be a morphism of graded schemes. If X is locally of Krull type and q is a graded (quasi-)characteristic space then the following hold with K denoting L (resp. L^{fin}):*

(i) *$\widehat{X}$ is locally of Krull type,*
(ii) *q is a good quotient and the pullback $\mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ is an isomorphism of partially ordered groups,*
(iii) *we have $\deg((\mathcal{K}(\widehat{X})^{\mathrm{hom}})^*) + gr(\mathcal{O}_X) = gr(\mathcal{O}_{\widehat{X}})$, $\mathrm{div}_{\widehat{X}}$ is surjective onto $\mathrm{WDiv}(\widehat{X})$ (resp. $\mathrm{WDiv}^{\mathrm{fin}}(\widehat{X})$), and $(\mathcal{O}(\widehat{X})^{\mathrm{hom}})^* = (\mathcal{O}(X)^{\mathrm{hom}})^*$.*

If K is free then the converse holds.

Proof. By Proposition V.3.1.3 q is a graded (quasi-)characteristic space if and only if condition (ii) of Theorem V.2.2.4 is satisfied. This condition is there shown to be neccessary for $q_*\mathcal{O}_{\widehat{X}}$ to be a Cox sheaf, and in case K is free also sufficient. □

Remark V.3.1.5. For each Cox sheaf $\mathcal{R}$ on X, the corresponding graded characteristic space $q\colon \widehat{X} \to X$ and each $w \in \mathrm{Cl}(X)$ we have

$$\mathrm{Bas}(w)(X) = \bigcap_{\substack{d\in pr^{-1}(w)\\ 0\neq f\in\mathcal{R}(X)_d}} |\mathrm{div}_{\mathcal{R},X}(f)| = q(V(\bigcup_{d\in pr^{-1}(w)} \mathcal{O}(\widehat{X})_d)).$$

Remark V.3.1.6. Let $\mathcal{R}$ be a Cox sheaf on $(X, \Omega_{X,H})$. Then for a closed irreducible subset $A \subseteq X$ and $U \in \Omega_{X,A}$ we have

$$S_{\mathrm{Cl}(X),A}(U) = c_X(\mathrm{div}_{\mathcal{R},X}(\mathcal{R}(U)^{\mathrm{hom}} \cap (\mathcal{R}_A^{\mathrm{hom}})^*)) = pr(\deg(\mathcal{R}(U)^{\mathrm{hom}} \cap (\mathcal{R}_A^{\mathrm{hom}})^*)).$$

V.3.2. Characterizations under finite generation conditions.

Proposition V.3.2.1. *Let $q\colon \widehat{X} \to X$ be a good quotient of graded schemes of Krull type. If there exists a q-saturated open set $\widehat{U} \subseteq \widehat{X}$ intersecting every prime divisor such that we have $\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{x}} = (\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{x}})^* \mathcal{O}^{\mathrm{hom}}_{X,q(\widehat{x})}$ for each point $\widehat{x} \in \widehat{U}$, then the pullback $q_X^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ is an isomorphism of partially ordered groups. The converse is true if $\deg((\mathcal{K}(\widehat{X})^{\mathrm{hom}})^*)) + gr(\mathcal{O}_X)/gr(\mathcal{O}_X)$ is finitely generated. In these cases each $\widehat{x} \in \widehat{U}$ in particular satisfies*

$$\deg((\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{x}})^*) + gr(\mathcal{O}_X) = \deg((\mathcal{K}(\widehat{X})^{\mathrm{hom}})^*) + gr(\mathcal{O}_X).$$

Proof. For $\widehat{U} \subseteq \widehat{X}$ as above the open set $U = q(\widehat{U}) \subseteq X$ intersects every prime divisor of X by Proposition V.1.1.12. For $Y = \overline{\{\eta\}} \in \mathcal{Y}(X)$ let $\widehat{\eta}$ be the special point in $q^{-1}(\eta)$. Since we have $\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{\eta}} = (\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{\eta}})^* \mathcal{O}_{X,\eta}$ and $\mathcal{O}_{X,\eta}$ is a discrete graded valuation ring, so is $\mathcal{O}_{\widehat{X},\widehat{\eta}}$, and a homogeneously prime element in the former is homogeneously prime in the latter. $\widehat{Y} = \overline{\{\widehat{\eta}\}} \in \mathcal{Y}(\widehat{X})$ is then the only prime divisor with image in Y and thus equals $q_X^*(Y)$. If $\widehat{Y}$ is an arbitrary prime divisor of $\widehat{X}$ then its generic point $\widehat{\eta}$ lies in $\widehat{U}$.

Conversely, suppose that $q_X^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ is an isomorphism. For $Y \in \mathcal{Y}(X)$ and $\widehat{Y} = q_X^*(Y)$, $\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{Y}}$ has a uniformizer in $\mathcal{O}^{\mathrm{hom}}_{X,Y}$ which means that

$$\deg((\mathcal{K}(\widehat{X})^{\mathrm{hom}})^*) + gr(\mathcal{O}_X) = \deg((\mathcal{O}_{\widehat{X},\widehat{Y}})^{\mathrm{hom}})^*) + gr(\mathcal{O}_X).$$

Due to the finite generation assumption there exist $f_1, \dots, f_m \in (\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{Y}})^*$ with

$$\deg((\mathcal{K}(\widehat{X})^{\mathrm{hom}})^*) + gr(\mathcal{O}_X) = \langle \deg(f_1), \dots, \deg(f_m)\rangle + gr(\mathcal{O}_X).$$

With $W^Y := X \setminus q(|\mathrm{div}_{\widehat{X}}(\prod_i f_i)|)$ we then have $f_i \in (\mathcal{O}(q^{-1}(W^Y))^{\mathrm{hom}})^*$ for each i. Consequently, each $W \in \mathcal{B}_{W^Y}$ satisfies $\mathcal{O}(q^{-1}(W))^{\mathrm{hom}} = (\mathcal{O}(q^{-1})^{\mathrm{hom}})^* \mathcal{O}(W)^{\mathrm{hom}}$ which in particular means that we have $\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{x}} = (\mathcal{O}^{\mathrm{hom}}_{\widehat{X},\widehat{x}})^* \mathcal{O}^{\mathrm{hom}}_{X,q(\widehat{x})}$ for each point $\widehat{x} \in \widehat{X}$. The union $\widehat{U}$ over all the sets $q^{-1}(W^Y)$ is then as desired. $\square$

Proof of Theorem V.2.2.5, Part II. Now, let K/G be finitely generated, and suppose $R_G \subseteq R$ satisfies conditions (i) - (iii). Let F be a system of representatives for the K-prime classes in $R^{\mathrm{hom}}/(R^{\mathrm{hom}})^*$. By finite generation of K/G there are $f_1, \dots, f_m \in F$ with $\langle \deg(f_1), \dots, \deg(f_m)\rangle + G = K$. Condition (iii) for $\mathfrak{p}_j = \langle f_j \rangle$ gives finite sets $F_j \subseteq F \setminus \{f_j\}$ with

$$\langle \deg(f) | f \in F_j\rangle + G = \deg(((R_{\mathfrak{p}_j})^{\mathrm{hom}})^*) + G = K.$$

The union $\{f_1, \dots, f_r\}$ of $\{f_1, \dots, f_m\}$ and all F_j satisfies $\langle \deg(f_j) | j \neq k\rangle + G = K$ for every $k = 1, \dots, r$.

For $j = 1, \dots, r$ let R_j be the localization by the product of all f_k with $k \neq j$. Let $\widehat{X}$ be the union of the sets $\widehat{X}_j := \mathrm{Spec}_{gr}(R_j) \subseteq \mathrm{Spec}_{gr}(R) =: \overline{X}$. By choice of $f_1, \dots, f_r$ all $X_j = \mathrm{Spec}_{gr}((R_j)_G)$ contain $X' = \mathrm{Spec}((R_{f_1 \cdots f_r})_G)$ as a principal open subset and thus glue to a graded scheme X. The maps $\widehat{X}_j \to X_j$ glue along $\widehat{X}' = \mathrm{Spec}(R_{f_1 \cdots f_r}) \to X'$ to a good quotient $q\colon \widehat{X} \to X$.

We verify that R is the Cox ring of X by showing that q is a graded characteristic space. $\widehat{X}$ is a K-Krull scheme because every R_j is a K-Krull ring (they are even K-factorial). By construction each R_j satisfies $\deg((R_j^{\mathrm{hom}})^*) + G = K$. Firstly, this yields $\deg((\mathcal{K}(\widehat{X})^{\mathrm{hom}})^*) + G = K$. Secondly, each pullback

$$q_{X_j}^*\colon \mathrm{WDiv}(X_j) = \mathrm{Div}_{gr}((R_j)_G) \to \mathrm{Div}_{gr}(R_j) = \mathrm{WDiv}(\widehat{X}_j)$$

is an isomorphism and hence, so is $q_X^*\colon \operatorname{WDiv}(X) \to \operatorname{WDiv}(\widehat{X})$. Moreover, we have $\mathcal{Y}_{\overline{X}}(\widehat{X}) = \mathcal{Y}_{\overline{X}}(\overline{X})$ because each $\mathcal{Y}_{\overline{X}}(\overline{X}) \setminus \mathcal{Y}_{\overline{X}}(\widehat{X}_j)$ is the set of the prime divisors corresponding to f_k where $k \neq j$. Thus, we have $\mathcal{O}(\widehat{X}) = R$ and assertions (i) and (ii) give $\operatorname{Cl}(\widehat{X}) = \operatorname{Cl}_{gr}(R) = 0$ and $(\mathcal{O}(\widehat{X})^{\mathrm{hom}})^* = (\mathcal{O}(\widehat{X})_G^{\mathrm{hom}})^*$. Thus, Theorem V.3.1.4 gives the assertion. □

V.3.3. Graded characteristic spaces of $\mathbb{F}_1$-schemes of finite type. We know from Proposition IV.3.4.3 that for a graded scheme X of finite type over $\mathbb{F}_1$ the (finite) set of points is in bijection with the basis $\mathcal{B}_X$ of open affine subsets. We show that if X is of Krull type then Weil divisors and class groups are finitely generated, as are sections of Cox sheaves and of $\mathcal{O}(\operatorname{WDiv}(X))$. We also give a simplified formula for the semigroups $S^{\mathrm{aff}}_{\operatorname{Cl}(X),x}(U)$, see Proposition V.3.3.4.

Proposition V.3.3.1. *For an $\mathbb{F}_1$-scheme X of finite and Krull type the following hold:*

(i) *Each* $\operatorname{WDiv}(U)$ *and* $\operatorname{Cl}(U)$ *is finitely generated.*
(ii) $\operatorname{Cl}(U)$ *is finite if and only if* $\{\nu_{Y,U}\}_{Y\in\mathcal{Y}(U)}$ *is linearly independent.*
(iii) $\operatorname{Cl}(U) = 0$ *if and only if* $\{\nu_{Y,U}\}_{Y\in\mathcal{Y}(U)}$ *may be completed to a basis of* $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}(X)^*, \mathbb{Z})$.

Proof. Assertion (i) follows from finiteness of X. In (ii) suppose that $\{\phi_j\}_{j\in J}$ is a (finite) basis of $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{N}^{-1}\mathcal{K}(X)^*, \mathbb{Q})$ containing $\{\mathbb{N}^{-1}\nu_{Y,U}\}_{Y\in\mathcal{Y}(U)}$. The element f_Y/n_Y of the dual basis corresponding to $\mathbb{N}^{-1}\nu_{Y,U}$ satisfies $\operatorname{div}_U(f_Y) = n_Y Y$. Conversely, if $\operatorname{Cl}(U)$ is finite then for $Y \in \mathcal{Y}(U)$ there are $f_Y \in \mathcal{K}(X)^*$ and $n_Y \in \mathbb{N}$ with $\operatorname{div}_U(f_Y) = n_Y Y$. Then $\mathbb{N}^{-1}(\mathcal{O}(U)\setminus\{0\}) = \mathbb{N}^{-1}\mathcal{O}(U)^* \oplus \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{Q}_{\geq 0}(f_Y/1)$ and $\mathbb{N}^{-1}\mathcal{K}(X)^* = \mathbb{N}^{-1}\mathcal{O}(U)^* \oplus \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{Q}(f_Y/1)$ hold. Let $\mathcal{O}(U)^* = G \oplus E$ be the composition into a finite and a free $\mathbb{Z}$-module. Now, $\{f_Y/n_Y\}_{Y\in\mathcal{Y}(U)}$ extends to a $\mathbb{Q}$-basis of $\mathbb{N}^{-1}E \oplus \bigoplus_Y \mathbb{Q}(f_Y/1)$. In the dual basis of $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{N}^{-1}\mathcal{K}(X)^*, \mathbb{Q})$ the element corresponding to f_Y/n_Y is $\mathbb{N}^{-1}\nu_{Y,U}$.

For (iii) suppose that $\{\phi_j\}_{j\in J}$ is a (finite) basis of $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}(X)^*, \mathbb{Z})$ which contains $\{\nu_{Y,U}\}_{Y\in\mathcal{Y}(U)}$. Let $\mathcal{K}(X)^* = G \oplus F$ be a sum of a finite subgroup G and a free subgroup F. Let $\{f_j\}_{j\in J} \subseteq F$ be the dual basis. Then the basis element f_Y corresponding to $\nu_{Y,U}$ satisfies $\operatorname{div}_U(f_Y) = Y$. Conversely, if $\operatorname{Cl}(U) = 0$ then there exist $f_Y \in \mathcal{K}(X)^*$ with $\operatorname{div}_U(f_Y) = Y$ for each $Y \in \mathcal{Y}(U)$. Then we have $\mathcal{O}(U) \setminus \{0\} = \mathcal{O}(U)^* \oplus \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{N}_0 f_Y$ and $\mathcal{K}(X)^* = \mathcal{O}(U)^* \oplus \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{Z} f_Y$. Let $\mathcal{O}(U)^* = G \oplus E$ be the composition into a finite group G and a free group E. Now, $\{f_Y\}_{Y\in\mathcal{Y}(U)}$ extends to a basis of $E \oplus \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{Z} f_Y$. Then in the dual basis of $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{K}(X)^*, \mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(E \oplus \bigoplus_{Y\in\mathcal{Y}(U)} \mathbb{Z} f_Y, \mathbb{Z})$ the element corresponding to f_Y is $\nu_{Y,U}$. □

Proposition V.3.3.2. *Let X be a graded $\mathbb{F}_1$-scheme of finite and Krull type. Then the following hold:*

(i) *For each divisorial $\mathcal{O}_X$-algebra and hence each Cox sheaf $\mathcal{R}$ on X, $\mathcal{R}(U)$ is finitely generated for each open U.*
(ii) *If $\phi\colon \widehat{X} \to X$ is a dominant morphism between graded $\mathbb{F}_1$-schemes of finite and Krull type such that the pullback $\phi_X^*\colon \operatorname{WDiv}(X) \to \operatorname{WDiv}(\widehat{X})$ is an isomorphism of partially ordered groups, e.g. ϕ is a graded characteristic space over X, then for a generic point $\widehat{\eta} \in \widehat{X}$ of a prime divisor we have $\phi^{-1}(U_{\phi(\widehat{\eta})}) = \widehat{U}_{\widehat{\eta}}$ for the respective minimal affine open sets containing $\widehat{\eta}$ resp. $\phi(\widehat{\eta})$.*

Consider an affine graded scheme X over $\mathbb{F}_1$ and let $U, V \in \mathcal{B}_X$. Then $U \subseteq V$ is *K-saturated* if there exists $f \in \mathcal{O}(V)_K$ with $U = V_f$. More generally, if $q\colon X \to Z$ is a good quotient of graded $\mathbb{F}_1$-schemes then open sets of the form $q^{-1}(U)$ with $U \subseteq Z$ open are called *q-saturated*.

Proposition V.3.3.3. *Let X be an integral canonically graded affine $\mathbb{F}_1$-scheme, let $K \subseteq gr(\mathcal{O}_X)$ be a subgroup and let $\pi\colon gr(\mathcal{O}_X) \to gr(\mathcal{O}_X)/K$ denote the canonical epimorphism. Then for $U, V \in \mathcal{B}_X \setminus \{\emptyset\}$ the following hold:*

(i) *If $U \subseteq V$ then $\pi(\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ \subseteq \pi(\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ$ if and only if U is K-saturated in V.*

(ii) *If $\pi(\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ \cap \pi(\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ \neq \emptyset$ then $U \cap V$ is K-saturated in U and V, and the canonical map $\mathcal{O}(U)_K \otimes_{\mathbb{F}_1} \mathcal{O}(V)_K \to \mathcal{O}(U \cap V)_K$ is then surjective. The converse holds if $\mathcal{O}(X)$ is factorial.*

Proof. For (i) suppose that U is K-saturated in V, i.e. there exists $w \in \mathcal{O}(V)_K$ with $U = V_w$. Let $v \in \mathcal{O}(X)$ with $V = X_v$. Then $w = u - kv$ holds with $u \in (\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ$ and $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\pi(u) = \pi(kv) &\in \pi((\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ) \cap \pi((\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ)\\ &\subseteq \pi(\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ \cap \pi(\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ.\end{aligned}$$

and consequently, $\pi(\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ$ is contained in $\pi(\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ$.

In (ii) suppose there exist $v \in (\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ, u \in (\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ$ with $w := v - u \in K$. Then we have $U = X_u$, $V = X_v$ and $U \cap V = X_{u+v} = U_w = V_{-w}$. Moreover,

$$\begin{aligned}\mathcal{O}(U)_K + \mathcal{O}(V)_K \subseteq \mathcal{O}(U \cap V)_K &= \mathcal{O}(U)_K - \mathbb{N}_0 w = \mathcal{O}(V)_K + \mathbb{N}_0 w\\ &\subseteq \mathcal{O}(U)_K + \mathcal{O}(V)_K\end{aligned}$$

which means that $U \cap V$ is saturated in both U and V, and the canonical map $\mathcal{O}(U)_K \otimes_{\mathbb{F}_1} \mathcal{O}(V)_K \to \mathcal{O}(U \cap V)_K$ is surjective. If the converse holds then by Remark I.1.3.26 there exists $w \in \mathcal{O}(U)_K \cap -\mathcal{O}(V)_K$ with

$$\mathcal{O}(U)_K + \mathcal{O}(V)_K = \mathcal{O}(U \cap V)_K = \mathcal{O}(U)_K - \mathbb{N}_0 w = \mathcal{O}(V)_K + \mathbb{N}_0 w.$$

Due to Example I.1.3.27 we must have $w \in (\mathcal{O}(X) \cap \mathcal{O}(V)^*)^\circ - (\mathcal{O}(X) \cap \mathcal{O}(U)^*)^\circ$ as required. □

Proposition V.3.3.4. *Let X be a $\mathbb{F}_1$-scheme of finite and Krull type and let $x \in X$. If $S^{\mathrm{aff}}_{\mathrm{Cl}(X),x}(U)$ is non-empty, i.e. $\omega^{\mathrm{aff}}_{\mathrm{Cl}(X),x}(U)$ is non-empty, then $S^{\circ}_{\mathrm{Cl}(X),x}(U)$ is the saturation of $S^{\mathrm{aff}}_{\mathrm{Cl}(X),x}(U)$ and we have $\omega^{\mathrm{aff}}_{\mathrm{Cl}(X),x}(U) = \omega^{\circ}_{\mathrm{Cl}(X),x}(U)$.*

Proof. Let $q\colon \widehat{X} \to X$ be a graded characteristic space, set $L := \mathrm{WDiv}(X)$ and let $\pi\colon \mathcal{K}(\widehat{X})^* \to L \to \mathrm{Cl}(X)$ denote the canonical map. Consider $U \in \Omega_{X,x}$ and set $\widehat{U} := q^{-1}(U)$. Then $S^{\mathrm{aff}}_{\mathrm{Cl}(X),x}(U)$ is the union over all the semigroups $c_X(S_{L,y}(U)^\circ)$ where x specializes to y and U_y has a purely one-codimensional complement in U. If such an y exists then U_x has purely one-codimensional complement in U_y, and hence it also has a purely one-codimensional complement in U. Let $\widehat{x}$ and $\widehat{y}$ be the special points over x resp. y. Since $\widehat{U}_{\widehat{x}}$ is $\mathcal{K}(X)^*$-saturated in $\widehat{U}_{\widehat{y}}$, Proposition V.3.3.3 gives

$$\begin{aligned}\mathrm{sat}(c_X(S^\circ_{L,y}(U))) = S^\circ_{\mathrm{Cl}(X),y}(U) &= \pi(\mathcal{O}(\widehat{U}) \cap \mathcal{O}(\widehat{U}_{\widehat{y}})^*)^\circ \subseteq \pi(\mathcal{O}(\widehat{U}) \cap \mathcal{O}(\widehat{U}_{\widehat{x}})^*)^\circ\\ &= S^\circ_{\mathrm{Cl}(X),x}(U) = \mathrm{sat}(c_X(S^\circ_{L,x}(U)))\end{aligned}$$

which establishes the assertion. □

Remark V.3.3.5. Let X be an $\mathbb{F}_1$-scheme of finite and Krull type. Then due to the above Proposition X is covered by the set $\mathcal{U}$ of those $U \in \mathcal{B}_X$ for which $X \setminus U$ is purely one-codimensional if and only if $\mathcal{U} = \mathcal{B}_X$.

V.3.4. $\mathbb{F}_1$-schemes of finite Krull type and combinatorial schematic functors. We continue in the notation of Section IV.3.4. After performing a dualizing operation on the cofunctors defined there we obtain *combinatorial schematic functors* whose equivalence with $\mathbb{F}_1$-schemes of finite and Krull type is shown in Corollary V.3.4.9. Using the results of the previous section we are then able to give a characterization of graded characteristic spaces in terms of the corresponding morphisms of combinatorial schematic functors in Proposition V.3.4.13.

DEFINITION V.3.4.1. Let $\mathfrak{K}$ be the full subcategory of those $\mathfrak{O}_{\mathrm{int}}^{\mathrm{covar}}$-objects $(\mathcal{O}\colon J \to \mathbf{Alg}_{\mathbb{F}_1}, B)$ for which each $\mathcal{O}(j)$ is of Krull type and B has a free unit group.

DEFINITION V.3.4.2. The category $\mathfrak{I}$ of *combinatorial schematic functors* is defined as follows: A $\mathfrak{I}$-object is a functor $\sigma\colon I \to \mathbf{Mon}$ from a partially ordered set together with a free finitely generated abelian group N such that

(i) I has a least element,
(ii) σ maps to the category of pointed, finitely generated and saturated submonoids of N,
(iii) for each $i \in I$, $\sigma_{|I_{\leq i}}$ defines an isomorphism to $\mathrm{faces}(\sigma(i))$.

An $\mathfrak{I}$-morphism from (σ, N) to the pair formed by $\sigma'\colon I' \to \mathbf{Mon}$ and N' is a morphism $\alpha\colon I \to I'$ of partially ordered sets together with a homomorphism $\phi\colon N \to N'$ such that $\mathrm{face}(\phi(\sigma(i))) = \sigma'(\alpha(i))$ holds for each $i \in I$.

REMARK V.3.4.3. Note that for an $\mathfrak{I}$-object $\sigma\colon I \to \mathbf{Mon}$ the set I is finite. Moreover, the composition of $\mathfrak{I}$-morphisms is again a morphism because under monoid homomorphisms $\phi\colon M \to M'$ and $\phi'\colon M' \to M''$ we have

$$\mathrm{face}(\phi'(\mathrm{face}(\phi(\mathrm{face}(u))))) = \mathrm{face}(\phi'(\phi(u))) = \mathrm{face}(\phi'(\phi(\mathrm{face}(u)))).$$

EXAMPLE V.3.4.4. Let $\phi\colon N \to N'$ be a homomorphism of free finitely generated abelian groups and let $M \subseteq N$ and $M' \subseteq N'$ be saturated submonoids with $\phi(M) \subseteq M'$. Then ϕ together with $\alpha\colon \mathrm{faces}(M) \to \mathrm{faces}(M'), \tau \mapsto \mathrm{face}(\phi(\tau))$ defines an $\mathfrak{I}$-morphism from $\mathrm{id}_{\mathrm{faces}(M)}$ to $\mathrm{id}_{\mathrm{faces}(M')}$.

REMARK V.3.4.5. If (α, ϕ) is a morphism from (σ, N) to (σ', N') then for $j \in I_{\leq i}$, $\alpha(j)$ is the unique minimal $j' \in I'_{\leq \alpha(i)}$ with $\phi(\sigma(j)) \subseteq \sigma'(j')$. In other words, for $i \in I$ the canonical isomorphism $(\mathrm{faces}(\sigma(i)), N) \to (\sigma_{|I_{\leq i}}, N)$ composed with the restriction $(\sigma_{|I_{\leq i}}, N) \to (\sigma'_{|I'_{\leq \alpha(i)}}, N')$ is equal to the canonical morphism $(\mathrm{id}_{\mathrm{faces}(\sigma(i))}, N) \to (\mathrm{id}_{\mathrm{faces}(\sigma'(\alpha(i)))}, N')$ composed with the canonical isomorphism $(\mathrm{faces}(\sigma'(\alpha(i))), N') \to (\sigma'_{|I'_{\leq \alpha(i)}}, N')$.

CONSTRUCTION V.3.4.6. For an $\mathfrak{O}_{\mathrm{int}}^{\mathrm{covar}}$-object $(\mathcal{O}\colon J \to \mathfrak{C}, B)$ sending $j \in J$ to

$$\sigma(j) := (\mathcal{O}(j) \setminus 0)^{\vee} \subseteq \mathrm{Hom}_{\mathbb{Z}}(B \setminus 0, \mathbb{Z})$$

as defined in Section I.1.3 constitutes an $\mathfrak{I}$-object. For a morphism (α, ψ) from $(\mathcal{O}, B)$ to $(\mathcal{O}'\colon J' \to \mathfrak{C}, B')$, the induced map $\phi\colon \mathrm{Hom}_{\mathbb{Z}}(B \setminus 0, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(B' \setminus 0, \mathbb{Z})$ together with α forms an $\mathfrak{I}$-morphism. This defines a functor to $\mathfrak{I}$.

PROOF. Due to Proposition I.1.3.23 each $\sigma(j)$ is pointed and all faces of $\sigma(j)$ are duals of principal localizations of $\mathcal{O}(j) \setminus 0$. Let $\tau' := \sigma'(\alpha(j)) \cap u'^{\perp}$, where $\eta' = \mathrm{face}(u')$ holds with some $u' \in \mathcal{O}'(\alpha(j)) \setminus \{0\}$, such that $\phi(\sigma(j)) \subseteq \tau' = \sigma'(\imath'_{\tau'})$. Then we have $\mathcal{O}'(i'_{\tau'}) = \eta'^{-1}\mathcal{O}'(\alpha(j))$ which means $\rho^{\alpha(j)}_{i'_{\tau'}}(\eta') \subseteq \mathcal{O}'(i'_{\tau'})^*$. Then $\psi(u') = \psi(\rho^{\alpha(j)}_{i'_{\tau'}}(u'))$ is a unit and locality implies that u' is a unit in $\mathcal{O}(j)$ and hence $\tau' = \sigma'(\alpha(j))$. $\square$

CONSTRUCTION V.3.4.7. For an $\mathfrak{I}$-object $(\sigma\colon I \to \mathbf{Mon}, N)$ sending $i \in I$ to

$$\mathcal{O}(i) := \mathbb{F}_1[\sigma(i)^\vee] \subseteq \mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})]$$

as defined in Section I.1.3 constitutes a $\mathfrak{K}$-object. For a morphism (α, ϕ) from (σ, N) to $(\sigma'\colon I' \to \mathbf{Mon}, N')$, the induced map $\psi\colon \mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(N', \mathbb{Z})] \to \mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})]$ together with forms α an $\mathfrak{K}$-morphism.

PROOF. To show that (α, ψ) is a morphism consider $i \in I$ and $j' \leq \alpha(i)$, and let $\tau' \preceq \mathcal{O}'(\alpha(i)) \setminus \{0\}$ be the corresponding face. Set $\tau := \mathrm{face}(\psi(\tau')) \preceq \mathcal{O}(i)$ and let j be the corresponding element below i. Since $\psi(\tau') \subseteq \tau$ we calculate $\psi(\mathcal{O}'(j')) = \psi(\tau'^{-1}\mathcal{O}'(\alpha(i))) \subseteq \tau^{-1}\mathcal{O}(i) = \mathcal{O}(j)$ and hence Remark I.1.3.24 gives $\phi(\sigma(j)) \subseteq \sigma'(j')$. Minimality gives $\alpha(j) \leq j'$ as required. □

PROPOSITION V.3.4.8. *The above constructions define mutually essentially inverse (covariant) equivalences of categories between $\mathfrak{I}$ and $\mathfrak{K}$.*

PROOF. Let $\sigma\colon I \to \mathbf{Mon}$ be an $\mathfrak{I}$-object. Then due to Proposition I.1.3.23(ii) For an $\mathfrak{I}$-object (σ, N), id_I together with the canonical isomorphism

$$N \longrightarrow \mathrm{Hom}_\mathbb{Z}(\mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})] \setminus 0, \mathbb{Z})$$

forms an $\mathfrak{I}$-isomorphism from (σ, N) to the pair consisting of $(\mathbb{F}_1[\sigma^\vee] \setminus 0)^\vee$ and $\mathrm{Hom}_\mathbb{Z}(\mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})] \setminus 0, \mathbb{Z})$. These isomorphisms form a natural isomorphism between $\mathrm{id}_\mathfrak{I}$ and $(- \setminus 0)^\vee \circ \mathbb{F}_1[(-)^\vee]$.

Likewise, for an $\mathfrak{K}$-object $(\mathcal{O}\colon J \to \mathbf{Alg}_{\mathbb{F}_1}, B)$ Proposition I.1.3.23(ii) implies that id_J together with the canonical isomorphism

$$B \longrightarrow \mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(\mathrm{Hom}_\mathbb{Z}(B \setminus 0, \mathbb{Z}), \mathbb{Z})]$$

forms an $\mathfrak{K}$-isomorphism from $(\mathcal{O}, B)$ to the pair consisting of $\mathbb{F}_1[((\mathcal{O} \setminus 0)^\vee)^\vee]$ and $\mathbb{F}_1[\mathrm{Hom}_\mathbb{Z}(\mathrm{Hom}_\mathbb{Z}(B \setminus 0, \mathbb{Z}), \mathbb{Z})]$. These morphisms form a natural isomorphism from $\mathrm{id}_\mathfrak{K}$ to $\mathbb{F}_1[(-)^\vee] \circ (- \setminus 0)^\vee$. □

Together with Proposition IV.3.4.5 the above gives the following.

COROLLARY V.3.4.9. *We have mutually essentially inverse covariant equivalences between $\mathfrak{I}$ and the category of $\mathbb{F}_1$-schemes X of finite and Krull type for which $\mathcal{K}(X)^*$ is free and dominant morphisms; one sending (σ, N) to* $\mathrm{Spec}(\mathbb{F}_1[\sigma^\vee \setminus \{0\}])$, *the other sending $(X, \mathcal{O}_X)$ to* $((\mathcal{O}_{X|\mathcal{B}_X \setminus \{\emptyset\}} \setminus \{0\})^\vee, \mathrm{Hom}_\mathbb{Z}(\mathcal{O}_{X,\xi} \setminus \{0\}, \mathbb{Z}))$.

DEFINITION V.3.4.10. For an $\mathfrak{I}$-object $(\sigma\colon I \to \mathbf{Mon}, N)$, I has a unique minimal element $0 \in I$. A minimal element of $I_{>0}$ is called a *ray* and the set of all rays is denoted $I^{(1)}$.

REMARK V.3.4.11. For $i \in I^{(1)}$ we have $\sigma(i) = \mathbb{N}_0 v_i$ with a unique $v_i \in N$ called the generator of the ray i. The corresponding element of the double dual of N is denoted ν_i. Moreover, for each $j \in I$, $\sigma(j)$ is the saturation of the monoid generated by all v_i for $i \in I^{(1)}_{\leq j}$, and $\{\nu_i\}_{i \in I^{(1)}_{\leq j}}$ are the essential valuations of $\sigma(j)$.

REMARK V.3.4.12. For an $\mathbb{F}_1$-scheme X of finite and Krull type such that $\mathcal{K}(X)^*$ is free consider the $\mathfrak{I}$-object consisting of the assignment $[x \mapsto (\mathcal{O}_{X,x} \setminus \{0\})^\vee]$ and the group $\mathrm{Hom}_\mathbb{Z}(\mathcal{K}(X)^*, \mathbb{Z})$. Then the minimal element of X with respect to specialization is the generic point ξ of X and the rays are precisely the generic points of the prime divisors of X.

PROPOSITION V.3.4.13. *Let (α, ϕ) be an $\mathfrak{I}$-morphism from $(\widehat{\sigma}\colon \widehat{I} \to \mathbf{Mon}, \widehat{N})$ to $(\sigma\colon I \to \mathbf{Mon}, N)$. Let $q\colon \widehat{X} \to X$ be the corresponding morphism of $\mathbb{F}_1$-schemes. Then the following hold:*

(i) *q is affine if and only if each non-empty $\alpha^{-1}(I_{\leq i})$ has a greatest element.*

(ii) *q is a good quotient if and only if ϕ has full rank, each $\alpha^{-1}(i)$ has a greatest element $\widehat{i}$ and it satisfies $\sigma(i) = \phi(\widehat{\sigma}(\widehat{i}))^{\vee\vee}$.*

(iii) *The pullback $q^*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ is an isomorphism of partially ordered groups if and only if α restricts to a bijection $\widehat{I}^{(1)} \to I^{(1)}$ and $\phi(\widehat{v}_{\widehat{i}}) = v_{\alpha(\widehat{i})}$ holds for the respective ray generators.*

(iv) *Equipping $\mathcal{O}_{\widehat{X}}$ with the canonical $\mathcal{K}(\widehat{X})^*/q_X^*(\mathcal{K}(X)^*)$-grading turns q into a graded characteristic space if and only if ϕ has full rank, each $\alpha^{-1}(i)$ has a greatest element $\widehat{i}$ and it satisfies $\sigma(i) = \phi(\widehat{\sigma}(\widehat{i}))^{\vee\vee}$, α restricts to a bijection $\widehat{I}^{(1)} \to I^{(1)}$, $\phi(\widehat{v}_{\widehat{i}}) = v_{\alpha(\widehat{i})}$ and $\{\widehat{v}_{\widehat{i}} | \widehat{i} \in \widehat{I}^{(1)}\}^{\perp} = \phi^*\{v_i | i \in I^{(1)}\}^{\perp}$ hold for the ray generators, and $\{\widehat{v}_{\widehat{i}}\}_{\widehat{i} \in \widehat{I}^{(1)}}$ may be completed to a $\mathbb{Z}$-basis.*

Proof. Assertion (ii) follows from Remark IV.3.3.2 and Remark I.1.3.24. Assertion (iii) is due to Corollary V.1.2.13. In (iv) we relate the conditions of Theorem V.3.1.4 to the present setting. The criterion for factoriality of $\widehat{X}$ is due to Proposition V.3.3.1. The unit condition is due to $\mathcal{O}(\widehat{X})^* = \{\widehat{v}_{\widehat{i}} | \widehat{i} \in \widehat{I}^{(1)}\}^{\perp}$ and $\mathcal{O}(X)^* = \{v_i | i \in I^{(1)}\}^{\perp}$. $\square$

Definition V.3.4.14. An $\mathfrak{J}$-morphism (α, ϕ) whose associated morphism of $\mathbb{F}_1$-schemes carries the structure of a graded characteristic space is called a *(combinatorial) Cox construction.*

Remark V.3.4.15. Let $\mathcal{O}\colon J \to \mathfrak{C}$ be a $\mathfrak{J}$-object and let X be the associated graded scheme. Then X is locally of Krull type if and only if each two $j, k \in J$ satisfy $J_{\leq j} \cap J_{\leq k} \neq \emptyset$ and each $\mathcal{O}(j)$ is of Krull type. We then have a canonical isomorphism $\mathrm{WDiv}(X) \cong \mathrm{colim}_{j \in J} \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(j))$. The divisor homomorphism $\mathrm{div}_U\colon (\mathcal{K}(U)^{\mathrm{hom}})^* \to \mathrm{WDiv}(U)$ is then the map induced by all the canonical maps $(Q_{\mathrm{gr}}(\mathcal{O}(j))^{\mathrm{hom}})^* \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(j))$ for $U_j \subseteq U$.

Let $(\mathfrak{f}, \mathfrak{f}^*)$ be a $\mathfrak{J}$-morphism from $\mathcal{O}$ to $\mathcal{O}'\colon J' \to \mathfrak{C}$ such that all $\mathcal{O}(j)$ and $\mathcal{O}'(j')$ are of Krull type and both J and J' are finitely generated. Suppose the associated morphism $(\phi, \phi^*)\colon (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ of graded schemes is dominant, i.e. each $\mathfrak{f}(j)$ is non-empty. Then the pullback is the map induced by the canonical maps $\mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(j)) \to \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}'(j'))$ where $j' \in \mathfrak{f}(j)$.

CHAPTER VI

Cox sheaves of quasi-torus actions and their characteristic spaces

In this chapter we translate the results on graded characteristic spaces into the equivalent category of quasi-torus actions. For such a (morphical) action $H \circlearrowright X$ on a prevariety over $\mathbb{K}$ with affine H-invariant cover we consider the invariant topology $\Omega_{X,H}$ on X and the invariant structure sheaf $\mathcal{O}_{X,H} := (\mathcal{O}_X)_{|\Omega_{X,H}}$, which is naturally graded by the character group of H. Provided that the sections of $\mathcal{O}_{X,H}$ are of Krull type (i.e. normally graded) one may define invariantly prime divisors, their associated graded valuations, invariant class groups $\mathrm{Cl}_H(X)$, Cox sheaves on $\Omega_{X,H}$ and characteristic spaces of $H \circlearrowright X$.

After recalling facts on quasi-tori in Section VI.1.3 and a brief discussion of the invariant structure and good quotients of actions in the general case in Section VI.1 we establish the equivalence between reduced graded schemes of finite type over $\mathbb{K}$ and quasi-torus actions on prevarieties with affine invariant cover over $\mathbb{K}$ in Section VI.2. As a further preparation we discuss generic isotropy groups which turn out to have a useful description in terms of the degrees of homogeneous units of the stalks of the invariant structure sheaf in Section VI.3.

Characteristic spaces are then characterized in Section VI.4 as good quotients $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ such that $\widehat{H} \circlearrowright \widehat{X}$ is of Krull type with $\mathrm{Cl}_{\widehat{H}}(\widehat{X}) = 0$, the rings $\mathcal{O}(\widehat{X})$ and $\mathcal{O}(X)$ have the same homogeneous units, $\ker(\theta)$ acts with constant isotropy on a big saturated $\widehat{H}$-invariant open subset of $\widehat{X}$ and θ restricts to an isomorphism $\widehat{H}_{\widehat{X}} \to H_X$. We also relate various cones of invariant divisor classes of X to orbit cones of the actions of $\widehat{H}$ resp. $\ker(\theta)$.

VI.1. Invariant geometry of algebraic actions

VI.1.1. Invariant topology and structure sheaf of an action. By the category of *affine algebraic groups* over $\mathbb{K}$ we mean the category of group objects in the category of affine varieties over $\mathbb{K}$. The forgetful functor from affine varieties to their underlying sets induces a functor from affine algebraic groups to groups. In general, we consider only *abelian* affine algebraic groups H. The corresponding *cogroup objects* in the category of affine $\mathbb{K}$-algebras are affine Hopf $\mathbb{K}$-algebras.

Throughout, we will study *morphical* actions of affine algebraic groups on prevarieties over $\mathbb{K}$ and correspondingly, coactions of affine Hopf $\mathbb{K}$-algebras on affine $\mathbb{K}$-algebras. A morphical action $\mu\colon H \times X \to X$ of an affine algebraic group H on a prevariety X will also be denoted $H \circlearrowright X$. We will then also speak of an H-prevariety X. For any $\mathbb{K}$-prevariety X we denote the Zariski topology by Ω_X and the basis of affine open subsets by $\mathcal{B}_X$.

Definition VI.1.1.1. Let H be an affine algebraic group and X an H-prevariety.

(i) The *H-invariant topology* on X is the subtopology $\Omega_{X,H} \subseteq \Omega_X$ of those Zariski open subsets which are H-invariant. Its intersection with the set $\mathcal{B}_X$ of affine open sets is denoted $\mathcal{B}_{X,H}$.

(ii) A subset of X is *H-open/-closed/-irreducible* if it is H-invariant and $\Omega_{X,H}$-open/-closed/-irreducible.

(iii) The H-*invariant dimension* of X is defined as the topological dimension
$$\dim_H(X) := \dim(X, \Omega_{X,H})$$
of the topological space $(X, \Omega_{X,H})$.

(iv) The H-*invariant codimension* of an H-irreducible subspace $Y \subseteq X$ in X is the codimension
$$\operatorname{codim}_{X,H}(Y) := \operatorname{codim}_{(X,\Omega_{X,H})}(Y)$$
in the topological space $(X, \Omega_{X,H})$.

Remark VI.1.1.2. For an H-prevariety X the following hold:

(i) For $A \subseteq X$ we have $\overline{A}^{\Omega_{X,H}} = \overline{HA}^{\Omega_X}$ because each $\Omega_{X,H}$-closed set which contains A also contains HA and conversely, $\overline{HA}^{\Omega_X}$ is H-invariant because whenever an Ω_X-closed set Y contains HA, so does the H-invariant set $\bigcap_{h\in H} hY$.

(ii) Minimal H-irreducible subsets are just orbits and minimal closed H-irreducible subsets are closed orbits. Every H-closed subset contains closed orbits.

(iii) If $X = \overline{Hx}$ then Hx is the unique minimal element of $\Omega_{X,H}$.

(iv) If $Y \subset X$ is irreducible, then $H \cdot Y$ is H-irreducible.

Remark VI.1.1.3. Equivariant morphisms are continuous with respect to the invariant topologies, because under equivariant maps preimages of invariant sets are invariant.

Remark VI.1.1.4. A morphism $(\theta, \phi)\colon H \circlearrowright X \to G \circlearrowright Y$ is called *equivariantly dominant* if the induced continuous map $(X, \Omega_{X,H}) \to (Y, \Omega_{Y,G})$ is dominant.

Remark VI.1.1.5. For an H-prevariety X the topology $\Omega_{X,H}$ is noetherian.

Proposition VI.1.1.6. *For an H-closed subset $Y \subset X$ the following hold:*

(i) *Any irreducible component $Z \subset Y$ is invariant under the connected component H_e containing the unit element e. The (different) products of the form $H \cdot Z$ form a decomposition of Y into H-irreducible components.*

(ii) *If Y is H-irreducible then Y is equidimensional and all its irreducible components Z satisfy $Y = H \cdot Z$.*

Proof. Let $H = H_e \sqcup h_1 H_e \sqcup \ldots \sqcup h_d H_e$ and $Y = Z_1 \cup \ldots \cup Z_m$ be decompositions into connected resp. irreducible components.

$H_e Z_i$ is the image of the irreducible set $H_e \times Z_i$ under the action and hence $\overline{H_e \cdot Z_i}$ is contained in some Z_j which gives $H_e Z_i = Z_i$, and the first part of (i) is shown. We also infer that $HZ_i = H_e Z_i \cup h_1 H_e Z_i \cup \ldots \cup h_d H_e Z_i$ is closed. Removing duplicates we obtain a subset $I \subseteq \{1, \ldots, m\}$ such that
$$Y = H \cdot Y = \bigcup_{i\in I} HZ_i = \bigcup_{i\in I} \bigcup_{j=0}^{d} h_j H_e Z_i$$
is a decomposition into irreducible components $h_j H_e Z_i$ where $h_0 = e$. Therefore, the sets HZ_i are irreducible components with respect to $\Omega_{X,H}$. This shows the remainder of assertion (i).

If Y is H-irreducible, then $Y = H \cdot Y = H \cdot (Z_1 \cup \ldots \cup Z_{m-1}) \cup H \cdot Z_m$ is a decomposition into H-invariant closed subsets, so by H-irreducibility of Y we obtain $Y = HZ_i$ for some i and the above decomposition of HZ_i implies that the irreducible components $h_j H_e Z_i$ of $Y = HZ_i$ are pairwise isomorphic. □

Remark VI.1.1.7. In an H-prevariety X translation by an element $h \in H$ defines a automorphism $\mu_h\colon X \xrightarrow{x\mapsto(h,x)} \{h\} \times X \xrightarrow{\mu_{|\{h\}\times X}} X$ of prevarieties with inverse $\mu_{h^{-1}}$. μ_h is H-equivariant if h lies in the center of H.

DEFINITION VI.1.1.8. Let X be a H-prevariety.

(i) Its *invariant structure sheaf* is the restriction $\mathcal{O}_{X,H} := \mathcal{O}_{X|\Omega_{X,H}}$. Via
$$H \times \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U), \quad (h,f) \longmapsto h \cdot f := (\mu^*_{h^{-1}})_U(f),$$
H acts by automorphisms on the sections of $\mathcal{O}_{X,H}$ in a way which is compatible with restriction maps.

(ii) For any H-irreducible subset $Y \subseteq X$ we define the *stalk* of $\mathcal{O}_{X,H}$ at Y as
$$(\mathcal{O}_{X,H})_Y := \operatorname*{colim}_{U \in (\Omega_{X,H})_Y} \mathcal{O}_{X,H}(U).$$
It carries an induced H-action by automorphisms.

(iii) For an affine algebraic subgroup $G \subseteq H$ the subsheaf $\mathcal{O}^G_{X,H} \subseteq \mathcal{O}_{X,H}$ of G-*invariants* sends $U \in \Omega_{X,H}$ to the G-fixpoint algebra
$$\mathcal{O}^G_{X,H}(U) := \{f \in \mathcal{O}_{X,H}(U) \mid \mu^*_U(f) = pr^*_U(f)\}.$$

REMARK VI.1.1.9. For a morphism $(\theta,\phi)\colon G \circlearrowright X \to H \circlearrowright Y$ the induced homomorphism $\phi^*\colon \mathcal{O}_{Y,H} \to \phi_*\mathcal{O}_{X,G}$ is compatible with the actions. Consequently, for a G-irreducible subset $Z \subseteq X$ the map $(\mathcal{O}_{Y,H})_{\overline{\phi(Z)}^{\Omega_{Y,H}}} \to (\mathcal{O}_{X,H})_Z$ is compatible.

VI.1.2. Invariant Zariski topology. Until further notice we suppose that $H \circlearrowright X$ is an action on an affine variety and set $R := \mathcal{O}(X)$. In this section we relate H-invariant subsets of X to H-invariant ideals of R.

DEFINITION VI.1.2.1. An ideal $\mathfrak{a} \trianglelefteq R$ is called H-*invariant* if $H\mathfrak{a} = \mathfrak{a}$. A proper H-invariant ideal $\mathfrak{q} \trianglelefteq R$ is called H-prime if for any H-invariant ideals $\mathfrak{a}, \mathfrak{b}$, $\mathfrak{ab} \subseteq \mathfrak{q}$ implies $\mathfrak{a} \subseteq \mathfrak{q}$ or $\mathfrak{b} \subseteq \mathfrak{q}$. $\mathfrak{p}$ is called H-maximal, if it is maximal among H-invariant ideals.

REMARK VI.1.2.2. Sums, intersections and finite products of invariant ideals are invariant. Thus, Zorn's Lemma implies the existence of H-maximal ideals. Clearly, H-invariant prime ideals are also H-prime. The converse need not be true in general, though it holds for certain groups, e.g. connected algebraic groups acting on affine varieties and their coordinate rings.

PROPOSITION VI.1.2.3. *Let $\mathfrak{p} \trianglelefteq R$ be prime. Then $\bigcap_{h\in H} h\mathfrak{p}$ is H-prime.*

PROOF. Set $\mathfrak{q} := \bigcap_{h\in H} h\mathfrak{p}$ and let $\mathfrak{a}, \mathfrak{b} \trianglelefteq R$ be H-invariant with $\mathfrak{ab} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. Then we may assume $\mathfrak{a} \subseteq \mathfrak{p}$ and for all $h \in H$ we get $\mathfrak{a} = h\mathfrak{a} \subseteq h\mathfrak{p}$. Thus, $\mathfrak{a} \subseteq \bigcap_{h\in H} h\mathfrak{p} = \mathfrak{q}$. □

PROPOSITION VI.1.2.4. *Let $\mathfrak{a} \trianglelefteq R$ be H-invariant. Then $R/\mathfrak{a}$ has the induced H-action and it respects $\phi\colon R \to R/\mathfrak{a}$. Furthermore, the order preserving bijection*
$$\{\mathfrak{a} \subseteq \mathfrak{b} \trianglelefteq R;\ H-invariant\} \to \{\mathfrak{c} \trianglelefteq R/\mathfrak{a};\ H-invariant\}$$
$$\mathfrak{b} \mapsto \phi(\mathfrak{b})$$
$$\phi^{-1}(\mathfrak{c}) \leftarrow\!\shortmid \mathfrak{b}$$
induces bijections of the respective sets of H-invariant ideals and H-prime ideals.

COROLLARY VI.1.2.5. *A H-invariant ideal $\mathfrak{p}$ is H-maximal if and only if $R/\mathfrak{p}$ has only trivial invariant ideals, and it is H-prime if and only if the zero ideal in $R/\mathfrak{p}$ is H-prime. In particular, H-maximal ideals are H-prime.*

PROPOSITION VI.1.2.6. *For any ideal $\mathfrak{a} \trianglelefteq \mathcal{O}(X)$, any set $Y \subseteq X$ and any $h \in H$ we have $V(h\cdot\mathfrak{a}) = h \cdot V(\mathfrak{a})$ and $I(h\cdot Y) = h \cdot I(Y)$. Thus, the following hold:*

(i) *$V(H\cdot\mathfrak{a}) = \bigcap_{h\in H} h\cdot V(\mathfrak{a})$;*
(ii) *$I(H\cdot Y) = \bigcap_{h\in H} h\cdot I(Y)$;*
(iii) *if $\mathfrak{a}$ is H-invariant then $V(\mathfrak{a})$ is H-invariant;*
(iv) *if $Y \subseteq X$ is H-invariant then $I(Y)$ is H-invariant.*

In particular, if Y is closed then Y is H-invariant if and only if $I(Y)$ is H-invariant.

Proof. We first show $V(h \cdot f) = h \cdot V(f)$. Let $x \in X$. Then $x \in V(h \cdot f)$ iff $h^{-1} \cdot x \in V(f)$ iff $x \in h \cdot V(f)$. Moreover, we have $I(h \cdot x) = h \cdot I(x)$: For any $f \in \mathcal{O}(X)$ we have $f \in I(h \cdot x)$ iff $h^{-1} \cdot f \in I(x)$ iff $f \in h \cdot I(x)$. Now, the other assertions follow. $\square$

A consequence of the above and Hilberts Nullstellensatz is that H-prime ideals are radical ideals:

Proposition VI.1.2.7. *Let $\mathfrak{q} \trianglelefteq \mathcal{O}(X)$ a H-prime ideal. Then $\sqrt{\mathfrak{q}} = \mathfrak{q}$.*

Proof. Since $\mathcal{O}(X)$ is noetherian, $\sqrt{\mathfrak{q}}$ is finitely generated, so there is an $n > 0$ with $\sqrt{\mathfrak{q}}^n \subseteq \mathfrak{q}$. Because $\sqrt{\mathfrak{q}}$ is H-invariant, H-primality of $\mathfrak{q}$ gives $\sqrt{\mathfrak{q}} \subseteq \mathfrak{q}$. $\square$

Proposition VI.1.2.8. *Let $Y \subseteq X$ be a closed H-invariant subset. Then Y is H-irreducible if and only if $I(Y)$ is H-prime.*

Proof. Let Y be H-irreducible and let $\mathfrak{a}, \mathfrak{b} \trianglelefteq \mathcal{O}(X)$ be H-invariant ideals with $\mathfrak{a}\mathfrak{b} \subseteq I(Y)$. Then $Y \subseteq V(\mathfrak{b}) \cup V(\mathfrak{b})$ is a decomposition into closed H-invariant sets, so we may assume $Y \subseteq V(\mathfrak{a})$. Thus, $\mathfrak{a} \subseteq I(V(\mathfrak{a})) \subseteq I(Y)$.

Conversely, if $I(Y)$ is H-prime, consider $Y = A \cup B$ with closed H-invariant sets $A, B \subseteq X$. Then $I(A)I(B) = I(A \cup B) = I(Y)$ and we may assume $I(A) \subseteq I(Y)$, so $Y \subseteq A$. $\square$

VI.1.3. Quasi-tori, characters and one-parameter-groups. Here, we list well-known facts on quasi-tori, also called diagonalizable groups, with some (sketches of) proofs added for convenience.

Example VI.1.3.1. For the multiplicative group $(\mathbb{K}^*)^n$ the multiplication map and the canonical map $\{1\} \to (\mathbb{K}^*)^n$ to $(1, \dots, 1)$ are morphisms of affine varieties. To see that the map $inv\colon a \mapsto a^{-1}$ is a morphism first note that the homomorphism $\mathbb{K}[T_1, \dots, T_{2n}] \to \mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}] = \mathcal{O}((\mathbb{K}^*)^n)$ which sends T_i to T_i and T_{i+n} to $T_i^{\pm 1}$ has kernel $\langle T_1T_{n+1} - 1, \dots, T_nT_{2n} - 1\rangle$ and induces an isomorphism ϕ from the factor ring to $\mathcal{O}((\mathbb{K}^*)^n)$. Then the projection

$$pr_1\colon X := V_{(\mathbb{K}^*)^{2n}}(T_1T_{n+1} - 1, \dots, T_nT_{2n} - 1) \longrightarrow (\mathbb{K}^*)^n$$

onto the first n coordinates is an isomorphism because its corresponding ring homomorphism is ϕ^{-1}. inv is now the composition of the projection $pr_2\colon X \to (\mathbb{K}^*)^n$ onto the last n coordinates and pr_1^{-1}. Thus, $(\mathbb{K}^*)^n$ is an affine algebraic group called the *(standard) n-torus* over $\mathbb{K}$. With respect to the canonical Hopf algebra structure group algebras from Section IV.2.3 the isomorphism $\mathcal{O}((\mathbb{K}^*)^n) \to \mathbb{K}[\mathbb{Z}^n]$ is an isomorphism of Hopf algebras.

Example VI.1.3.2. Consider $G = \mathbb{Z}/m_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_n\mathbb{Z}$. Since $\mathbb{Z}/m_i\mathbb{Z}$ maps bijectively onto the group of m_i-th roots of unity G is in canonical bijection with the closed algebraic subgroup $H := V(T_1^{m_1} - 1, \dots, T_n^{m_n} - 1) \subseteq (\mathbb{K}^*)^n$. We have $I(H) = \langle T_1^{m_1} - 1, \dots, T_n^{m_n} - 1\rangle$ because each of the generators $T_i^{m_i}$ decomposes into m_i pairwise different prime factors. Thus, the epimorphism of Hopf algebras $\mathbb{K}[T_1^{\pm 1}, \dots, T_n^{\pm 1}] \to \mathbb{K}[G], T_i \mapsto \chi^{[m_i]}$ induces an isomorphism of $\mathcal{O}(H) \to \mathbb{K}[G]$.

Remark VI.1.3.3. The canonical full functor $\mathbb{K}[-]$ of Proposition IV.2.3.16 from abelian groups to Hopf algebras over $\mathbb{K}$restricts to a full functor from finitely generated abelian groups to affine Hopf algebras. Indeed, the group algebra over a finitely generated abelian group $K = \mathbb{Z}^{\mathrm{rk}\,(K)} \oplus t(K)$ is the tensor product the of affine algebras $\mathbb{K}[\mathbb{Z}^{\mathrm{rk}\,(K)}]$ and $\mathbb{K}[t(K)]$. Composing with $\mathrm{Spec}_{\max}$ then yields a full contravariant functor from finitely generated abelian groups to affine algebraic groups.

Definition VI.1.3.4. An affine algebraic group which is isomorphic to (an affine algebraic subgroup of) some standard torus is a *(quasi-)torus*.

Construction VI.1.3.5. Let H be an abelian affine algebraic group. The group $\mathbb{X}(H)$ of *rational characters* of H is defined as the abelian group of morphisms of affine algebraic groups from H to $\mathbb{K}^*$ with multiplication being defined point-wise. $\mathbb{X}$ canonically defines a contravariant functor from abelian affine algebraic groups to abelian groups by sending a morphism $\theta\colon H \to G$ to the *pullback*

$$\mathbb{X}(\theta) := \theta^*\colon \mathbb{X}(G) \longrightarrow \mathbb{X}(H), \quad \chi \longmapsto \chi \circ \theta.$$

Remark VI.1.3.6. $\mathbb{X}(H)$ may be considered as a (multiplicative) subgroup of $\mathcal{O}(H)$ because every rational character gives rise to a unique morphism to $\mathbb{K}$. Then $\mathbb{X}(H)$ equals the set of group-like elements of $\mathcal{O}(H)$.

Proposition VI.1.3.7. *The contravariant functor $\mathfrak{f}$ from finitely generated abelian groups to the category of quasi-tori is essentially inverse to $\mathbb{X}$.*

Proof. For $H = \mathrm{Spec}_{\max}(\mathbb{K}[K])$ consider a surjection $\pi\colon \mathbb{Z}^n \to K$. Then the corresponding morphism of algebraic groups $H \to \mathbb{T}$ is a closed embedding. The map $\mathbb{X}(\mathrm{Spec}_{\max}(\mathbb{K}[K])) \to K, \phi \mapsto \deg(\phi^*(\chi^1))$ is an isomorphism, with the inverse mapping w to the morphism from H to $\mathbb{K}^*$ defined by $\mathbb{K}[\mathbb{Z}] \to \mathbb{K}[K], \chi^1 \mapsto \chi^w$. The homomorphism property of the first map follows from the point-wise definition of multiplication in the character group. These isomorphisms now define an isomorphism between $\mathbb{X} \circ \mathfrak{f}$ and the identity.

For well-definedness of $\mathfrak{f}$ consider a closed embedding $H \to \mathbb{T} = (\mathbb{K}^*)^n$. Then the injection $\mathbb{K}[\mathbb{X}(\mathbb{T})] \to \mathcal{O}(\mathbb{T}) \cong \mathbb{K}[\mathbb{Z}^n]$ is an isomorphism because the induced map of the group-like elements is an isomorphism by Remark VI.1.3.6 and Example VI.1.3.1. Using surjectivity of $\mathcal{O}(\mathbb{T}) \to \mathcal{O}(H)$ we see that $\mathcal{O}(H)$ is also generated by its characters and the restriction $\mathbb{X}(\mathbb{T}) \to \mathbb{X}(H)$ is surjective. In particular, the injection $\mathbb{K}[\mathbb{X}(H)] \to \mathcal{O}(H)$ is bijective and the induced morphism $H \to \mathrm{Spec}_{\max}(\mathbb{K}[\mathbb{X}(H)])$ is an isomorphism. These isomorphisms now define an isomorphism, i.e. a natural transformation, between the identity and $\mathfrak{f} \circ \mathbb{X}$ where the relevant commutativity condition is due to Proposition IV.2.3.16. □

Corollary VI.1.3.8. *For two closed algebraic subgroups H_1, H_2 of a quasi-torus H we have $\mathbb{X}(H_1 \cap H_2) = \mathbb{X}(H)/(K_1 + K_2)$ where K_i is the kernel of the pullback $\mathbb{X}(H) \to \mathbb{X}(H_i)$.*

Corollary VI.1.3.9. *An affine algebraic group H is quasi-torus if and only if it is isomorphic to a product of a torus and a finite abelian affine algebraic group.*

Construction VI.1.3.10. Let H be an abelian affine algebraic group. The group $\Lambda(H)$ of *one parameter groups* of H is the abelian group of morphisms of affine algebraic groups from $\mathbb{K}^*$ to H with multiplication being defined point-wise. If $\phi\colon H \to G$ is a morphism of abelian affine algebraic groups, then there is an induced *push-forward* homomorphisms

$$\Lambda(\phi) := \phi_*\colon \Lambda(H) \longrightarrow \Lambda(G), \quad \chi \longmapsto \phi \circ \chi.$$

This turns Λ into a covariant functor from abelian affine algebraic groups to abelian groups.

Remark VI.1.3.11. Let H be an abelian affine algebraic group.

(i) We have a $\mathbb{Z}$-bilinear canonical map

$$\langle\ ,\ \rangle\colon \mathbb{X}(H) \times \Lambda(H) \longrightarrow \mathbb{X}(\mathbb{K}^*) = \mathbb{Z}, \quad (\chi, \lambda) \longmapsto \chi_*(\lambda) = \lambda^*(\chi) = m_{\chi\circ\lambda}$$

(ii) Every $\lambda \in \Lambda(H)$ maps into the unit component H_e of H, and hence the push-forward $\Lambda(H_e) \to \Lambda(H)$ is an isomorphism.

Example VI.1.3.12. For a quasi-torus H the homomorphism

$$\Lambda(H) \longrightarrow \operatorname{Hom}(\mathbb{X}(H), \mathbb{Z}), \quad \lambda \longmapsto [\chi \mapsto \langle \chi, \lambda \rangle]$$

is an isomorphism by virtue of the contravariant equivalence of quasi-tori and finitely generated abelian groups.

VI.1.4. Semi-invariants of characters. In this section we observe that semi-invariants are well-behaved under morphisms and form a direct sum. Moreover, we consider vanishing sets of semi-invariants and the Zariski topology defined by principal open subsets associated to semi-invariants.

Definition VI.1.4.1. Let $H \circlearrowright X$ be an action.

(i) For every character $\chi \in \mathbb{X}(H)$ the sheaf of *χ-semi-invariants* is the subsheaf of $\mathbb{K}$-vector spaces $(\mathcal{O}_{X,H})_\chi \subseteq \mathcal{O}_{X,H}$ defined by

$$(\mathcal{O}_{X,H})_\chi(U) := \{f \in \mathcal{O}_{X,H}(U) \mid h^{-1} \cdot f = \chi(h)f \text{ for all } h \in H\}.$$

(ii) The sheaf of $\mathbb{F}_1$-algebras $(\mathcal{O}_{H,X})_\mathbb{X} := \bigcup_{\chi \in \mathbb{X}(H)} (\mathcal{O}_{X,H})_\chi \subseteq \mathcal{O}_{X,H}$ is called the sheaf of *semi-invariants*.

Remark VI.1.4.2. Let X be a H-prevariety with an abelian affine algebraic group H. Let $U \subseteq X$ be open and invariant. Then $f \in \mathcal{O}_X(U)$ is a χ-semi-invariant if and only if $\mu_U^*(f) = pr_H^*(\chi)pr_U^*(f)$ because $((\mu_h^*)_U(f))(x) = \mu_U^*(f)(h,x)$ holds for all $x \in X$ and $h \in H$.

Example VI.1.4.3. With respect to the action of a quasi-torus H on itself via the multiplication morphism we have $\mathcal{O}(H)_\chi = \mathbb{K}\chi$ for each $\chi \in \mathbb{X}(H)$.

Proposition VI.1.4.4. *The sum $\sum_{\chi \in \mathbb{X}(H)} (\mathcal{O}_{X,H})_\chi \subseteq \mathcal{O}_{X,H}$ is direct.*

Proof. Let $U \in \Omega_{X,H}$ and let $n \in \mathbb{N}_0$ be the minimal number for which there exist non-zero semi-invariants $f_1, \ldots, f_n \in \mathcal{O}(U)$ of different characters $\chi_1, \ldots, \chi_n$ such that $\sum_{i=1}^n f_i = 0$. If $n \neq 0$ then we must have $n > 1$. Each $h \in H_{\chi_1 - \chi_n}$ then produces an equation $0 = \sum_{i=1}^n (\chi_1(h) - \chi_i(h))f_i$ in which the number of non-zero summands is greater than 0 but smaller than n - a contradiction. □

Remark VI.1.4.5. The sheaf of invariants $\mathcal{O}_{X,H}^H$ coincides with $(\mathcal{O}_{X,H})_{0_{\mathbb{X}(H)}}$.

Proposition VI.1.4.6. *Let $(\theta, \phi)\colon H \circlearrowright X \to G \circlearrowright Z$ be a morphism of actions of affine algebraic groups and let $\chi \in \mathbb{X}(G)$. Then $\phi^*\colon \mathcal{O}_{Z,G} \to \phi_*\mathcal{O}_{X,H}$ restricts to a homomorphism $\phi^*\colon (\mathcal{O}_{Z,G})_\chi \to \phi_*(\mathcal{O}_{X,H})_{\theta^*(\chi)}$. Likewise, for an H-irreducible subset $B \subseteq X$, the homomorphism $\phi_B^*\colon (\mathcal{O}_{Z,G})_{\overline{\phi(B)}} \to (\mathcal{O}_{X,H})_B$ restricts to a homomorphism $\phi_B^*\colon ((\mathcal{O}_{Z,G})_{\overline{\phi(B)}})_\chi \to ((\mathcal{O}_{X,H})_B)_{\theta^*(\chi)}$.*

Proof. Let $U \in \Omega_{Z,G}$ and let $f \in \mathcal{O}_Z(U)_\chi$ where $\chi \in \mathbb{X}(G)$. Then for every $h \in H$ and every $x \in \phi^{-1}(U)$ we have

$$\phi_U^*(f)(hx) = f(\phi(hx)) = f(\theta(h)\phi(x)) = \chi(\theta(h))f(\phi(x)) = \theta^*(\chi)(h)\phi_U^*(f)(x)$$

which means that $\phi_U^*(f) \in \phi_*(\mathcal{O}_{X,H})_{\theta^*(\chi)}(U)$. □

Proposition VI.1.4.7. *Let $H \circlearrowright X$ be an action of an affine algebraic group on an affine variety. Let $\Omega_{X,H,\mathbb{X}}$ be the subtopology of $\Omega_{X,H}$ generated by the principal open sets at semi-invariants. Let $Y \subseteq X$ be a $\Omega_{X,H,\mathbb{X}}$-closed $\Omega_{X,H,\mathbb{X}}$-irreducible sbuset. Then the following hold:*

(i) *With $S := (\mathcal{O}_{X,H})_\mathbb{X}(X) \setminus I(Y)$ the canonical map*

$$S^{-1}\mathcal{O}(X) = \operatorname*{colim}_{s \in S} \mathcal{O}(X)_s \longrightarrow ((\mathcal{O}_{X,H})_{|\Omega_{X,H,\mathbb{X}}})_Y = \operatorname*{colim}_{s \in S} \mathcal{O}(X_s),$$

$$f/g \longmapsto (g_{|X_g})_Y^{-1} f_Y$$

to the stalk at Y is an isomorphism which respects the respective H-actions (and hence maps semi-invariants to semi-invariants).

(ii) *If* $(\theta, \phi)\colon H \circlearrowright X \to G \circlearrowright Z$ *is a morphism between actions then under* $((\mathcal{O}_{G,Z})_{|\Omega_{Z,G,\mathbb{X}}})_{\overline{\phi(Y)}} \to ((\mathcal{O}_{X,H})_{|\Omega_{X,H,\mathbb{X}}})_Y$ *semi-invariant preimages of units are units.*

Proof. Assertion (i) is due to the fact that the sets $X_s, s \in S$ form a basis for the $\Omega_{X,H,\mathbb{X}}$-neighbourhoods of Y. Assertion (ii) follows from the fact that we have $\phi^{-1}((\mathcal{O}_{X,H})_{\mathbb{X}}(X) \setminus I(Y)) = (\mathcal{O}_{Z,G})_{\mathbb{X}}(Z) \setminus I(\overline{\phi(Y)})$ and $(\mathcal{O}_{Z,G})_{\mathbb{X}}(Z) \setminus I(\overline{\phi(Y)})$ is a face of the multiplicative monoid $(\mathcal{O}_{Z,G})_{\mathbb{X}}(Z)$. □

We close the section with variants of identity theorem and Krulls principal ideal theorem for semi-invariants.

Lemma VI.1.4.8. *Let X be a H-irreducible prevariety. Then $(\mathcal{O}_{X,H})_{\mathbb{X}}$ has injective restriction maps. In particular, if $f \in \mathcal{O}_X(U)_\chi$ and $f' \in \mathcal{O}_X(U')_\chi$ coincide on some open H-invariant $V \subset U \cap U'$, then they coincide on all of $U \cap U'$.*

Proof. Let $f \in \mathcal{O}_X(U)$ with $f_{|V} = 0$. $f^{-1}(0)$ is closed by continuity and H-invariant by semi-invariance of f. Using H-irreducibility of U we conclude $U = \overline{V}^{\Omega_{X,H}} \subseteq f^{-1}(0)$. □

Lemma VI.1.4.9. *Let X be an affine H-irreducible variety and let f be a χ-semi-invariant which is neither zero nor a unit. Then $V_X(f)$ is equidimensional of codimension one in X.*

Proof. We already know that X is equidimensional and $X = H \cdot X'$ for all components X' of X. Since $f \neq 0$ it vanishes on no X', because if otherwise $f_{|X'} = 0$ for some X', then as semi-invariant, f would vanish on all $h \cdot X'$, i.e. on all of X. By the same argument, f is not a unit on any X'. Now, Krulls principal ideal theorem applied to the restriction of f to the components X' gives the assertion. □

VI.1.5. Good quotients of actions and minimal closed sets. Next to known properties of good quotients in the setting of good quotients between actions, we generalize the existens of unique closed orbits in the fibres of points and prove that each a invariant closed set in the quotient space contains a special closed set in its preimage with similar properties, see Proposition VI.1.5.4.

Definition VI.1.5.1. A morphism $(\theta, \phi)\colon G \circlearrowright X \to H \circlearrowright Y$ of algebraic group actions is a *good quotient* if ϕ is affine, θ is surjective and the canonical homomorphism $\phi^*\colon \mathcal{O}_{Y,H} \to \phi_* \mathcal{O}_{X,G}^{\ker(\theta)}$ is an isomorphism.

Remark VI.1.5.2. (θ, ϕ) is a good quotient if and only if θ is surjective, and ϕ is a good quotient by $\ker(\theta)$. In particular, good quotients are surjective.

Proposition VI.1.5.3. *Let $(\theta, \phi)\colon G \circlearrowright X \to H \circlearrowright Y$ be a good quotient. Then the following hold:*

(i) *If Z is G-closed then $\phi(Z)$ is H-closed and $(\theta, \phi_{|Z})\colon G \circlearrowright Z \to H \circlearrowright \phi(Z)$ is a good quotient.*
(ii) *For every $V \in \Omega_{Y,H}$ the restriction $(\theta, \phi_{|\phi^{-1}(V)})\colon G \circlearrowright \phi^{-1}(V) \to H \circlearrowright V$ is a good quotient.*
(iii) *If $Z_j, j \in J$ are G-closed subsets then $\phi(\bigcap_{j \in J} Z_j) = \bigcap_{j \in J} \phi(Z_j)$.*
(iv) *Given a morphism $(\theta, \kappa)\colon G \circlearrowright X \to H \circlearrowright W$ there exists a unique morphism $(\mathrm{id}_H, \psi)\colon H \circlearrowright Y \to H \circlearrowright W$ with $(\theta, \kappa) = (\mathrm{id}_H, \psi) \circ (\theta, \phi)$.*

Proof. All assertions are consequences of the properties of classical good quotients by actions. □

Proposition VI.1.5.4. *Let $(\theta, \phi)\colon G \circlearrowright X \to H \circlearrowright Y$ be a good quotient and let $B \subseteq Y$ be a closed subset. Then the closure A of the union over all closed* $\ker(\theta)$*-orbits in $\phi^{-1}(B)$ is called the* special set over B *and has the following properties.*

(i) *A is the minimal closed* $\ker(\theta)$*-invariant subset mapping onto B.*
(ii) *A is G-invariant if and only if B is H-invariant.*
(iii) *A is $\Omega_{X,G}$-irreducible if and only if B is $\Omega_{Y,H}$-irreducible.*
(iv) *B is an H-orbit closure $\overline{Hy}$ if and only if A is a G-orbit closure $\overline{Gx}$, and in this case any $x \in A$ with $\phi(x) = y$ satisfies $A = \overline{Gx}$.*
(v) *B is a closed H-orbit if and only if A is a closed G-orbit.*

Proof. In (i) note that firstly A is contained in $\phi^{-1}(B)$ and secondly A contains fibres of all points of B so we conclude $\phi(A) = B$. If A' is closed and $\ker(\theta)$-invariant with $B \subseteq \phi(A')$. Let $\ker(\theta)x$ be a closed orbit in $\phi^{-1}(B)$. Then $\phi(x) = \phi(z)$ holds with some $z \in A'$ and we conclude $\ker(\theta)x \subseteq \overline{\ker(\theta)z} \subseteq A'$. In (ii) suppose that B is H-invariant. Let $\ker(\theta)x$ be a closed orbit with $\phi(x) \in B$ and let $g \in G$. Then $\phi(gx) = \theta(g)\phi(x) \in B$ by assumption. Furthermore,

$$\ker(\theta)gx = g\ker(\theta)x = g\overline{\ker(\theta)x} = \overline{g\ker(\theta)x} = \overline{\ker(\theta)gx}$$

shows $gx \in A$.

In (iii) suppose that B is $\Omega_{Y,H}$-irreducible and that C and D are closed G-invariant subsets of X such that $A \subseteq C \cup D$. Then we may assume that $B \subseteq \phi(C)$ and by (i) we conclude $A \subseteq C$. In (iv) suppose that $B = \overline{Hy}$. For any $x \in A$ with $\phi(x) = y$ we then have $B = \overline{\phi(Gx)} \subseteq \phi(\overline{Gx}) \subseteq B$. By minimality of A this implies $\overline{Gx} = A$.

In (v) suppose that $B = Hy$. For any $x \in A$ with $\phi(x) = y$ we then have $A = \overline{Gx}$ by (iv). Furthermore, the closed G-orbit Gx' of A satisfies $\phi(Gx') = Hy = B$ which implies $A = Gx'$ by minimality of A. $\square$

Remark VI.1.5.5. A $\ker(\theta)$-invariant set $U \subseteq X$ is *ϕ-saturated*, i.e. it satisfies $\phi^{-1}(\phi(U)) = U$ if and only if for $z \in U$ and $x \in X$, $\overline{\ker(\theta)z} \cap \overline{\ker(\theta)x} \neq \emptyset$ implies $x \in U$. Moreover, such U is open if and only if $\phi(U)$ is open.

Remark VI.1.5.6. For an action $G \circlearrowright X$ on an affine X and an epimorphism $\theta\colon G \to H$ between reductive affine algebraic groups the algebra $\mathcal{O}(X)^{\ker(\theta)}$ is finitely generated by Hilberts Invariant Theorem. Then the inclusion $\mathcal{O}(X)^{\ker(\theta)} \subseteq \mathcal{O}(X)$ induces a morphism $\phi\colon X \to Y := \mathrm{Spec}_{\max}(\mathcal{O}(X)^{\ker(\theta)})$ which is a good quotient by the action of $\ker(\theta)$.

Moreover, Y has an induced H-action such that $(\theta, \phi)\colon G \circlearrowright X \to H \circlearrowright Y$ is a good quotient of actions. Indeed, in the diagram

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d, "\theta\times\phi"] & X \arrow[d, "\phi"] \\ H \times Y \arrow[r, dashed] & Y \end{tikzcd}$$

the morphism $G \times X \to H \times Y$ is a good quotient by $\ker(\theta) \times \ker(\theta)$, the morphism $G \times X \to X \to Y$ is $\ker(\theta) \times \ker(\theta)$-invariant and by the universal property of the quotient $H \times Y$ we obtain the desired morphism $H \times Y \to Y$. By similar considerations, this is indeed an action.

Proposition VI.1.5.7. *Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be an affine morphism. If X is of affine intersection then so is $\widehat{X}$. The converse holds if (θ, q) is a good quotient.*

Proof. Suppose that X is of affine intersection. Then $\widehat{X}$ is covered by the sets $q^{-1}(U)$ for $U \in \mathcal{B}_{X,H}$ whose pairwise intersections are affine by affinenes of q. Thus, the diagonal morphism of $\widehat{X}$ is affine, i.e. $\widehat{X}$ is of affine intersection. If q is a good quotient and $\widehat{X}$ is of affine intersection, then for $U, V \in \mathcal{B}_{X,H}$ the set $q^{-1}(U \cap V) \in \mathcal{B}_{\widehat{X},\widehat{H}}$ is saturated and uniqueness of good quotients gives $U \cap V \cong \mathrm{Spec}_{\max}(\mathcal{O}(q^{-1}(U \cap V)))$. $\square$

Lastly, we observe that *geometric quotients*, i.e. good quotients where all fibres are orbits, have a number of desirable properties:

REMARK VI.1.5.8. Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be a geometric quotient. Then the following hold:

(i) We have mutually inverse inclusion preserving bijections

$$\begin{aligned} \{\widehat{H}\text{-invariant subsets of } \widehat{X}\} &\longleftrightarrow \{H\text{-invariant subsets of } X\} \\ \widehat{A} &\longmapsto q(\widehat{A}) \\ q^{-1}(A) &\longmapsto A \end{aligned}$$

which restrict to mutually inverse inclusion preserving bijections between $\Omega_{\widehat{X},\widehat{H}}$ and $\Omega_{X,H}$.

(ii) For each H-closed set A the minimal closed $\ker(\theta)$-invariant set mapping onto A is $q^{-1}(A)$. Thus, the above mappings also define bijections between the set of $\widehat{H}$-closed $\widehat{H}$-irreducible subsets of $\widehat{X}$ and the set of H-closed H-irreducible subsets of X. These preserve invariant dimension and codimension.

Moreover, for $\widehat{H}$-irreducible $\widehat{H}$-closed $\widehat{A}$ with image $A := q(\widehat{A})$ the canonical map $(q_*\mathcal{O}_{\widehat{X},\widehat{H}})_A \to (\mathcal{O}_{\widehat{X},\widehat{H}})_{\widehat{A}}$ is an isomorphism which restricts to an isomorphism $(\mathcal{O}_{X,H})_A = ((q_*\mathcal{O}_{\widehat{X},\widehat{H}})_A)^{\ker(\theta)} \to ((\mathcal{O}_{\widehat{X},\widehat{H}})_{\widehat{A}})^{\ker(\theta)}$.

VI.2. Quasi-torus actions and graded schemes

From now on, all algebraic groups under consideration are quasi-tori and our general requirement of all quasi-torus actions is that they have an affine invariant cover, i.e. we consider actions $H \circlearrowright X$ for which $\mathcal{B}_{X,H}$ is a basis of $\Omega_{X,H}$. We will only occasionally stress this assumption. In this Section, we discuss the connection between graded schemes and quasi-torus actions. In Subsection VI.2.1 we show that the invariant structure sheaf of a quasi-torus action is graded, it decomposes into the direct sum of the semi-invariants of characters. As an alternative to the well-known proof using representation theory we offer a purely algebraic-geometric proof. Subsection VI.2.3 recalls the *soberification functor* $\mathfrak{t}$ which assigns to a topological space with structure sheaf the space of closed irreducible subsets equipped with the induced sheaf. In Subsection VI.2.4 we show that the soberification functor applied to the invariant structure of quasi-torus actions defines an equivalence between quasi-torus actions on prevarieties over $\mathbb{K}$ (with affine invariant cover) and reduced graded schemes of finite type over $\mathbb{K}$. This extends the well-known equivalence from prevarieties over $\mathbb{K}$ to reduced schemes of finite type over $\mathbb{K}$.

VI.2.1. Canonical grading of the invariant structure sheaf.

THEOREM VI.2.1.1. *Let $H \circlearrowright X$ be a quasi-torus action such that X has an affine invariant cover. Then $(X, \Omega_{X,H}, \mathcal{O}_{X,H})$ is a space with stalkwise homogeneously local $\mathbf{GrAlg}_{\mathbb{K}}^{\mathrm{fix}}$-structure sheaf where*

$$\mathcal{O}_{X,H} = \bigoplus_{\chi \in \mathbb{X}(H)} (\mathcal{O}_{X,H})_\chi$$

is canonically $\mathbb{X}(H)$-graded. Moreover, sending $H \circlearrowright X$ to $(X, \Omega_{X,H}, \mathcal{O}_{X,H})$ defines a covariant functor $\mathfrak{inv}$ from (morphical) quasi-torus actions to the category of topological spaces with stalkwise homogeneously local $\mathbf{GrAlg}_{\mathbb{K}}^{\mathrm{fix}}$-structure sheaves.

PROOF. For each $U \in \mathcal{B}_{X,H}$ we may apply Construction IV.2.3.19 to the coaction $\mathcal{O}(U) \to \mathcal{O}(H) \otimes_{\mathbb{K}} \mathcal{O}(U)$ to obtain a grading $\mathcal{O}(U) = \bigoplus_{\chi \in \mathbb{X}(H)} (\mathcal{O}_{X,H})_\chi(U)$. For an arbitrary $U \in \Omega_{X,H}$ let $U = U_1 \cup \ldots \cup U_m$ and $U_i \cap U_j = U_{i,j}^{(1)} \cup \ldots \cup U_{i,j}^{(d_{i,j})}$ be affine invariant covers. Due to Proposition II.1.3.8 $\mathcal{O}(U)$ is the limit in $\mathrm{GrRing}^{\mathbb{X}(H)}$ of the diagram defined by the restrictions $\mathcal{O}(U_i) \to \mathcal{O}(U_{i,j}^{(k)})$. Since $(\mathcal{O}_{X,H})_\chi(U)$ is the limit of the diagram defined by the restrictions $(\mathcal{O}_{X,H})_\chi(U_i) \to (\mathcal{O}_{X,H})_\chi(U_{i,j}^{(k)})$ we conclude $\mathcal{O}(U) = \bigoplus_{\chi \in \mathbb{X}(H)} (\mathcal{O}_{X,H})_\chi(U)$.

The stalks of $\mathcal{O}_{X,H}$ are K-local due to Proposition VI.2.1.4(iii). For a morphism $(\theta, \phi)\colon H \circlearrowright X \to G \circlearrowright Z$ the map of graded sheaves $\mathcal{O}_{Z,G} \to \phi_* \mathcal{O}_{X,H}$ is local because of Proposition VI.1.4.7 (ii). □

Example VI.2.1.2. Consider a quasi-torus action $H \circlearrowright \mathbb{K}^n$ and the corresponding $\mathbb{X}(H)$-grading of $\mathbb{K}[T_1, \ldots, T_n]$. Then for $h \in H$ and $x \in \mathbb{K}^n$ we have

$$hx = (\deg(T_1)(h)x_1, \ldots, \deg(T_n)(h)x_n).$$

Definition VI.2.1.3. Let X be an affine H-variety. Then $\mathcal{B}^{\mathrm{pr}}_{X,H} \subseteq \mathcal{B}_{X,H}$ denotes the collection of principal open subsets defined by semi-invariants.

Proposition VI.2.1.4. *Let $H \circlearrowright X$ be an action of a quasi-torus on an affine variety. Then the following hold:*

(i) *For each ideal $\mathfrak{a}$ of $\mathcal{O}(X)$ we have $\bigcap_{h \in H} h\mathfrak{a} = \mathfrak{a}^{\mathrm{gr}}$. In particular, $\mathfrak{a}$ is H-invariant if and only if it is $\mathbb{X}(H)$-graded. Consequently, the radical of a $\mathbb{X}(H)$-graded ideal is again $\mathbb{X}(H)$-graded, and an ideal $\mathfrak{a}$ is H-prime if and only it is $\mathbb{X}(H)$-prime.*

(ii) *$\mathcal{B}^{\mathrm{pr}}_{X,H}$ is a basis for $\Omega_{X,H}$. Consequently, sections of $\mathcal{O}_{X,H}$ are locally fractions with homogeneous denominators.*

(iii) *The canonical map*

$$\mathcal{O}(X)_{I(Y)} \longrightarrow (\mathcal{O}_{X,H})_Y, \quad f/g \longmapsto (g_{|X_g})_Y^{-1} f_Y$$

is a graded isomorphism. In particular, if X is H-irreducible than there is a canonical isomorphism $Q_{\mathrm{gr}}(\mathcal{O}(X)) \cong (\mathcal{O}_{X,H})_X$.

Proof. For (i), first suppose that $\mathfrak{a}$ is H-invariant. Let $f = f_{\chi_1} + \ldots + f_{\chi_n} \in \mathfrak{a}$. We proceed by induction on n to show that all f_{χ_i} lie in $\mathfrak{a}$. If $n > 1$ then each $h_0 \in H_{\chi_1 - \chi_n}$ satisfies

$$\mathfrak{a} \ni h_0^{-1} \cdot f - \chi_1(h_0) f = \sum_{i=2}^{n} (\chi_i(h_0) - \chi_1(h_0)) f_{\chi_i}.$$

By induction, $\chi_2(h_0) - \chi_1(h_0))f_{\chi_1}, \ldots, (\chi_n(h_0) - \chi_1(h_0))f_{\chi_n} \in \mathfrak{a}$. By choice of h_0 we have $f_{\chi_n} \in \mathfrak{a}$. Thus, $f - f_{\chi_n} \in \mathfrak{a}$ holds and another application of the induction hypothesis yields $f_{\chi_1}, \ldots, f_{\chi_{n-1}} \in \mathfrak{a}$. Conversely, let $\mathfrak{a}$ be $\mathbb{X}(H)$-graded and let $f = f_{\chi_1} + \ldots + f_{\chi_n} \in \mathfrak{a}$. Then $f_{\chi_i} \in \mathfrak{a}$ and for every $h \in H$ we have $h^{-1} \cdot f = \chi_1(h) f_{\chi_1} + \ldots \chi_n(h) f_{\chi_n} \in \mathfrak{a}$.

For the general statement it now suffices to note that $\mathfrak{a}^{\mathrm{gr}}$ is the maximal $\mathbb{X}(H)$-graded subideal of $\mathfrak{a}$ and $\bigcap_{h \in H} h\mathfrak{a}$ is the maximal H-invariant subideal of $\mathfrak{a}$. Assertion (ii) follows from (i). Assertion (iii) is a special case of Proposition VI.1.4.7(i). □

Proposition VI.2.1.5. *Let R be a finitely generated K-graded $\mathbb{K}$-algebra. Then each K-radical ideal $\mathfrak{a}$ of R is radical. In particular, if R is K-reduced, then R is reduced.*

Proof. Let $\pi\colon \mathbb{K}[T_1, \ldots, T_n] \to R/\mathfrak{a}$ be a degree-preserving epimorphism and set $H := \mathrm{Spec}_{\max}(\mathbb{K}[K])$. Since $V(\ker(\pi))$ is H-invariant, $I(V(\ker(\pi))) = \sqrt{\ker(\pi)}$ is H-invariant and hence graded, meaning

$$\sqrt{\ker(\pi)} = \sqrt{\ker(\pi)}^{\mathrm{gr}} = \pi^{-1}(\sqrt{\langle 0 \rangle}^{\mathrm{gr}}) = \pi^{-1}(\langle 0 \rangle) = \ker(\pi).$$

Consequently, $R/\mathfrak{a}$ is reduced, i.e. $\mathfrak{a}$ is radical. □

Corollary VI.2.1.6. *Let X be an affine H-variety. Then $\mathbb{X}(H)$-prime ideals of $\mathcal{O}(X)$ are radical.*

Proof. This follows from Proposition VI.2.1.4 and Proposition VI.1.2.7. □

Proposition VI.2.1.7. *Let H be quasi-torus and let X be a H-prevariety with affine invariant cover. For a closed H-irreducible $Y \subseteq X$ be we then have* $\operatorname{codim}_{X,H}(Y) = \dim_{\mathbb{X}(H)}((\mathcal{O}_{X,H})_Y)$.

Proof. We may assume that X is affine. Then the assignments $A \mapsto I(A)$ and $\mathfrak{a} \mapsto V(\mathfrak{a})$ define order reversing mutually inverse bijections between the closed H-irreducible subsets of X and the $\mathbb{X}(H)$-prime ideals of $\mathcal{O}(X)$. Since the latter are in natural bijection with the $\mathbb{X}(H)$-prime ideals of $(\mathcal{O}_{X,H})_Y = \mathcal{O}(X)_{I(Y)}$ the assertion follows. □

Proposition VI.2.1.8. *Let H be quasi-torus and let X be a H-prevariety with affine invariant cover. Let Y be a closed and H-irreducible subset of X. If Z is an irreducible component of Y then* $\operatorname{codim}_{X,H}(Y) = \operatorname{codim}_X(Z)$.

Lemma VI.2.1.9. *Let $H = \mathbb{T}$ be a torus and let X be an irreducible H-prevariety with affine invariant cover. Let $Y \subseteq X$ be a non-empty H-irreducible subset and set $d := \operatorname{codim}_X(Y)$. Then there exists a (maximal) ascending chain*

$$Y = Y_0 \subsetneq \ldots \subsetneq Y_d \subseteq X$$

of H-irreducible subsets Y_i of X containing Y. Therefore,

$$\operatorname{codim}_{X,H}(Y) = \operatorname{codim}_X(Y) = \dim(X) - \dim(Y).$$

Proof. We argue by induction over d. If $d = 0$ there is nothing to show. If $d > 0$ then $I(Y)$ contains a non-zero homogeneous element f. In particular, f is no unit. Let Z be an irreducible component of $V_X(f)$ which contains Y. In particular, Z is also H-invariant. By choice, the number $d' := \operatorname{codim}_Z(Y)$ is smaller than d. By induction, there exits a chain

$$Y = Y_0 \subsetneq \ldots \subsetneq Y_{d'} = Z$$

which is maximal among all chains between Y and Z. Hence, the chain

$$Y = Y_0 \subsetneq \ldots \subsetneq Y_{d'} = Z \subsetneq X$$

is maximal among all chains between Y and X and thus has length d.

The above shows that $\operatorname{codim}_X(Y) \leq \operatorname{codim}_{X,H}(Y)$. The other inequality is immediate since every chain of H-invariant irreducible subsets is in particular a chain of irreducible subsets. □

Proof of Proposition VI.2.1.8. Let $H = \mathbb{T} \times G$ where $\mathbb{T}$ is the unit component of H. Let Z be an irreducible component of Y and set $d := \operatorname{codim}_X(Z)$. Then Z is $\mathbb{T}$-invariant and by the above lemma there exists a chain

$$Z = Z_0 \subsetneq \ldots \subsetneq Z_d \subseteq X$$

of closed $\mathbb{T}$-invariant subsets of X. Set $Y_i := GZ_i$. Then

$$Y = Y_0 \subsetneq \ldots \subsetneq Y_d \subseteq X$$

is the desired proper chain. Indeed, if we had $Y_i = Y_{i+1}$ at some index i then equidimensionality of Y_i, Y_{i+1} implies that their components Z_i, Z_{i+1} have the same dimension - a contradiction.

This shows that $\operatorname{codim}_{X,H}(Y) \leq \operatorname{codim}_X(Z)$. For the converse inequality let

$$Y_0 \subsetneq \ldots \subsetneq Y_m \subseteq X$$

be an ascending chain of closed H-irreducible subsets and set $Z_0 := Z$. Inductively, we choose irreducible components Z_i of Y_i such that $Z_i \subseteq Z_{i+1}$. Then

$$Z_0 \subsetneq \ldots \subsetneq Z_m \subseteq X$$

is an ascending chain of closed $\mathbb{T}$-invariant irreducible subsets which shows

$$\operatorname{codim}_{X,H}(Y) \geq \operatorname{codim}_X(Z).$$

□

PROPOSITION VI.2.1.10. *Let $(\theta, \phi)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be a morphism where $\widehat{X}$ is $\widehat{H}$-irreducible. Then the preimage of a closed H-irreducible subset Y of (invariant) codimension one contains an $\widehat{H}$-irreducible component of (invariant) codimension one. Thus, if a ϕ-saturated open set $\widehat{U}$ is big in $\widehat{X}$ then $\phi(\widehat{U})$ is big in X.*

PROOF. We may assume that X is affine. By Lemma VI.1.4.9 $V(f)$ is of pure codimension one in X for $f \in I(Y)^{\mathrm{hom}} \setminus 0$. Let $Y_1, \ldots, Y_d$ be the H-irreducible components of $V(f)$ which differ from Y. Then there exist $g_i \in I(Y_i)^{\mathrm{hom}} \setminus I(Y)$ and with $g := \prod_{i=1}^d g_i$ we have $V_{X_g}(f) = Y \cap X_g$. Since ϕ is affine we have $\phi^{-1}(X_g) = \widehat{X}_{\phi^*(g)}$. Thus,

$$\phi^{-1}_{|\widehat{X}_{\phi^*(g)}}(Y \cap X_g) = \phi^{-1}_{|\widehat{X}_{\phi^*(g)}}(V_{X_g}(f)) = V_{\widehat{X}_{\phi^*(g)}}(\phi^*(f))$$

is of pure codimension one in $\widehat{X}$. □

VI.2.2. Quasi-coherent modules and algebras over $\mathcal{O}_{X,H}$. In this section, we consider a fixed quasi-torus action $H \circlearrowright X$ action with affine invariant cover. $\mathcal{O}_{X,H}$ being a $\mathbf{GrAlg}^{\mathrm{fix}}_{\mathbb{K}}$-sheaf on $\Omega_{X,H}$, we consider algebras over $\mathcal{O}_{X,H}$ in that category, as well as modules over $\mathcal{O}_{X,H}$ in terminology of Definition III.3.0.2. A canonical example is the following.

DEFINITION VI.2.2.1. Let X be an H-irreducible H-prevariety. Then the constant (pre-)sheaf $\mathcal{K}_{X,H}$ assigning $(\mathcal{O}_{X,H})_X$ is called the *sheaf of graded fraction rings* of X.

We now list the main facts on quasi-coherent $\mathcal{O}_{X,H}$-algebras and their relative spectra. All proofs are analogous to those of Section IV.1.3 and are therefore omitted.

DEFINITION VI.2.2.2. An $\mathcal{O}_{X,H}$-algebra/-module is *quasi-coherent* if for each (affine) $U \in \Omega_{X,H}$ and each $f \in \mathcal{O}(U)^{\mathrm{hom}}$ the canonical map $\mathcal{A}(U)_f \to \mathcal{A}(U_f)$ is an isomorphism.

CONSTRUCTION VI.2.2.3. For an affine quasi-torus action $H \circlearrowright X$ set $R := \mathcal{O}(X)$ and denote $\mathbf{GrAlg}^{\lambda}_R$ or $\mathbf{GrMod}^{\lambda}_R$ by $\mathfrak{D}$ where λ is a fixed accompanying $gr(R)$-algebra/-module structure. For $W \in \mathcal{B}^{\mathrm{pr}}_{X,H}$ set $S_W := (\rho^X_W)^{-1}_{|R^{\mathrm{hom}}}((\mathcal{O}(W)^{\mathrm{hom}})^*)$. The $\mathcal{O}_{X,H}$-algebra/-module $\mathcal{A} := A^{\sim}$ associated to a $\mathfrak{D}$-object A is defined via $\mathcal{A}(U) := \lim_{W \in \mathcal{B}^{\mathrm{pr}}_U} S_W^{-1} A$ for $U \in \Omega_X$ where restriction maps are induced by the universal property. If A was an R-algebra then the maps $S_W^{-1} R \to S_W^{-1} A$ induce maps $\mathcal{O}_X(U) \to \mathcal{A}(U)$ which form a homomorphism $\mathcal{O}_X \to \mathcal{A}$ of presheaves of constantly graded A-algebras with accompaniment λ.

In the case of an R-module let $w \in gr(R)$ and $v \in gr(A)$ and consider for all $W \in \mathcal{B}^{\mathrm{pr}}_U$ the maps

$$\mathcal{O}_X(U)_w \times \mathcal{A}(U)_v \longrightarrow (S_W^{-1} R)_w \times (S_W^{-1} A)_v \longrightarrow (S_W^{-1} A)_{\lambda(w,v)}.$$

The universal property of $\mathcal{A}(U)_{\lambda(w,v)} = \lim_{W \in \mathcal{B}^{\mathrm{pr}}_U} (S_W^{-1} A)_{\lambda(w,v)}$ induces a map $\mathcal{O}_X(U)_w \times \mathcal{A}(U)_v \to \mathcal{A}(U)_{\lambda(w,v)}$. These maps fit together to a scalar multiplication giving $\mathcal{A}(U)$ an $\mathcal{O}_X(U)$-structure with accompaniment λ.

The next statements may be proven analogously to that of Section IV.1.3.

Proposition VI.2.2.4. *In the setting of the above proposition, let $\mathcal{D}$ denote the category of quasi-coherent $\mathcal{O}_{X,H}$-algebras/-modules. Then sending a $\mathcal{D}$-object $\mathcal{A}$ to $\mathcal{A}(X)$ is essentially inverse to the functor sending A to $A^{\sim}$.*

Remark VI.2.2.5. A quasi-coherent $\mathcal{O}_{X,H}$-algebra/-module $\mathcal{A}$ is locally of finite type if and only if $\mathcal{A}(U)$ is finitely generated over $\mathcal{O}(U)$ (and hence over $\mathbb{K}$) for every $U \in \mathcal{B}_{X,H}$.

Remark VI.2.2.6. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_{X,H}$-algebra. Then $\mathcal{A}$ extends uniquely to a quasi-coherent $\mathcal{O}_X$-algebra $\overline{\mathcal{A}}$. Specifically, for each $U \in \mathcal{B}_{X,H}$ consider the $\mathcal{O}_U$-module defined by $\mathcal{A}(U)$. Then all these sheaves are compatible and thus glue to an $\mathcal{O}_X$-module $\overline{\mathcal{A}}$ which is quasi-coherent because quasi-coherence only needs to be checked on an affine cover.

Remark VI.2.2.7. Sending a reduced graded algebra R of finite type over $\mathbb{K}$ to the coaction

$$R \longrightarrow \mathbb{K}[gr(R)] \otimes_{\mathbb{K}} R, \quad R_w \ni f \longmapsto \chi^w \otimes f$$

from Construction IV.2.3.17 defines a functor $\mathfrak{ca}$ from reduced graded algebras of finite type over $\mathbb{K}$ to coactions of affine Hopf $\mathbb{K}$-algebras. $\mathfrak{ca}$ is compatible with graded localizations.

Construction VI.2.2.8. Let $\mathcal{R}$ be a quasi-coherent reduced $\mathcal{O}_{X,H}$-algebra which is locally of finite type over $\mathbb{K}$. Denote the extension of $\mathcal{R}$ to Ω_X by $\overline{\mathcal{R}}$ and set $G := \mathrm{Spec}_{\max}(\mathbb{K}[gr(\mathcal{R})])$. For $U \in \mathcal{B}_{X,H}$ we obtain a morphism

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathbb{K}[\mathbb{X}(H)] \otimes \mathcal{O}(U) \arrow[d] \\ \mathcal{R}(U) \arrow[r] & \mathbb{K}[gr(\mathcal{R})] \otimes \mathcal{R}(U) \end{tikzcd}$$

of coactions of affine Hopf $\mathbb{K}$-algebras. Applying $\mathrm{Spec}_{\max}$ gives a morphism

$$G \circlearrowright \mathrm{Spec}_{\max}(\mathcal{R}(U)) \to H \circlearrowright U$$

of quasi-torus actions. For all $V \in \mathcal{B}^{\mathrm{pr}}_{U,H}$ and hence for all $V \in \mathcal{B}_{U,H}$ the inclusion $\mathrm{Spec}_{\max}(\mathcal{R}(V)) \to \mathrm{Spec}_{\max}(\mathcal{R}(U))$ is G-equivariant due to Proposition VI.2.1.4. The *relative spectrum* of $\mathcal{R}$ is then the colimit $\mathrm{Spec}_{X,H}(\mathcal{R}) := G \circlearrowright \mathrm{Spec}_X(\overline{\mathcal{R}})$ of the diagram defined by all these actions for $U \in \mathcal{B}_{X,H}$.

Proposition VI.2.2.9. *Let $H \circlearrowright X$ be a quasi-torus action. Then the functor sending a quasi-coherent reduced $\mathcal{O}_{X,H}$-algebra $\mathcal{A}$ which is locally of finite type to $\mathrm{Spec}_{X,H}(\mathcal{A}) \to H \circlearrowright X$ is essentially inverse to the functor sending an affine quasi-torus action over $H \circlearrowright X$ with structure morphism $(\theta, \phi)\colon G \circlearrowright Z \to H \circlearrowright X$ to $\mathcal{O}_{X,H} \to \phi_* \mathcal{O}_{Z,G}$.*

VI.2.3. The soberification functor.

Construction VI.2.3.1. Let (X, Ω_X) be a topological space. Denote by $\mathfrak{t}(X)$ the set of closed irreducible (in particular, non-empty) subsets of X. Then we have a canonical map between the power sets

$$t := t_{(X,\Omega_X)} \colon \mathcal{P}(X) \longrightarrow \mathcal{P}(\mathfrak{t}(X)), \quad B \longmapsto \{A \in \mathfrak{t}(X) \mid \overline{B \cap A} = A\}$$

and a canonical *singleton closure map*

$$sc := sc_X := sc_{(X,\Omega_X)} \colon X \longrightarrow \mathfrak{t}(X), \quad x \longmapsto \overline{\{x\}}.$$

Proposition VI.2.3.2. *Let $B, B' \subseteq X$ be any subsets. Then the following hold:*

(i) *If B is open then $t(B)$ is the set of all $A \in \mathfrak{t}(X)$ with $B \cap A \neq \emptyset$.*

(ii) *If B is closed then $t(B)$ is the set of all $A \in \mathfrak{t}(X)$ with $A \subseteq B$.*
(iii) *t is injective and preserves inclusions.*
(iv) *t commutes with finite intersections of arbitrary sets, and with arbitrary intersections of open sets.*
(v) *We have $t(B \cap B') \subseteq t(B) \cap t(B')$ with equality holding if of B and B' at least one is closed or both are open.*
(vi) *If B' is constructible then $t(B) \subseteq t(B')$ implies $B \subseteq B'$. If B is also constructible then $t(B \setminus B') = t(B) \setminus t(B')$.*

Proof. In (i) note that if $A \cap B$ is a non-empty then it is dense in A. Conversely, if $A \cap B$ is dense in A then $A \cap B \neq \emptyset$ because the empty set is dense only in itself. For (ii) note that if A is contained in B then $A \cap B = A$ is dense in A. Conversely, if $A \cap B$ is dense in A then $A = \overline{A \cap B} \subseteq \overline{B} = B$. Concerning (iii) note that if there exists $x \in B \setminus B'$ then we have $\overline{\{x\}} \subseteq \overline{\overline{\{x\}} \cap B} \subseteq \overline{\{x\}}$ but $\overline{\{x\}} \notin t(B')$. In (iv) note that if $A \cap (B \cup B')$ is dense in A then $A = \overline{A \cap B} \cup \overline{A \cap B'}$ and by irreducibility of A we may assume that $A = \overline{A \cap B}$, i.e. $A \in t(B)$. Secondly, $A \in \mathfrak{t}(X)$ intersects a union $\bigcup_{i \in I} U_i$ of open sets non-trivially if and only if it intersects some U_i non-trivially.

In (v) assume that B is closed and B' is any set. If $A \subseteq B$ and $A \cap B'$ is dense in A then $A \cap B \cap B' = A \cap B'$ remains dense in A, i.e. $A \in t(B \cap B')$. Now, assume that B, B' are open. If the open subsets $A \cap B$ and $A \cap B'$ of A are non-empty then irreducibility of A implies that their intersection $A \cap B \cap B'$ is also non-empty.

For (vi) first observe that if $t(B) \subseteq t(B')$ and $B' = \bigcup_{i=1}^n U_i \cap A_i$ is constructible then for $x \in B$ the closure $\overline{\{x\}}$ is an element of $t(B') = \bigcup_{i=1}^n t(U_i) \cap t(A_i)$, i.e. there exists i such that $\overline{\{x\}}$ intersects U_i non-trivially and is contained in A_i. If x were no element of U_i then $\overline{\{x\}}$ would be a subset of $X \setminus U_i$ - a contradiction. Thus, $x \in U_i \cap A_i$ and we conclude $B \subseteq B'$. Lastly, note that if B' is open or closed then (i) and (ii) give $t(X \setminus B') = t(X) \setminus t(B')$. Using this as well as assertions (iii) and (iv) one calculates $t(B \setminus B') = t(B) \setminus t(B')$. □

Construction VI.2.3.3. For a topological space (X, Ω_X), setting $\Omega_{\mathfrak{t}(X)} := t(\Omega_X)$ turns $(\mathfrak{t}(X), \Omega_{\mathfrak{t}(X)})$ into topological space.

Proposition VI.2.3.4. *For a topological space X the following hold:*
(i) *t and the assignment $W \mapsto sc_X^{-1}(W)$ define mutually inverse inclusion-preserving bijections between the collections of open/closed/constructible sets of X and $\mathfrak{t}(X)$. In particular, sc_X is continuous.*
(ii) *$\Omega_{\mathfrak{t}(X)}$ is noetherian if and only if Ω_X is noetherian.*
(iii) *If $B \subseteq X$ is open/closed then $t(B)$ is the smallest open/closed set containing $sc_X(B)$.*
(iv) *For every set $B \subseteq X$ we have $t(\overline{B}) = \overline{t(B)}$.*
(v) *$B \subseteq X$ is open/closed/irreducible/quasi-compact if and only if $t(B)$ is so.*
(vi) *For a closed irreducible $B \subseteq X$ the bijections of (i) restrict to mutually inverse bijectsions between $\Omega_{X,B}$ and $\Omega_{\mathfrak{t}(X),t(B)}$.*
(vii) *$(\mathfrak{t}(X), \Omega_{\mathfrak{t}(X)})$ is sober.*

Proof. In assertion (ii) note that $sc_X(B) \subseteq t(B)$ always holds. Let $x \in sc_X^{-1}(t(B))$. If B is open then $x \notin X \setminus B$, because otherwise $\overline{x} \subseteq X \setminus B$. If B is closed then $x \in B$ because $\overline{\overline{x} \cap B} = \overline{x} \cap B$. The remaining statements follow from Lemma VI.2.3.2.

Assertion (iii) is a consequence of assertion (ii). In assertion (iv) note that if $B \subseteq X$ and $W \subseteq \mathfrak{t}(X)$ are open/closed with $B \subseteq sc_X^{-1}(W)$ then $t(B) \subseteq t(sc_X^{-1}(W)) = W$.

As a consequence of (ii) a set $B \subseteq X$ is contained in an open or closed set C if and only if $t(B)$ is contained in $t(C)$. Indeed, if $t(B) \subseteq t(C)$ then $B \subseteq sc_X^{-1}(t(B)) \subseteq sc_X^{-1}(t(C)) = C$. This observation yields assertion (v) and the statements on irreducibility and quasi-compactness in assertion (vi). The statements on open and closed sets in (vi) follow directly from assertions (i) and (ii).

In (vi) note that if $U \in \Omega_{X,B}$ then $t(U) \in \Omega_{\mathfrak{t}(X),t(B)}$ due to Proposition VI.2.3.2 (v). For (vii) let $t(B) \subseteq \mathfrak{t}(X)$ be an irreducible closed subset. Then $B \subseteq X$ is irreducible and closed. We claim that B is the unique generic point of $t(B)$. Firstly, $B \in t(B)$ by definition. Thus, $\overline{\{B\}} \subseteq t(B)$. For the converse, consider $A \in t(B)$ and any closed set W containing $t(B)$. Then we have $A \subseteq B \subseteq sc_X^{-1}(W)$ and hence $A \in t(sc_X^{-1}(W)) = W$. For uniqueness let $B' \in t(B)$ be a generic point. Then $t(B') = \overline{\{B'\}} = t(B)$ and since B, B' are both closed we deduce $B = B'$. $\square$

Proposition VI.2.3.5. *Sending a continuous map $\phi\colon X \to X'$ to the map*

$$\mathfrak{t}(\phi)\colon \mathfrak{t}(X) \to \mathfrak{t}(X'), \quad A \mapsto \overline{\phi(A)}$$

turns $\mathfrak{t}$ into a functor from topological spaces to sober topological spaces with the following properties:

(i) *We have $sc_{X'} \circ \phi = \mathfrak{t}(\phi) \circ sc_X$.*
(ii) *We have $\overline{\mathrm{im}(\mathfrak{t}(\phi))} = t(\overline{\mathrm{im}(\phi)})$. In particular, ϕ is dominant if and only if $\mathfrak{t}(\phi)$ is dominant.*
(iii) *If ϕ is an open embedding of topological spaces then $\mathfrak{t}(\phi)\colon \mathfrak{t}(W) \to \mathfrak{t}(X)$ is an open embedding.*
(iv) *$\mathfrak{t}$ is canonically left adjoint to the inclusion of sober topological spaces into* **Top**. *Thus, $\mathfrak{t}$ preserves colimits and in particular gluing.*

Proof. We have to show that if $\phi\colon X \to X'$ and $\phi'\colon X' \to X''$ are continuous maps then we have $\mathfrak{t}(\phi' \circ \phi) = \mathfrak{t}(\phi') \circ \mathfrak{t}(\phi)$ and moreover we have $\mathfrak{t}(\mathrm{id}_X) = \mathrm{id}_{\mathfrak{t}(X)}$.

For the equation $\overline{\mathrm{im}(\mathfrak{t}(\phi))} = t(\overline{\mathrm{im}(\phi)})$ let $A' \in \mathfrak{t}(X')$. If $A' = \overline{\phi(A)} \in \mathrm{im}(\mathfrak{t}(\phi))$ with $A \in \mathfrak{t}(X)$ then $\mathrm{im}(\phi) \cap \phi(A) = \phi(A)$ is dense in A' and hence $A' \in t(\overline{\mathrm{im}(\phi)})$. This shows $\mathrm{im}(\mathfrak{t}(\phi)) \subseteq t(\overline{\mathrm{im}(\phi)})$. Conversely, consider a closed subset $B' \subseteq X'$ such that $t(B')$ contains $\mathrm{im}(\mathfrak{t}(\phi))$. If $A' \in t(\overline{\mathrm{im}(\phi)})$ then we have to show $A' \in t(B')$, i.e. $A' \subseteq B'$. Let $x' = \phi(x) \in A' \cap \mathrm{im}(\phi)$ with some $x \in X$. Then $\overline{\{x'\}} = \overline{\{\phi(\overline{\{x\}})\}} = \mathfrak{t}(\phi)(\overline{\{x\}})$ which means that $\overline{\{x'\}} \in t(B')$ and hence $\overline{\{x'\}} \subseteq B'$, in particular, $x' \in B'$. Thus, we obtain $A' \cap \mathrm{im}(\phi) \subseteq B'$ and hence $A' = \overline{A' \cap \mathrm{im}(\phi)} \subseteq B'$.

Suppose that $\phi\colon X \to X'$ is dominant. Then $\overline{\mathrm{im}(\mathfrak{t}(\phi))} = t(\overline{\mathrm{im}(\phi)}) = t(X')$. Conversely, if $\mathfrak{t}(\phi)$ is dominant then $t(X') = \overline{\mathrm{im}(\mathfrak{t}(\phi))} = t(\overline{\mathrm{im}(\phi)})$ and since both X' and $\overline{\mathrm{im}(\phi)}$ are closed they coincide.

In (iv) we apply Lemma A.0.0.2, noting that we have $\overline{\{B\}} = \overline{\{\overline{\{x\}} | x \in B\}}$ for each closed irreducible $B \subseteq X$. $\square$

Definition VI.2.3.6. Let X be a topological spaces. The *inverse image under soberification* is the functor $t^{-1} := t_X^{-1} := t_{(X,\Omega_X)}^{-1}$ which sends a $\mathfrak{C}$-presheaf $\mathcal{G}$ on $\mathfrak{t}(X)$ to the presheaf $\mathcal{G} \circ t_{|\Omega_X}$ and a homomorphism $\phi\colon \mathcal{G} \to \mathcal{F}$ of $\mathfrak{C}$-presheaves on $\mathfrak{t}(X)$ to the homomorphism given by $\phi_{t(U)}$ for $U \in \Omega_X$.

Proposition VI.2.3.7. *t^{-1} constitutes a functor $\mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(\mathfrak{t}(X)) \to \mathcal{P}r\mathcal{S}h_{\mathfrak{C}}(X)$ which is isomorphic to sc_X^{-1}, and which is inverse to $(sc_X)_*$. Moreover, t^{-1} and $(sc_X)_*$ restrict to mutually inverse functors between $\mathcal{S}h_{\mathfrak{C}}(\mathfrak{t}(X))$ and $\mathcal{S}h_{\mathfrak{C}}(X)$. Furthermore, for $\mathfrak{C}$-presheaves $\mathcal{F}$ and $\mathcal{G}$ on X resp. $\mathfrak{t}(X)$ and a closed irreducible $A \subseteq X$ we have canonical isomorphisms*

$$((sc_X)_*\mathcal{G})_{t(A)} \longrightarrow \mathcal{G}_A, \quad (t^{-1}\mathcal{F})_A \cong (c_X^{-1}\mathcal{F})_A \longrightarrow \mathcal{F}_{t(A)}.$$

Proof. The first two statements follow from Proposition VI.2.3.4 (iii) and (i), the stalk formulae from (vi). The supplement on sheaf-categories is a direct consequence. $\square$

REMARK VI.2.3.8. Let $\mathcal{F}$ be a presheaf on $\mathfrak{t}(X)$ and let $\phi\colon X \to X'$ be a continuous map. Then we have $t_{X'}^{-1}\mathfrak{t}(\phi)_*\mathcal{F} = \phi_* t_X^{-1}\mathcal{F}$ by Proposition VI.2.3.4.

CONSTRUCTION VI.2.3.9. Let $\mathfrak{D}$ denote the category of topological spaces (X, Ω_X) with $\mathfrak{C}$-structure (pre-)sheaves $\mathcal{O}_X$, where morphisms are consist of a topological map $\phi\colon X \to X'$ and a homomorphism $\phi^*\colon \mathcal{O}_{X'} \to \phi_*\mathcal{O}_X$. For a $\mathfrak{D}$-object $(X, \Omega_X, \mathcal{O}_X)$ setting $\mathcal{O}_{\mathfrak{t}(X)} := (sc_X)_*\mathcal{O}_X$ gives an object $(\mathfrak{t}(X), \Omega_{\mathfrak{t}(X)}, \mathcal{O}_{\mathfrak{t}(X)})$ of the full subcategory $\mathfrak{D}^s$ of sober topological spaces with $\mathfrak{C}$-structure (pre-)sheaves. For a $\mathfrak{D}$-morphism $(\phi\colon X \to X', \phi^*)$ we obtain a $\mathfrak{D}^s$-morphism via

$$\mathcal{O}_{\mathfrak{t}(X')} \xrightarrow{(sc_{X'})_*\phi^*} (sc_{X'})_*\phi_*\mathcal{O}_X = \mathfrak{f}(\phi)_*(sc_X)_*\mathcal{O}_X.$$

PROPOSITION VI.2.3.10. *The above construction turns $\mathfrak{t}$ into a functor from topological spaces with $\mathfrak{C}$-structure (pre-)sheaves to sober topological spaces with $\mathfrak{C}$-structure (pre-)sheaves which is left adjoint to the inclusion functor.*

REMARK VI.2.3.11. In the situation of Construction VI.2.3.9 let $A \subseteq X$ be closed and irreducible and let $(\phi\colon X \to X', \phi^*)$ be a $\mathfrak{D}$-morphism. Then we have

$$\begin{aligned} t_{X'}(\overline{\phi(A)}) &= \overline{sc_{X'}(\overline{\phi(A)})} = \overline{sc_{X'}(\phi(A))} = \overline{\mathfrak{t}(\phi)(sc_X(A))} = \overline{\mathfrak{t}(\phi)(\overline{sc_X(A)})} \\ &= \overline{\mathfrak{t}(\phi)(t_X(A))}, \end{aligned}$$

and due to Proposition VI.2.3.4 the canonical isomorphisms from Proposition VI.2.3.7 give a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{X',\overline{\phi(A)}} \arrow[r, "\phi_A^*"] \arrow[d, "\cong"] & \mathcal{O}_{X,A} \arrow[d, "\cong"] \\ \mathcal{O}_{\mathfrak{t}(X'),\overline{\mathfrak{t}(\phi)(t(A))}} \arrow[r, "\mathfrak{t}(\phi)_{t(A)}^*"] & \mathcal{O}_{\mathfrak{t}(X),t(A)}. \end{tikzcd}$$

COROLLARY VI.2.3.12. *Let $\mathfrak{C}$ be the category of graded $\mathbb{F}_1$-algebras/rings. Then $\mathfrak{t}$ constitutes a functor from locally $\mathfrak{C}^{\mathrm{fix}}$-ed spaces to sober locally $\mathfrak{C}^{\mathrm{fix}}$-ed spaces which is left adjoint to the inclusion functor.*

VI.2.4. Equivalence of graded schemes and quasi-torus actions. This section generalizes the equivalence between reduced schemes of finite type over $\mathbb{K}$ and prevarieties over $\mathbb{K}$, which we first briefly recapitulate. The functor $\mathfrak{pv}$ sends a reduced scheme $(X, \Omega_X, \mathcal{O}_X)$ of finite type over $\mathbb{K}$ to the subset X_{cl} of closed points of X equipped with the subspace topology and the structure sheaf sending $U \in \Omega_{X_{cl}}$ to the $\mathcal{O}_X$-sections of the set V of all points of X which specialize to points of U. Here, we identify $f \in \mathcal{O}_X(V)$ with the function sending $x \in U$ to $[f_x] \in \mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbb{K}$. The soberification functor $\mathfrak{t}$ restricted to $\mathbb{K}$-prevarieties has image in the category of reduced schemes of finite type over $\mathbb{K}$ and is essentially inverse to $\mathfrak{pv}$.

In order to formulate the theorem on the equivalence of graded schemes and quasi-torus actions we fix notation and recollect the various functors we are going to employ. In one direction, we compose the functor $\mathfrak{inv}$ to sending a quasi-torus action to its invariant structure with the soberification functor $\mathfrak{t}$.

In the other direction the first step is to apply the functor $\mathfrak{a}^{\mathrm{gr}}$ which sends a graded scheme W over $\mathbb{K}$ to the canonical $\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[gr(\mathcal{O}_W)])$-action on W, see Construction IV.2.3.18. The functor $\mathfrak{s}^0$ from Proposition IV.1.5.4 sends a graded scheme W to the relative spectrum of the trivially 0-graded $\mathcal{O}_W$-algebra defined via coarsening of $\mathcal{O}_W$. Since $\mathfrak{s}^0$ preserves products we obtain an induced functor, also denoted $\mathfrak{s}^0$, from actions of graded group schemes over $\mathbb{K}$ to actions of group schemes over $\mathbb{K}$. This latter functor $\mathfrak{s}^0$ forms the second step. The third step is the equivalence from actions of reduced group schems of finite type over $\mathbb{K}$ on reduced schems of finite type over $\mathbb{K}$ which is induced by $\mathfrak{pv}$ and which is also denoted $\mathfrak{pv}$.

THEOREM VI.2.4.1. *Let $\mathbb{K}$ be an algebraically closed field. Denote by $\mathfrak{G}$ resp. $\mathfrak{G}^{\mathrm{aff}}$ the category of (affine) homogeneously reduced graded schemes of finite type over $\mathbb{K}$, and by $\mathfrak{A}$ resp. $\mathfrak{A}^{\mathrm{aff}}$ the category of quasi-torus action on (affine) $\mathbb{K}$-prevarieties with affine invariant cover. Then the following hold:*

(i) *For a $\mathfrak{G}$-object W each of the rings $\mathcal{O}_W(U)$ is reduced.*
(ii) *$\mathfrak{ac}_{qt} := \mathfrak{pv} \circ (\mathfrak{s}^0 \circ \mathfrak{a}^{gr})_{|\mathfrak{G}}$ and $\mathfrak{s}_{gr} := \mathfrak{t} \circ \mathfrak{inv}$ resp. $(\mathfrak{ac}_{qt})_{|\mathfrak{G}^{\mathrm{aff}}}$ and $(\mathfrak{s}_{gr})_{|\mathfrak{A}^{\mathrm{aff}}}$ are mutually essentially inverse covariant equivalences. Moreover, the two latter functors commute with taking principal open subsets.*
(iii) *$\mathfrak{ac}_{qt}$ sends a $\mathfrak{G}^{\mathrm{aff}}$-object W to the canonical $\mathrm{Spec}_{\max}(\mathbb{K}[gr(\mathcal{O}_W)])$-action on $\mathrm{Spec}_{\max}(\mathcal{O}(W))$ given by the coaction $f_v \mapsto \chi^v \otimes f_v$.*
(iv) *$(\mathfrak{s}_{gr})_{|\mathfrak{A}^{\mathrm{aff}}}$ is isomorphic to the functor $\mathfrak{g}$ sending $H \circlearrowright X$ to $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$, via the isomorphism mapping $Y \in \mathfrak{s}_{gr}(H \circlearrowright X)$ to $\mathcal{I}_Y(X)$.*

PROOF. For (i) note that the topology of a $\mathfrak{G}$-object W is noetherian by Example IV.2.1.17 and thus $\mathcal{O}_W$ is a sheaf of $\mathbb{K}$-algebras by Remark III.2.0.13. For each $U \in \mathcal{B}_W$, $\mathcal{O}(U)$ is of finite type and hence reduced by Proposition VI.2.1.5. By Remark IV.2.1.1 this means that each $\mathcal{O}_W(V)$ is reduced. In assertion (iv) well-definedness of the map $\eta_{H \circlearrowright X} \colon \mathfrak{s}^0(H \circlearrowright X) \to \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$ follows from Propositions VI.2.1.4 and VI.1.2.8. Bijectivity is due to Proposition VI.1.2.6, with the inverse map sending $\mathfrak{p} \in \mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))$ to $V(\mathfrak{p}) \subseteq X$. The preimage of $\mathrm{Spec}_{\mathrm{gr}}(\mathcal{O}(X))_f$ is $t(X_f)$ and vice versa. The canonical isomorphisms $\mathcal{O}(\mathrm{Spec}_{\mathrm{g}}(\mathcal{O}(X))_f) \to \mathcal{O}(t(X_f))$ define an isomorphism of sheaves. Thus, $\eta_{H \circlearrowright X}$ is an isomorphism of graded schemes over $\mathbb{K}$. These isomorphisms define an isomorphism from $\mathfrak{s}^0_{|\mathfrak{A}^{\mathrm{aff}}}$ to $\mathfrak{g}$. Assertion (iii) is due to the general fact that $\mathfrak{pv}(\mathrm{Spec}(R)) = \mathrm{Spec}_{\max}(R)$.

In (ii) well-definedness of $\mathfrak{ac}_{qt}$ follows from the fact that firstly, $\mathfrak{s}^0(W)$ is reduced by (i) and secondly, $\mathfrak{s}^0$ and $\mathfrak{pv}$ commute with products and hence they map actions to actions. For well-definedness of $\mathfrak{s}_{gr}$ note that $\mathfrak{s}_{gr}$ preserves open embeddings, which means well-definedness follows from (iv). In the following we use that $\mathfrak{s}_{gr}$ and $\mathfrak{ac}_{qt}$, and hence also their compositions, commute with open embeddings and gluing of these. For a $\mathfrak{G}^{\mathrm{aff}}$-morphism $\phi \colon U \to U'$ (iii) and (iv) define a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\sim} & \mathfrak{s}_{gr}(\mathfrak{ac}_{qt}(U)) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathfrak{s}_{gr}(\mathfrak{ac}_{qt}(\phi))} \\ U' & \xrightarrow{\sim} & \mathfrak{s}_{gr}(\mathfrak{ac}_{qt}(U')). \end{array}$$

For a $\mathfrak{G}$-morphism $\phi \colon W \to W'$ the family of diagrams given by $\phi_{|U} \colon U \to U'$, where for $U' \in \mathcal{B}_{W'}$ and $U \in \mathcal{B}_{\phi^{-1}(U')}$, glue together to a commutative diagram. Thus, a natural isomorphism between $\mathrm{id}_{\mathfrak{G}}$ and $\mathfrak{s}_{gr} \circ \mathfrak{ac}_{qt}$ is formed.

Likewise, for a $\mathfrak{A}^{\mathrm{aff}}$-morphism $(\theta, \phi) \colon H \circlearrowright U \to H' \circlearrowright U'$ (iii) and (iv) define a commutative diagram

$$\begin{array}{ccc} H \circlearrowright U & \xrightarrow{\sim} & \mathfrak{ac}_{qt}(\mathfrak{s}_{gr}(H \circlearrowright U)) \\ \downarrow{\scriptstyle (\theta,\phi)} & & \downarrow{\scriptstyle \mathfrak{ac}_{qt}(\mathfrak{s}_{gr}(\theta,\phi))} \\ H' \circlearrowright U' & \xrightarrow{\sim} & \mathfrak{ac}_{qt}(\mathfrak{s}_{gr}(H' \circlearrowright U')) \end{array}$$

and gluing gives such a diagram for general $\mathfrak{A}$-morphisms. Thus, a natural isomorphism between $\mathrm{id}_{\mathfrak{A}}$ and $\mathfrak{ac}_{qt} \circ \mathfrak{s}_{gr}$ is formed, which establishes (ii). □

PROPOSITION VI.2.4.2. *Let X be an H-prevariety and let W be the associated graded scheme, with $sc_X \colon X \to W$ denoting the singleton closure map. Then the following hold:*

(i) *The induced bijection $\Omega_{X,H} \to \Omega_W$ restricts to a bijection $\mathcal{B}_{X,H} \to \mathcal{B}_W$.*

(ii) *We have mutually inverse isomorphisms* $(sc_X)_*$ *and* t^{-1} *between algebras/modules over* $\mathcal{O}_{X,H}$ *and* $\mathcal{O}_W$. *These all preserve quasi-coherence and the property of being of finite type.*

(iii) X *is* H*-irreducible if and only if* W *is homogeneously integral, and we then have canonical isomorphisms* $\mathcal{K}_{X,H} \cong sc_X^{-1}\mathcal{K}_W$ *and* $\mathcal{K}_W \cong (sc_X)_*\mathcal{K}_{X,H}$.

Lemma VI.2.4.3. *Let* $q\colon \widehat{Z} \to Z$ *be a good quotient of graded schemes over* $\mathbb{K}$ *and suppose that* $\widehat{Z}$ *is of finite type over* $\mathbb{K}$ *and reduced. Then* Z *is also of finite type over* $\mathbb{K}$ *and reduced.*

Proof. It suffices to consider the affine case. Let $\widehat{H} \circlearrowright \widehat{X}$ be the quasi-torus action associated to $\widehat{Z}$ and let $\psi\colon gr(\mathcal{O}_Z) \to gr(\mathcal{O}_{\widehat{Z}}) = \mathbb{X}(\widehat{H})$ be the accompanying map of the graded morphism $\mathcal{O}_Z \to q_*\mathcal{O}_{\widehat{Z}}$. Then Hilberts Invariant Theorem says that $\mathcal{O}(Z) = \mathcal{O}(\widehat{Z})_{\ker(\psi)} = \mathcal{O}(\widehat{X})_{\ker(\psi)}$ is a finitely generated $\mathbb{K}$-algebra. $\square$

Proposition VI.2.4.4. *A morphism of quasi-torus actions is affine resp. a good quotient if and only if the corresponding morphism of graded schemes is so.*

Remark VI.2.4.5. Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be a good quotient, let $A \subseteq X$ be closed and H-irreducible and let $\widehat{A}$ be the special set over A. Then by Proposition IV.2.2.3 $q^*_{\widehat{A}}\colon (\mathcal{O}_{X,H})_A \to (\mathcal{O}_{\widehat{X},\widehat{H}})_{\widehat{A}}$ is Veronesean.

Proposition VI.2.4.6. *Let* $(\jmath, \imath)\colon H \circlearrowright X \to G \circlearrowright Z$ *be an affine morphism between quasi-torus actions such that* $\mathcal{O}_{Z,G} \to \imath_*\mathcal{O}_{X,H}$ *is a CBE. Then the following hold:*

(i) *The assignments* $A \mapsto G\imath(A)$ *and* $B \mapsto \imath^{-1}(B)$ *define mutually inverse bijections between the sets of invariant subsets of* X *and* Z, *with both respecting orbits, openness, closedness, invariant irreducibility, as well as inclusions, unions and intersections.*

(ii) *The canonical map* $\mathcal{B}_{Z,G} \to \mathcal{B}_{X,H}$ *is bijective. Moreover, for each* $U \in \mathcal{B}_{Z,G}$ *the canonical map* $\mathcal{B}^{\mathrm{pr}}_{U,G} \to \mathcal{B}_{\imath^{-1}(U),H}$ *is bijective.*

Proof. First note that by Proposition IV.1.4.13 the induced morphism of graded schemes is a homeomorphism. We now show that $\imath$ induces an a bijection between H-orbits of X and G-orbits of Z. Indeed, for $z \in Z$ there exists $x \in X$ with $\overline{G\imath(x)} = \overline{\imath(\overline{Hx})}^{\Omega_{Z,G}} = \overline{Gz}$ by our initial observation. Thus, Proposition VI.3.2.5 gives $G\imath(Hx) = G\imath(x) = Gz$. For injectivity, consider $x, y \in X$ with $G\imath(x) = G\imath(y)$. Then we have $\overline{G\imath(\overline{Hx})} = \overline{G\imath(x)} = \overline{G\imath(y)} = \overline{G\imath(\overline{Hy})}$ and the initial observation gives $\overline{Hx} = \overline{Hy}$ which means $Hx = Hy$.

Next, we show that $Gz = G\imath(\imath^{-1}(Gz))$. Let $x \in X$ with $Hx = \imath^{-1}(Gz)$. Then $G\imath(\imath^{-1}(Gz)) = G\imath(Hx) \subseteq Gz$ is an equality. Moreover, we have $Hx = \imath^{-1}(G\imath(Hx))$. Indeed, for $y \in X$ with $Hy \subseteq \imath^{-1}(G\imath(Hx))$ we have $G\imath(Hy) = G\imath(Hx)$ and hence $Hy = Hx$. Thus, the assignments specified in the assertion are mutually inverse on orbits, and hence on all invariant subsets.

For a family $A_i, i \in I$ of H-invariant subsets of X we have

$$\imath^{-1}(\bigcap_i G\imath(A_i)) = \bigcap_i \imath^{-1}(G\imath(A_i)) = \bigcap_i A_i = \imath^{-1}(G\imath(\bigcap_i A_i))$$

which gives $\bigcap_i G\imath(A_i) = G\imath(\bigcap_i A_i)$. Finally, for an H-invariant closed $A \subseteq X$ consider the unique G-invariant closed $B \subseteq Z$ with $A = \imath^{-1}(B)$. Then we have

$$\overline{G\imath(A)} = \overline{G\imath(\imath^{-1}(B))} = \overline{B} = B = G\imath(A).$$

$\square$

VI.3. Isotropy groups and orbit closures

From this section onwards all occuring affine algebraic groups H are understood to be quasi-tori, and all actions $H \circlearrowright X$ are assumed to have affine invariant covers. In the following two sections, we connect weight monoid functors at a point x and their colimits, weight groups with the orbit closure of x resp. the isotropy group of x. Generalizing from x to an H-closed, H-irreducible subset B we calculate the generic isotropy of B in using the generic weight group at B.

VI.3.1. orbit closures and weight monoids. We observe basic properties of the weight monoid functor at a given point x of an H-prevariety X (or more generally, at a given H-irreducible subset of X). In the case of an affine H-variety X, we recall the isomorphism $\mathcal{O}(\overline{Hx}) \to \mathbb{K}[S_{\mathbb{X}(H),x}(X)$. This connects the orbit closure $\overline{Hx}$ with $\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[S_{\mathbb{X}(H),x}(X)])$ and ultimately, with $\mathrm{Spec}(\mathbb{F}_1[S_{\mathbb{X}(H),x}(X)])$, see Proposition VI.3.1.3. Consequently, affine orbit closures inherit several topological properties from affine $\mathbb{F}_1$-schemes of finite type which are featured in Proposition VI.3.1.4.

Definition VI.3.1.1. Let $H \circlearrowright X$ be an action and let $B \subseteq X$ be closed and H-irreducible. The functor $S_{\mathbb{X}(H),B} \colon (\Omega_{X,H})_B \to \mathbf{Mon}$ of *weight monoids at B* sends U to $\deg((\rho_B^U)^{-1}_{|\mathcal{O}(U)^{\mathrm{hom}}}((\mathcal{O}_{X,H}^{\mathrm{hom}})_B^*))$.

Proposition VI.3.1.2. *For a closed H-irreducible $B \subseteq X$ and $U \in (\Omega_{X,H})_B$ the following hold:*

(i) *For each $f \in \mathcal{O}(U)_\chi \cap (\rho_B^U)^{-1}(((\mathcal{O}_{X,H})_B^{\mathrm{hom}})^*)$ we have*
$$S_{\mathbb{X}(H),B}(U_f) = S_{\mathbb{X}(H),B}(U)_\chi.$$
(ii) *If $B \subseteq B' \subseteq X$ is a closed and H-irreducible then*
$$S_{\mathbb{X}(H),B}(U) \subseteq S_{\mathbb{X}(H),B'}(U).$$
(iii) *If $\mathcal{O}(U) = \mathbb{K}[f_1, \ldots, f_s]$ with homogeneous elements $f_1, \ldots, f_s$ then*
$$S_{\mathbb{X}(H),B}(U) = \sum_{f_j \notin I(U \cap B)} \mathbb{N}_0 \deg(f_j).$$
If U is also affine then
$$S_{\mathbb{X}(H),B}(U) = \deg(\mathcal{O}(X)^{\mathrm{hom}} \setminus I(B)) = \mathrm{degsupp}(\mathcal{O}(B)).$$
(iv) *If the monoid $S_{\mathbb{X}(H),B}(U)$ is finitely generated then there exists an element $f \in (\rho_B^U)^{-1}_{|\mathcal{O}(U)^{\mathrm{hom}}}((\mathcal{O}_{X,H}^{\mathrm{hom}})_B^*)$ such that $S_{\mathbb{X}(H),\overline{Hx}}(U) = S_{\mathbb{X}(H),B}(U)$ holds for each $x \in U_f \cap B$. In this sense, $S_{\mathbb{X}(H),B}(U)$ is the* generic *weight monoid of B.*

Proof. For (i) let $g/(f^n) \in \mathcal{O}(U_f)_{\chi'}$ be a fraction whose stalk at B is a unit. Then $g \in \mathcal{O}(U)_{\chi'+n\chi}$ and g_B is also a unit which means
$$\chi' = (\chi' + n\chi) - (n\chi) \in S_{\mathbb{X}(H),B}(U)_\chi.$$

In (iv) consider $f_1, \ldots, f_m \in \mathcal{O}(U)^{\mathrm{hom}} \setminus 0$ whose degrees $w_j := \deg(f_j)$ generate $S_{\mathbb{X}(H),B}(U)$, and let f be the product over all f_j. For $x \in B \cap U_f$ we then have $w_1, \ldots, w_m \in S_{\mathbb{X}(H),\overline{Hx}}(U)$ which gives $S_{\mathbb{X}(H),B}(U) \subseteq S_{\mathbb{X}(H),\overline{Hx}}(U)$. □

Proposition VI.3.1.3. *Let $H \circlearrowright X$ be an affine orbit closure with $X = \overline{Hx}$ and set $M := \mathrm{degsupp}(\mathcal{O}(X)) \subseteq \mathbb{X}(H)$. Then the following hold:*

(i) *We have an isomorphism $\mathcal{O}(X) \to \mathbb{K}[M], f_w \mapsto f_w(x)\chi^w$.*
(ii) *We have order-preserving bijections between $\mathfrak{t}(X, \Omega_{X,H})$, i.e. the set of orbit closures, $\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[M])$ and $\mathrm{Spec}(M)$, and an order-reversing bijection from $\mathrm{Spec}(M)$ to $\mathrm{faces}(M)$.*

(iii) *For $U \in \mathcal{B}_{X,H}$ and $f \in \mathcal{O}(X)^{\mathrm{hom}} \setminus 0$ we have $U = X_f$ if and only if $\deg(f) \in S_{\mathbb{X}(H),\overline{O_U}}(X)^\circ$.*

Proof. For (i) note that $w \in M$ holds if and only if $\mathcal{O}(X)_w \nsubseteq I(x)$ because $I(x)^{\mathrm{gr}} = I(X) = \{0\}$. In (ii) we first use (i) and the equivalence between quasi-torus actions and graded schemes, then Example IV.1.5.10 and lastly Proposition I.1.4.2. Assertion (iii) follows from $\mathcal{O}(U) = (\mathcal{O}_{X,H})_{\overline{O_U}}$, which is shown in Proposition VI.3.1.4 and assertion (i). □

For an action $H \circlearrowright X$ the set of H-orbits carries the *specialization* relation, where we say that Hx *specializes* to Hy if Hy is contained in the closure of Hx.

Proposition VI.3.1.4. *Let $H \circlearrowright X$ have a dense orbit. Then the following hold:*

(i) *X consists of finitely many orbits, in particular, the dense orbit is the minimal element of $\Omega_{X,H}$ and $\mathcal{B}_{X,H}$. Consequently, the set of H-orbits is partially ordered by specialization and the canonical map from orbits to orbit closures is an isomorphism.*
(ii) *For an orbit O the union U_O of all orbits that specialize to O belongs to $\mathcal{B}_{X,H}$. Thus, U_O is the smallest open invariant neighbourhood of O and satisfies $(\mathcal{O}_{X,H})_{\overline{O}} = \mathcal{O}(U_O)$.*
(iii) *Each $U \in \mathcal{B}_{X,H}$ contains a unique orbit O_U which is maximal with respect to specialization among the orbits of U, in particular, O_U is closed in U.*
(iv) *The assignments $U \mapsto O_U$ and $O \mapsto U_O$ define mutually inverse isomorphisms between $\mathcal{B}_{X,H}$ and the H-orbits of X, equipped with inclusion resp. specialization order.*

Proof. In (i) finiteness of the set of orbits follows from the affine case. If X is affine then finiteness and soberness of $\mathfrak{t}(X, \Omega_{X,H})$ implies injectivity of $O \mapsto \overline{O}$. In general, if $\overline{O} = \overline{O'}$ then consider neighbourhoods $U, U'\mathcal{B}_{X,H}$ of O resp. O'. By irreducibility there exists $V \in \mathcal{B}_{X,H}$ with $O, O' \subseteq V$, and $\overline{O}^V = \overline{O'}^V$ implies $O = O'$. Assertions (ii) and (iii) follow from the affine case using Proposition VI.3.1.3(ii) and the respective statements for affine $\mathbb{F}_1$-schemes. For (iv) observe that O_{U_O} specializes to O because it is contained in U_O, and conversely, O belongs to U_O and thus specializes to O_{U_O}. Therefore, both orbits have the same closure and hence coincide. Moreover, U a neighbourhood of O and it is contained in U_{O_U} because all orbits in U specialize to O_U, hence $U = U_{O_U}$. □

Remark VI.3.1.5. Let $(\theta, \phi) \colon G \circlearrowright X \to H \circlearrowright Y$ be a morphism of actions such that $Y = \overline{H\phi(X)}$. If $X = \overline{Gx}$ is a G-orbit closure then $Y = \overline{H\phi(x)}$ is an H-orbit closure and in that case the following hold for $U \in \mathcal{B}_{X,G}$:

(i) The set $V_{H\phi(O_U)} \in \mathcal{B}_{Y,H}$ corresponding to the orbit $H\phi(O_U)$ is the intersection over all $W \in \mathcal{B}_{Y,H}$ which contain $\phi(U)$.
(ii) Then U is ϕ-saturated if and only if it is the union over all $V \in \mathcal{B}_{X,G}$ with $\phi(V) \subseteq V_{H\phi(O_U)}$. In this case we have $\phi(U) = V_{H\phi(O_U)}$.
(iii) $\phi^{-1}(V_{H\phi(O_U)})$ is the unique minimal ϕ-saturated set containing U.

VI.3.2. generic isotropy and weight groups in terms of graded stalks. In this section, we calculate the isotropy group of x in terms of its weight group. Generalizing leads to the concept of a generic weight group at a closed H-irreducible subset whose generic isotropy may then be expressed in terms of the former. As an application, Proposition VI.3.2.5 characterizes orbits via homogeneous simplicity of their sections, and orbit closures via finiteness of their collection of closed H-irreducible subsets. Moreover, this shows that $\overline{Hx} = \overline{Hy}$ implies $x = y$.

CONSTRUCTION VI.3.2.1. Let $H \circlearrowright X$ be a quasi-torus action. The *generic weight group* of a closed H-irreducible subset $B \subseteq X$ is

$$\mathbb{X}(H)_B := \deg((\mathcal{O}_{X,H}^{\mathrm{hom}})_B^*) = \operatorname*{colim}_{U \in (\Omega_{X,H})_B} S_{\mathbb{X}(H),B}(U) = \bigcup_{U \in (\Omega_{X,H})_B} S_{\mathbb{X}(H),B}(U).$$

The *weight group* of a point x is $\mathbb{X}(H)_x := \mathbb{X}(H)_{\overline{Hx}}$.

By definition, the generic weight group is a local object, i.e. it may be calculated in any invariant neighbourhood.

PROPOSITION VI.3.2.2. *For closed H-irreducible subsets B, C of X the following hold:*

(i) $\mathbb{X}(H)_B = \operatorname{degsupp}(\mathcal{K}_{B,H}(B))$
(ii) *If $C \subseteq B$ of X we have $\mathbb{X}(H)_C \subseteq \mathbb{X}(H)_B$.*
(iii) *If X is affine then*

$$\begin{aligned}\mathbb{X}(H)_B &= \deg((\mathcal{O}(X)_{I(B)}^{\mathrm{hom}})^*) = \operatorname{degsupp}(Q_{\mathrm{gr}}(\mathcal{O}(B)))\\ &= \langle \operatorname{degsupp}(\mathcal{O}(B))\rangle = \langle S_{\mathbb{X}(H),B}(X)\rangle \subseteq \langle \operatorname{degsupp}(\mathcal{O}(X))\rangle.\end{aligned}$$

Applied to a point x this means $\mathbb{X}(H)_x = \langle \mathcal{O}(\overline{Hx})\rangle = \langle S_{\mathbb{X}(H),\overline{Hx}}(X)\rangle$. Explicitely, if $\mathcal{O}(X) = \mathbb{K}[f_1, \ldots, f_s]$ for certain $f_j \in \mathcal{O}(X)^{\mathrm{hom}} \setminus 0$ and B is a closed H-irreducible subset then

$$\mathbb{X}(H)_B = \langle \deg(f_j) \mid f_j \notin I(B)\rangle.$$

PROOF. Assertion (ii) follows from injectivity of the map $(\mathcal{O}_C^{\mathrm{hom}})^* \to (\mathcal{O}_B^{\mathrm{hom}})^*$. The first set of equations in (iii) follow from Proposition VI.3.1.2 and Remark II.1.8.5 Lastly, in (i) we use that we may calculate $(\mathcal{O}_{X,H})_B$ in any affine chart which intersects B non-trivially. $\square$

If X is not affine then the above equations in (iii) may fail to hold:

EXAMPLE VI.3.2.3. Consider the projective space $X = \mathbb{P}$ with its canonical action by the torus $\mathbb{T} = \mathbb{K}^*$. Then $\mathbb{X}(\mathbb{T})_X = \mathbb{X}(\mathbb{T})_{[1:1]} = \mathbb{Z}$ because $\mathbb{T}$ acts freely on an open subset of X but $\mathcal{O}(X) = \mathbb{K}$ means $S_{\mathbb{X}(\mathbb{T}),[1:1]}(X) = \operatorname{degsupp}(\mathcal{O}(X)) = 0$ does not generate $\mathbb{X}(\mathbb{T})_X$.

PROPOSITION VI.3.2.4. *Let X be an H-prevariety with affine invariant cover and let x be a point. Then*

$$\begin{aligned}\mathcal{O}(H_x) &= \mathcal{O}(H)/\langle \chi - 1 \mid \chi \in \mathbb{X}(H)_x\rangle \cong \mathbb{K}[\mathbb{X}(H)/\mathbb{X}(H)_x]\\ H_x &= V_H(\chi - 1 \mid \chi \in \mathbb{X}(H)_x)\end{aligned}$$

PROOF. Fix $U \in (\mathcal{B}_{X,H})_x$. Let $h \in H_x$. For any $\chi \in S_{\mathbb{X}(H),\overline{Hx}}(U)$ consider an element $f \in \mathcal{O}(U)_\chi \setminus I(Hx)$. Then $\chi(h)f(x) = f(hx) = f(x) \neq 0$ implies $\chi(h) = 1$. For each two $\chi, \chi' \in S_{\mathbb{X}(H),\overline{Hx}}(U)$ we thus have $(\chi\chi'^{-1})(h) = \chi(h)\chi'(h)^{-1} = 1$ and since $\mathbb{X}(H)_x = \langle S_{\mathbb{X}(H),\overline{Hx}}(U)\rangle$ we have $h \in V_H(\chi - 1 \mid w \in \mathbb{X}(H)_x)$.

Conversely, if $\chi(h) = 1$ for all $\chi \in \mathbb{X}(H)_x$ then we claim that $f(hx) = f(x)$ for all homogeneous and hence all $f \in \mathcal{O}(U)$ which implies $hx = x$. If $f(x) = 0$ then $f(hx) = \deg(f)(h)f(x) = 0$ and if $f(x) \neq 0$ then $\deg(f) \in S_{\mathbb{X}(H),\overline{Hx}}(U) \subseteq \mathbb{X}(H)_x$ which means that $\deg(f)(h) = 1$ and hence $f(hx) = \deg(f)(h)f(x) = f(x)$.

By Example II.2.1.11 the ideal $\langle \chi - 1 \mid \chi \in \mathbb{X}(H)_x\rangle$ is $\mathbb{X}(H)/\mathbb{X}(H)_x$-prime, in particular radical, and

$$\mathcal{O}(H)/\langle \chi - 1 \mid \chi \in \mathbb{X}(H)_x\rangle \cong \mathbb{K}[\mathbb{X}(H)/\mathbb{X}(H)_x].$$

$\square$

PROPOSITION VI.3.2.5. *Let X be an affine H-variety.*

(i) *X is an H-orbit if and only if $\mathcal{O}(X)$ is $\mathbb{X}(H)$-simple.*

(ii) *A closed H-irreducible $A \subseteq X$ is an orbit closure if and only if the set $t(A)$ of closed H-irreducible subsets of A is finite.*
(iii) *We have $\overline{Hx} = \overline{Hy}$ if and only if $Hx = Hy$.*

PROOF. In (i) first note that $\mathcal{O}(Hx) \cong \mathcal{O}(H/H_x) = \mathbb{K}[\mathbb{X}(H)_x]$ is indeed $\mathbb{X}(H)$-simple. For the converse fix $x \in X$. The orbit map $\mu_x \colon H \to X$ induces a morphism $\overline{\mu}_x \colon H/H_x \to X$ of H-actions. Since $I(Hx) = \{0\}$ we have $X = \overline{Hx}$ and hence $\overline{\mu}_x^* \colon \mathcal{O}(X) \to \mathcal{O}(H/H_x)$ is injective. Its image is $\mathbb{K}[\mathrm{degsupp}(\mathcal{O}(X))] = \mathcal{O}(H/H_x)$ because we have

$$\mathrm{degsupp}(\mathcal{O}(X)) = S_{\mathbb{X}(H),x}(X) = \mathbb{X}(H)_x$$

and therefore, $\overline{\mu}_x$ must be bijective, i.e. μ_x is surjective.

In (ii) we may assume that $A = X$ holds. If $t(A) = \{A, A_1, \ldots, A_n\}$ holds with pairwise different A_i then there exist $f_i \in \mathcal{I}_{A_i}(X)^{\mathrm{hom}} \setminus \{0\}$. For $f := \prod_{i=1}^n f_i$ the ring $\mathcal{O}(X_f)$ must be $\mathbb{X}(H)$-simple, which by (i) means that the dense subset X_f is an H-orbit. The converse follows from Proposition VI.3.1.3.

For (iii) suppose that $X = \overline{Hx} = \overline{Hy}$ holds and let $A_1, \ldots, A_n$ be the orbit closures which are strictly contained in $\overline{Hx}$. Then there exist $f_i \in \mathcal{I}_{A_i}(X)^{\mathrm{hom}} \setminus \{0\}$ and for $f := \prod_{i=1}^n f_i$ the ring $\mathcal{O}(X_f)$ must be $\mathbb{X}(H)$-simple, which by (i) means that X_f is an H-orbit. Since y is not contained in any A_i we have $y \in X_f = Hx$. □

DEFINITION VI.3.2.6. For a closed H-irreducible subset B of an H-prevariety X with affine invariant cover the *generic isotropy group* of B is

$$H_B := V_H(\chi - 1 \mid \chi \in \mathbb{X}(H)_B).$$

REMARK VI.3.2.7. In the situation of the above definition, Example II.2.1.11 gives $\mathcal{O}(H_B) \cong \mathbb{K}[\mathbb{X}(H)/\mathbb{X}(H)_B]$ and hence $\mathbb{X}(H)_B$ is the kernel of the pullback $\mathbb{X}(H) \to \mathbb{X}(H_B)$.

REMARK VI.3.2.8. For two closed H-irreducible subsets $C \subseteq B$ of X we have $H_B \subseteq H_C$.

The names *generic* weight and isotropy group are explained below.

PROPOSITION VI.3.2.9. *Let X be a H-prevariety with affine invariant cover and let B be a closed H-irreducible subset. Then there exists $U \in (\mathcal{B}_{X,H})_B$, which may be chosen from $(\mathcal{B}^{\mathrm{pr}}_{X,H})_B$ if X is affine, such that the following hold:*
(i) $\deg((\mathcal{O}(U)^{\mathrm{hom}})^*) = \mathrm{degsupp}(\mathcal{O}(U \cap B)) = \mathbb{X}(H)_B$,
(ii) *$H_x = H_B$ and $\mathbb{X}(H)_x = \mathbb{X}(H)_B$ for every $x \in U \cap B$.*

Moreover, whenever there exists $W \in \Omega_{B,H}$ such that we have $H_x = H'$ resp. $\mathbb{X}(H)_x = K'$ holds for all $x \in W$, then $H' = H_B$ resp. $K' = \mathbb{X}(H)_B$.

PROOF. Let $u_1, \ldots, u_s \in \mathbb{X}(H)_B$ such that $\mathbb{X}(H)_B = \langle u_1, \ldots, u_s \rangle$ and consider $V \in (\mathcal{B}_{X,H})_B$. Then there exist elements $f_i, g_i \in \mathcal{O}(V)^{\mathrm{hom}} \setminus \mathcal{I}_B(V)$ such that $\deg(f_i/g_i) = u_i$. Set $w_i := \deg(f_i)$ and $v_i := \deg(g_i)$ as well as $h := \prod_{i=1}^s f_i g_i$. We claim that $U := V_h$ has the desired properties. Firstly, note that by definition we have

$$\mathbb{X}(H)_B = \langle u_1, \ldots, u_s \rangle \subseteq \langle w_i, v_i \mid i = 1, \ldots, s \rangle \subseteq \mathbb{X}(H)_B$$

Since f_i and g_i remain units on $U \cap B$ we have $\mathbb{X}(H)_B \subseteq \mathrm{degsupp}(\mathcal{O}(U \cap B))$. The converse inclusion follows from Proposition VI.3.2.2. For (ii) note that f_i and g_i remain units on Hx for every $x \in U \cap B$ and thus $\mathbb{X}(H)_B \subseteq \mathbb{X}(H)_x \subseteq \mathbb{X}(H)_B$. The supplement follows from H-irreducibility of B. □

PROPOSITION VI.3.2.10. *Let $(\theta, q) \colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be a morphism, let $\widehat{A} \subseteq \widehat{X}$ be closed and $\widehat{H}$-irreducible and set $A := \overline{q(\widehat{A})}$. Then we have $\theta(\widehat{H}_{\widehat{A}}) \subseteq H_A$ and $\mathbb{X}(H)_A \subseteq (\theta^*)^{-1}(\mathbb{X}(\widehat{H})_{\widehat{A}})$. If q is a good quotient and $\widehat{A}$ is the special set over A*

then equality holds in both cases. In particular, $\widehat{H}_{\widehat{A}} \to H_A$ is then an isomorphism if and only if $\mathbb{X}(\widehat{H}) = \theta^*(\mathbb{X}(H)) + \mathbb{X}(\widehat{H})_{\widehat{A}}$.

Proof. let $U \in (\Omega_{X,H})_A$ such that $H_x = H_A$ holds for each $x \in U \cap A$. Then there exists $V \in (\Omega_{q^{-1}(U),\widehat{H}})_{\widehat{A}\cap q^{-1}(U)}$ such that $\widehat{H}_y = \widehat{H}_{\widehat{A}}$ holds for each $y \in V \cap \widehat{A}$ and we conclude

$$\theta(\widehat{H}_{\widehat{A}}) = \theta(\widehat{H}_y) \subseteq H_{q(y)} = H_A.$$

If q is a good quotient then the second equality is due to Remark VI.2.4.5 and the first follows from the second via

$$(\theta_H^*)^{-1}(\langle \widehat{\chi} - 1 | \widehat{\chi} \in \mathbb{X}(\widehat{H})_{\widehat{A}} \rangle) = \langle \chi - 1 | \chi \in \mathbb{X}(H)_A \rangle.$$

For the last supplement note that the pullback $\mathbb{X}(H)/\mathbb{X}(H)_A \to \mathbb{X}(\widehat{H})/\mathbb{X}(\widehat{H})_{\widehat{A}}$ is already injective, and surjectivity is equivalent to $\mathbb{X}(\widehat{H}) = \theta^*(\mathbb{X}(H)) + \mathbb{X}(\widehat{H})_{\widehat{A}}$. $\square$

If X itself is H-irreducible then H_X is also called the *kernel of ineffectivity*.

Proposition VI.3.2.11. *Let X be H-irreducible and let V be the invariant open subset of X whereon H acts with constant isotropy H_X. Then the following hold:*

(i) *V is covered by certain $U \in \mathcal{B}_{X,H}$ such that* $\deg((\mathcal{O}(U)^{\text{hom}})^*) = \mathbb{X}(H)_X$*. If X is affine then these U may be chosen from $\mathcal{B}_{X,H}^{\text{pr}}$.*
(ii) *Let $k \in \{0, \ldots, \dim_H(X)\}$. Then $X \setminus V$ has codimension greater than k if and only if for every closed H-irreducible $B \subseteq X$ of codimension k we have $\mathbb{X}(H)_B = \mathbb{X}(H)_X$, i.e. $H_B = H_X$.*

Proof. In (ii) first suppose that H acts with constant isotropy H_X on an open H-invariant subset X' with $\operatorname{codim}_X(X \setminus X') > k$. For each closed H-irreducible $B \subseteq X$ of codimension k we then have $X' \cap B \neq \emptyset$. Thus, Proposition VI.3.2.9 gives $H_B = H_X$ and $\mathbb{X}(H)_B = \mathbb{X}(H)_X$.

Conversely, suppose that $\mathbb{X}(H)_B = \mathbb{X}(H)_X$ for each closed H-irreducible subset B of codimension k. For each such B Proposition VI.3.2.9 gives $U^{(B)} \in (\mathcal{B}_{X,H})_B$ with

$$\deg((\mathcal{O}(U^{(B)})^{\text{hom}})^*) = \operatorname{degsupp}(\mathcal{O}(U^{(B)} \cap B)) = \mathbb{X}(H)_x = \mathbb{X}(H)_B = \mathbb{X}(H)_X$$

for every $x \in U^{(B)} \cap B$. Let X' be the union over all the sets $U^{(B)}$. Then the H-irreducible components of $X \setminus X'$ have codimension at least $k+1$. In the situation of (i) we note that V is the union over all $U^{(\overline{Hx})}$ where $x \in V$. $\square$

Remark VI.3.2.12. If H acts with constant isotropy H_X on X then every orbit has dimension $\operatorname{rk}(\mathbb{X}(H)_X)$. Thus, all orbits are closed. Hence, if the action allows a good quotient then this is already a geometric quotient. In particular, the above holds for the case of free actions.

Definition VI.3.2.13. Let $H \circlearrowright X$ be an action and let $H' \subseteq H$ be a closed algebraic subgroup. Let A be a closed H-irreducible subset of X. Then the *generic isotropy group* of the H'-action on A is $H'_A := H_A \cap H'$.

Remark VI.3.2.14. In the situation of the above definition, the kernel of the pullback $\mathbb{X}(H) \to \mathbb{X}(H'_A)$ is the sum of $\mathbb{X}(H)_A$ and the kernel of the pullback $\mathbb{X}(H) \to \mathbb{X}(H')$.

Proposition VI.3.2.15. *Let $H \circlearrowright X$ be an action and let $H' \subseteq H$ be a closed algebraic subgroup. Let A be a closed H-irreducible subset of X. Then for any H'-irreducible component B of A the generic isotropy group H'_B equals H'_A and hence $\mathbb{X}(H)_A$ is the preimage of $\mathbb{X}(H')_B$ under the pullback $\mathbb{X}(H) \to \mathbb{X}(H')$.*

Proof. Let U be an open H-invariant subset of X such that all points of the non-empty set $U \cap A$ have isotropy group H_A. Then $U \cap B$ is also non-empty, and for each of its points x we have

$$H'_x = H_x \cap H' = H_A \cap H' = H'_A.$$

Therefore, H'_A equals H'_B. □

Remark VI.3.2.16. For an action with constant isotropy all orbits have the same dimension are a therefore closed.

Proposition VI.3.2.17. *Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be a good quotient of invariantly irreducible actions. Then $\ker(\theta)$ acts with constant isotropy on $\widehat{X}$ if and only if X has an affine $\widehat{H}$-invariant cover by open q-saturated sets V such that $\mathcal{O}(V)$ is $\mathbb{X}(H)$-associated.*

Proof. If $\ker(\theta)$ acts with constant isotropy on $\widehat{X}$ holds consider a closed orbit Hx and the closed orbit $\widehat{H}\widehat{x}$ in its fibre. Let U be an affine invariant neighbourhood of Hx. Let $f_1, \ldots, f_m \in \mathcal{O}(q^{-1}(U))^{\mathrm{hom}} \setminus I(\widehat{H}\widehat{x})$ be homogeneous elements whose degrees generate $\deg(((\mathcal{O}_{\widehat{X},\widehat{H}})^{\mathrm{hom}}_{\widehat{H}\widehat{x}})^*)$ and set $f := f_1 \cdots f_m$. Then $q(q^{-1}(U)_f)$ is an affine invariant neighbourhood of Hx and $\mathcal{O}(q^{-1}(U)_f)$ is $\mathbb{X}(H)$-associated. Choosing such an open set U_f for every closed orbit we obtain the desired cover.

For the converse consider $\widehat{x} \in \widehat{X}$ and an affine invariant neighbourhood U of $q(\widehat{x})$. By assumption $\mathcal{O}(q^{-1}(U))_{I_{q^{-1}(U)}(\widehat{H}\widehat{x})}$ is $\mathbb{X}(H)$-associated. For each homogeneous fraction $f/g \in Q_{\mathrm{gr}}(\mathcal{O}(q^{-1}(U))_{I_{q^{-1}(U)}(\widehat{H}\widehat{x})}) = \mathcal{K}_{\widehat{H}}(\widehat{X})$ we find homogeneous units $a, b \in \mathcal{O}(q^{-1}(U))_{I_{q^{-1}(U)}(\widehat{H}\widehat{x})}$ such that af and bg have degrees in $\mathbb{X}(H)$. Thus,

$$\deg(f/g) = \deg(b/a) + \deg(af/bg) \in \mathbb{X}(\widehat{H})_{\overline{\widehat{H}\widehat{x}}} + \mathbb{X}(H)$$

which means $\mathbb{X}(\widehat{H})_{\widehat{X}} + \mathbb{X}(H) = \mathbb{X}(\widehat{H})_{\overline{\widehat{H}\widehat{x}}} + \mathbb{X}(H)$, i.e. $\ker(\theta)_{\widehat{X}} = \ker(\theta)_{\widehat{x}}$. □

VI.4. Characteristic spaces of actions

Again, all actions $H \circlearrowright X$ are morphical actions of quasi-tori on prevarieties with affine invariant covers. After developing the theory of Weil divisors, class groups, divisorial algebras and Cox sheaves with respect to $\Omega_{X,H}$ we translate properties of a graded characteristic space into properties of the corresponding characteristic space $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ of actions. Specifically, order-preserving invertibility of the pullback of Weil divisors translates into the existence of a big $\widehat{H}$-invariant q-saturated subset on which $\ker(\theta)$ acts with constant isotropy, see Proposition VI.4.2.6, while the condition on the occuring grading groups translates into the restriction $\theta\colon \widehat{H}_{\widehat{X}} \to H_X$ being an isomorphism. This gives the criterion of characteristic spaces in Theorem VI.4.2.6. Finally, Section VI.4.3 shows that all notions of cones of divisors (or their classes) and base loci are well-behaved under the equivalence of graded schemes and actions.

VI.4.1. Invariant Weil divisors and Cox sheaves of actions. Here, we define invariant Weil divisors for actions $H \circlearrowright X$ of Krull type, meaning that for some affine invariant cover the section rings are $\mathbb{X}(H)$-Krull rings. All claims made during the development of the theory of invariant Weil divisors, divisorial $\mathcal{O}_{X,H}$-algebras and Cox sheaves on $\Omega_{X,H}$ canonically follow from their analoga in the Chapter on graded schemes of Krull type via the equivalence from Section VI.2.4.

Definition VI.4.1.1. An action $H \circlearrowright X$ is of *Krull type*, *H-normal* or *invariantly normal* if $\mathcal{O}(U)$ is of Krull type or equivalently, $\mathbb{X}(H)$-normal for every $U \in \Omega_{X,H}$.

Remark VI.4.1.2. An action is of Krull type if and only if the corresponding graded scheme is of Krull type. In particular, $H \circlearrowright X$ is of Krull type if the sections of $\mathcal{O}_{X,H}$ are of Krull type for an affine invariant cover of X.

Definition VI.4.1.3. For an action $H \circlearrowright X$ of Krull type an *H-prime divisor* is a closed H-irreducible subset $Y \subset X$ of codimension one. The presheaf which assigns to $U \in \Omega_{X,H}$ the set of H-prime divisors of X which intersect U non-trivially is denoted $\mathcal{Y}_{X,H}$ resp. $\mathcal{Y}_H$ if there is only one H-prevariety under consideration.

Construction VI.4.1.4. Let $H \circlearrowright X$ be of Krull type. For $Y \in \mathcal{Y}_H(X)$ denote by $\mathbb{Z}^{(Y)}$ the discrete value sheaf on $(X, \Omega_{X,H})$ assigning $\mathbb{Z}$ on $(\Omega_{X,H})_Y$ and 0 on $\Omega_{X,H} \setminus (\Omega_{X,H})_Y$. Then one obtains a graded valuation $\nu_Y \colon (\mathcal{K}^{\mathrm{hom}}_{X,H})^* \to \mathbb{Z}^{(Y)}$ by defining $\nu_{Y,U}$ as the canonical map

$$(\mathcal{K}_{X,H}(U)^{\mathrm{hom}})^* \longrightarrow (\mathcal{K}_{X,H}(U)^{\mathrm{hom}})^* / ((\mathcal{O}_{X,H})^{\mathrm{hom}}_Y)^* \cong \mathbb{Z}$$

if $U \in (\Omega_{X,H})_Y$ (and as the zero map otherwise).

Proposition VI.4.1.5. *For an action $H \circlearrowright X$ of Krull type $\mathcal{O}_{X,H}$ is of Krull type in $\mathcal{K}_{X,H}$ with essential graded valuations $\{\nu_Y\}_{Y \in \mathcal{Y}_H(X)}$.*

Construction VI.4.1.6. For an action $H \circlearrowright X$ of Krull type one obtains the *invariant divisor homomorphism*

$$\mathrm{div}_{X,H} = \sum_{Y \in \mathcal{Y}_H(X)} \nu_Y \colon (\mathcal{K}^{\mathrm{hom}}_{X,H})^* \longrightarrow \mathrm{WDiv}_{X,H} := \bigoplus_{Y \in \mathcal{Y}_H(X)} \mathbb{Z}^{(Y)}$$

to the sheaf of *H-Weil divisors* or *invariant Weil divisors* on $(X, \Omega_{X,H})$. Image $\mathrm{PDiv}_{X,H} := \mathrm{im}(\mathrm{div}_{X,H})$ and cokernel $\mathrm{Cl}_{X,H} := \mathrm{coker}\,(\mathrm{div}_{X,H})$ of $\mathrm{div}_{X,H}$ are called the presheaves of *invariant principal divisors* and *invariant class groups* respectively. The quotient $\mathrm{CaDiv}_{X,H} / \mathrm{PDiv}_{X,H}$ is the presheaf $\mathrm{Pic}_{X,H}$ of *Picard groups*. We then have $\mathcal{O}^{\mathrm{hom}}_{X,H} \setminus \{0\} = \mathrm{div}^{-1}_{X,H}(\mathrm{WDiv}_{\geq 0})$ and $(\mathcal{O}^{\mathrm{hom}}_{X,H})^* = \ker(\mathrm{div}_{X,H})$. In situations where we consider only a single action the subscript will feature only H instead of X and H.

Construction VI.4.1.7. Let $H \circlearrowright X$ be of Krull type and let $A \subseteq X$ be H-closed and -irreducible. Consider the canonical homomorphism ϕ from $\mathrm{WDiv}_{X,H}$ to the skyscraper sheaf assigning $(\mathrm{WDiv}_{X,H})_A$ at A. The preimage under ϕ of the skyscraper sheaf assigning $(\mathrm{PDiv}_{X,H})_A$ at A is the sheaf $\mathrm{PDiv}^{(A)}_{X,H}$ of *principal divisors near* A. The image of $\mathrm{PDiv}^{(A)}_{X,H}$ under $c\colon \mathrm{WDiv}_{X,H} \to \mathrm{Cl}_{X,H}$ is the kernel $\mathrm{Cl}^{(A)}_{X,H}$ of the canonical homomorphism from $\mathrm{Cl}_{X,H}$ to the skyscraper sheaf assigning $(\mathrm{Cl}_{X,H})_A$ at A. In this notation we have

$$\mathrm{CaDiv}_{X,H} = \bigcap_{x \in X} \mathrm{PDiv}^{(\overline{Hx})}_{X,H} = \bigcap_{\substack{x \in X \\ \overline{Hx} = Hx}} \mathrm{PDiv}^{(Hx)}_{X,H},$$

$$\mathrm{Pic}_{X,H} = \bigcap_{x \in X} \mathrm{Cl}^{(\overline{Hx})}_{X,H} = \bigcap_{\substack{x \in X \\ \overline{Hx} = Hx}} \mathrm{Cl}^{(Hx)}_{X,H}.$$

Construction VI.4.1.8. Let $(\theta, \phi)\colon H \circlearrowright X \to G \circlearrowright Z$ be an equivariantly dominant morphism between actions of Krull type. Then we have a homomorphism $\mathrm{PDiv}_{Z,G} \to \phi_* \mathrm{PDiv}_{X,H}$ which sends $\mathrm{div}_{U,G}(f)$ to $\mathrm{div}_{\phi^{-1}(U),H}(\phi^*_U(f))$.

For each $Y \in \mathcal{Y}_{X,H}(X)$ we obtain a homomorphism $\mathrm{WDiv}_{Z,G} \to \phi_* \mathbb{Z}^{(Y)}$ as follows: If $(\mathrm{Cl}_{Z,G})_{\overline{\phi(Y)}} \neq 0$ or $Y \notin \mathcal{Y}_{X,H}(\phi^{-1}(U))$ then $\mathrm{WDiv}_{Z,G}(U) \to \mathbb{Z}^{(Y)}(\phi^{-1}(U))$ is the zero map. Otherwise, it is the canonical homomorphism

$$\mathrm{WDiv}_{Z,G}(U) \longrightarrow \mathrm{WDiv}_{Z,\overline{\phi(Y)}} \longrightarrow (\mathrm{PDiv}_{X,H})_Y \cong \mathbb{Z}^{(Y)}(\phi^{-1}(U)).$$

The induced homomorphism $\mathrm{WDiv}_{Z,G} \to \phi_* \prod_{Y \in \mathcal{Y}_{X,H}(X)} \mathbb{Z}^{(Y)}$ then has image in $\mathrm{WDiv}_{X,H}$ and hence defines a homomorphism $(\theta, \phi)^* \colon \mathrm{WDiv}_{Z,G} \to \phi_* \mathrm{WDiv}_{X,H}$ called the *pullback homomorphism*.

Remark VI.4.1.9. In the situation of the above construction, each $Y \in \mathcal{Y}_H(X)$ with $(\mathrm{Cl}_{Z,G})_{\overline{\phi(Y)}} = 0$ satisfies

$$\mathrm{pr}_{Y,X}((\theta,\phi)^*_Z(\mathrm{div}_{Z,G}(f))) = \nu_{Y,X}(\phi^*_Z(f))$$

for all $f \in (\mathcal{K}_G(Z)^{\mathrm{hom}})^*$. Thus, if $(\mathrm{Cl}_{Z,G})_{\overline{\phi(Y)}} = 0$ holds for all $Y \in \mathcal{Y}_H(X)$ then we also have a pullback homomorphism $\phi^* \colon \mathrm{Cl}_{Z,G} \to \mathrm{Cl}_{X,H}$ of presheaves.

Example VI.4.1.10. Consider $H \circlearrowright X$ with X of Krull type. Then H is of Krull type and the pullback induced by the morphism $(\imath, \mathrm{id}_X) \colon 1 \circlearrowright X \to H \circlearrowright X$ is an isomorphism onto its image, the $\Omega_{X,H}$-presheaf WDiv^H of H-invariant Weil divisors. The images of the prime H-Weil divisors are just the sums of their irreducible components, and they are called *H-prime* Weil divisors.

Indeed, for each $Y \in \mathcal{Y}(X)$ the set $\overline{HY}$ is either X or of (invariant) codimension one. In the latter case Y is an irreducible component of $\overline{HY} = HY$.

Proposition VI.4.1.11. *Let X be an affine H-variety. Then $H \circlearrowright X$ is of Krull type with $\mathrm{Cl}_H(X) = 0$ if and only if every $\mathbb{X}(H)$-prime ideal of $\mathbb{X}(H)$-height one in $\mathcal{O}(X)$ is $\mathbb{X}(H)$-principal.*

Proof. For a non-unit $f \in \mathcal{O}(X)^{\mathrm{hom}} \setminus 0$ let $Y_1, \ldots, Y_m$ be the H-irreducible components of $V(f)$ and let $p_1, \ldots, p_m$ be $\mathbb{X}(H)$-prime elements with $I(Y_i) = \langle p_i \rangle$. Due to noetherianity of $\mathcal{O}(X)$ the set of all $k > 0$ such that $p_i^k | f$ is finite for every i. Thus, f is a product of a homogeneous unit and powers of the p_i. $\square$

Remark VI.4.1.12. Let $H \circlearrowright X$ be a quasi-torus action of Krull type. If X is of affine intersection (e.g. separated or even quasi-projective), then the proof of Proposition V.1.3.5 shows that the complement of every affine (invariant) open subset is of pure codimension one.

Definition VI.4.1.13. Let $H \circlearrowright X$ be of Krull type. For $D \in \mathrm{WDiv}_H(X)$ the corresponding $\mathcal{O}_{X,H}$-module $\mathcal{O}_{X,H}(D) = \mathcal{O}(D)$ on $(X, \Omega_{X,H})$ is defined via

$$\mathcal{O}(D)(U) := \bigoplus_{w \in \mathbb{X}(H)} \mathcal{K}_{X,H}(U)_w \cap (\{0\} \cup \mathrm{div}_U^{-1}(-D_{|U} + \mathrm{WDiv}_H(U)_{\geq 0})).$$

Remark VI.4.1.14. Let $(\theta, \phi) \colon H \circlearrowright X \to G \circlearrowright Z$ be a morphism between actions of Krull type such that $(\mathrm{Cl}_{Z,G})_{\overline{\phi(Y)}} = 0$ holds for all $Y \in \mathcal{Y}_H(X)$. Then due to Remark VI.4.1.9 for $D \in \mathrm{WDiv}_G(Z)$ the canonical homomorphism $\mathcal{K}_{Z,G} \to \phi_* \mathcal{K}_{X,H}$ restricts to a homomorphism $\mathcal{O}(D) \to \phi_* \mathcal{O}((\theta,\phi)^* D)$.

Definition VI.4.1.15. Let $H \circlearrowright X$ be of Krull type. The *divisorial $\mathcal{O}_{X,H}$-algebra* associated to a subgroup $L \leq \mathrm{WDiv}_H(X)$ is the $\mathbb{X}(H) \oplus L$-graded $\mathcal{O}_{X,H}$-subalgebra

$$\mathcal{O}_{X,H}(L) := \mathcal{O}(L) := \bigoplus_{D \in L} \mathcal{O}(D) \chi^D \subseteq \mathcal{K}_H[L]$$

of the constant $\mathcal{O}_{X,H}$-algebra $\mathcal{K}_H[L]$.

In the following let $L := \mathrm{WDiv}_H(X)$.

Definition VI.4.1.16. Let $H \circlearrowright X$ be of Krull type. A *Cox sheaf* on $(X, \Omega_{X,H})$ resp. an *invariant Cox sheaf* on X is an $\mathcal{O}_{X,H}$-algebra $\mathcal{R}$ which allows a CBE $\pi \colon \mathcal{O}_{X,H}(L) \to \mathcal{R}$ of $\mathcal{O}_{X,H}$-algebras whose accompanying group homomorphism $\psi \colon \mathbb{X}(H) \oplus L \to gr(\mathcal{R})$ satisfies $L \cap \psi^{-1}(\mathbb{X}(H)) = \mathrm{PDiv}_H(X)$.

Next, we show that the inverse image under soberification and the direct image under the singleton closure map induce correspondences of (invariant) Weil divisors, their $\mathcal{O}_X$-modules and -algebras, as well as (invariant) Cox sheaves.

PROPOSITION VI.4.1.17. *For a quasi-torus action $H \circlearrowright X$ of Krull type and its corresponding graded Krull scheme $W = \mathfrak{t}(X, \Omega_{X,H}, \mathcal{O}_{X,H})$ the following hold:*

(i) *For $Y \in \mathcal{Y}_H(X)$ and $t(Y) = \overline{\{Y\}}$ we have commutative diagrams*

$$\begin{tikzcd} t^{-1}(\mathcal{K}_W^{\mathrm{hom}})^* \arrow[r, "t^{-1}\nu_{t(Y)}"] \arrow[d, "\cong"] & t^{-1}\mathbb{Z}^{(t(Y))} \arrow[d, "="] \\ (\mathcal{K}_{X,H}^{\mathrm{hom}})^* \arrow[r, "\nu_Y"] & \mathbb{Z}^{(Y)} \end{tikzcd}$$

which together with the bijection $\mathcal{Y}_H(X) \to \mathcal{Y}(W), Y \mapsto t(Y)$ induce commutative diagrams with exact rows

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & t^{-1}(\mathcal{O}_W^{\mathrm{hom}})^* \arrow[r] \arrow[d, "="] & t^{-1}(\mathcal{K}_W^{\mathrm{hom}})^* \arrow[r, "t^{-1}\mathrm{div}"] \arrow[d, "\cong"] & t^{-1}\,\mathrm{WDiv}_W \arrow[r] \arrow[d, "\cong"] & t^{-1}\,\mathrm{Cl}_W \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d] \\ 0 \arrow[r] & (\mathcal{O}_{X,H}^{\mathrm{hom}})^* \arrow[r] & (\mathcal{K}_{X,H}^{\mathrm{hom}})^* \arrow[r, "\mathrm{div}_H"] & \mathrm{WDiv}_{X,H} \arrow[r] & \mathrm{Cl}_{X,H} \arrow[r] & 0 \end{tikzcd}$$

(ii) *For $D \in \mathrm{WDiv}_H(X)$ and the corresponding $E \in \mathrm{WDiv}(W)$ we have an isomorphism $t^{-1}\mathcal{O}_W(E) \to \mathcal{O}_{X,H}(D)$ which is induced by the canonical isomorphism $t^{-1}\mathcal{K}_W \to \mathcal{K}_{X,H}$. Consequently, for corresponding subgroups $M \leq \mathrm{WDiv}_H(X)$ and $N \leq \mathrm{WDiv}(W)$ the canonical isomorphism $t^{-1}\mathcal{K}_W[N] \to \mathcal{K}_{X,H}[M]$ induces an isomorphism $t^{-1}\mathcal{O}_W(N) \to \mathcal{O}_{X,H}(M)$.*

(iii) *the isomorphism of $\mathbf{Alg}_{\mathcal{O}_{X,H}}$ and $\mathbf{Alg}_{\mathcal{O}_W}$ from Proposition VI.2.4.2 restricts to an isomorphism of Cox sheaves on $(X, \Omega_{X,H})$ and Cox sheaves on W.*

PROOF. For (i) we use the canonical isomorphism $(\mathcal{O}_{X,H})_Y \to \mathcal{O}_{W,A}$. Assertion (ii) is a consequence of (i). In (iii) note that upto the isomorphisms from (ii) each defining CBE of a Cox sheaf on W is mapped to a defining CBE of a Cox sheaf on $(X, \Omega_{X,H})$ under t^{-1}, and the reverse holds under $(sc_X)_*$. □

REMARK VI.4.1.18. In the situation of Proposition VI.4.1.17 consider an equivariantly dominant morphism $(\theta, \phi)\colon H \circlearrowright X \to H' \circlearrowright X'$ and the corresponding dominant morphism $\eta\colon W \to W'$. Then with the notations $t^{-1} := t^{-1}_{(X,\Omega_{X,H})}$ and $t'^{-1} := t^{-1}_{(X',\Omega_{X',H'})}$ the isomorphisms from the above Proposition have the following properties:

(i) With we have a commutative diagram

$$\begin{tikzcd} t'^{-1}\,\mathrm{WDiv}_{W'} \arrow[r, "t'^{-1}\eta^*"] \arrow[d, "\cong"] & t'^{-1}\eta_*\,\mathrm{WDiv}_W \arrow[r, "="] & \phi_* t^{-1}\,\mathrm{WDiv}_W \arrow[d, "\cong"] \\ \mathrm{WDiv}_{X',H'} \arrow[rr, "(\theta,\phi)^*"] & & \phi_*\,\mathrm{WDiv}_{X,H}\,. \end{tikzcd}$$

(ii) Suppose that each $Y \in \mathcal{Y}_H(X)$ satisfies $\mathrm{Cl}_{H',\overline{\phi(Y)}} = 0$, i.e. that each $A \in \mathcal{Y}(W)$ satisfies $\mathrm{Cl}_{\overline{\eta(A)}} = 0$. Then we have a commutative diagram

$$\begin{tikzcd} t'^{-1}\,\mathrm{Cl}_{W'} \arrow[r, "t'^{-1}\eta^*"] \arrow[d, "\cong"] & t'^{-1}\eta_*\,\mathrm{Cl}_W \arrow[r, "="] & \phi_* t^{-1}\,\mathrm{Cl}_W \arrow[d, "\cong"] \\ \mathrm{Cl}_{X',H'} \arrow[rr, "(\theta,\phi)^*"] & & \phi_*\,\mathrm{Cl}_{X,H}\,. \end{tikzcd}$$

Furthermore, in that case for $D' \in \mathrm{WDiv}_{H'}(X')$ and the corresponding $E' \in \mathrm{WDiv}(W')$ we have a commutative diagram

$$\begin{tikzcd} t'^{-1}\mathcal{O}_{W'}(E') \arrow[r, "t'^{-1}\eta^*"] \arrow[d, "\cong"] & t'^{-1}\eta_*\mathcal{O}_W(\eta^*E') \arrow[r, "="] & \phi_* t^{-1}\mathcal{O}_W(\eta^*E') \arrow[d, "\cong"] \\ \mathcal{O}_{X',H'}(D') \arrow[rr, "(\theta,\phi)^*"] & & \phi_*\mathcal{O}_{X,H}((\theta,\phi)^*D'). \end{tikzcd}$$

VI.4.2. Characteristic spaces of actions. By the last section, all results on Cox sheaves of graded schemes of Krull type also hold (analogously) for Cox sheaves of quasi-torus actions of Krull type. The same holds for the characterizing properties of (graded) characteristic spaces. We will apply our observations from Section VI.3.2 to translate the properties concerning grading groups and the pullback of Weil divsiors into more geometric statements on generic isotropy groups and obtain the criterion in Theorem VI.4.2.7.

Lemma VI.4.2.1. *Let $\pi\colon \mathcal{O}_{X,H}(K) \to \mathcal{R}, \psi$ be a CBE of $\mathcal{O}_{X,H}$-algebras where the classes of $K \leq \mathrm{WDiv}_H(X)$ generate $\mathrm{Cl}_H(X)$. Then each $U \in \Omega_{X,H}$ such that $\mathcal{R}(U)$ is finitely generated over $\mathbb{K}$ satisfies the following:*

(i) *for every $V \in \Omega_{U,H}$, $\mathcal{R}(V)$ is also finitely generated over $\mathbb{K}$,*
(ii) *$gr(\mathcal{R})$ and $\mathrm{Cl}_H(X)$ are finitely generated,*
(iii) *if U is affine then $\mathcal{R}(U)$ is affine, i.e. reduced.*

Consequently, if $\mathcal{R}$ is locally of finite type then it is a sheaf of reduced $\mathbb{K}$-algebras.

Proof. In (i) note that there exists $f \in \mathcal{R}(U)^{\mathrm{hom}}\backslash 0$ with $\mathcal{Y}_{U,H}(V) = \mathcal{Y}_{U,H}(U_f)$. Then $\mathcal{R}(V) = \mathcal{R}(U_f) \cong \mathcal{R}(U)_f = \mathcal{R}(U)[1/f]$ is finitely generated over $\mathbb{K}$. Assertion (ii) follows from the equation $gr(\mathcal{R}) = \mathbb{X}(H) + \langle \mathrm{degsupp}(\mathcal{R}(U))\rangle$. Assertion (iii) follows from graded integrality of $\mathcal{R}(U)$ and Proposition VI.2.1.5.

Under the conditions of the supplement the sections of $\mathcal{R}$ over an affine cover each are reduced $\mathbb{K}$-algebras by (iii). $\mathcal{R}$ is a subsheaf of the sheaf of rings $\mathcal{K}_{\mathcal{R}}$. Therefore, the 0-preideal $\mathcal{I}$ of $\mathcal{R}$ is sheaf. Now, Remark III.3.0.6 gives radicality of $\mathcal{I}$, i.e. reducedness of $\mathcal{R}$. □

Example VI.4.2.2. Suppose that $\pi\colon \mathcal{O}(L) \to \mathcal{R}$ is a defining CBE onto an invariant Cox sheaf on $H \circlearrowleft X$. Then $\mathcal{R}$ is locally of finite type if and only if $\mathcal{O}(L)$ is so, and in this case the embedding $\mathrm{Spec}(\mathbb{K}[gr(\mathcal{R})]) \circlearrowleft \mathrm{Spec}_X(\mathcal{R}) \to \mathrm{Spec}(\mathbb{K}[\mathbb{X}(H) \oplus L]) \circlearrowleft \mathrm{Spec}_X(\mathcal{O}(L))$ induces a homeomorphism of graded schemes.

Definition VI.4.2.3. A morphism $(\theta, q)\colon \widehat{H} \circlearrowleft \widehat{X} \to H \circlearrowleft X$ of actions is a *characteristic space* if $H \circlearrowleft X$ is of Krull type and (θ, q) is isomorphic as an action over $H \circlearrowleft X$ to $\mathrm{Spec}_{X,H}(\mathcal{R}) \to X$ for some Cox sheaf $\mathcal{R}$ on $H \circlearrowleft X$ which is locally of finite type over $\mathbb{K}$. Equivalently, we require that q is affine and $q_*\mathcal{O}_{\widehat{X},\widehat{H}}$ is isomorphic to a Cox sheaf on $H \circlearrowleft X$. We also say that (θ, q) is a characteristic space of $H \circlearrowleft X$.

Remark VI.4.2.4. For a characteristic space $(\theta, q)\colon \widehat{H} \circlearrowleft \widehat{X} \to H \circlearrowleft X$ and a big $U \in \Omega_{X,H}$ the restriction $(\theta, q_{|q^{-1}(U)})\colon \widehat{H} \circlearrowleft q^{-1}(U) \to H \circlearrowleft U$ is also a characteristic space.

Remark VI.4.2.5. Let $H \circlearrowright X$ be an action with an invariant Cox sheaf which is locally of finite type and let W be the associated graded scheme. By Proposition VI.4.1.17 the canonical equivalence of actions over $H \circlearrowright X$ and graded schemes of finite type over W restricts to an equivalence of characteristic spaces of $H \circlearrowright X$ and graded characteristic spaces of finite type over W.

Proposition VI.4.2.6. *For a good quotient $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ invariant normality of $\widehat{X}$ implies invariant normality of X and if both are present the following are equivalent:*

(i) $\ker(\theta)$ *acts with constant isotropy on a $\widehat{H}$-saturated big open set $\widehat{X}'$ of $\widehat{X}$,*

(ii) *the pullback $q^*\colon \mathrm{WDiv}_H(X) \to \mathrm{WDiv}_{\widehat{H}}(\widehat{X})$ is an isomorphism of partially ordered groups.*

Moreover, for each $\widehat{X}'$ as in (i) the open set $q(\widehat{X}')$ is big in X. Furthermore, if the above are satisfied then $\widehat{X}$ is the special set over X and $|(\theta, q)^*Y|$ is the special set over $Y \in \mathcal{Y}_H(X)$. If additionally $\mathrm{Cl}_H(\widehat{X})$ is a torsion module then X is of affine intersection if and only if $\widehat{X}$ is quasi-affine.

Proof. The equivalence of the two conditions follows from Proposition V.3.2.1 and Proposition VI.3.2.17. Concerning the supplement, note that if $\widehat{X}$ admits an $\widehat{H}$-equivariant open embedding into an affine $\widehat{H}$-action $\overline{X}$ then the former is of affine intersection because the latter is. By Proposition VI.1.5.7 X is of affine intersection.

If X is of affine intersection we fix $U_1, \dots, U_n \in \mathcal{B}_{X,H}$ which cover X. For each i we fix a $f_i \in \mathcal{O}(\widehat{X})^{\mathrm{hom}}$ for which the support of $\mathrm{div}_{\widehat{X},\widehat{H}}(f_i)$ consists precisely of $\mathcal{Y}_{\widehat{H}}(\widehat{X}) \setminus \mathcal{Y}_{\widehat{H}}(q^{-1}(U_i))$. Then there exist $g_1^{(i)}, \dots, g_{m_i}^{(i)} \in \mathcal{O}(\widehat{X})^{\mathrm{hom}}$ which together with f_i^{-1} generate $\mathcal{O}(q^{-1}(U_i))$ as a $\mathbb{K}$-algebra. Let R be the graded $\mathbb{K}$-subalgebra of $\mathcal{O}(\widehat{X})$ generated by all $g_j^{(i)}$ and f_i. Then the corresponding affine $\widehat{H}$-variety $\overline{X}$ contains each $q^{-1}(U_i)$ as a principal open subset. $\square$

As a consequence of Remark VI.4.2.5 and Proposition VI.4.2.6 we obtain the following.

Theorem VI.4.2.7. *A morphism $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ of quasi-torus actions is a characteristic space, if and only if the following hold:*

(i) *(θ, q) is a good quotient,*

(ii) *$\widehat{H} \circlearrowright \widehat{X}$ is of Krull type,*

(iii) $\ker(\theta)$ *acts with constant isotropy on a big $\widehat{H}$-saturated open subset of $\widehat{X}$,*

(iv) *θ restricts to an isomorphism $\widehat{H}_{\widehat{X}} \to H_X$,*

(v) *we have $\mathrm{Cl}_{\widehat{H}}(\widehat{X}) = 0$,*

(vi) *we have $(\mathcal{O}(\widehat{X})^{\mathrm{hom}})^* = (\mathcal{O}(X)^{\mathrm{hom}})^*$.*

In the special case $H = \{1\}$ we obtain a criterion for characteristic spaces of normal prevarieties. The difference to our model from [**4**] is that there, irreducibility and normality are neccessary conditions for a morphism to be a characteristic space. We therefore obtain the following.

Corollary VI.4.2.8. *Let $q\colon \widehat{H} \circlearrowright \widehat{X} \to X$ be a good quotient by the quasi-torus H and suppose that $\widehat{H} \circlearrowright \widehat{X}$ is of Krull type, $\widehat{H}$ acts freely on a big open saturated subset of $\widehat{X}$, and we have $\mathrm{Cl}_{\widehat{H}}(\widehat{X}) = 0$ as well as $(\mathcal{O}(\widehat{X})^{\mathrm{hom}})^* = \mathcal{O}(X)^*$. Then $\widehat{X}$ is irreducible and normal.*

VI.4.3. cones of divisors in terms of characteristic spaces. In this section, we relate the various cones and (stable) base loci of invariant divisors of actions to the corresponding notions of graded schemes. In the following we use the notation

$$\mathrm{Cl}_H(X)_{\mathbb{Q}} := \mathbb{N}^{-1}\,\mathrm{WDiv}_X(X)/\mathbb{N}^{-1}\,\mathrm{PDiv}_H(X) \cong \mathbb{N}^{-1}\,\mathrm{Cl}_H(X)$$

for the vector space of *rational divisor classes*.

DEFINITION VI.4.3.1. Let $H \circlearrowright X$ be of Krull type and let $A \subseteq X$ be H-closed and -irreducible.

(i) $S_{\mathrm{WDiv}_H(X),A}$ resp. $\omega_{\mathrm{WDiv}_H(X),A}$ is the functor on $\Omega_{X,A}$ which assigns to U the submonoid of those D in $\mathrm{WDiv}_H(X)$ resp. $\mathbb{N}^{-1}\,\mathrm{WDiv}_H(X)$ with $D_{|U} \geq 0$ and $D_A = 0_A$. The composition with c_X resp. $\mathbb{N}^{-1}c_X$ is denoted $S_{\mathrm{Cl}_H(X),A}$ resp. $\omega_{\mathrm{Cl}_H(X),A}$.

(ii) For $U \in \Omega_{X,A}$ let $S^{\mathrm{aff}}_{\mathrm{WDiv}_H(X),A}(U)$ resp. $\omega^{\mathrm{aff}}_{\mathrm{WDiv}_H(X),A}(U)$ be the set of those D in $\mathrm{WDiv}_H(X)$ resp. $\mathbb{N}^{-1}\,\mathrm{WDiv}_H(X)$ with $U \setminus |D| \in \mathcal{B}_{X,A}$ and $D_{|U} \geq 0$. Again, the composition with c_X resp. $\mathbb{N}^{-1}c_X$ is denoted $S^{\mathrm{aff}}_{\mathrm{Cl}_H(X),A}$ resp. $\omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),A}$.

REMARK VI.4.3.2. Let $H \circlearrowright X$ be of Krull type, let W be the corresponding graded scheme and consider a H-closed, -irreducible $A \subseteq X$. Then the isomorphisms $\mathrm{WDiv}_H(X) \to \mathrm{WDiv}(W)$ and $\mathrm{Cl}_H(X) \to \mathrm{Cl}(W)$ induce isomorphisms of functors

$$S_{\mathrm{WDiv}_H(X),A} \longrightarrow S_{\mathrm{WDiv}(X),\overline{\{A\}}} \circ t_{|\Omega_{X,H,A}}, \quad S_{\mathrm{Cl}_H(X),A} \longrightarrow S_{\mathrm{Cl}(W),\overline{\{A\}}} \circ t_{|\Omega_{X,H,A}},$$

$$\omega_{\mathrm{WDiv}_H(X),A} \longrightarrow \omega_{\mathrm{WDiv}(X),\overline{\{A\}}} \circ t_{|\Omega_{X,H,A}}, \quad \omega_{\mathrm{Cl}_H(X),A} \longrightarrow \omega_{\mathrm{Cl}(W),\overline{\{A\}}} \circ t_{|\Omega_{X,H,A}}.$$

$$\omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),A} \longrightarrow \omega^{\mathrm{aff}}_{\mathrm{Cl}(W),\overline{\{A\}}} \circ t_{|\Omega_{X,H,A}}$$

as well as canonical isomorphisms

$$S^{\mathrm{aff}}_{\mathrm{WDiv}_H(X),A}(U) \longrightarrow S^{\mathrm{aff}}_{\mathrm{WDiv}(W),\overline{\{A\}}}(t(U)), S^{\mathrm{aff}}_{\mathrm{Cl}_H(X),A}(U) \longrightarrow S^{\mathrm{aff}}_{\mathrm{Cl}(W),\overline{\{A\}}}(t(U)),$$

$$\omega^{\mathrm{aff}}_{\mathrm{WDiv}_H(X),A}(U) \longrightarrow \omega^{\mathrm{aff}}_{\mathrm{WDiv}(W),\overline{\{A\}}}(t(U))$$

for each $U \in \Omega_{X,H,A}$.

DEFINITION VI.4.3.3. Let $H \circlearrowright X$ be of Krull type and let w be an element of $\mathrm{Cl}_H(X)$ resp. $\mathrm{Cl}_H(X)_{\mathbb{Q}}$. The $\mathcal{P}(X)^{op}$-presheaf $\mathrm{Bas}(w)$ resp. $\mathrm{StBas}(w)$ of *base loci* resp. *stable base loci* assigns to $U \in \Omega_{X,H}$ the (closed) set of those $x \in U$ with $w \notin S_{\mathrm{Cl}_H(X),\overline{\{Hx\}}}(U)$ resp. $w \notin \omega_{\mathrm{Cl}_H(X),\overline{\{Hx\}}}(U)$.

REMARK VI.4.3.4. Let $H \circlearrowright X$ be of Krull type and let W be the corresponding graded scheme. Let $\overline{\phi}$ denote the canonical isomorphism $\mathrm{Cl}_H(X) \to \mathrm{Cl}(W)$ resp. $\mathrm{Cl}_H(X)_{\mathbb{Q}} \to \mathrm{Cl}(W)_{\mathbb{Q}}$. Then we have isomorphisms of functors

$$t \circ \mathrm{Bas}(w) = \mathrm{Bas}(\overline{\phi}(w)) \circ t_{|\Omega_{X,H}}, \quad t \circ \mathrm{StBas}(w) = \mathrm{StBas}(\overline{\phi}(v)) \circ t_{|\Omega_{X,H}}.$$

for $w \in \mathrm{Cl}_H(X)$ and $v \in \mathrm{Cl}_H(X)_{\mathbb{Q}}$.

DEFINITION VI.4.3.5. Let $H \circlearrowright X$ be of Krull type. The presheaves $\mathrm{SAmple}_{X,H}$ and $\mathrm{Ample}_{X,H}$ of *semiample* resp. *ample* rational divisor classes are defined via

$$\mathrm{SAmple}_{X,H}(U) := \bigcap_{x \in U} \omega_{\mathrm{Cl}_H(X),\overline{\{Hx\}}}(U), \quad \mathrm{Ample}_{X,H}(U) := \bigcap_{x \in U} \omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),\overline{\{Hx\}}}(U).$$

for $U \in \Omega_{X,H}$. The presheaf $\mathrm{Mov}_{X,H}$ of *moving* rational divisor classes is defined via

$$\mathrm{Mov}_{X,H}(U) := \bigcap_{Y \in \mathcal{Y}_H(U)} \omega_{\mathrm{Cl}_H(X),Y}(U).$$

REMARK VI.4.3.6. Let $H \circlearrowright X$ be of Krull type and let W be the corresponding graded scheme. Then the isomorphism $\mathrm{Cl}_H(X) \to \mathrm{Cl}(W)$ induces isomorphisms of functors

$$\mathrm{SAmple}_{X,H} \longrightarrow \mathrm{SAmple}_W \circ t_{|\Omega_{X,H}}, \quad \mathrm{Ample}_{X,H} \longrightarrow \mathrm{Ample}_W \circ t_{|\Omega_{X,H}},$$

$$\mathrm{Mov}_{X,H} \longrightarrow \mathrm{Mov}_W \circ t_{|\Omega_{X,H}}.$$

PROPOSITION VI.4.3.7. *Let* $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ *be a characteristic space, and write* $pr\colon \mathbb{X}(\widehat{H}) \to \mathbb{X}(\ker(\theta))$ *for the pullback. Let* $\overline{\phi}$ *denote the canonical isomorphism* $\mathrm{Cl}_H(X) \to \mathbb{X}(\ker(\theta))$. *Then the following hold:*

(i) *Let* $A \subseteq X$ *be* H*-closed and -irreducible, and let* $\widehat{A} \subseteq \widehat{X}$ *be the special set over* A. *Then* $\overline{\phi}$ *restricts to an isomorphism*

$$\mathrm{Cl}^{(A)}_{X,H}(X) \cong pr(\deg(((q_*\mathcal{O}_{\widehat{X},\widehat{H}})^{\mathrm{hom}}_A)^*)) = \mathbb{X}(\ker(\theta))_{\widehat{A}}.$$

Moreover, we have a canonical isomorphism of functors

$$S_{\mathrm{Cl}_H(X),A} \longrightarrow q_*pr(S_{\mathbb{X}(\widehat{H}),\widehat{A}}).$$

(ii) $\overline{\phi}$ *restricts to an isomorphism*

$$\mathrm{Pic}_{X,H}(X) \cong \bigcap_{\widehat{H}\widehat{x} \subseteq \widehat{X}} \mathbb{X}(\ker(\theta))_{\widehat{H}\widehat{x}} = \bigcap_{\substack{\widehat{H}\widehat{x} \subseteq \widehat{X} \\ \overline{\widehat{H}\widehat{x}} = \widehat{H}\widehat{x}}} \mathbb{X}(\ker(\theta))_{\widehat{H}\widehat{x}}.$$

PROOF. Assertion (i) is due to Proposition V.2.1.12. Assertion (ii) is a consequence of (i). $\square$

REMARK VI.4.3.8. One may show that $\mathrm{Ample}_H(X)$ is non-empty if and only if $H \circlearrowright X$ allows an equivariant closed embedding into an open toric subvariety of a projective space.

CHAPTER VII

Very neat embeddings into toric characteristic spaces and Cox algebras of finite type

In this chapter, we first explore the relation between quasi-toric prevarieties and $\mathbb{F}_1$-schemes of finite type and show that it preserves Weil divisors, their modules and cones, as well as Cox sheaves and characteristic spaces. Moreover, in Theorem VII.3.1.1 we prove that a characteristic space $\widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ of actions admits a *very neat embedding* into a toric characteristic space if and only if $\mathcal{O}(\widehat{X})$ is of finite type over $\mathbb{K}$. As an application, Theorem VII.3.3.1 gives a neccessary and sufficient criterion for a graded algebra of finite type over $\mathbb{K}$ to be a Cox ring of some quasi-torus action. Together with previous knowledge on Cox rings this provides an integrality and normality criterion for graded rings. Section VII.2 gives a more general view on very neat embeddings of (characteristic spaces of) actions with arbitrary ambient objects.

VII.1. Quasi-toric prevarieties

In the following we proceed to step by step to establish a canonical equivalence between quasi-toric prevarieties over $\mathbb{K}$, quasi-toric graded schemes over $\mathbb{K}$ and over $\mathbb{F}_1$, and integral schemes over $\mathbb{F}_1$. In Section VII.1.2 we turn to the description of invariantly normal quasi-toric prevarieties via convergent one parameter groups and relate the combinatorial schematic functors obtained from the latter to those corresponding to $\mathbb{F}_1$-schemes. Finally, we provide canonical correspondences between (invariant) Cox sheaves on quasi-toric prevarieties and those of $\mathbb{F}_1$-schemes, and between quasi-toric characteristic spaces and graded characteristic spaces over $\mathbb{F}_1$-schemes, see Section VII.1.3. A comprehensive reference on separated (normal) toric varieties is [**11**], the non-separated case has been studied in [**1**].

VII.1.1. Equivalence of quasi-toric prevarieties and $\mathbb{F}_1$-schemes. In this section, we show that the categories of quasi-toric prevarieties over $\mathbb{K}$, quasi-toric graded schemes over $\mathbb{K}$ and over $\mathbb{F}_1$, as well as integral schemes over $\mathbb{F}_1$ are all canonically equivalent to one another.

Remark VII.1.1.1. Let $\mathfrak{K}$ be a category with finite products. Denote by $\mathfrak{gr}(\mathfrak{K})$ the category of group objects of $\mathfrak{K}$ and by $\mathfrak{ac}(\mathfrak{K})$ the category of group object actions on $\mathfrak{K}$-objects. Then the group objects of $\mathfrak{ac}(\mathfrak{K})$ are actions of type $G \circlearrowright H \to H$ where G, H are $\mathfrak{K}$-group objects whose multiplications are compatible with the action. Moreover, the canonical functor sending a $\mathfrak{K}$-group object to its multiplication morphism is full.

Proposition VII.1.1.2. *Sending a quasi-torus multiplication action $H \circlearrowright H$ over $\mathbb{K}$ considered as a group object to its soberification $\mathfrak{t}(H \circlearrowright H)$ is essentially inverse to the functor sending a graded quasi-torus Q of finite type over $\mathbb{K}$ to the group object defined by the quasi-torus action $\mathfrak{ac}_{qt}(\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[gr(\mathcal{O}_Q)]))$.*

Proof. It suffices to note that $\mathrm{Spec}_{\mathrm{gr}} \circ \mathbb{K}[gr(\mathcal{O}_-)]$ defines an auto-equivalence on the category of graded quasi-tori over $\mathbb{K}$. $\square$

Definition VII.1.1.3. A *(quasi-)toric prevariety* is an equivariantly dominant open embedding of (quasi-)torus actions $(\mathrm{id}_H, \imath_H)\colon H \circlearrowright H \to H \circlearrowright Z$ where $H \circlearrowright H$ is the group object in the category of quasi-torus actions determined by the multiplication morphism of H. $H \circlearrowright Z$ is called the *underlying action* and $(\mathrm{id}_H, \imath_H)$ is also called a *(quasi-)toric structure* on $H \circlearrowright Z$.

A *(quasi-)toric morphism* or a morphism of (quasi-)toric prevarieties is a morphism of actions $(\theta, \phi)\colon H \circlearrowright Z \to H' \circlearrowright Z'$ such that $\imath_{H'} \circ \theta = \phi \circ \imath_H$, i.e. the diagram of actions

$$\begin{tikzcd}
H \circlearrowright H \arrow[r, "(\mathrm{id}_H,\imath_H)"] \arrow[d, "(\theta,\theta)"] & H \circlearrowright Z \arrow[d, "(\theta,\phi)"] \\
H' \circlearrowright H' \arrow[r, "(\mathrm{id}_{H'},\imath_{H'})"'] & H' \circlearrowright Z'
\end{tikzcd}$$

commutes.

Remark VII.1.1.4. Quasi-toric morphisms are equivariantly dominant. Moreover, due to Remark VII.1.1.1 to give a morphism of (quasi-)toric prevarieties is the same as to give a pair of morphisms $H \circlearrowright Z \to H' \circlearrowright Z'$ and $H \circlearrowright H \to H' \circlearrowright H'$ of quasi-torus actions resp. group objects thereof such that the resulting diagram commutes.

Remark VII.1.1.5. To give an H-equivariant open embedding of a quasi-torus H into $H \circlearrowright Z$ is the same as to give a *base point* $z_0 \in Z$ such that the orbit map $h \mapsto hz_0$ is an open embedding. Indeed, z_0 is the image of e_H under the embedding. Equivariant dominance of the embedding now means that Z is the closure of Hz_0. Quasi-toric morphisms are then equivariant morphisms that map base points to base points.

Definition VII.1.1.6. Let A denote $\mathbb{F}_1$ or $\mathbb{K}$. A *quasi-toric graded scheme over* A is a degree-preserving open embedding $Q \to W$ of a graded quasi-torus of finite type over A into an integral graded scheme of finite type over A. A morphism in the category $\mathbf{QTGrSch}_A$ of quasi-toric graded schemes A is a pair consisting of a morphism of graded quasi-tori and a morphism of graded schemes such that the resulting diagram commutes.

Remark VII.1.1.7. Morphisms of quasi-toric graded schemes over A are dominant.

Proposition VII.1.1.8. *Sending a quasi-toric graded scheme $Q \to W$ over $\mathbb{K}$ to the morphism of actions associated to* $\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[gr(\mathcal{O}_Q)]) \cong Q \to W$ *defines a functor $\mathfrak{f}$ which is essentially inverse to the functor $\mathfrak{g}$ obtained by applying the soberification functor $\mathfrak{t}$ from Section VI.2.3 to the category of quasi-toric prevarieties over $\mathbb{K}$. Moreover, for a quasi-toric prevariety* $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright Z$ *there is a canonical map*

$$c_Z\colon Z \longrightarrow \mathfrak{t}(Z), \quad z \longmapsto \overline{Hz}$$

so $\mathfrak{t}(H \circlearrowright Z)$ may be considered as an orbit space of $H \circlearrowright Z$ with a graded structure sheaf.

Proof. First note that the quasi-torus action $\mathfrak{ac}_{qt}(\mathrm{Spec}_{\mathrm{gr}}(\mathbb{K}[gr(\mathcal{O}_Q)]))$ is the multiplication action of $\mathrm{Spec}_{\max}(\mathbb{K}[gr(\mathcal{O}_Q)])$ on itself. By Proposition VII.1.1.2 both $\mathfrak{ac}_{qt}$ and $\mathfrak{t}$ preserve structure morphisms of group objects which completes the proof of well-definedness of $\mathfrak{f}$ and $\mathfrak{g}$. □

Remark VII.1.1.9. The category of integral schemes (of finite type) over $\mathbb{F}_1$ with dominant morphisms embedds naturally into the category of integral *graded* schemes (of finite type) over $\mathbb{F}_1$ with dominant morphisms. For each such $\mathbb{F}_1$-scheme X the structure sheaf is canonically $\mathcal{K}(X)^*$-graded via the inclusion $\mathcal{O}_X \setminus 0 \subseteq \mathcal{K}^*$.

REMARK VII.1.1.10. For an integral canonically graded scheme X of finite type over $\mathbb{F}_1$ the canonical morphism $\mathrm{Spec}_{\mathrm{gr},X}(\mathcal{K}_X) \to X$ is a quasi-toric graded scheme over $\mathbb{F}_1$. This defines an equivalence between integral canonically graded schemes of finite type over $\mathbb{F}_1$ with dominant morphisms and quasi-toric graded schemes over $\mathbb{F}_1$.

PROPOSITION VII.1.1.11. *Applying the functor $\mathbb{K}[-]$ to structure sheaves defines a functor $\mathfrak{j}$ from quasi-toric graded schemes over $\mathbb{F}_1$ to quasi-toric graded schemes over $\mathbb{K}$ which is essentially inverse to the functor $\mathfrak{k}$ defined by applying $(-)^{\mathrm{hom}}/\mathbb{K}^*$ to structure sheaves. Moreover, $\mathfrak{k}$ is isomorphic to the functor $\mathfrak{l}$ defined by applying $\mathbb{F}_1[\mathrm{degsupp}(-)]$ to structure sheaves (while keeping the grading groups).*

PROOF. Let $\mathfrak{f}$ denote the functor which sends a graded scheme X over $\mathbb{F}_1$ to $(X, \mathbb{K}[\mathcal{O}_X])$, which by Example IV.1.5.10 is a graded scheme over $\mathbb{K}$. Let $\imath\colon Q \to X$ be a quasi-toric graded scheme over $\mathbb{F}_1$. By Proposition IV.2.3.16 $(Q, \mathbb{K}[\mathcal{O}_Q])$ is a quasi-torus over $\mathbb{K}$ and since $\mathfrak{f}$ preserves open embeddings, $\mathfrak{f}(\imath)$ is a graded quasi-torus over $\mathbb{K}$.

Let $\mathfrak{g}$ denote the functor which sends a topological space (Z, Ω_Z) with a structure sheaf $\mathcal{O}_Z$ of graded $\mathbb{K}$-algebras to $(Z, \Omega_Z, \mathcal{O}_Z^{\mathrm{hom}}/\mathbb{K}^*)$. For a quasi-toric graded scheme $\imath\colon Q \to X$ over $\mathbb{K}$ the canonical isomorphism $\mathbb{K}[\mathcal{O}_Q^{\mathrm{hom}}/\mathbb{K}^*] \to \mathcal{O}_Q$ from Proposition IV.2.3.16 induces an isomorphism

$$\mathbb{K}[\mathcal{O}_X^{\mathrm{hom}}/\mathbb{K}^*] \cong \mathbb{K}[\mathrm{im}(\imath_*)^{\mathrm{hom}}/\mathbb{K}^*] \longrightarrow \mathrm{im}(\imath_*) \cong \mathcal{O}_X.$$

and together these constitute an isomorphism $\tau_\imath$ from $\imath$ to $\mathfrak{f}(\mathfrak{g}(\imath))$. Moreover, Example IV.1.5.10 shows that $(U, \mathcal{O}(U)^{\mathrm{hom}}/\mathbb{K}^*)$ is an affine graded scheme over $\mathbb{F}_1$ for each $U \in \mathcal{B}_Z$. Consequently, $\mathfrak{g}(\imath)$ is indeed a quasi-toric graded scheme over $\mathbb{F}_1$. The isomorphisms $\tau_\imath$ constitute a natural isomorphism between the identity functor and $\mathfrak{j} \circ \mathfrak{k}$.

For a quasi-toric graded scheme $\imath\colon Q \to X$ over $\mathbb{F}_1$ the canonical isomorphism $\mathcal{O}_Q \to \mathbb{K}[\mathcal{O}_Q]^{\mathrm{hom}}/\mathbb{K}^*$ from Proposition IV.2.3.16 gives rise to an isomorphism

$$\mathcal{O}_X \cong \mathrm{im}(\imath_*) \longrightarrow \mathbb{K}[\mathrm{im}(\imath_*)]^{\mathrm{hom}}/\mathbb{K}^* \cong \mathbb{K}[\mathcal{O}_X]^{\mathrm{hom}}/\mathbb{K}^*.$$

This defines an isomorphism $\epsilon_X : \mathfrak{g}(\mathfrak{f}(X)) \to X$ which together with the isomorphism $\mathfrak{g}(\mathfrak{f}(Q)) \to Q$ forms an isomorphism of quasi-toric graded schemes over $\mathbb{F}_1$. These isomorphisms constitute a natural isomorphism from $\mathfrak{k} \circ \mathfrak{j}$ to the identity functor.

For the supplement, denote by $\mathfrak{h}$ the functor sending a graded scheme Z over $\mathbb{K}$ to $(Z, \Omega_Z, \mathbb{F}_1[\mathrm{degsupp}(\mathcal{O}_Z)])$. For a quasi-toric graded scheme $Q \to Z$ the respective degree maps then induce homomorphisms $\mathcal{O}_Q^{\mathrm{hom}}/\mathbb{K}^* \to \mathbb{F}_1[\mathrm{degsupp}(\mathcal{O}_Q)]$ and $\mathcal{O}_Z^{\mathrm{hom}}/\mathbb{K}^* \to \mathbb{F}_1[\mathrm{degsupp}(\mathcal{O}_Z)]$, where the second one is an isomorphism because the first one is an isomorphism due to Proposition IV.2.3.16. This defines an isomorphism of morphisms of graded schemes over $\mathbb{F}_1$ from $\mathfrak{h}(Q) \to \mathfrak{h}(Z)$ to $\mathfrak{g}(Q) \to \mathfrak{g}(Z)$. These now constitute an isomorphism from $\mathfrak{l}$ to $\mathfrak{k}$. $\square$

Composing the above equivalences now gives the following.

THEOREM VII.1.1.12. *Let $\mathbb{K}$ be an algebraically closed field. Then we have canonical equivalences $\mathfrak{f}$ and $\mathfrak{g}$ from quasi-toric prevarieties over $\mathbb{K}$ to the category of integral $\mathbb{F}_1$-schemes of finite type with dominant morphisms.*

REMARK VII.1.1.13. Let $Y = \mathfrak{s}(Z)$ be the scheme over $\mathbb{K}$ associated to the quasi-toric H-prevariety Z. Then the orbit (closure) map gives rise to a natural morphism $Y \to X$ in the category of sesquiad schemes which is affine and surjective and has the initial topology $\Omega_{Z,H}$.

DEFINITION VII.1.1.14. A (quasi-)toric morphism which is also a good quotient is called a *(quasi-)toric good quotient.*

Remark VII.1.1.15. Let $(\theta, \phi)\colon H \circlearrowright Z \to H' \circlearrowright Z'$ be a quasi-toric morphism and let $q\colon X \to X'$ be the corresponding morphism of $\mathbb{F}_1$-schemes. Then (θ, ϕ) is a good quotient if and only if the morphism of canonically graded integral $\mathbb{F}_1$-schemes defined by q is a good quotient, i.e. if $q^*\colon \mathcal{O}_{X'} \to q_* \mathcal{O}_X$ together with the canonical map $\mathcal{K}(X')^* \to \mathcal{K}(X)^*$ is Veronesean.

VII.1.2. Invariantly normal quasi-toric prevarieties. In this section, we show that an invariantly normal quasi-toric prevariety is a product of a normal toric prevariety and a finite abelian group, see Proposition VII.1.2.6. We also relate convergent one parameter groups to the category $\mathfrak{I}$ of combinatorial schematic functors from Section V.3.4 and the latter's connection to $\mathbb{F}_1$-schemes, see Proposition VII.1.2.10. Furthermore, we describe separatedness in terms of convergent one parameter groups, see Proposition VII.1.2.11.

Definition VII.1.2.1. A (quasi-)toric prevariety $(\mathrm{id}_H, \imath_H)\colon H \circlearrowright H \to H \circlearrowright Z$ is *invariantly normal* or *H-normal* or *of Krull type* of Z is invariantly normal resp. $H \circlearrowright Z$ is of Krull type.

Remark VII.1.2.2. By Proposition II.2.7.9 a quasi-toric prevariety is invariantly normal if and only if the corresponding $\mathbb{F}_1$-scheme is normal, i.e. of Krull type.

Remark VII.1.2.3. By Theorem II.2.5.15 a toric prevariety is invariantly normal if and only it is normal.

Sumihiros well-known theorem states that a toric variety, i.e. a *separated* normal toric prevariety, is covered by its affine invariant open subsets [**26**]. The same is true also in the non-separated case:

Proposition VII.1.2.4. [**1**, Prop.1.3] *A normal toric prevariety is covered by its affine invariant open subsets.*

Remark VII.1.2.5. Let H be a quasi-torus and let $t(\mathbb{X}(H))$ denote the torsion elements. Then $\mathbb{T} := V_H(\chi - 1 | \chi \in t(\mathbb{X}(H)))$ is the maximal subtorus and also the unit component of H.

Proposition VII.1.2.6. *Let H be a quasi-torus with unit component $\mathbb{T}$ and let $G \subseteq H$ be a finite subgroup such that H is the direct product of $\mathbb{T}$ and G. For a quasi-toric prevariety $(\mathrm{id}_H, \imath_H)\colon H \circlearrowright H \to H \circlearrowright Z$ set $X := \overline{\imath_H(\mathbb{T})}$. Then the restriction $\mathbb{T} \circlearrowright \mathbb{T} \to \mathbb{T} \circlearrowright X$ is a toric prevariety and G permutes the irreducible components of Z. Moreover, Z is invariantly normal if and only if the canonical quasi-toric morphism $X \times G \to Z$ is an isomorphism and X is invariantly normal.*

Proof. First, note that the pullback $\mathbb{X}(H) \to \mathbb{X}(G)$ restricts to an isomorphism $t(\mathbb{X}(H)) \to \mathbb{X}(G)$. If Z is H-normal then for each $U \in \mathcal{B}_{Z,H}$ the group $t(\mathbb{X}(H))$ is a subgroup of the saturated monoid $M := \mathrm{degsupp}(\mathcal{O}(U))$ and the quotient is again saturated. Thus, $\mathcal{O}(X \cap U) \cong \mathbb{K}[M]/\langle \chi - 1 | \chi \in t(\mathbb{X}(H))\rangle$ is normally graded and X is $\mathbb{T}$-normal. The induced map $M \to M/t(\mathbb{X}(H)) \times \mathbb{X}(G)$ is an isomorphism and gives the desired canonical isomorphism $\mathcal{O}(U) \cong \mathcal{O}(X \cap U) \otimes_{\mathbb{K}} \mathcal{O}(G)$. □

Definition VII.1.2.7. Let $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright X$ be an affine quasi-toric prevariety. Then the monoid $\sigma_{\mathbb{Z}}(X) \subseteq \Lambda(H)$ of *convergent one parameter groups* is the set of those $\lambda \in \Lambda(H)$ which extend to a quasi-toric morphism (λ, ϕ) from the canonical toric variety $\mathbb{K}^* \circlearrowright \mathbb{K}^* \to \mathbb{K}^* \circlearrowright \mathbb{K}$ to $(\mathrm{id}_H, \imath)$.

Remark VII.1.2.8. Let $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright X$ be a quasi-toric prevariety and consider $\chi \in \mathbb{X}(H)$ and $\lambda \in \Lambda(H)$. Let $n \in \mathbb{Z}$ be the element corresponding to $\chi \circ \lambda \in \mathbb{X}(\mathbb{K}^*)$ under the canonical isomorphism $\mathbb{Z} \to \mathbb{X}(\mathbb{K}^*)$. Let $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ be the element corresponding to $\chi \circ \lambda \in \Lambda(\mathbb{K}^*)$ under the canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \to \Lambda(\mathbb{K}^*)$.

Then $n = \mu(1)$ holds and thus we have $n \geq 0$ if and only if $\mu(\mathbb{N}_0) \subseteq \mathbb{N}_0$, which in turn holds if and only if there exists a $\mathbb{K}^*$-equivariant morphism $\psi\colon \mathbb{K} \to \mathbb{K}$ with $\imath_{\mathbb{K}^*} \circ \chi \circ \lambda = \psi \circ \imath_{\mathbb{K}^*}$.

Proposition VII.1.2.9. *Let $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright X$ be a quasi-toric prevariety. Then the following hold:*

(i) *If $(\lambda, \phi)\colon \mathbb{K}^* \circlearrowright \mathbb{K} \to H \circlearrowright X$ is a morphism of quasi-torus actions such that $\phi_{|\mathbb{K}^*} = \imath \circ \lambda$ then for each $\chi \in \mathrm{degsupp}(\mathcal{O}(X))$ the unique $f \in \mathcal{O}(X)_\chi$ with $\imath_X^*(f) = \chi$ sends $\phi(0)$ to 1 if $\chi \circ \lambda$ is constant and to 0 otherwise. In particular, ϕ is closed. Consequently, if X is affine then a convergent $\lambda \in \Lambda(H)$ allows only one ϕ as required in the definition.*

(ii) *We have $\sigma_{\mathbb{Z}}(X) \subseteq \mathrm{degsupp}(\mathcal{O}(X))^\vee$ in $\Lambda(H)$ with respect to the canonical pairing $\langle \cdot, \cdot \rangle\colon \mathbb{X}(H) \times \Lambda(H) \to \mathbb{Z}$. If X is affine then this is an equality.*

Proof. Fix $\lambda \in \Lambda(H)$ and first suppose that there exists a morphism of quasi-torus actions $(\lambda, \phi)\colon \mathbb{K}^* \circlearrowright \mathbb{K} \to H \circlearrowright X$ with $\phi \circ \imath_{\mathbb{K}^*} = \imath \circ \lambda$. For $\chi \in \mathrm{degsupp}(\mathcal{O}(X))$ there exists a unique $f_\chi \in \mathcal{O}(X)_\chi$ with $\imath_{\mathbb{K}^*} \circ \chi = f_\chi \circ \imath$ and we have

$$\imath_{\mathbb{K}^*} \circ \chi \circ \lambda = f_\chi \circ \imath \circ \lambda = (f_\chi \circ \phi) \circ \imath_{\mathbb{K}^*}.$$

By Remark VII.1.2.8 we have $\langle \chi, \lambda \rangle \geq 0$, i.e. $\lambda \in \mathrm{degsupp}(\mathcal{O}(X))^\vee$, and the first part of (ii) is shown. Furthermore, if $\chi \circ \lambda$ is constant then $f_\chi(\phi(a)) = (\chi \circ \lambda)(a) = 1$ holds for each $a \in \mathbb{K}^*$, and continuity gives $\mathbb{K} = \overline{\mathbb{K}^*} \subseteq (f_\chi \circ \phi)^{-1}(1)$. If $\chi \circ \lambda$ is non-constant then $f \circ \phi \in \mathcal{O}(\mathbb{K})_{\chi \circ \lambda}$ is a non-constant monomial and hence $f(\phi(0)) = 0$, which shows (i).

For the remainder of (ii) suppose that X is affine and $\langle \chi, \lambda \rangle \geq 0$ holds for each $\chi \in \mathrm{degsupp}(\mathcal{O}(X))$, which by Remark VII.1.2.8 means there exists a $\mathbb{K}^*$-equivariant $\psi^\chi\colon \mathbb{K} \to \mathbb{K}$ with $\imath_{\mathbb{K}^*} \circ \chi \circ \lambda = \psi^\chi \circ \imath_{\mathbb{K}^*}$. Let $\chi_1, \ldots, \chi_n \in \mathrm{degsupp}(\mathcal{O}(X))$ be a generating subset and let $f_i \in \mathcal{O}(X)_{\chi_i}$ be the element with $\imath_{\mathbb{K}^*} \circ \chi_i = f_i \circ \imath$. Then we have a closed embedding of actions $\jmath\colon X \to \mathbb{K}^n, x \mapsto (f_1(x), \ldots, f_n(x))$ with accompanying map $H \to (\mathbb{K}^*)^n, h \mapsto (\chi_1(h), \ldots, \chi_n(h))$. Now, the morphism $(\psi^{\chi_1}, \ldots, \psi^{\chi_n})\colon \mathbb{K} \to \mathbb{K}^n$ together with $(\chi_1, \ldots, \chi_n)\colon \mathbb{K}^* \to (\mathbb{K}^*)^n$ induces a morphism $(\lambda, \psi)\colon \mathbb{K}^* \circlearrowright \mathbb{K} \to H \circlearrowright X$ with $\phi \circ \imath_{\mathbb{K}^*} = \imath \circ \lambda$. $\square$

Proposition VII.1.2.10. *Sending a quasi-toric prevariety $H \circlearrowright Z$ to $([\mathcal{B}_{Z,H} \ni U \mapsto \sigma_{\mathbb{Z}}(U)], \Lambda(H))$ defines a functor to the category $\mathfrak{I}$ of combinatorial schematic functors from Section V.3.4. Restricting this functor to toric prevarieties is isomorphic to the composition of the equivalence with $\mathbb{F}_1$ of finite type and the functor sending $(X, \mathcal{O}_X)$ to $((\mathcal{O}_{X|\mathcal{B}_X \setminus \{\emptyset\}} \setminus \{0\})^\vee, \mathrm{Hom}_{\mathbb{Z}}(\mathcal{O}_{X,\xi} \setminus \{0\}, \mathbb{Z}))$ from Corollary V.3.4.9.*

Proposition VII.1.2.11. *Let $H \circlearrowright H \to H \circlearrowright Z$ be an invariantly normal quasi-toric prevariety of affine intersection. Then the following are equivalent:*

(i) *Z is separated.*

(ii) *For all $U, V \in \mathcal{B}_{H,Z} \setminus \{\emptyset\}$ we have $\sigma_{\mathbb{Z}}(U \cap V) = \sigma_{\mathbb{Z}}(U) \cap \sigma_{\mathbb{Z}}(V)$.*

(iii) *For all $U, V \in \mathcal{B}_{H,Z} \setminus \{\emptyset\}$, the monoids $\sigma_{\mathbb{Z}}(U)$ and $\sigma_{\mathbb{Z}}(V)$ are separable by an element of $\mathbb{X}(H)$.*

(iv) *$\omega_{\mathrm{Cl}_H(Z), \overline{Hx}}(Z)^\circ \cap \omega_{\mathrm{Cl}_H(Z), \overline{Hy}}(Z)^\circ$ is non-empty for all $x, y \in Z$.*

Proof. If Z is separated, then $U \cap V$ is principal in $U, V \in \mathcal{B}_{Z,H}$ because it is affine. For $f \in \mathcal{O}(U)^{\mathrm{hom}}$ and $g \in \mathcal{O}(V)^{\mathrm{hom}}$ with $U \cap V = U_f = V_g$ we then have

$$\begin{aligned}\deg(\mathcal{O}(U)^{\mathrm{hom}} \setminus 0) - \deg(f) &= \deg(\mathcal{O}(V)^{\mathrm{hom}} \setminus 0) - \deg(g) = \deg(\mathcal{O}(U \cap V)^{\mathrm{hom}} \setminus 0) \\ &= \deg(\mathcal{O}(U)^{\mathrm{hom}} \setminus 0) + \deg(\mathcal{O}(V)^{\mathrm{hom}} \setminus 0).\end{aligned}$$

By Remark I.1.3.26 there exists $w \in \deg(\mathcal{O}(U)^{\mathrm{hom}} \setminus 0) \cap -\deg(\mathcal{O}(V)^{\mathrm{hom}} \setminus 0)$ with

$$\begin{aligned}\deg(\mathcal{O}(U)^{\mathrm{hom}} \setminus 0) - w &= \deg(\mathcal{O}(V)^{\mathrm{hom}} \setminus 0) + w = \deg(\mathcal{O}(U \cap V)^{\mathrm{hom}} \setminus 0) \\ &= \deg(\mathcal{O}(U)^{\mathrm{hom}} \setminus 0) + \deg(\mathcal{O}(V)^{\mathrm{hom}} \setminus 0).\end{aligned}$$

Due to Remark I.1.3.22 dualizing now gives $\sigma_{\mathbb{Z}}(U \cap V) = \sigma_{\mathbb{Z}}(U) \cap \sigma_{\mathbb{Z}}(V)$ and we have shown that w separates $\sigma_{\mathbb{Z}}(U)$ and $\sigma_{\mathbb{Z}}(V)$.

If (iii) holds then for $U, V \in \mathcal{B}_{Z,H}$ there exists $w \in \deg(\mathcal{O}(U)^{\text{hom}} \setminus 0) \cap -\deg(\mathcal{O}(V)^{\text{hom}} \setminus 0)$ with

$$\begin{aligned}\deg(\mathcal{O}(U)^{\text{hom}} \setminus 0) - w &= \deg(\mathcal{O}(V)^{\text{hom}} \setminus 0) + w \\ &= \deg(\mathcal{O}(U)^{\text{hom}} \setminus 0) + \deg(\mathcal{O}(V)^{\text{hom}} \setminus 0) \\ &\subseteq \deg(\mathcal{O}(U \cap V) \setminus 0).\end{aligned}$$

Consequently, we have $U_w = V_{-w}$ and both sets are contained in $U \cap V$. Since w is a unit in $\deg(\mathcal{O}(U \cap V) \setminus 0)$ the converse inclusion also holds. Applying $(-)^\vee$ now realizes $\sigma_{\mathbb{Z}}(U \cap V)$ as the face of $\sigma_{\mathbb{Z}}(U)$ resp. $\sigma_{\mathbb{Z}}(V)$ defined by w.

If (ii) holds then $\sigma_{\mathbb{Z}}(U) \cap \sigma_{\mathbb{Z}}(V) = \sigma_{\mathbb{Z}}(U \cap V)$ is a face of $\sigma_{\mathbb{Z}}(U)$ and $\sigma_{\mathbb{Z}}(V)$ for $U, V \in \mathcal{B}_{Z,H}$ which by Lemma I.1.3.25 means

$$\deg(\mathcal{O}(U \cap V)^{\text{hom}} \setminus 0) = \deg(\mathcal{O}(U)^{\text{hom}} \setminus 0) + \deg(\mathcal{O}(V)^{\text{hom}} \setminus 0).$$

Lastly, let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{Z} \to H \circlearrowright Z$ be a quasi-toric characteristic space and let $Q\colon \mathbb{N}^{-1}\mathbb{X}(\widehat{H}) \to \mathbb{N}^{-1}(\mathbb{X}(\ker(\theta)))$ denote the localized pullback. Then $\widehat{H} \circlearrowright \widehat{Z}$ is an open toric subvariety of $\overline{Z} := \operatorname{Spec}_{\max}(\mathcal{O}(\widehat{Z}))$ and for $U \in \mathcal{B}_{Z,H}$ we have

$$\begin{aligned}\omega_{\operatorname{Cl}_H(Z), \overline{O_U}}(Z) &= Q(\omega_{\mathbb{X}(\widehat{H}), \overline{O_{q^{-1}(U)}}}(\widehat{Z})) \\ &= Q(\mathbb{N}^{-1} \deg(\mathcal{O}(\overline{Z})^{\text{hom}} \cap (\mathcal{O}(q^{-1}(U))^{\text{hom}})^*)).\end{aligned}$$

Due to Proposition V.3.3.3 the relative interiors of these cones intersect pairwise non-trivially if and only if Z is separated. $\square$

Remark VII.1.2.12. Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{Z} \to H \circlearrowright Z$ be a quasi-toric good quotient. Then Z is separated if and only if for all maximal $U, V \in \mathcal{B}_{\widehat{Z},\widehat{H}}$ the intersection $U \cap V$ is affine, and $\sigma(U)$ and $\sigma(V)$ are separable by an element of $(\theta, q)^*\mathbb{X}(H)$.

VII.1.3. Correspondence of Cox sheaves of toric prevarieties and $\mathbb{F}_1$-schemes. In this section, we relate (invariant) Weil divisors, class groups, divisorial algebras and Cox sheaves on toric prevarieties to those of the corresponding $\mathbb{F}_1$-schemes, see Proposition VII.1.3.3. Likewise, we establish an equivalence between *quasi-toric* characteristic spaces over a quasi-toric prevariety $H \circlearrowright Z$ and graded characteristic spaces over the corresponding $\mathbb{F}_1$-scheme, see Proposition VII.1.3.7. We close the section with an observation on the behaviour of smoothness under the correspondence of $\mathbb{F}_1$-schemes and quasi-toric prevarieties.

Remark VII.1.3.1. Let X be an integral scheme of finite type over $\mathbb{F}_1$ and let $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright Z$ be the associated quasi-toric prevariety over $\mathbb{K}$.

(i) Let $\alpha\colon \Omega_{Z,H} \to \Omega_X$ denote the canonical bijection (induced by soberification) and let $\operatorname{orb}\colon Z \to X$ be the canonical map. Then composition with α defines a functor from $\mathcal{P}r\mathcal{S}h_{\mathfrak{D}}(X, \Omega_X)$ to $\mathcal{P}r\mathcal{S}h_{\mathfrak{D}}(Z, \Omega_{Z,H})$ which is inverse to orb_* and isomorphic to orb^{-1}. Here, all functors preserve the sheaf-property.

(ii) We have $\mathcal{O}_{Z,H} = \mathbb{K}[\mathcal{O}_X \circ \alpha]$ and this equation induces an isomorphism $\mathcal{K}_{Z,H} \cong \mathbb{K}[\mathcal{K}_X \circ \alpha]$.

(iii) Z is invariantly normal if and only if X is of Krull type.

Remark VII.1.3.2. Let $\phi\colon X \to X'$ be a dominant morphism of schemes of finite and Krull type over $\mathbb{F}_1$ and let $(\theta, \eta)\colon H \circlearrowright Z \to H' \circlearrowright Z'$ be the corresponding quasi-toric morphism. Then the following hold:

(i) the canonical map $\operatorname{orb}\colon Z \to X$ induces a bijection

$$\mathcal{Y}_H(Z) \longrightarrow \mathcal{Y}(X), \quad Y \longmapsto \overline{\{\operatorname{orb}(Y)\}}.$$

(ii) For each $Y \in \mathcal{Y}_H(Z)$ and $D = \overline{\{\mathrm{orb}(Y)\}} \in \mathcal{Y}(X)$ the canonical homomorphism $\mathcal{K}_X^* \to \mathrm{orb}_*(\mathcal{K}_{Z,H}^{\mathrm{hom}})^* \to \mathrm{orb}_*\mathbb{Z}^{(Y)} = \mathbb{Z}^{(D)}$ equals the valuation corresponding to D.

(iii) The bijection of (i) gives rise to isomorphisms $\mathrm{WDiv}_X \to \mathrm{orb}_* \mathrm{WDiv}_{Z,H}$, $\mathrm{PDiv}_X \to \mathrm{orb}_* \mathrm{PDiv}_{Z,H}$ and $\mathrm{Cl}_X \to \mathrm{orb}_* \mathrm{Cl}_{Z,H}$ which we all denote by orb^*. These are compatible with the pullbacks given by ϕ and (θ, η) (so far as they are well-defined), i.e. we have $\phi_* \mathrm{orb}^* \circ \phi^* = \mathrm{orb}'_*(\theta,\eta)^* \circ \mathrm{orb}'^*$.

(iv) For each $D \in \mathrm{WDiv}(X)$ the isomorphism from Remark VII.1.3.1 restricts to an isomorphism $\mathrm{orb}_*\mathcal{O}_{Z,H}(\mathrm{orb}_X^*(D)) \cong \mathbb{K}[\mathcal{O}_X(D)]$ and these give rise to an isomorphism $\mathrm{orb}_*\mathcal{O}_{Z,H}(\mathrm{WDiv}_{Z,H}(Z)) \cong \mathbb{K}[\mathcal{O}_X(\mathrm{WDiv}_X(X))]$.

(v) For $v \in \mathrm{Cl}(X)$ and $w \in \mathrm{Cl}(X)_{\mathbb{Q}}$ and each $U \in \Omega_X$ we have

$$\mathrm{Bas}(\mathrm{orb}^*(v))(\mathrm{orb}^{-1}(U)) = \mathrm{orb}^{-1}(\mathrm{Bas}(v)(U)),$$
$$\mathrm{StBas}(\mathrm{orb}^*(w))(\mathrm{orb}^{-1}(U)) = \mathrm{orb}^{-1}(\mathrm{StBas}(w)(U)).$$

(vi) Let $B \subseteq X$ be closed and H-irreducible, let G_Z denote $\mathrm{WDiv}_H(Z)$ or $\mathrm{Cl}_H(Z)$ and correspondingly, let G_X denote $\mathrm{WDiv}(X)$ or $\mathrm{Cl}(X)$. For each $U \in \Omega_X$ we then have

$$S_{G_Z,B}(\mathrm{orb}^{-1}(U)) = \mathrm{orb}_X^* \circ S_{G_X,\overline{\mathrm{orb}(B)}}(U),$$
$$S^{\mathrm{aff}}_{G_Z,B}(\mathrm{orb}^{-1}(U)) = \mathrm{orb}_X^* \circ S^{\mathrm{aff}}_{G_X,\overline{\mathrm{orb}(B)}}(U).$$

Consequently, we have

$$\mathrm{orb}_* \mathrm{WDiv}_{Z,H,\geq 0} = \mathrm{orb}_X^* \circ \mathrm{WDiv}_{X,\geq 0},$$
$$\mathrm{orb}_*\mathrm{Mov}_{Z,H} = \mathrm{orb}_X^* \circ \mathrm{Mov}_X,$$
$$\mathrm{orb}_*\mathrm{SAmple}_{Z,H} = \mathrm{orb}_X^* \circ \mathrm{SAmple}_X,$$
$$\mathrm{orb}_*\mathrm{Ample}_{Z,H} = \mathrm{orb}_X^* \circ \mathrm{Ample}_X.$$

Proposition VII.1.3.3. *Let X be a scheme of finite and Krull type over $\mathbb{F}_1$ and let $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright Z$ be the associated H-normal quasi-toric prevariety over $\mathbb{K}$. Then sending $\mathcal{R}$ to $\mathbb{K}[\mathcal{R} \circ \alpha]$ endowed with the canonical $\mathcal{R}_X^*$-grading defines a functor $\mathfrak{f}$ from Cox sheaves on X to invariant Cox sheaves on Z which is full and an isomorphism onto its image category* $\mathrm{im}(\mathfrak{f})$*. In particular, all invariant Cox sheaves on Z are of finite type over $\mathbb{K}$. Conversely, sending $\mathcal{S}$ to* $\mathrm{orb}_*\mathcal{S}^{\mathrm{hom}}/\mathbb{K}^*$ *endowed with the canonical grading by* $\mathrm{Cl}_H(Z)$ *defines a functor $\mathfrak{g}$ from invariant Cox sheaves on Z to Cox sheaves on X which on* $\mathrm{im}(\mathfrak{f})$ *is isomorphic to the inverse $\mathfrak{f}^{-1}$ from Remark III.3.0.12.*

Proof. For well-definedness of $\mathfrak{f}$ consider a Cox sheaf $\mathcal{R}$ on X with defining CBE $\pi\colon \mathcal{O}_X(\mathrm{WDiv}(X)) \to \mathcal{R}, c_X\colon \mathrm{WDiv}(X) \to \mathrm{Cl}(X)$. Then with respect to the canonical gradings by $\mathcal{O}_X(\mathrm{WDiv}(X))_X^* = \mathcal{K}(X)^* \oplus \mathrm{WDiv}(X)$ resp. $\mathcal{R}_X^*$ and the restriction ψ of π_X as the accompanying map π remains a CBE. For $D \in \mathrm{WDiv}(X)$ with $\pi_X(\chi^D) \in \mathcal{K}(X)^*$ we have $[D] = [\mathrm{div}_{\mathcal{R},X}(\pi_X(\chi^D))] = [0]$ as required. Using the isomorphism from Remark VII.1.3.2 we obtain the desired CBE $\mathcal{O}_{Z,H}(\mathrm{WDiv}_H(Z)) \to \mathbb{K}[\mathcal{R} \circ \alpha]$ with accompanying map ψ.

For well-definedness of $\mathfrak{g}$ let $\pi\colon \mathcal{O}_{Z,H}(\mathrm{WDiv}_H(Z)) \to \mathcal{S}, \psi$ be a defining CBE for a Cox sheaf on $\Omega_{Z,H}$. Let $p\colon gr(\mathcal{R}) \to \mathrm{Cl}_H(Z) \cong \mathrm{Cl}(X)$ be the canonical map. Then (π, p) remains a CBE and using the canonical isomorphism from Remark VII.1.3.2 we obtain a CBE

$$\mathcal{O}_X(\mathrm{WDiv}(X)) \cong \mathrm{orb}_*\mathcal{O}_{Z,H}(\mathrm{WDiv}_H(Z))^{\mathrm{hom}}/\mathbb{K}^* \longrightarrow \mathrm{orb}_*\mathcal{S}^{\mathrm{hom}}/\mathbb{K}^*$$

with accompanying map $c_X\colon \mathrm{WDiv}(X) \cong \mathrm{WDiv}_H(Z) \xrightarrow{p \circ \psi} \mathrm{Cl}(X)$ as desired.

Concerning fullness let $\mathcal{S}$ be a Cox sheaf on $\Omega_{Z,H}$ and let $\pi\colon \mathcal{O}_{Z,H}(L) \to \mathcal{S}$ be a defining CBE where $L := \mathrm{WDiv}^H(Z)$. Let $D_1, \ldots, D_k$ be a basis of L for which there exist $n_1, \ldots, n_k \in \mathbb{N}$ such that $n_1 D_1, \ldots, n_k D_k$ form a basis of $\mathrm{PDiv}^H(Z)$. For each i there exists a unique $a_i \chi^{u_i + n_i D_i} \in \mathcal{O}_{Z,H}(L)(Z)_{n_i D_i}$ (in the notation from Remark VII.1.3.2) with image 1 under π_Z and we may choose a n_i-the root $b_i \in \mathbb{K}^*$ of a_i. The invariant Cox sheaf with kernel relation $\chi^{u_i+n_i D_i} \sim 1$ is in the image of $\mathfrak{f}$ and is isomorphic to $\mathcal{S}$ via the isomorphism induced by the assignment $\chi^{D_i} \mapsto b_i \chi^{D_i}$. $\square$

Definition VII.1.3.4. Let $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright Z$ be a quasi-toric prevariety. A *quasi-toric prevariety over* $(\mathrm{id}_H, \imath)$ is a morphism $(\mathrm{id}_{\widehat{H}}, \widehat{\imath}) \to (\mathrm{id}_H, \imath)$ of quasi-toric prevarieties over $\mathbb{K}$. A *morphism* of quasi-toric prevarieties over $(\mathrm{id}_H, \imath)$ is a morphism of the underlying quasi-toric prevarieties over $\mathbb{K}$ such that the resulting triangle of morphism of quasi-toric prevarieties over $\mathbb{K}$ commutes.

Remark VII.1.3.5. For a quasi-toric prevariety $(\mathrm{id}_H, \imath)\colon H \circlearrowright H \to H \circlearrowright Z$ and its corresponding $\mathbb{F}_1$-scheme X we have an induced equivalence of quasi-toric prevarieties over $(\mathrm{id}_H, \imath)$ and $\mathbb{F}_1$-schemes over X.

Definition VII.1.3.6. A quasi-toric prevariety over $H \circlearrowright Z$ whose structure morphism is a characteristic space of actions is called a *quasi-toric characteristic space* over $H \circlearrowright Z$.

Proposition VII.1.3.7. *Let $q\colon \widehat{X} \to X$ be a morphism of integral schemes of finite type over $\mathbb{F}_1$ and let $(\theta_Z, q_Z)\colon \widehat{H} \circlearrowright \widehat{Z} \to H \circlearrowright Z$ be the corresponding morphism of quasi-toric prevarieties. Then (θ_Z, q_Z) is a characteristic space if and only if q becomes a graded characteristic space once $\mathcal{O}_{\widehat{X}}$ is endowed with the canonical grading by $\mathcal{K}(\widehat{X})^*/\mathcal{K}(X)^*$. Moreover, the canonical equivalence induces an equivalence between graded characteristic spaces over X and quasi-toric characteristic spaces over $H \circlearrowright H \to H \circlearrowright Z$.*

Proof. By Remark VII.1.3.1 $\widehat{X}$ and X are of Krull type if and only if $\widehat{H} \circlearrowright \widehat{Z}$ and $H \circlearrowright Z$ are, which we from now on assume to be the case. By Remark VII.1.3.2 the pullback $\mathrm{WDiv}(X) \to \mathrm{WDiv}(\widehat{X})$ is an isomorphism of partially ordered groups if and only if the pullback $\mathrm{WDiv}_H(Z) \to \mathrm{WDiv}_{\widehat{H}}(\widehat{Z})$ is one, and we have an isomorphism $\mathrm{Cl}(\widehat{X}) \cong \mathrm{Cl}_{\widehat{H}}(\widehat{Z})$. Moreover, the homomorphism $(\mathcal{O}(Z)^{\mathrm{hom}})^* \to (\mathcal{O}(\widehat{Z})^{\mathrm{hom}})^*$ is an equality if and only if $\mathcal{O}(X)^* \to \mathcal{O}(\widehat{X})^*$ is an equality. The degree map of $\mathcal{K}(\widehat{X})$ has kernel $\mathcal{K}(X)^*$ if and only if $\mathcal{O}_X \to q_* \mathcal{O}_{\widehat{X}}$ is Veronesean, i.e. if and only if $\mathcal{O}_{Z,H} \to (q_Z)_* \mathcal{O}_{\widehat{Z},\widehat{H}}$ is Veronesean, and it is surjective if and only if the degree support set condition of graded characteristic spaces is satisfied. Thus, q (with $gr(\mathcal{O}_X) = 0$ and $gr(\mathcal{O}_{\widehat{X}})$ as stated) satisfies the conditions of Theorem V.3.1.4 if and only if (θ_Z, q_Z) satisfies the conditions of Theorem VI.4.2.7.

The induced functor in one direction sends a graded characteristic space over X to the underlying of non-graded $\mathbb{F}_1$-scheme over X and then to the latter's corresponding quasi-toric prevariety over Z. In the other direction we send a quasi-toric characteristic space over $H \circlearrowright Z$ to the associated $\mathbb{F}_1$-scheme $q\colon \widehat{X} \to X$ over X and endow $\mathcal{O}_{\widehat{X}}$ with the canonical grading by $\mathcal{K}(\widehat{X})^*/\mathcal{K}(X)^* \cong \mathrm{Cl}(X)$ using the isomorphism provided by Remark V.2.1.13. $\square$

Remark VII.1.3.8. Consider a quasi-toric characteristic space $\widehat{H} \circlearrowright \widehat{Z} \to H \circlearrowright Z$. Then by Proposition V.2.2.10 $\widehat{H}$ is a torus if and only if H is a torus.

We close the section with observations on regularity of quasi-toric prevarieties and their corresponding $\mathbb{F}_1$-schemes.

Definition VII.1.3.9. A $\Omega_{Z,H}$-irreducible set $Y \subseteq Z$, e.g. an orbit, is said to be *regular*, if its stalk is regularly graded in the sense of Definition II.2.1.13.

Remark VII.1.3.10. Consider an integral $\mathbb{F}_1$-scheme X, its corresponding quasi-toric prevariety $H \circlearrowleft Z$ and a point $z \in Z$. Then $\overline{Hz}$ is regular if and only if $\overline{\operatorname{orb}(z)}$ is regular in X.

Proposition VII.1.3.11. *A toric prevariety $\mathbb{T} \circlearrowleft Z$ is smooth if and only if the corresponding $\mathbb{F}_1$-scheme X is regular.*

Proof. $U \in \mathcal{B}_X$ is regular if and only if its closed point p is regular, which by Remark IV.2.1.19 is equivalent to factoriality of $\mathcal{O}_{X,p} = \mathcal{O}(U)$. This holds if and only if $\mathcal{O}(\operatorname{orb}^{-1}(U))$ is a principal localization of a polynomial ring by a monomial, i.e. if $\operatorname{orb}^{-1}(U)$ is smooth. □

Corollary VII.1.3.12. *A quasi-toric prevariety Z is smooth if and only if all of its closed orbits are smooth.*

VII.2. Embedded characteristic spaces and Cox data

In this section we discuss closed embeddings of characteristic spaces of actions such that the pullback of invariant Weil divisors preserves invariant primality, the invariant class groups coincide and the pullback of invariant Cox rings is surjective. In Section VII.2.1 we show that in the case of such *irredundant very neat* embeddings, Picard group and various cones of divisor classes coincide for the embedded and its ambient object. In Section VII.2.2 we also study commutative squares made up of characteristic spaces and very neat embeddings, which we call very neat embeddings of characteristic spaces. These turn out to be determined by the ambient characteristic space and some global data from the other morphisms, see Theorem VII.2.2.8.

VII.2.1. Basic properties of very neat embeddings. Generalizing the embeddings of prevarieties into toric varieties from [**17**] we consider arbitrary actions of Krull type for the moment, with no restrictions on their Cox algebras as of yet. We list first observations and introduce the notion of *irredundance* of a very neat embedding.

Definition VII.2.1.1. A morphism $(\jmath, \imath)\colon H \circlearrowleft X \to G \circlearrowleft Z$ of actions of Krull type is a *very neat embedding* if

(i) $\imath$ is affine,
(ii) the induced morphism $\imath\colon (X, \Omega_{X,H}) \to (Z, \Omega_{Z,G})$ of topological spaces is dominant,
(iii) the pullback $(\jmath, \imath)^*_Z\colon \operatorname{WDiv}_G(Z)_{\geq 0} \to \operatorname{WDiv}_H(X)_{\geq 0}$ is injective and preserves primality, and each $Y \in \mathcal{Y}_H(X)$ satisfies $\operatorname{Cl}_{G,\overline{\imath(Y)}}\Omega_{Z,G} = 0$,
(iv) the pullback $(\jmath, \imath)^*_Z\colon \operatorname{Cl}_G(Z) \to \operatorname{Cl}_H(X)$ is an isomorphism,
(v) the induced homomorphism $\imath^*_Z\colon \mathcal{O}_{Z,G}(D)(Z) \to \mathcal{O}_{X,H}((\jmath, \imath)^* D)(X)$ is surjective for each $D \in \operatorname{WDiv}_G(Z)$.

$G \circlearrowleft Z$ is then called the *ambient space* of $H \circlearrowleft X$.

A *morphism* from a very neat embedding $(\jmath, \imath)\colon H \circlearrowleft X \to G \circlearrowleft Z$ to a very neat embedding $(\jmath', \imath')\colon H' \circlearrowleft X' \to G' \circlearrowleft Z'$ is pair of morphisms $H \circlearrowleft X \to H' \circlearrowleft X'$ and $G \circlearrowleft Z \to G' \circlearrowleft Z'$ such that the resulting diagram commutes.

Proposition VII.2.1.2. *For a very neat embedding $(\jmath, \imath)\colon H \circlearrowleft X \to G \circlearrowleft Z$ the induced homomorphism $\imath^*_U\colon \mathcal{O}_{Z,G}(D)(U) \to \mathcal{O}_{X,H}((\jmath, \imath)^* D)(\imath^{-1}(U))$ is surjective for each $U \in \Omega_{Z,G}$ and $D \in \operatorname{WDiv}_G(Z)$. In particular, $\imath$ is a closed embedding.*

Proof. Let S be the submonoid of all χ^D where $D \in \mathrm{WDiv}_G(Z)$ satisfies $|D| \subseteq Z \setminus U$. Then we have a canonical epimorphism

$$\begin{aligned}\imath_U^*\colon \mathcal{O}_{Z,G}(\mathrm{WDiv}_G(Z))(U) &\xrightarrow{\sim} S^{-1}\mathcal{O}_{Z,G}(\mathrm{WDiv}_G(Z))(Z)\\ &\longrightarrow \imath_Z^*(S)^{-1}\mathcal{O}_{X,H}((\jmath,\imath)^*\,\mathrm{WDiv}_G(Z))(X)\\ &\xrightarrow{\sim} \mathcal{O}_{X,H}((\jmath,\imath)^*\,\mathrm{WDiv}_G(Z))(\imath^{-1}(U)).\end{aligned}$$

Lastly, note that the assertion holds in particular for affine U and $D = 0$ which means that $\imath$ is a closed embedding. □

Proposition VII.2.1.3. *Compositions of very neat embeddings are very neat embeddings.*

Proof. Let $(\jmath,\imath)\colon H \circlearrowright X \to H' \circlearrowright X'$ and $(\jmath',\imath')\colon H' \circlearrowright X' \to H'' \circlearrowright X''$ be very neat embeddings. Under $\imath$ and $\imath'$ and hence also under their composition invariant closures of images of invariantly prime divisors are of codimension at most one. Thus, we have $(\jmath,\imath)_{X'}^* \circ (\jmath',\imath')_{X''}^* = (\jmath' \circ \jmath, \imath' \circ \imath)_{X''}^*$ for the respective pullbacks of invariant Weil divisors and axioms (iii) and (iv) are verified. For $D \in \mathrm{WDiv}_{H''}(X'')$ the pullback of divisorial modules

$$\begin{aligned}(\imath' \circ \imath)_{X''}^*\colon \mathcal{O}_{X'',H''}(D)(X'') &\xrightarrow{\imath'^*_{X''}} \mathcal{O}_{X',H'}((\jmath',\imath')_{X''}^* D)(X')\\ &\xrightarrow{\imath^*_{X'}} \mathcal{O}_{X,H}((\jmath' \circ \jmath, \imath' \circ \imath)_{X''}^* D)(X)\end{aligned}$$

is the a composition of surjections. □

Definition VII.2.1.4. A very neat embedding $(\jmath,\imath)\colon H \circlearrowright X \to G \circlearrowright Z$ is *irredundant* if the following hold:

(i) $\imath(X)$ intersects each closed G-orbit (equivalently, each $\Omega_{Z,G}$-closed subset) of Z,

(ii) for $U, V \in \mathcal{B}_{Z,G}$ such that $\imath^{-1}(U) = \imath^{-1}(V)$ holds and $\mathcal{O}(U) = \mathcal{O}(V)$ holds in $\mathcal{K}_G(Z)$ we have $U = V$.

Example VII.2.1.5. For a characteristic space $q\colon \widehat{X} \to X$ and a defining CBE of $\mathcal{O}_X$-algebras $\pi\colon \mathcal{O}(L) \to q_*\mathcal{O}_{\widehat{X}}$ the induced morphism $\widehat{X} \to \mathrm{Spec}_X(\mathcal{O}(L))$ is an irredundant very neat embedding.

Remark VII.2.1.6. A very neat embedding $(\jmath,\imath)\colon H \circlearrowright X \to G \circlearrowright Z$ into a G-orbit closure gives rise to a unique irredundant embedding, which is obtained by gluing all the G-invariant affine open subsets of Z whose closed orbit intersects X non-trivially along those $U, V \in \mathcal{B}_{Z,G}$ with $\imath^{-1}(U) = \imath^{-1}(V)$ and $\mathcal{O}(U) = \mathcal{O}(V)$ in $\mathcal{K}_G(Z)$.

Remark VII.2.1.7. Due to Proposition V.1.3.15 and Remark VI.4.3.2 a very neat embedding $(\jmath,\imath)$ has the following properties:

(i) We have $\mathrm{Bas}((\jmath,\imath)^* w)(\imath^{-1}(U)) = \imath^{-1}(\mathrm{Bas}(w)(U))$ for $w \in \mathrm{Cl}_G(Z)$ and $U \in \mathcal{B}_{Z,G}$,

(ii) For $A \in \mathfrak{t}(X, \Omega_{X,H})$ and $U \in \Omega_{X,H,A}$ we have

$$\begin{aligned}(\jmath,\imath)_Z^* \circ S_{\mathrm{Cl}_G(Z),\overline{G\imath(A)}}(U) &= S_{\mathrm{Cl}_H(X),A}(\imath^{-1}(U)),\\ (\jmath,\imath)_Z^* \circ \mathrm{sat}(S^{\mathrm{aff}}_{\mathrm{Cl}_G(Z),\overline{G\imath(A)}}(U)) &\subseteq \mathrm{sat}(S^{\mathrm{aff}}_{\mathrm{Cl}_H(X),A})(\imath^{-1}(U)).\end{aligned}$$

In particular, we also have $(\jmath,\imath)_Z^* \circ \mathrm{WDiv}_{G,\geq 0} = \imath_*\,\mathrm{WDiv}_{H,\geq 0}$ as well as $(\jmath,\imath)_Z^* \circ \mathrm{Mov}_G = \imath_*\mathrm{Mov}_H$.

(iii) If $\imath(X)$ intersects every G-orbit closure then the inclusions of presheaves $(\jmath,\imath)_Z^* \circ \mathrm{SAmple}_G \subseteq \imath_*\mathrm{SAmple}_H$ and $(\jmath,\imath)_Z^* \circ \mathrm{Pic}_G(Z) \subseteq \mathrm{Pic}_H(X)$ are equations.

Construction VII.2.1.8. Let $(\theta, \phi)\colon H \circlearrowright X \to G \circlearrowright Z$ be an equivariantly dominant morphism of actions of Krull type. Let K be the subgroup of $\mathrm{WDiv}_H(X)$ generated by all $Y \in \mathcal{Y}_H(X)$ occuring in the support of some element of $(\theta,\phi)^* \,\mathrm{PDiv}_G(Z)$. Suppose that K maps onto $\mathrm{Cl}_H(X)$ and $K \cap \mathrm{PDiv}_H(X) = (\theta,\phi)^* \,\mathrm{PDiv}_G(Z)$.

Let $\mathcal{R}$ be a Cox sheaf on $G \circlearrowright Z$. Let $\pi\colon \mathcal{O}_Z(L) \to \mathcal{R}$ be a defining CBE of $\mathcal{O}_Z$-algebras where $L = \mathrm{WDiv}_G(Z)$ and let $\kappa\colon \mathrm{PDiv}_G(Z) \to (\mathcal{O}_Z(L)^{\mathrm{hom}})^*$ be the kernel character of π. Let $\mathcal{I}$ be the $\mathcal{O}_{X,H}(K)$-ideal defined by

$$\mathcal{I}(U) := \langle 1 - \phi^*(\kappa(D)) \mid D \in \mathrm{PDiv}_G(Z) \rangle_{\mathcal{O}(K)(U)}.$$

Then the presheaf $\mathcal{S} := \mathcal{O}_X(K)/\mathcal{I}$ is a Cox sheaf on $H \circlearrowright X$ and $\mathcal{O}_X(K) \to \mathcal{S}$ is a CBE of $\mathcal{O}_X$-algebras. Moreover, if (θ, ϕ) is a very neat embedding then the induced map $\mathcal{R} \to \imath_* \mathcal{S}$ is surjective.

VII.2.2. very neat embeddings of characteristic spaces and Cox triples. In the setting of a square made up of vertical characteristic spaces and arbitrary horizontal morphisms we show that the lower morphism is a very neat embedding if and only if the upper one is and the kernels of the characteristic quasi-torus homomorphisms coincide. Such squares are called very neat embeddings of characteristic spaces. We show that they are determined by the ambient characteristic space and a reduced set of (global) data from the other morphisms, see Theorem VII.2.2.8.

Proposition VII.2.2.1. *For characteristic spaces (θ, q) and (θ_Z, q_Z) a commutative diagram*

$$\begin{array}{ccc} \widehat{H} \circlearrowright \widehat{X} & \xrightarrow{(\widehat{\jmath},\widehat{\imath})} & \widehat{G} \circlearrowright \widehat{Z} \\ \Big\downarrow {\scriptstyle (\theta, q)} & & \Big\downarrow {\scriptstyle (\theta_Z, q_Z)} \\ H \circlearrowright X & \xrightarrow{(\jmath,\imath)} & G \circlearrowright Z \end{array}$$

is a very neat embedding if and only if $\widehat{\jmath}$ maps $\ker(\theta)$ isomorphically onto $\ker(\theta_Z)$ and $(\widehat{\jmath},\widehat{\imath})$ is a very neat embedding. Here, surjectivity of $\mathcal{O}(\widehat{Z}) \to \mathcal{O}(\widehat{X})$ suffices for surjectivity of all the pullbacks $\mathcal{O}_{\widehat{Z},\widehat{G}}(D)(\widehat{Z}) \to \mathcal{O}_{\widehat{X},\widehat{H}}((\widehat{\jmath},\widehat{\imath})^*_{\widehat{Z}} D)(\widehat{X})$.

Proof. Firstly, note that if $\imath$ is affine then considering an affine $\widehat{G}$-invariant and q_Z-saturated cover shows that $\widehat{\imath}$ is affine. Conversely, if $\widehat{\imath}$ is affine then for $U \in \mathcal{B}_{Z,G}$, $q^{-1}(\imath^{-1}(U)) = \widehat{\imath}^{-1}(q_Z^{-1}(U))$ is affine and hence so is $\imath^{-1}(U)$.

If $(\jmath,\imath)$ satisfies axiom (ii) then so does $(\widehat{\jmath},\widehat{\imath})$ because $\widehat{Z}$ is the minimal closed subset over Z and we have $q_Z(\overline{\widehat{\imath}(\widehat{X})}^{\Omega_{\widehat{Z},\widehat{G}}}) = \overline{\imath(q(\widehat{X}))}^{\Omega_{Z,G}} = Z$. The converse follows from surjectivity of q and q_Z.

From now on suppose that $(\jmath,\imath)$ and $(\widehat{\jmath},\widehat{\imath})$ satisfy axioms (i) and (ii). If $(\jmath,\imath)$ satisfies axiom (iii) then for $\widehat{Y} \in \mathcal{Y}_{\widehat{H}}(\widehat{X})$ the set $q_Z(\overline{\widehat{\imath}(\widehat{Y})}^{\Omega_{\widehat{Z},\widehat{G}}}) = \overline{\imath(q(\widehat{Y}))}^{\Omega_{Z,G}}$ has codimension smaller than 2 and so do the minimal closed set over it and $\overline{\widehat{\imath}(\widehat{Y})}^{\Omega_{\widehat{Z},\widehat{G}}}$ because the latter contains the former. In particular, we have $\mathrm{Cl}_{\widehat{G},\overline{\widehat{\imath}(\widehat{Y})}^{\Omega_{\widehat{Z},\widehat{G}}}} = 0$ and by Proposition V.1.2.14 the diagram of pullbacks of invariant Weil divisors commutes. Now, injectivity and preservation of primality follow from axiom (iii). Conversely, if $(\widehat{\jmath},\widehat{\imath})$ satisfy axiom (iii) then for $D \in \mathcal{Y}_H(X)$ set $\widehat{D} := (\theta, q)^* D \in \mathcal{Y}_{\widehat{H}}(\widehat{X})$. Then $\overline{\widehat{\imath}(\widehat{D})}^{\Omega_{\widehat{Z},\widehat{G}}}$ has codimension smaller than 2 and thus, so does its image $\overline{\imath(D)}^{\Omega_{Z,G}}$ under q_Z. Consequently, we have $\mathrm{Cl}_{G,\overline{\imath(D)}^{\Omega_{Z,G}}} = 0$ and by Proposition V.1.2.14 the diagram of pullbacks of invariant Weil divisors commutes. Now, injectivity and preservation of primality follow from axiom (iii).

Suppose that $(\jmath,\imath)$ and $(\widehat{\jmath},\widehat{\imath})$ satisfy axioms (i) - (iii). Then the restriction $\widehat{\jmath}\colon \ker(\theta) \to \ker(\theta_Z)$ factors into the canonical isomorphism $\ker(\theta) \cong \mathrm{Cl}_H(X)$, the homomorphism $\mathrm{Cl}_H(X) \to \mathrm{Cl}_G(Z)$ and the isomorphism $\mathrm{Cl}_G(Z) \cong \ker(\theta_Z)$. Thus, $(\jmath, \imath)$ satisfies axiom (iv) if and only if $(\widehat{\jmath}, \widehat{\imath})$ does.

From now on suppose that $(\jmath, \imath)$ and $(\widehat{\jmath}, \widehat{\imath})$ satisfy axioms (i) - (iv). Consider $E \in \mathrm{WDiv}_G(Z)$ and $f \in (\mathcal{K}(\widehat{Z})^{\mathrm{hom}})^*$ with $D := (\theta_Z, q_Z)^* E = \mathrm{div}_{\widehat{G},\widehat{Z}}(f)$. Then the restriction $\mathcal{O}(\widehat{Z})_{[D]} \xrightarrow{\widehat{\imath}^*_{\widehat{Z}}} \mathcal{O}(\widehat{X})_{(\jmath,\imath)^*_Z[D]}$ is the composition of the isomorphism $\mathcal{O}(\widehat{Z})_{[D]} \xrightarrow{\cdot f} \mathcal{O}_{Z,G}(E)(Z)$, with the pullback $\imath^*_Z \colon \mathcal{O}_{Z,G}(E)(Z) \to \mathcal{O}_{X,H}((\jmath,\imath)^*E)(X)$ and the isomorphism $\mathcal{O}_{X,H}((\jmath,\imath)^*E)(X) \xrightarrow{\cdot \imath^*_Z(f^{-1})} \mathcal{O}(\widehat{X})_{(\jmath,\imath)^*_Z([D])}$. In the present circumstances, surjectivity of $\mathcal{O}(\widehat{Z}) \to \mathcal{O}(\widehat{X})$ is thus equivalent to that of all $\mathcal{O}(D)(\widehat{Z}) \to \mathcal{O}((\jmath,\imath)^*D)(\widehat{X})$. The supplement is due to invariant factoriality of $\widehat{Z}$ and $\widehat{X}$. $\square$

Definition VII.2.2.2. If the conditions of the above proposition are satisfied then we say that (θ, q) is *very neatly embedded* into (θ_Z, q_Z), the latter being called the *ambient* characteristic space of the former, and the diagram is called a *very neat embedding* of characteristic spaces.

A *morphism* of very neat embeddings is a quadruple of equivariantly dominant morphisms of actions from the vertices of one square to those of the other such that a commutative cube is formed.

Remark VII.2.2.3. In the situation of Construction VII.2.1.8, if the Cox sheaf on $G \circlearrowright Z$ is locally of finite type over $\mathbb{K}$ then so is the induces Cox sheaf on $H \circlearrowright X$ and in this case forming the relative spectra gives a very neat embedding of characteristic spaces.

Remark VII.2.2.4. In the notation of Definition VII.2.2.2 $\widehat{\imath}(\widehat{X})$ is the special $\Omega_{\widehat{Z},\widehat{H}}$-closed and -irreducible set over $\imath(X)$ because $\widehat{X}$ is the special $\Omega_{\widehat{X},\widehat{H}}$-closed and -irreducible set over X.

We now show that very neat embeddings of characteristic spaces are determined by the ambient space (θ_Z, q_Z), the graded homomorphisms $\widehat{\imath}^*_Z \colon \mathcal{O}(\widehat{Z}) \to \mathcal{O}(\widehat{X})$ and $q^*_X \colon \mathcal{O}(X) \to \mathcal{O}(\widehat{X})$ and their properties. More precisely, we establish an equivalence of the category defined by such triples and the category of very neat embeddings of characteristic spaces.

Definition VII.2.2.5. A *Cox triple* $((\theta_Z, q_Z), (\imath_A, \imath_{gr(A)}), (\pi, \psi))$ consists of

- a characteristic space $(\theta_Z, q_Z) \colon \widehat{G} \circlearrowright \widehat{Z} \to G \circlearrowright Z$,
- a Veronesean $\mathbb{K}$-algebra $\imath_A \colon A \to R, \imath_{gr(A)} \colon gr(A) \to gr(R)$ such that $(R^{\mathrm{hom}})^* = (A^{\mathrm{hom}})^*$ and R is factorially graded,
- a graded surjection $\pi \colon \mathcal{O}(\widehat{Z}) \to R, \psi$ with trivial graded kernel inducing a primality preserving injection $\mathcal{O}(\widehat{Z})^{\mathrm{hom}}/(\mathcal{O}(\widehat{Z})^{\mathrm{hom}})^* \to R^{\mathrm{hom}}/(R^{\mathrm{hom}})^*$ and a bijection $\mathbb{X}(\widehat{G})/\mathbb{X}(G) \to gr(R)/gr(A)$.

A *morphism* from a Cox triple $((\theta_Z, q_Z), (\imath_A, \imath_{gr(A)}), (\pi, \psi))$ to a Cox triple $((\theta_{Z'}, q_{Z'}), (\imath_{A'}, \imath_{gr(A')}), (\pi', \psi'))$ consists of an equivariantly dominant morphism $(\zeta, \phi) \colon \widehat{G} \circlearrowright \widehat{Z} \to \widehat{G}' \circlearrowright \widehat{Z}'$, together with a morphism $\alpha \colon R' \to R, \beta \colon gr(R') \to gr(R)$ of graded $\mathbb{K}$-algebras such that $\beta(\imath_{gr(A')}(gr(A'))) \subseteq \imath_{gr(A)}(gr(A))$, $\ker(\alpha)^{\mathrm{gr}} = \{0\}$, $\pi \circ \phi^*_{Z'} = \alpha \circ \pi'$ and $\psi \circ \mathbb{X}(\zeta) = \beta \circ \psi'$.

Remark VII.2.2.6. Due to Proposition I.2.6.9 (ii) the requirement in the above that $\mathcal{O}(\widehat{Z})^{\mathrm{hom}}/(\mathcal{O}(\widehat{Z})^{\mathrm{hom}})^* \to R^{\mathrm{hom}}/(R^{\mathrm{hom}})^*$ is a primality preserving injection is equivalent to the condition that the canonical map $\mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(\widehat{Z})) \to \mathrm{Div}_{\mathrm{gr}}(R)$ is a primality preserving injection and $\mathrm{Cl}_{\mathrm{gr}}(\mathcal{O}(\widehat{Z})_{\pi^{-1}(\mathfrak{p})^{\mathrm{gr}}}) = 0$ holds for each $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$. Moreover, having an induced bijection $\mathbb{X}(\widehat{G})/\mathbb{X}(G) \to gr(R)/gr(A)$ spells out as $\mathrm{im}(\psi) + \mathrm{im}(\imath_{gr(A)}) = gr(R)$ and $\psi^{-1}(\imath_{gr(A)}) = \mathrm{im}(\mathbb{X}((\theta_Z)^*_G))$.

CONSTRUCTION VII.2.2.7. Let $((\theta_Z, q_Z), (\imath_A, \imath_{gr(A)}), (\pi, \psi))$ be a Cox triple. Let $\theta\colon \widehat{H} \to H$, $\widehat{\jmath}\colon \widehat{H} \to \widehat{G}$ and $\jmath\colon H \to G$ be the morphisms of quasi-tori associated to $\imath_{gr(A)}$, ψ and $\psi \circ \mathbb{X}((\theta_Z)^*_G)\colon \mathbb{X}(G) \to gr(A)$, respectively.

Then $(\theta, q_Z)\colon \widehat{H} \circlearrowright \widehat{Z} \to H \circlearrowright Z$ is a good quotient and $\widehat{X} := V_{\widehat{Z}}(\ker(\pi))$ is closed and $\widehat{H}$-invariant which means $X := q_Z(\widehat{X})$ is closed and H-invariant. Moreover, the following diagram is a very neat embedding of characteristic spaces:

$$\begin{tikzcd}
\widehat{H} \circlearrowright \widehat{X} \arrow[r, "(\widehat{\jmath}, \subseteq)"] \arrow[d, "(\theta, (q_Z)_{|\widehat{X}})"'] & \widehat{G} \circlearrowright \widehat{Z} \arrow[d, "(\theta_Z, q_Z)"] \\
H \circlearrowright X \arrow[r, "(\jmath, \subseteq)"] & G \circlearrowright Z
\end{tikzcd}$$

THEOREM VII.2.2.8. *Construction VII.2.2.7 defines a covariant functor* $\mathfrak{f}$ *from the category of Cox triples to the category of very neat embeddings into characteristic spaces, which is essentially inverse to the functor* $\mathfrak{g}$ *which sends a very neat embedding* $((\widehat{\jmath}, \widehat{\imath}), (\jmath, \imath))$ *of* $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ *into* $(\theta_Z, q_Z)\colon \widehat{G} \circlearrowright \widehat{Z} \to G \circlearrowright Z$ *to the Cox triple* $((\theta_Z, q_Z), (q_X^*, \mathbb{X}(\theta_H^*)), (\widehat{\imath}^*_{\widehat{Z}}, \mathbb{X}(\widehat{\jmath}^*_{\widehat{G}})))$.

PROOF. For a characteristic space $(\theta_Z, q_Z)\colon \widehat{G} \circlearrowright \widehat{Z} \to G \circlearrowright Z$ fix $f^{(Y)} \in \mathcal{O}(\widehat{Z})$ with $\mathrm{div}_{\widehat{G},\widehat{Z}}(f^{(Y)}) = Y$ for each $Y \in \mathcal{Y}_{\widehat{G}}(\widehat{Z})$ and set $f^{(U)} := \prod_{Y \in \mathcal{Y}_{\widehat{G}}(\widehat{Z}) \setminus \mathcal{Y}_{\widehat{G}}(U)} f^{(Y)}$ for $U \in \mathcal{B}_{\widehat{Z},\widehat{G}} \setminus \{\emptyset\}$. First, consider a very neat embedding $((\widehat{\jmath}, \widehat{\imath}), (\jmath, \imath))$ of (θ, q) into (θ_Z, q_Z). Then the map induced by $\widehat{\imath}^*_{\widehat{Z}}$ canonically factors into

$$\begin{aligned}
\mathcal{O}(\widehat{Z})^{\mathrm{hom}}/(\mathcal{O}(\widehat{Z})^{\mathrm{hom}})^* &\cong \mathbb{F}_1[\mathrm{WDiv}_{\widehat{G}}(\widehat{Z})_{\geq 0}] \longrightarrow \mathbb{F}_1[\mathrm{WDiv}_{\widehat{H}}(\widehat{X})_{\geq 0}] \\
&\cong \mathcal{O}(\widehat{X})^{\mathrm{hom}}/(\mathcal{O}(\widehat{X})^{\mathrm{hom}})^*
\end{aligned}$$

and is thus a primality preserving injection, where the isomorphisms are induced by $\mathrm{div}_{\widehat{G},\widehat{Z}}$ resp. $\mathrm{div}_{\widehat{H},\widehat{X}}$. This shows that $((\theta_Z, q_Z), (q_X^*, \mathbb{X}(\theta_H^*)), (\widehat{\imath}^*_{\widehat{Z}}, \mathbb{X}(\widehat{\jmath}^*_{\widehat{G}})))$ is a Cox triple. For $U \in \mathcal{B}_{\widehat{Z},\widehat{G}}$, $\ker(\widehat{\imath}^*_U)$ is the image of $\ker(\widehat{\imath}^*_{\widehat{Z}})_{f^{(U)}}$ under the natural isomorphism and hence equals $\langle \rho_U^{\widehat{Z}}(\ker(\widehat{\imath}^*_{\widehat{Z}})) \rangle$. Thus, we calculate

$$\mathrm{im}(\widehat{\imath}) \cap U = V_U(\ker(\widehat{\imath}^*_U)) = V_U(\langle \rho_U^{\widehat{Z}}(\ker(\widehat{\imath}^*_{\widehat{Z}})) \rangle) = V_{\widehat{Z}}(\ker(\widehat{\imath}^*_{\widehat{Z}})) \cap U.$$

and obtain that Construction VII.2.2.7 applied to the above triple gives the inclusion of the quotient

$$\mathrm{Spec}_{\max}(\mathbb{K}[\mathbb{X}(\widehat{H})]) \circlearrowright \mathrm{im}(\widehat{\imath}) \xrightarrow{(\mathrm{Spec}_{\max}(\mathbb{K}[\mathbb{X}(\theta)]), q_Z)} \mathrm{Spec}_{\max}(\mathbb{K}[\mathbb{X}(H)]) \circlearrowright \mathrm{im}(\imath)$$

into (θ_Z, q_Z). Now, $\widehat{\imath}$ and $\imath$ together with the canonical isomorphisms of quasi-tori $\widehat{H} \to \mathrm{Spec}_{\max}(\mathbb{K}[\mathbb{X}(\widehat{H})])$ and $H \to \mathrm{Spec}_{\max}(\mathbb{K}[\mathbb{X}(H)])$ form an isomorphism of very neat embeddings.

For the quotient constructed from a Cox triple $((\theta_Z, q_Z), (\imath_A, \imath_{gr(A)}), (\pi, \psi))$ let $\widehat{\imath}\colon \widehat{X} \to \widehat{Z}$ and $\imath\colon X \to Z$ denote the inclusion maps and set $q := (q_Z)_{|\widehat{X}}$. For each $U \in \mathcal{B}_{\widehat{Z},\widehat{G}} \setminus \{\emptyset\}$ the graded ring $\mathcal{O}(\widehat{X} \cap U) \cong R_{\pi(f^{(U)})}$ is of Krull type and hence $\widehat{X} \cap U$ is $\Omega_{\widehat{X},\widehat{H}}$-irreducible and also $\Omega_{\widehat{Z},\widehat{G}}$-dense in U because $\ker(\pi)^{\mathrm{gr}}_{f^{(U)}} = \{0\}_{f^{(U)}}$. For each further $U' \in \mathcal{B}_{\widehat{Z},\widehat{G}} \setminus \{\emptyset\}$ there exists $U'' \in \mathcal{B}_{U \cap U',\widehat{G}} \setminus \{\emptyset\}$ and we have $\emptyset \neq U'' \cap \widehat{X} \subseteq U \cap U' \cap \widehat{X}$, which gives irreducibility of $\Omega_{\widehat{X},\widehat{H}}$. Consequently, $\widehat{H} \circlearrowright \widehat{X}$ is of Krull type and $\widehat{X}$ is $\Omega_{\widehat{Z},\widehat{G}}$-dense in $\widehat{Z}$.

Under π and thus also under $\widehat{\imath}_U^*$ for $U \in \mathcal{B}_{\widehat{Z},\widehat{G}}$, preimages of homogeneously prime divisors are zero or homogeneously prime divisors. Consequently, the canonical maps form a commutative diagram

$$\begin{tikzcd} \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(\widehat{Z})) \arrow[r] \arrow[d] & \mathrm{Div}_{\mathrm{gr}}(R) \arrow[d, "\epsilon_{U\cap\widehat{X}}"'] \\ \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U)) \arrow[r] & \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(U\cap\widehat{X})) \end{tikzcd}$$

which induces a commutative diagram

$$\begin{tikzcd} \mathrm{Div}_{\mathrm{gr}}(\mathcal{O}(\widehat{Z})) \arrow[r] \arrow[d] & \mathrm{Div}_{\mathrm{gr}}(R) \arrow[d, "\epsilon_{\widehat{X}}"'] \\ \mathrm{WDiv}_{\widehat{G}}(\widehat{Z}) \arrow[r] & \mathrm{WDiv}_{\widehat{H}}(\widehat{X}). \end{tikzcd}$$

Note that $\epsilon_{U\cap\widehat{X}}$ restricts to a bijection between the sets of homogeneously prime divisors whose preimage under π resp. $\widehat{\imath}_U^*$ is zero. Thus, under $\epsilon_{\widehat{X}}$ the former set of homogeneously prime divisors is in bijection with those $D \in \mathcal{Y}_{\widehat{H}}(\widehat{Y})$ with $\overline{D}^{\Omega_{\widehat{Z},\widehat{G}}} = \widehat{Z}$. Moreover, $(\widehat{\jmath},\widehat{\imath})_{\widehat{Z}}^*\colon \mathrm{WDiv}_{\widehat{G}}(\widehat{Z}) \to \mathrm{WDiv}_{\widehat{H}}(\widehat{X})$ is a primality preserving injection because $(\widehat{\jmath},\widehat{\imath})_U^*\colon \mathrm{WDiv}_{\widehat{G}}(U) \to \mathrm{WDiv}_{\widehat{H}}(\widehat{X}\cap U)$ is one for each $U \in \mathcal{B}_{\widehat{Z},\widehat{G}}$. Thus, $\epsilon_{\widehat{X}}$ is an isomorphism. Consequently, the isomorphism $Q_{\mathrm{gr}}(R) \to \mathcal{K}(\widehat{X})$ with accompanying map $\beta\colon gr(R) \to \mathbb{X}(\mathrm{Spec}_{\max}(\mathbb{K}[gr(R)]))$ restricts to an isomorphism $\alpha\colon R \to \mathcal{O}(\widehat{X})$. This gives surjectivity of $\widehat{\imath}_{\widehat{Z}}^*\colon \mathcal{O}(\widehat{Z}) \to \mathcal{O}(\widehat{X})$, triviality of $\mathrm{Cl}_{\widehat{H}}(\widehat{X}) \cong \mathrm{Cl}_{\mathrm{gr}}(R)$ and $(\mathcal{O}(\widehat{X})^{\mathrm{hom}})^* = q_X^*((\mathcal{O}(X)^{\mathrm{hom}})^*)$. Moreover, $(\mathrm{id}_{\widehat{G}}, \mathrm{id}_{\widehat{Z}})$ and (α,β) together form an isomorphism of Cox triples.

Let $\widehat{Z}' \subseteq \widehat{Z}$ be the big open $\widehat{G}$-saturated set of those points of $\widehat{Z}$ on which $\ker(\theta_Z)$ acts freely. $\jmath$ restricts to an isomorphism $\ker(\theta) \to \ker(\theta_Z)$ because ψ induces an isomorphism $gr(\mathcal{O}(\widehat{Z}))/gr(\mathcal{O}(Z)) \to gr(R)/gr(A)$. In particular, $\ker(\theta)$ acts freely on $\widehat{Z}' \cap \widehat{X}$ because $\ker(\theta_Z)$ acts freely on $\widehat{Z}'$. To see that $\widehat{Z}' \cap \widehat{X}$ is big in $\widehat{X}$ note that $\widehat{H}$-prime divisors which are not pullbacks of $\widehat{G}$-prime divisors of $\widehat{Z}$ intersect $\widehat{\imath}^{-1}(U)$ non-trivially for each $U \in \Omega_{\widehat{Z},\widehat{G}}$ while those that are pullbacks are also preimages of $\widehat{G}$-prime divisors. Thus, the criteria for characteristic spaces in Theorem VI.4.2.7 and very neat embeddings thereof in Proposition VII.2.2.1 are satisfied.

The canonical isomorphisms of very neat embeddings resp. Cox triples obtained in the above define isomorphisms of $\mathfrak{f}\circ\mathfrak{g}$ resp. $\mathfrak{g}\circ\mathfrak{f}$ and the respective identity functors as required. $\square$

VII.3. Embeddings into toric characteristic spaces

From now on we only consider neat embeddings into *toric* prevarieties resp. toric characteristic spaces. *Morphisms* of such objects are morphisms of very neat embeddings of characteristic spaces for which the morphisms between ambient toric prevarieties are toric morphisms. In Section VII.3.1 we prove that existence of a very neat embedding into a tori character space is equivalent to finite generation of the Cox algebra. Section VII.3.2 studies additional connections between properties of embedded and ambient space which occur when the latter is toric. Namely, the ambient space encodes the affine intersection and A_2-properties of the embedded space and both have similar formulae for the cones of divisor classes defined in terms of affine neighbourhoods. Lastly, Section VII.3.3 we apply the existence result to prove an algebraic criterion for Cox algebras of finite type over $\mathbb{K}$, see Theorem VII.3.3.1. Together with the results from [**4**] on algebraic properties of

Cox rings this gives conditions on the graded structure of a ring which guarantee its being integral and normal.

VII.3.1. Construction of embeddings into toric characteristic spaces. The main result of this section, stated below, generalizes a similar result from [**17**] which stated that normal A_2-*varieties* has a finitely generated Cox ring if and only if it allows a very neat embedding into a toric variety.

THEOREM VII.3.1.1. *An action of Krull type allows a very neat embedding into a toric prevariety if and only if it has a Cox ring of finite type over* $\mathbb{K}$.

One direction of the above theorem follows directly from finite generation of polynomial rings, the other is proven constructively below.

CONSTRUCTION VII.3.1.2. Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be a characteristic space such that $\mathcal{O}(\widehat{X})$ is of finite type over $\mathbb{K}$, and let $X = X_1 \cup \ldots \cup X_m$ be an affine invariant cover.

1) Choose pairwise non-associated primes $f_1, \ldots, f_s$ and units $f_{s+1}, \ldots, f_r$ of $\mathcal{O}(\widehat{X})^{\mathrm{hom}}$ such that $\mathcal{O}(\widehat{X}) = \mathbb{K}[f_1, \ldots, f_s, f_{s+1}^{\pm 1}, \ldots, f_r^{\pm 1}]$ holds, each $\mathcal{Y}(X_k)$ contains $\mathcal{Y}(X_{\prod_i f_i})$, and each $X_k \cap X_l$ is covered by $U \in \mathcal{B}_{X,H}$ with $\mathcal{Y}(X_{\prod_i f_i}) \subseteq \mathcal{Y}(U)$. Define a morphism of actions by

$$\begin{aligned} \widehat{H} &\xrightarrow{\widehat{\jmath}} \widehat{\mathbb{T}} := (\mathbb{K}^*)^r, & \widehat{h} &\longmapsto (\deg(f_1)(\widehat{h}), \ldots, \deg(f_r)(\widehat{h})), \\ \overline{X} := \operatorname{Spec}_{\max}(\mathcal{O}(\widehat{X})) &\xrightarrow{\overline{\imath}} \overline{Z} := \mathbb{K}^r_{T_{s+1}\cdots T_r}, & \overline{x} &\longmapsto (f_1(\overline{x}), \ldots, f_r(\overline{x})). \end{aligned}$$

Let $\theta_Z\colon \widehat{\mathbb{T}} \to \mathbb{T}$ be the cokernel of the induced morphism $\ker(\theta) \to \widehat{\mathbb{T}}$.

2) For $k = 1, \ldots, m$ let $\widehat{Z}_k \subseteq \overline{Z}$ be the union of all $\widehat{\mathbb{T}}$-orbits whose closure intersect $q^{-1}(X_k)$ non-trivially in $\overline{Z}$. Then we have a toric good quotient $(\theta_Z, (q_Z)_k)\colon \widehat{\mathbb{T}} \circlearrowright \widehat{Z}_k \to \mathbb{T} \circlearrowright Z_k$ and an embedding $(\widehat{\jmath}, \widehat{\imath}_k)\colon \widehat{H} \circlearrowright q^{-1}(X_k) \to \widehat{\mathbb{T}} \circlearrowright \widehat{Z}_k$ which induces an embedding $(\jmath, \imath_k)\colon H \circlearrowright X_k \to \mathbb{T} \circlearrowright Z_k$.

For each two $k, l = 1, \ldots, s$ let $\widehat{Z}_{k,l}$ be the union over all $U \in \mathcal{B}_{\widehat{Z}_k \cap \widehat{Z}_l, \widehat{\mathbb{T}}}$ which are $\ker(\theta_Z)$-saturated in $\widehat{Z}_k$ and $\widehat{Z}_l$ such that $U \cap \overline{X} \subseteq q^{-1}(X_k \cap X_l))$. Equivariant gluing yields an irredundant very neat embedding

$$\begin{tikzcd} \widehat{H} \circlearrowright \widehat{X} \arrow[r, "(\widehat{\jmath}, \widehat{\imath})"] \arrow[d, "(\theta, q)"] & \widehat{\mathbb{T}} \circlearrowright \widehat{Z} \arrow[d, "(\theta_Z, q_Z)"] \\ H \circlearrowright X \arrow[r, "(\jmath, \imath)"] & \mathbb{T} \circlearrowright Z \end{tikzcd}$$

into a toric characteristic space.

PROOF. For step 2) set $G := \ker(\theta) = \ker(\theta_Z)$ and note that $\overline{\imath}$ is a closed embedding into $\overline{Z} := \mathbb{K}^r_{T_{s+1}\cdots T_r}$. We first show that q_Z is a well-defined toric characteristic space. For every maximal affine $\widehat{\mathbb{T}}$-invariant subset $V \subseteq \widehat{Z}_k$ consider an $\widehat{H}$-orbit $\widehat{H}\widehat{x}$ lying in the closed orbit O_V of V. Then $\widehat{H}\widehat{x}$ is closed because the closed orbit $\widehat{H}\widehat{y}$ in the closure of $\widehat{H}\widehat{x}$ also lies in O_V and is hence of the form $\widehat{H}\widehat{y} = \widehat{H}\widehat{t}\widehat{x}$. Denote by g_k the product of those f_j whose corresponding $Y \in \mathcal{Y}_H(X)$ does not intersect X_k. Then

$$\begin{aligned} pr_{\mathbb{X}(G)}(S_{\mathbb{X}(\widehat{\mathbb{T}}), \overline{O_V}}(\overline{Z})) - \mathbb{N}_0 \deg_{\mathbb{X}(G)}(g_k) &= S_{\mathbb{X}(G), \widehat{H}\widehat{x}}(\widehat{X}) - \mathbb{N}_0 \deg_{\mathbb{X}(G)}(g_k) \\ &= S_{\mathbb{X}(G), \widehat{H}\widehat{x}}(q^{-1}(X_k)) \end{aligned}$$

is a group, i.e. $\deg_{\mathbb{X}(G)}(g_k)$ lies in the relative interior of $S_{\mathbb{X}(G), \overline{O_V}}(\overline{Z})$. Since this is true of all maximal affine $\widehat{\mathbb{T}}$-invariant $V \subseteq \widehat{Z}_k$ Remark VII.1.1.15 and Proposition V.3.3.3 tells us that the quotients $\widehat{\mathbb{T}} \circlearrowright V \to \mathbb{T} \circlearrowright V /\!\!/ G$ glue to a separated good quotient $(\theta_Z, (q_Z)_k)\colon \widehat{\mathbb{T}} \circlearrowright \widehat{Z}_k \to \mathbb{T} \circlearrowright Z_k$.

Lemma VII.3.1.3 now tells us firstly that the sets $\widehat{Z}_{k,l}$ are non-empty and hence q is well-defined, and secondly that the union $\widehat{Z}'$ over all 0- or 1-codimensional $\widehat{\mathbb{T}}$-orbits of $\overline{Z}$ is a G-saturated open subset of $\widehat{Z}$ on which G acts freely. Consequently, we have $\mathrm{Cl}_{\widehat{\mathbb{T}}}(\widehat{Z}) = \mathrm{Cl}_{\widehat{\mathbb{T}}}(\overline{Z}) = 0$ and all homogeneous units of $\mathcal{O}(\widehat{Z}) = \mathcal{O}(\overline{Z})$ are of a degree in $\ker(Q)$ because the units $f_{s+1}, \ldots, f_r$ have degree in $\mathbb{X}(H)$. Thus, (θ_Z, q_Z) is a toric characteristic space.

By step 1) the set $q^{-1}(X_k \cap X_l)$ is a union in $\overline{X}$ of certain intersections of $\overline{X}$ with affine open $\widehat{\mathbb{T}}$-invariant subsets of $\overline{Z}$. Let $U \in \mathcal{B}_{\widehat{\mathbb{T}},\overline{Z}}$ be maximal with the property that the intersection of O_U with $q^{-1}(X_k \cap X_l)$ is non-empty and let $\widehat{H}\widehat{x}$ be an orbit contained in this intersection. As above, $\widehat{H}\widehat{x}$ is closed, and $\deg_{\mathbb{X}(G)}(g_k)$ is contained in the relative interior of $S_{\mathbb{X}(G),\overline{O_U}}(\overline{Z})$. Thus, U is G-saturated in every maximal $V \in \mathcal{B}_{\widehat{\mathbb{T}},\widehat{Z}_k}$ containing U and hence in $\widehat{Z}_k$. Consequently, U is contained in $\widehat{Z}_{k,l}$.

Therefore, $q^{-1}(X_k \cap X_l) = \overline{X} \cap \widehat{Z}_{k,l}$ and we conclude that the closed embeddings $q^{-1}(X_k) \to \widehat{Z}_k$ glue along $q^{-1}(X_k \cap X_l) \to \widehat{Z}_{k,l}$ to a closed embedding $\widehat{X} \to \widehat{Z}$. By Proposition VII.2.2.1 this constitutes a very neat embedding because firstly, the elements f_j are no zero divisors meaning that $\overline{X}$ intersects $\widehat{\mathbb{T}}$ non-trivially and secondly, the graded surjection π maps pairwise non-associated prime elements of $\mathcal{O}(\widehat{Z})^{\mathrm{hom}}$ to pairwise non-associated prime elements of $\mathcal{O}(\widehat{X})^{\mathrm{hom}}$. Irredundancy of the embedding follows directly from the definition of $\widehat{Z}_{k,l}$. □

LEMMA VII.3.1.3. *In the setting of Construction VII.3.1.2, each $U \in \mathcal{B}_{\widehat{\mathbb{T}},\overline{Z}}$ whose closed orbit O_U is 1-codimensional has the following properties:*

(i) *U is contained in some $\widehat{Z}_k$.*
(ii) *G acts freely on U.*
(iii) *whenever U is contained in $\widehat{Z}_k$, it is G-saturated in $\widehat{Z}_k$.*

Moreover, $\widehat{\mathbb{T}}$ is saturated in each $\widehat{Z}_k$.

PROOF. For assertion (i) note that $O_U \cap \overline{X}$ is one-codimensional in $\overline{X}$ and hence intersects some $q^{-1}(X_k)$. For (ii) let $\overline{O_U} = V(T_j)$ be the closure of O_U in $\overline{Z}$. Then we have

$$\begin{aligned}
\mathbb{X}(\widehat{\mathbb{T}})/\mathbb{X}(\mathbb{T}) &= \mathbb{X}(\widehat{H})/\mathbb{X}(H) = (\deg(((R_{\langle f_j \rangle})^{\mathrm{hom}})^*) + \mathbb{X}(H))/\mathbb{X}(H) \\
&= \left(\sum_{\substack{i=1 \\ i \neq j}}^{r} \mathbb{Z}\deg(f_i) + \mathbb{X}(H) \right) \Big/ \mathbb{X}(H) \\
&= \left(\sum_{\substack{i=1 \\ i \neq j}}^{r} \mathbb{Z}\deg(T_i) + \mathbb{X}(\mathbb{T}) \right) \Big/ \mathbb{X}(\mathbb{T}) \\
&= (\deg(((\mathcal{O}(\overline{Z})_{\overline{O_U}})^{\mathrm{hom}})^*) + \mathbb{X}(\mathbb{T}))/\mathbb{X}(\mathbb{T})
\end{aligned}$$

for every $j \in \{1, \ldots, s\}$. Concerning (iii) note that U is G-saturated in $\widehat{Z}_k$ by assertion (ii). □

REMARK VII.3.1.4. Let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be irredundantly and very neatly embedded into a toric characteristic space $(\theta_Z, q_Z)\colon \widehat{\mathbb{T}} \circlearrowright \widehat{Z} \to \mathbb{T} \circlearrowright Z$ Construction VII.3.1.2 returns the given embedding as output if in step 1) we choose the cover $U \cap X$, $U \in \mathcal{B}_{\mathbb{T},Z}$ and the graded surjection $\mathcal{O}(\widehat{Z}) \to \mathcal{O}(\widehat{X}), \mathbb{X}(\widehat{\mathbb{T}}) \to \mathbb{X}(\widehat{H})$.

VII.3.2. Toric ambient characteristic spaces. We show that a toric ambient space and its embedded space share the formula for the cones of divisor classes defined in terms of affine neighbourhoods. Furthermore, we study the affine intersection and A_2-properties. The simplest example of very neat embeddings into toric characteristic spaces are the characteristic spaces of the underlying prevarieties, see below. Other known examples of such embeddings are those of torus actions of complexity one, see [**18**].

Example VII.3.2.1. For any toric characteristic space $(\theta, q)\colon \widehat{\mathbb{T}} \circlearrowright \widehat{Z} \to \mathbb{T} \circlearrowright Z$ the diagram

$$\begin{array}{ccc} \ker(\theta) \circlearrowright \widehat{Z} & \longrightarrow & \widehat{\mathbb{T}} \circlearrowright \widehat{Z} \\ \big\downarrow & & \big\downarrow \\ 1 \circlearrowright Z & \xrightarrow{(1, \mathrm{id}_Z)} & \mathbb{T} \circlearrowright Z \end{array}$$

is a very neat embedding. Indeed, by Theorem VI.4.2.7 q is a characteristic space because $\mathrm{Cl}(\widehat{Z}) = \mathrm{Cl}_{\widehat{\mathbb{T}}}(\widehat{Z}) = 0$ and all units in $\mathcal{O}(\widehat{Z})$ are $\mathbb{X}(\widehat{\mathbb{T}})$-homogeneous of degree in $\mathbb{X}(H)$, in particular, they have degree zero in $\mathbb{X}(\ker(\theta))$.

Lemma VII.3.2.2. *For a very neat embedding $(\jmath, \imath)\colon H \circlearrowright X \to \mathbb{T} \circlearrowright Z$ the following hold:*

(i) *Consider $V \in \Omega_{Z,\mathbb{T}}$ such that $\imath(\imath^{-1}(V))$ intersects V each $\mathbb{T}$-orbit of V which is closed in V non-trivially. Then $U \in \mathcal{B}_{V,\mathbb{T}}$ has a purely one-codimensional complement in V if and only if $\imath^{-1}(V \setminus U)$ is of pure codimension one in $\imath^{-1}(V)$.*

(ii) *If X is of affine intersection and $(\jmath, \imath)$ is irredundant then $Z \setminus U$ is of pure codimension one for each $U \in \mathcal{B}_{Z,\mathbb{T}}$.*

(iii) *For a closed H-orbit $A \subseteq X$, its $\mathbb{T}$-invariant closure $B := \overline{\imath(A)}^{\Omega_{Z,\mathbb{T}}}$ and $U \in \Omega_{Z,\mathbb{T},B}$ such that $\omega^{\mathrm{aff}}_{\mathrm{Cl}_{\mathbb{T}}(Z),B}(U)$ is non-empty we have*

$$\begin{aligned}(\jmath, \imath)^*_Z(\omega^{\mathrm{aff}}_{\mathrm{Cl}_{\mathbb{T}}(Z),B}(U)) &= \omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),B}(\imath^{-1}(U)) = \omega_{\mathrm{Cl}_H(X),B}(\imath^{-1}(U))^{\circ} \\ &= (\jmath, \imath)^*_Z(\omega_{\mathrm{Cl}_{\mathbb{T}}(Z),B}(U)^{\circ}).\end{aligned}$$

Proof. In (i) assume that $\imath^{-1}(V \setminus U)$ is of pure codimension one and let $O \subseteq V \setminus U$ be a $\mathbb{T}$-orbit which is closed in V. Then $\imath^{-1}(O)$ is a non-empty subset of $\imath^{-1}(V \setminus U)$ and thus is contained in $\imath^{-1}(Y \cap V)$ for some $Y \in \mathcal{Y}_{\mathbb{T}}(V) \setminus \mathcal{Y}_{\mathbb{T}}(U)$. By Remark VI.4.1.12, assertion (ii) is a special case of (i). For (iii) we use Propositions V.2.1.15 and V.3.3.4 as well as Remarks VII.2.1.7, VII.1.3.2 and VI.4.3.2 to calculate

$$\begin{aligned}(\jmath, \imath)^*_Z(\omega^{\mathrm{aff}}_{\mathrm{Cl}_{\mathbb{T}}(Z),B}(U)) &\subseteq \omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),B}(\imath^{-1}(U)) \subseteq \omega_{\mathrm{Cl}_H(X),B}(\imath^{-1}(U))^{\circ} \\ &= (\jmath, \imath)^*_Z(\omega_{\mathrm{Cl}_{\mathbb{T}}(Z),B}(U)^{\circ}) = (\jmath, \imath)^*_Z(\omega^{\mathrm{aff}}_{\mathrm{Cl}_{\mathbb{T}}(Z),B}(U)).\end{aligned}$$

□

Proposition VII.3.2.3. *Let $H \circlearrowright X$ be of Krull type with Cox ring of finite type over $\mathbb{K}$. Let V be the union over the set $\mathcal{W}$ of all $W \in \mathcal{B}_{X,H}$ with purely one-codimensional complement. Then there exists an irredundant very neat embedding $(\jmath, \imath)\colon H \circlearrowright X \to \mathbb{T} \circlearrowright Z$ such that the complement of $U \in \mathcal{B}_{Z,\mathbb{T}}$ in Z is purely one-codimensional if and only if $\imath^{-1}(U) \subseteq V$.*

Proof. V is finitely covered $X_1, \ldots, X_n \in \mathcal{U}$ and we may complete these to an affine H-invariant cover $X = X_1 \cup \ldots \cup X_m$. Let $(\jmath, \imath)\colon H \circlearrowright X \to \mathbb{T} \circlearrowright Z$ be the embedding obtained from this input from Construction VII.3.1.2 where the system of generators was chosen arbitrarily. If $U \in \mathcal{B}_{Z,\mathbb{T}}$ satisfies $\imath^{-1}(U) \subseteq V$ then there exists $i \leq n$ with $\imath^{-1}(U) \subseteq X_i$. Then $\imath^{-1}(U)$ is purely one-codimensional in X_i and hence, in X. □

Proposition VII.3.2.4. *Let $H \circlearrowright X$ be of Krull type with Cox ring of finite type over $\mathbb{K}$ and consider a closed H-orbit $A \subseteq X$ and $V \in \Omega_{X,H,A}$. If $\omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),A}(V)$ is non-empty then it equals $\omega_{\mathrm{Cl}_H(X),A}(V)^{\circ}$.*

Proof. Suppose that there exists $W \in \mathcal{B}_{V,H,A}$ such that $V \setminus W$ is purely one-codimensional in V. Let $X_1, \ldots, X_n \in \mathcal{B}_{X,H}$ with $V = W \cup X_1 \cup \ldots \cup X_m$ and $X = W \cup X_1 \cup \ldots \cup X_n$. Applying Construction VII.3.1.2 then gives a very neat embedding $(\jmath, \imath)\colon H \circlearrowright X \to \mathbb{T} \circlearrowright Z$ such that $V = \imath^{-1}(U)$ holds with some $U \in \Omega_{Z,\mathbb{T},\overline{\imath(A)}^{\Omega_{Z,\mathbb{T}}}}$ whose closed $\mathbb{T}$-orbits intersect $\imath(V)$ non-trivially. Moreover, there exists $U' \in \mathcal{B}_{Z,\mathbb{T},\overline{\imath(A)}^{\Omega_{Z,\mathbb{T}}}}$ with $\imath^{-1}(U') \subseteq W$. Then $\imath^{-1}(U')$ has a purely one-codimensional complement in W and hence in V. Now, Lemma VII.3.2.2(i) implies that $U \setminus U'$ is of pure codimension one in U, in particular $\omega^{\mathrm{aff}}_{\mathrm{Cl}_{\mathbb{T}}(Z),\overline{\imath(A)}^{\Omega_{Z,\mathbb{T}}}}(U)$ is non-empty, and (iii) gives the assertion. $\square$

Lemma VII.3.2.5. *Let X be an irreducible prevariety of affine intersection. If for $U_1, U_2 \in \mathcal{B}_X$ and $V \in \Omega_{U_1 \cap U_2}$ the prevariety $Y := (U_1 \sqcup U_2)/\sim_V$ obtained by gluing along V is separated then $V = U_1 \cap U_2$.*

Proof. In the diagram of canonical homomorphisms

$$\begin{tikzcd} \mathcal{O}(U_1 \times U_2) \arrow[rr] \arrow[dr] & & \mathcal{O}(V) \\ & \mathcal{O}(U_1 \cap U_2) \arrow[ur, "\varrho_V^{U_1 \cap U_2}"'] & \end{tikzcd}$$

the horizontal arrow is a surjection by separatedness of Y and therefore, $\varrho_V^{U_1 \cap U_2}$ is also a surjection. Since X is irreducible, $\varrho_V^{U_1 \cap U_2}$ is also injective. We have $U_1 \cap U_2 \in \mathcal{B}_X$ and $V \in \mathcal{B}_Y$ because X and Y are of affine intersection. Thus, the inclusion $V \to U_1 \cap U_2$ is an isomorphism, i.e. $V = U_1 \cap U_2$. $\square$

Proposition VII.3.2.6. *Consider a very neat embedding $((\widehat{\jmath}, \widehat{\imath}), (\jmath, \imath))$ of the characteristic space $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ into $(\theta_Z, q_Z)\colon \widehat{\mathbb{T}} \circlearrowright \widehat{Z} \to \mathbb{T} \circlearrowright Z$ such that $\imath(X)$ intersects every closed $\mathbb{T}$-orbit of Z. Then X is of affine intersection if and only if for each two $U, V \in \mathcal{B}_{\widehat{Z},\widehat{\mathbb{T}}}$ whose closed $\widehat{\mathbb{T}}$-orbits intersect $\widehat{X}$ non-trivially, $\mathcal{O}(U) = \mathcal{O}(V)$ in $\mathcal{K}_{\widehat{\mathbb{T}}}(\widehat{Z})$ implies $U = V$.*

Proof. If $\widehat{X}$ is quasi-affine, then so is X and hence $\widehat{X}$ is an open subset of $\overline{X} = \mathrm{Spec}(\mathcal{O}(\widehat{X}))$. Consider $U, V \in \mathcal{B}_{\widehat{Z},\widehat{\mathbb{T}}}$ whose regular functions coincide and whose closed $\widehat{\mathbb{T}}$-orbits intersect $\widehat{\imath}(\widehat{X})$ non-trivially. Let $U' = \mathrm{Spec}_{\max}(\mathcal{O}(U))$ be the corresponding $\widehat{\mathbb{T}}$-invariant subset of $\overline{Z} = \mathrm{Spec}_{\max}(\mathcal{O}(\widehat{Z}))$. The set $(U \cup V) \cap \widehat{X} \subseteq \widehat{Z}$ is separated and is obtained by gluing two copies of $U' \cap \overline{X}$ along $W \cap \overline{X}$, where W is an open $\widehat{\mathbb{T}}$-invariant subset $W \subseteq U'$. Thus, Lemma VII.3.2.5 yields $W \cap \overline{X} = U' \cap \overline{X}$. Since $\overline{X}$ intersects the closed $\widehat{\mathbb{T}}$-orbit of U' this means $W = U'$, i.e. $U = V$.

For the converse consider two maximal $U, U' \in \mathcal{B}_{\widehat{Z},\widehat{\mathbb{T}}}$ and let V, V' be the corresponding subsets of $\overline{Z}$. Let $W \subseteq U$ be the subset corresponding to $V \cap V'$ and let $W' \subseteq U'$ be the subset corresponding to $V \cap V'$. For every affine $\widehat{\mathbb{T}}$-invariant subset $C \subseteq W$ whose closed orbit intersects $\widehat{X}$ non-trivially and the corresponding subset $C' \subseteq W'$ the assumption yields $C = C'$. Therefore, $U \cap U' \cap \widehat{X}$ equals $V \cap V' \cap \overline{X}$ in $\overline{Z}$ which means that $(U \cup U') \cap \widehat{X}$ equals $(V \cup V') \cap \overline{X}$ in $\overline{Z}$. Thus, $\widehat{X}$ is an open subset of $\overline{X}$. $\square$

The following statement lists results from [**28**] which we are going to use thereafter.

PROPOSITION VII.3.2.7. [**28**] *A normal prevariety X which satisfies the A_2-property is separated. If X is toric then the converse holds.*

PROPOSITION VII.3.2.8. *Let $X \subseteq Z$ be an irredundant very neat embedding of a normal prevariety into a normal toric prevariety. Then X satisfies the A_2-property if and only if Z does.*

PROOF. Suppose that X satisfies the A_2-property. By Proposition VII.3.2.7, X is then in particular separated. We show that Z is separated, which by Proposition VII.3.2.7 implies that Z satisfies the A_2-property. For two maximal $U, V \in \mathcal{B}_{\widehat{Z},\widehat{\mathbb{T}}}$ let $U', V' \in \mathcal{B}_{\overline{Z},\widehat{\mathbb{T}}}$ be the corresponding subsets of $\overline{Z} = \mathrm{Spec}_{\max}(\mathcal{O}(\widehat{Z}))$, and let $x \in U \cap \widehat{X}$ and $y \in V \cap \widehat{X}$ be points in the respective closed $\widehat{\mathbb{T}}$-orbits. By assumption there exists an affine neighbourhood $W \subseteq X$ of $q(x)$ and $q(y)$. Then the complement of W is a divisor D on X and by Proposition V.2.1.15 $[D]$ is contained in

$$\omega^{\mathrm{aff}}_{\mathrm{Cl}(X),x}(X) = \omega_{\mathbb{X}(\widehat{\mathbb{T}}),\overline{O_U}}(\widehat{Z})^{\circ} = \omega_{\mathbb{X}(\widehat{\mathbb{T}}),\overline{O_{U'}}}(\overline{Z})^{\circ}$$

and $\omega_{\mathbb{X}(\widehat{\mathbb{T}}),\overline{O_{V'}}}(\overline{Z})^{\circ}$. Using Remark VII.1.1.15 and Proposition V.3.3.3, we see that $U' \cap V'$ is $\ker(\theta_Z)$-saturated in both U' and V', and the canonical homomorphism $\mathcal{O}(U')_{\mathbb{X}(\mathbb{T})} \times \mathcal{O}(V')_{\mathbb{X}(\mathbb{T})} \to \mathcal{O}(U' \cap V')_{\mathbb{X}(\mathbb{T})}$ is surjective.

Let $U'' \subseteq U$ and $V'' \subseteq V$ be the subsets corresponding to $U' \cap V'$. Since $\widehat{X}$ is open in $\overline{X} = \mathrm{Spec}_{\max}(\mathcal{O}(\widehat{X}))$ we have $U'' \cap \widehat{X} = U' \cap V' \cap \overline{X} = V'' \cap \widehat{X}$ and hence $q(U'') \cap X = q(V'') \cap X$. Now, Proposition VII.3.2.6 implies $q_Z(U'') = q_Z(V'')$, i.e. $U'' = V''$. In particular, $U \cap V = U'' = V''$ is affine. Therefore, the canonical map

$$\mathcal{O}(q_Z(U)) \times \mathcal{O}(q_Z(V)) = \mathcal{O}(U)_{\mathbb{X}(\mathbb{T})} \times \mathcal{O}(V)_{\mathbb{X}(\mathbb{T})} \to \mathcal{O}(U \cap V)_{\mathbb{X}(\mathbb{T})} = \mathcal{O}(q_Z(U \cap V))$$

is surjective and we have shown that Z is separated. □

VII.3.3. Cox algebras of finite type. We now characterize Cox algebras of finite type over $\mathbb{K}$ in terms of graded algebra. Minimality of the set of characterizing conditions is shown in Remark VII.3.3.7.

THEOREM VII.3.3.1. *For a Veronesean algebra $A \to R$ of finite type over $\mathbb{K}$ there exists an action $H \circlearrowright X$ with a Cox sheaf $\mathcal{R}$ and an isomorphism of morphisms of graded $\mathbb{K}$-algebras from $A \to R$ to $\mathcal{O}(X) \to \mathcal{R}(X)$ if and only if R is factorially graded, $(R^{\mathrm{hom}})^* = (A^{\mathrm{hom}})^*$ and $\deg((R_{\mathfrak{p}}^{\mathrm{hom}})^*) + gr(A) = gr(R)$ holds for each $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$.*

Moreover, in this case X may be chosen to be of affine intersection and with the notations $pr\colon gr(R) \to gr(R)/gr(A)$ for the canonical epimorphism and $\omega_{\mathfrak{p}}$ for the cones $\mathbb{N}^{-1}pr(\deg(R^{\mathrm{hom}} \setminus \mathfrak{p})) \subseteq \mathbb{N}^{-1}(gr(R)/gr(A))$ associated to $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$ the following hold:

(i) *If $gr(A) = 0$ then X may be chosen with the A_2-property if and only if the sets $\omega_{\mathfrak{p}}^{\circ}$ for $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$ intersect pairwise non-trivially.*
(ii) *$H \circlearrowright X$ may be chosen such that $\mathrm{Ample}_H(X)$ is non-empty if and only if the intersection over all $\omega_{\mathfrak{p}}^{\circ}, \mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$ is non-empty.*

DEFINITION VII.3.3.2. A Veronesean algebra $A \to R$ of finite type over $\mathbb{K}$ is called a *Cox algebra of finite type over* $\mathbb{K}$ if $(R^{\mathrm{hom}})^* = (A^{\mathrm{hom}})^*$, R is factorially graded and $\deg((R_{\mathfrak{p}}^{\mathrm{hom}})^*) + gr(A) = gr(R)$ holds for each $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$.

REMARK VII.3.3.3. If $A \to R$ is a Cox algebra of finite type over $\mathbb{K}$ then $gr(R)$ is by definition finitely generated, and R is affine by Proposition VI.2.1.5. Consequently, A is finitely generated over $\mathbb{K}$ by Hilberts Invariant Theorem.

Since Cox rings of normal prevarieties with finitely generated class group are known to be integral and normal [**4, 7**] we obtain the following:

Corollary VII.3.3.4. *A finitely generated $\mathbb{K}$-algebra which is factorially graded with homogeneous units only in degree zero such that the localizations at graded prime divisors have units in every degree is integral and normal.*

Remark VII.3.3.5. Let $((\theta_Z, q_Z), A \to R, (\pi, \psi))$ be a Cox triple where (θ_Z, q_Z) is toric and let $(\theta, q)\colon \widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ be the induced characteristic space which is very neatly embedded into (θ_Z, q_Z). Fix a basis $e_1, \dots, e_r$ of $\mathbb{X}(\widehat{\mathbb{T}})$ and denote by $Q\colon \mathbb{Z}^r \to gr(R)/gr(A)$ the epimorphism sending e_i to the class of $\psi(\deg(\chi^{e_i}))$. For a H-closed -irreducible $B \subseteq X$ let $\widehat{B} \subseteq \widehat{X}$ be the special set over B. For $U \in \Omega_{Z,\mathbb{T}}$ we then have

$$S_{\mathrm{Cl}_H(X),B}(U) = \sum_{\chi^{e_i} \in \mathcal{O}(q_Z^{-1}(U)) \setminus I(\widehat{B})} \mathbb{N}_0 Q(e_i) + \sum_{\chi^{e_i} \notin \mathcal{O}(q_Z^{-1}(U))} \mathbb{Z} Q(e_i)$$

in $gr(R)/gr(A) = \mathrm{Cl}_H(X)$.

Construction VII.3.3.6. Let $A \to R$ be a Cox algebra of finite type over $\mathbb{K}$. By Remark II.2.2.3 R has a system $(f_1, \dots, f_r)$ of generators such that $f_1, \dots, f_s$ and $f_{s+1}, \dots, f_r$ are pairwise non-associated primes resp. units of R^{hom}. For $i = 1, \dots, s$ let $\widehat{Z}_i \subseteq \mathbb{K}^r$ be the principal open toric subvariety associated to the product of all coordinate functions χ^{e_j} other than χ^{e_i}, and let $\widehat{Z}$ be the union over all $\widehat{Z}_i$. Then we obtain a graded homomorphism

$$\mathcal{O}(\widehat{Z}) = \mathbb{K}[\mathbb{N}_0^s \oplus \mathbb{Z}^{r-s}] \xrightarrow{\pi} R, \quad \chi^{e_i} \longmapsto f_i,$$
$$\mathbb{X}((\mathbb{K}^*)^r) = \mathbb{Z}^r \xrightarrow{\psi} gr(R), \quad e_i \longmapsto \deg(f_i).$$

Moreover, $H' := V_{(\mathbb{K}^*)^r}(\chi^e - 1 | e \in \psi^{-1}(gr(A)))$ acts freely on $\widehat{Z}$ and the orbit space map is a geometric toric characteristic space $(\theta_Z, q_Z)\colon (\mathbb{K}^*)^r \circlearrowright \widehat{Z} \to (\mathbb{K}^*)^r/H' \circlearrowright Z$ which together with $A \to R$ and (π, ψ) forms a Cox triple.

Proof. Due to Remark II.2.5.19 each $i = 1, \dots, s$ satisfies

$$\langle \deg(f_j) | j \in \{1, \dots, s\} \setminus i \rangle + gr(A) = gr(R).$$

Consequently, H' acts freely on the principal subset defined by the product over all $\chi^{e_1}, \dots, \chi^{e_r}$ except χ^{e_i}, and $(\mathbb{K}^*)^r$ is H'-saturated in this set. Now, Theorem VI.4.2.7 implies that (θ_Z, q_Z) is a characteristic space. □

Proof of Theorem VII.3.3.1. The properties of Cox rings were given in Chapter V. If the algebra $A \to R$ satisfies the listed properties then by Construction VII.3.3.6 we may complete it to a Cox triple $((\theta_Z, q_Z), A \to R, (\pi, \psi))$. By Theorem VII.2.2.8 there exists an induced very neat embedding of a characteristic space $\widehat{H} \circlearrowright \widehat{X} \to H \circlearrowright X$ into (θ_Z, q_Z) such that $\mathcal{O}(X) \to \mathcal{O}(\widehat{X})$ is isomorphic to $A \to R$. Note that X and Z are of affine intersection by construction. By Remark VII.3.3.5 resp. Remark II.1.8.5 the set of all cones generated by at least $s-1$ of the values $pr(\deg(f_1)), \dots, pr(\deg(f_s))$ is equal to firstly, the set of all cones $\omega_{\mathrm{Cl}_H(X),Hx}(X)$ for all closed H-orbits Hx, and secondly, the set of all $\omega_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathfrak{P}_{\mathrm{gr}}(R)$.

Due to Remark VI.4.1.12 the affine intersection property of X means that complements of H-invariant affine open subsets are purely one-codimensional and by Proposition VII.3.2.4 we have $\omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),Hx}(X) = \omega_{\mathrm{Cl}_H(X),Hx}(X)^\circ$. Consequently, $\mathrm{Ample}_H(X)$ is the intersection over all $\omega_{\mathfrak{p}}^\circ$.

Suppose that $H = \{e_H\}$. For each $x \in X$ we have

$$\omega_{\mathrm{Cl}_H(X),\overline{Hx}}(X)^\circ = \omega^{\mathrm{aff}}_{\mathrm{Cl}_H(X),\overline{Hx}}(X) = \omega^{\mathrm{aff}}_{\mathrm{Cl}_{\mathbb{T}}(Z),\overline{\mathbb{T}x}}(Z) = \omega_{\mathrm{Cl}_{\mathbb{T}}(Z),\overline{\mathbb{T}x}}(Z)^\circ.$$

If X satisfies the A_2-property then all these cones intersect pairwise non-trivially. If the converse holds, then Z is separated by Proposition VII.1.2.11. □

Remark VII.3.3.7. Note that the above set of three conditions for Cox algebras is minimal. Firstly, consider the 0-graded monoid algebra $R = \mathbb{K}[M]$ over the monoid $M = \{0, 2, 3, \ldots\}$. Then $R \subseteq R$ satisfies all conditions apart from graded factoriality. Secondly, consider any toric variety $\mathbb{T} \circlearrowright Z$ with $\mathrm{Cl}(Z) \neq 0$. Then $S := \mathcal{O}(\mathrm{WDiv}^{\mathbb{T}}(Z))$ is factorial, satisfies the third condition, but has to many units. Finally, an example of a factorially graded $\mathbb{K}$-algebra with trivial units which does not satisfy the third condition is given in [**5**, Example 5.5].

APPENDIX A

Adjunction criteria

REMARK A.0.0.1. Recall that each category $\mathfrak{C}$ defines a category $\mathfrak{K} := \mathrm{Mor}(\mathfrak{C})$ whose objects are $\mathfrak{C}$-morphisms ϕ, and whose morphisms $(\alpha, \beta) \in \mathrm{Mor}_{\mathfrak{K}}(\phi, \psi)$ are pairs of $\mathfrak{C}$-morphisms such that $\beta \circ \phi = \psi \circ \alpha$. With respect to taking opposite categories we have $\mathrm{Mor}(\mathfrak{C}^{\mathrm{op}}) = \mathrm{Mor}(\mathfrak{C})^{\mathrm{op}}$.

LEMMA A.0.0.2. *Let $\mathfrak{i}\colon \mathfrak{C} \to \mathfrak{D}$ be a faithful functor and let $\mathfrak{f}\colon \mathfrak{D} \to \mathfrak{C}$ be a functor. Let $\mathfrak{g}\colon \mathfrak{D} \to \mathrm{Mor}(\mathfrak{D})$ be a functor sending an object X to a morphism $\mathfrak{g}(X) \in \mathrm{Mor}_{\mathfrak{D}}(X, \mathfrak{i}(\mathfrak{f}(X)))$ with $\mathfrak{g}(\mathfrak{i}(\mathfrak{f}(X))) = \mathfrak{i}(\mathfrak{f}(\mathfrak{g}(X)))$, and a morphism $\phi\colon X \to Y$ to the pair $(\phi, \mathfrak{i}(\mathfrak{f}(\phi)))$. Let $\mathfrak{h}\colon \mathfrak{C} \to \mathrm{Mor}(\mathfrak{C})$ be a functor sending an object X to a morphism $\mathfrak{h}(X) \in \mathrm{Mor}_{\mathfrak{C}}(\mathfrak{f}(\mathfrak{i}(X)), X)$ such that $\mathfrak{i}(\mathfrak{h}(X)) \circ \mathfrak{g}(\mathfrak{i}(X)) = \mathrm{id}_{\mathfrak{i}(X)}$, and a morphism $\phi\colon X \to Y$ to the pair $(\mathfrak{f}(\mathfrak{i}(\phi)), \phi)$.*

Then $(\mathfrak{f}, \mathfrak{i})$ is an adjoint pair. Specifically, $\mathfrak{i}$ and $\mathfrak{f}$ together with the natural transformations s and t defined below form an adjunction, where for $X \in \mathfrak{D}$ and $Y \in \mathfrak{C}$ we define $s_{X,Y}$ and $t_{X,Y}$ via

$$\begin{aligned} \mathrm{Mor}_{\mathfrak{C}}(\mathfrak{f}(X), Y) &\longleftrightarrow \mathrm{Mor}_{\mathfrak{D}}(X, \mathfrak{i}(Y)) \\ t_{X,Y}\colon \phi &\longmapsto \mathfrak{i}(\phi) \circ \mathfrak{g}(X) \\ \mathfrak{h}(Y) \circ \mathfrak{f}(\psi) &\longleftarrow\!\shortmid \psi\colon s_{X,Y} \end{aligned}$$

PROOF. Let $X \in \mathfrak{D}$ and $Y \in \mathfrak{C}$, and let $\psi \in \mathrm{Mor}_{\mathfrak{D}}(X, \mathfrak{i}(Y))$. Using functoriality of $\mathfrak{g}$ we calculate

$$t_{X,Y}(s_{X,Y}(\psi)) = \mathfrak{i}(\mathfrak{h}(Y)) \circ \mathfrak{i}(\mathfrak{f}(\psi)) \circ \mathfrak{g}(X) = \mathfrak{i}(\mathfrak{h}(Y)) \circ \mathfrak{g}(\mathfrak{i}(Y)) \circ \psi = \psi.$$

For $\phi \in \mathrm{Mor}_{\mathfrak{C}}(\mathfrak{f}(X), Y)$ we use the equation $\mathfrak{i}(\mathfrak{f}(\mathfrak{g}(X))) = \mathfrak{g}(\mathfrak{i}(\mathfrak{f}(X)))$ and functoriality of $\mathfrak{g}$ to calculate

$$\begin{aligned} \mathfrak{i}(s_{X,Y}(t_{X,Y}(\phi))) &= \mathfrak{i}(\mathfrak{h}(Y)) \circ \mathfrak{i}(\mathfrak{f}(\mathfrak{i}(\phi))) \circ \mathfrak{i}(\mathfrak{f}(\mathfrak{g}(X))) = \mathfrak{i}(\mathfrak{h}(Y)) \circ \mathfrak{i}(\mathfrak{f}(\mathfrak{i}(\phi))) \circ \mathfrak{g}(\mathfrak{i}(\mathfrak{f}(X))) \\ &= \mathfrak{i}(\mathfrak{h}(Y)) \circ \mathfrak{g}(\mathfrak{i}(Y)) \circ \mathfrak{i}(\phi) = \mathfrak{i}(\phi) \end{aligned}$$

and since $\mathfrak{i}$ is faithful we conclude $s_{X,Y}(t_{X,Y}(\phi)) = \phi$. For naturality of t and s we use functoriality of $\mathfrak{g}$ and $\mathfrak{h}$. Consider morphisms $\alpha\colon X \to X'$ in $\mathfrak{C}$ and $\beta\colon Y' \to Y$ in $\mathfrak{D}$. Then $\square$

EXAMPLE A.0.0.3. Let $\mathfrak{C}$ be a category with finite colimits. Then for each $\mathfrak{C}$-object A the category $A \setminus \mathfrak{C}$ of $\mathfrak{C}$-objects uder A has finite colimits. Suppose that we have chosen a coproduct functor for $A \setminus \mathfrak{C}$, whose output shall be written using the $\oplus_A$-sign. Let $\phi\colon A \to B$ be a $\mathfrak{C}$-morphism. Then the functor $A \setminus \mathfrak{C} \longrightarrow B \setminus \mathfrak{C}$ sending C to $C \oplus_A B$ is canonically left-adjoint to the faithful functor defined via composition with ϕ.

LEMMA A.0.0.4. *Let $\mathfrak{i}\colon \mathfrak{C} \to \mathfrak{D}$ be a faithful functor and let $\mathfrak{f}\colon \mathfrak{D} \to \mathfrak{C}$ be a functor. Let $\mathfrak{g}\colon \mathfrak{D} \to \mathrm{Mor}(\mathfrak{D})$ be a functor sending an object X to a morphism $\mathfrak{g}(X) \in \mathrm{Mor}_{\mathfrak{D}}(\mathfrak{i}(\mathfrak{f}(X)), X)$ with $\mathfrak{g}(\mathfrak{i}(\mathfrak{f}(X))) = \mathfrak{i}(\mathfrak{f}(\mathfrak{g}(X)))$, and a morphism $\phi\colon X \to Y$ to the pair $(\mathfrak{i}(\mathfrak{f}(\phi)), \phi)$. Let $\mathfrak{h}\colon \mathfrak{C} \to \mathrm{Mor}(\mathfrak{C})$ be a functor sending an object X to a morphism $\mathfrak{h}(X) \in \mathrm{Mor}_{\mathfrak{C}}(X, \mathfrak{f}(\mathfrak{i}(X)))$ such that $\mathfrak{g}(\mathfrak{i}(X)) \circ \mathfrak{i}(\mathfrak{h}(X)) = \mathrm{id}_{\mathfrak{i}(X)}$, and a morphism $\phi\colon X \to Y$ to the pair $(\mathfrak{f}(\phi, \mathfrak{i}(\phi)))$.*

Then $(\mathfrak{i}, \mathfrak{f})$ *is an adjoint pair. Specifically,* $\mathfrak{i}$ *and* $\mathfrak{f}$ *together with the natural transformations* s *and* t *defined below form an adjunction, where for* $X \in \mathfrak{C}$ *and* $Y \in \mathfrak{D}$ *we define* $s_{X,Y}$ *and* $t_{X,Y}$ *via*

$$\begin{aligned} \operatorname{Mor}_{\mathfrak{D}}(\mathfrak{i}(X), Y) &\longleftrightarrow \operatorname{Mor}_{\mathfrak{C}}(X, \mathfrak{f}(Y)) \\ t_{X,Y} \colon \phi &\longmapsto \mathfrak{f}(\phi) \circ \mathfrak{h}(X) \\ \mathfrak{g}(Y) \circ \mathfrak{i}(\psi) &\longleftarrow\!\shortmid \psi \colon s_{X,Y} \end{aligned}$$

Proof. The induced functors $\mathfrak{i}^{\mathrm{op}} \colon \mathfrak{C}^{\mathrm{op}} \to \mathfrak{D}^{\mathrm{op}}$ and $\mathfrak{f}^{\mathrm{op}} \colon \mathfrak{D}^{\mathrm{op}} \to \mathfrak{C}^{\mathrm{op}}$ as well as $\mathfrak{g}^{\mathrm{op}} \colon \mathfrak{D}^{\mathrm{op}} \to \operatorname{Mor}(\mathfrak{D}^{\mathrm{op}}) = \operatorname{Mor}(\mathfrak{D})^{\mathrm{op}}$ and $\mathfrak{h}^{\mathrm{op}} \colon \mathfrak{C}^{\mathrm{op}} \to \operatorname{Mor}(\mathfrak{C}^{\mathrm{op}}) = \operatorname{Mor}(\mathfrak{C})^{\mathrm{op}}$ satisfy the conditions of Lemma A.0.0.2. Thus, $(\mathfrak{f}^{\mathrm{op}}, \mathfrak{i}^{\mathrm{op}})$ is an adjoint pair and hence, so is $(\mathfrak{i}, \mathfrak{f})$. $\square$

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