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by

Manfred Stadler

Faculty of Economics and Social Sciences
www.wiwi.uni-tuebingen.de

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Manfred Stadler*

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Abstract

The paper studies a two-stage location-price duopoly game in a disk city with consumer concentration around the city center. When consumers are uniformly distributed over the plane, unconstrained firms locate outside of the city. Consumer concentration, however, induces firms to locate nearer to each other and, when the degree of concentration is sufficiently high, inside of the city. Prices and firm profits decrease in the degree of consumer concentration. We explicitly solve the model for classes of cone-shaped, dome-shaped, and bell-shaped consumer densities. In all cases we identify a loss of welfare due to the strategic effect which causes the firms' spatial differentiation being too large.

Keywords: Location strategies; disk city; concentration of consumer demand

JEL Classification: C72, L13, R32

* University of Tübingen, Department of Economics, Nauklerstr. 47, D-72074 Tübingen, Germany.
e-mail: manfred.stadler@uni-tuebingen.de

1 Introduction

Location is one of the most important business strategies of competing firms. On the one hand, demand drives firms to locate close to the consumers, on the other hand, competition acts as an incentive to locate far away from the rivals. These countervailing forces raise the question about firms' optimal location. Ever since the seminal model of Hotelling (1929), the game theoretical analysis of firms' location choices has attracted much attention in business economics. In the tradition of Hotelling, a large variety of strategic location-price models have emerged which build on the concept of a one-dimensional city line. In the nowadays established basic model, d'Aspremont et al. (1979) have shown that in a duopoly market, where consumers have to incur a quadratic transportation cost when buying one unit of the considered product, the two firms locate at the opposite ends of the city line. Due to that result, the principle of maximal (spatial) differentiation was proposed. However, as Lambertini (1994) has shown, this principle only holds if the city border is binding. If it is not binding, the firms locate outside of the city. A shortcoming of this important result of an unconstrained optimum is that there is no convincing empirical evidence of such a location configuration of firms. Usually, the stores can be found inside of the city, not outside.

This discrepancy between the theoretical prediction of the location configuration and anecdotal empirical observation has opened a wide research agenda where the d'Aspremont et al. (1979) and Lambertini (1994) assumptions of the basic location model have been relaxed or generalized in several directions, thereby identifying additional explanatory factors for location decisions. These extensions include the number of firms (Brenner 2005), the number of outlets (Tabuchi 2012), the sequence of entry (Tabuchi, Thisse 1995), the transportation technology (Economides 1984, Stadler 2012), the elasticity of market demand (Böckem 1994, Woeckener 2002), uncertainty about consumer distribution (Casado-Izaga 2000, Meagher, Zauner 2004, 2005) and - last but not least - consumer concentration around the city center (Anderson et al. 1997, Meagher et al. 2008). In our context, the effect of consumer concentration is of special interest because its consideration allows for the analysis of an unconstrained interior location optimum which balances the diverging effects of consumer demand and competition.

Observing real urban areas, one rarely finds cities which can be adequately described by a one-dimensional Hotelling line. Therefore, another strand of literature has emerged which studies location decisions in two-dimensional space. Under the assumption of a constrained location inside of a rectangular city, Tabuchi (1994) and Veendorp, Majeed (1995) have shown that two firms locate at the opposite ends of the city and argued that firms minimize their distance in one dimension but maximize it in the other one (see Irmen and Thisse 1998 and Ansari et al. 1998 for generalizations). A corresponding result has been derived by Mazalov and Sakaguchi (2003) and Feldin (2012) for the alternative geometrical area of a disk city. They conclude that the two firms locate on the disk's perimeter, symmetrically across the city center. Similar to the case of a one-dimensional city line, this result raises the question regarding centripetal economic forces leading the firms to locate closer to each other, and in particular inside of the city. Indeed, all the mechanisms at work in the context of a Hotelling line should also influence the firms' location strategies on the plane.

In a promising research direction, Feldin (2012) has analyzed the location configuration of three firms on the unit disk when the consumer distribution is uniform. He was able to show that the additional third firm is sufficient for all competitors to locate inside of the city. In our view, an extension of the spatial competition model to oligopoly markets is an important step in order to improve the link between the theoretical prediction of firms' location decisions and the empirical evidence.

The present paper aims to generalize the disk-city model in another direction. The focus is on consumer concentration around the city center. To keep the model analytically tractable, we have to restrict the analysis to the case of two firms and symmetric consumer densities. Since the uniform distribution is covered as a special case, we will be able to concurrently determine the unconstrained location optimum in the basic disk-city model. We will show that, similar to the case of the one-dimensional Hotelling line, the firms locate outside of the city. Starting with this benchmark solution, we will study the additional demand effect of consumer concentration around the city center. In the case of a sufficiently high degree of demand concentration, we expect locations inside of the city.

The rest of the paper is organized as follows: Section 2 presents the disk-city model and derives the subgame perfect equilibrium of the location-price game. Some relevant classes of quasi-concave consumer distributions over the disk are introduced and applied to the model in Section 3. In Section 4, a welfare analysis, being of interest for a regulator, is presented. An extension to an infinite city area, which allows for a reinterpretation of geographical space in terms of horizontal product space, is provided in Section 5. Section 6 concludes the paper.

2 The Disk-City Model

In order to study the location behavior of two firms $i = 1, 2$, we consider a city in two-dimensional space where consumers with mass 1 are continuously distributed over a unit disk. The consumer density $f(r)$ is assumed to be polar-symmetric to the city center such that it only depends on the radial distance $r \in (0, 1)$ to the hub. We assume that consumers have to incur a transportation cost that is quadratically increasing in the distance to the seller firms. The consumer surplus is assumed to be sufficiently high such that every consumer buys one (and only one) unit of the homogeneous good from one of the rivals. These features generally imply that the line of indifferent consumers is perpendicular to the line connecting the two firms' locations (see, e.g., Tabuchi 1994).

Let us rotate the disk so that all symmetric firm locations have the same value $y_1 = y_2 = 0$ on the y-axis. Then the connecting line between the firm locations is the x-axis and the indifferent consumers are located along a vertical line parallel to the y-axis. Without loss of generality, we assume that $x_1 > x_2$, i.e., firm 1 is located

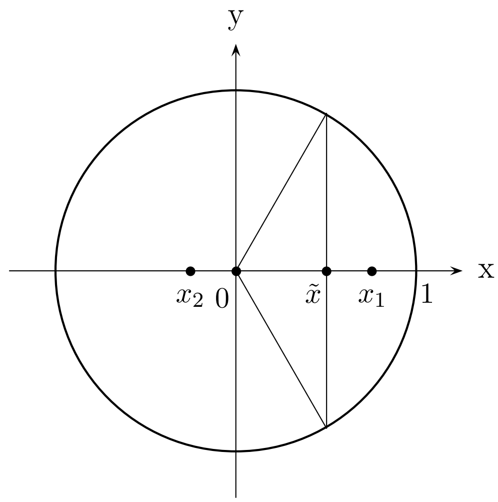

Figure 1: Vertical line of indifferent consumers at $x_2 < \tilde{x} < x_1$

to the right. When buying from firm $i = 1, 2$, a consumer located at the point (x, y) has to incur a total cost consisting of the (mill) price p_i, charged by firm i, and a transportation cost $t_i = y^2 + (x_i - x)^2$ which is quadratically increasing in the Euclidean distance between the location of the consumer and the location $(x_i, 0)$ of firm i. As is graphically shown in Figure 1, the indifferent consumers are then located along the vertical line at $\tilde{x} \in [x_2, x_1]$ which is derived from the consumers' indifference condition

$$p_1 + (x_1 - \tilde{x})^2 = p_2 + (x_2 - \tilde{x})^2$$

and is determined as

$$\tilde{x} = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2(x_2 - x_1)} \ . \tag{1}$$

Neglecting costs of production, the firms' profits are

$$\begin{aligned} \Pi^1(p_1, p_2, x_1, x_2) &= p_1 D(\tilde{x}) \\ \Pi^2(p_1, p_2, x_1, x_2) &= p_2(1 - D(\tilde{x})) \ , \end{aligned} \tag{2}$$

where the demand function of firm 1 is[1]

$$D(\tilde{x}) = 2\int_{\tilde{x}}^{1} f(r)\; r\; arccos(\tilde{x}/r)\; dr \tag{3}$$

and the demand function of firm 2 is $1 - D(\tilde{x})$.

We are interested in the subgame perfect equilibrium of the two-stage game, where firms $i = 1, 2$ strategically locate at x_i in stage 1 and charge prices p_i in stage 2. It proves useful to first solve the game for a general demand function $D(\tilde{x})$ and thereafter to empirically apply plausible specifications. This procedure corresponds to the one used by Anderson et al. (1997) and Meagher et al. (2008) for the case of a one-dimensional consumer distribution but extends it to a two-dimensional space.

In the second stage of the game, the firms maximize their profits (2) with respect to their prices. The first-order conditions read

$$\begin{aligned} D(\tilde{x}) + p_1 D'(\tilde{x})(d\tilde{x}/dp_1) &= 0 \\ 1 - D(\tilde{x}) - p_2 D'(\tilde{x})(d\tilde{x}/dp_2) &= 0\;. \end{aligned}$$

From (1), we obtain the derivatives $d\tilde{x}/dp_1 = -\; d\tilde{x}/dp_2 = -\; 1/(2(x_2 - x_1))$ and hence the implicit price equations

$$\begin{aligned} p_1 &= 2(x_2 - x_1)D(\tilde{x})/D'(\tilde{x}) \\ p_2 &= 2(x_2 - x_1)(1 - D(\tilde{x}))/D'(\tilde{x})\;. \end{aligned} \tag{4}$$

Substituting these expressions into the profit equations (2) gives

$$\begin{aligned} \Pi^1(x_1, x_2, \tilde{x}(x_1, x_2)) &= 2(x_2 - x_1)D(\tilde{x})^2/D'(\tilde{x}) \\ \Pi^2(x_1, x_2, \tilde{x}(x_1, x_2)) &= 2(x_2 - x_1)(1 - D(\tilde{x}))^2/D'(\tilde{x})\;, \end{aligned} \tag{5}$$

[1] The disk area on the right side of the vertical line $\tilde{x}$ is $2\int_{\tilde{x}}^{1}\; r\; arccos(\tilde{x}/r)\; dr$. Multiply at each point in this area by the local consumer density $f(r)$ to obtain the demand function $D(\tilde{x})$.

whereas the indifference condition (1) can be rewritten as

$$\tilde{x}(x_1, x_2) = \frac{x_1 + x_2}{2} + \frac{1 - 2D(\tilde{x})}{D'(\tilde{x})} \ . \tag{6}$$

Totally differentiating the implicit function (6) with respect to the firms' locations yields

$$\frac{d\tilde{x}}{dx_1} = \frac{d\tilde{x}}{dx_2} = \frac{D'(\tilde{x})^2}{2[3D'(\tilde{x})^2 + (1 - 2D(\tilde{x}))D''(\tilde{x})]} \ . \tag{7}$$

In the first stage of the game, the firms maximize their profit functions (5) with respect to their locations x_i. From the first-order conditions, we derive

$$\begin{aligned} \frac{D(\tilde{x})D'(\tilde{x})}{2D'(\tilde{x})^2 - D'(\tilde{x})D''(\tilde{x})} &= (x_2 - x_1)\frac{d\tilde{x}}{dx_1} \\ \frac{(1 - D(\tilde{x}))D'(\tilde{x})}{2D'(\tilde{x})^2 + (1 - D(\tilde{x}))D''(\tilde{x})} &= (x_2 - x_1)\frac{d\tilde{x}}{dx_2} \ . \end{aligned} \tag{8}$$

Rearranging terms and using (7) yields the fundamental equation

$$D''(\tilde{x}) = \frac{1 - 2D(\tilde{x})}{D(\tilde{x})(1 - D(\tilde{x}))} \ \ D'(\tilde{x})^2 \ , \tag{9}$$

which can be used to (numerically or even analytically) solve for $\tilde{x}$ in terms of the distribution parameters of consumer concentration. Substituting (7) and (9) into (8) gives the spatial distance between the rivals

$$x_1 - x_2 = 2(1 - D(\tilde{x}) + D(\tilde{x})^2)/D'(\tilde{x}) \ .$$

By additionally taking into account the indifference equation (6), we obtain the firms' locations

$$\begin{aligned} x_1 &= \tilde{x} - (1 - D(\tilde{x}))(2 - D(\tilde{x}))/D'(\tilde{x}) \\ x_2 &= \tilde{x} + (1 + D(\tilde{x}))D(\tilde{x})/D'(\tilde{x}) \end{aligned} \tag{10}$$

and from (4) and (5) the prices

$$p_1 = 4(1 - D(\tilde{x}) + D(\tilde{x})^2)D(\tilde{x})/D'(\tilde{x})^2$$
$$p_2 = 4(1 - D(\tilde{x}) + D(\tilde{x})^2)(1 - D(\tilde{x}))/D'(\tilde{x})^2 \tag{11}$$

and profits

$$\Pi^1 = 4(1 - D(\tilde{x}) + D(\tilde{x})^2)D(\tilde{x})^2/D'(\tilde{x})^2$$
$$\Pi^2 = 4(1 - D'(\tilde{x}) + D(\tilde{x}^2)(1 - D(\tilde{x}))^2/D'(\tilde{x})^2 \ . \tag{12}$$

In the symmetric subgame perfect equilibrium, the indifference line goes through the city center ($\tilde{x} = 0$) so that the rivals equally share the whole market demand. As is shown in the appendix, this implies that

$$D'(\tilde{x} = 0) = -2 \int_0^1 f(r)\, dr \ < 0 \tag{13}$$

and $D''(\tilde{x} = 0) = 0$. Note that $-D'(\tilde{x} = 0) > 0$ is the marginal density of the two-dimensional consumer distribution along the $\tilde{x} = 0$-line through the city center.

Then it follows from (10), (11), and (12), that in the subgame perfect equilibrium the firms locate at

$$x_1^* = -x_2^* = -3/(4D'(\tilde{x} = 0)) \ , \tag{14}$$

charge the prices

$$p^* = 3/(2D'(\tilde{x} = 0)^2) \ , \tag{15}$$

and realize the profits

$$\Pi^* = 3/(4D'(\tilde{x} = 0)^2) \ . \tag{16}$$

These equations are the two-dimensional counterparts to the solution equations derived by Anderson et al. (1997) for the case of a one-dimensional city. Interestingly, it proves sufficient to calculate the marginal density $2\int_0^1 f(r)\, dr$ of the two-dimensional consumer distribution at the city center. While Meagher et al. (2008) have presented a toolkit for special classes of one-dimensional consumer densities, we intend to provide explicit solutions for some two-dimensional classes of consumer distributions.

3 Classes of Feasible Distribution Functions

Since the mass of consumers is normalized to one, all feasible consumer densities at the distance r from the center with angle ϕ are restricted to

$$\int_0^{2\pi} \int_0^1 r f(r, \phi)\, dr\, d\phi \ = \ 1 \ .$$

In the special case of polar symmetry, this feasibility condition simplifies to

$$2\pi \int_0^1 r f(r)\, dr \ = \ 1 \ . \tag{17}$$

Now we are ready to apply some empirically plausible quasi-concave consumer densities $f(r)$ with the property $f'(r) < 0$ in order to study the impact of consumer concentration around the city center on the location policy of firms. The class of cone-shaped distributions satisfying the feasibility condition (17) is characterized by density functions of the type

$$f(r) = 1/\pi + \alpha(2/3 - r) \ ,$$

where $\alpha \in [0, 3/\pi]$ is an appropriate measure of the degree of consumer concentration around the city center.[2] Substituting $\int_0^1 f(r)\, dr = 1/\pi + \alpha/6$ into (13) gives

$$D'(\tilde{x} = 0) = -2/\pi - \alpha/3 \ .$$

Given this expression, the equations (14) to (16) imply the firms' location decisions

$$x_1^* = -x_2^* = \frac{9\pi}{4(6 + \alpha\pi)}$$

and price decisions

$$p^* = \frac{3}{2(2/\pi + \alpha/3)^2} \ ,$$

[2]This is a two-dimensional extension inter alia of the triangular density function as used by Tabuchi, Thisse (1995). A corresponding specification can be found in Mazalov and Sakaguchi (2003).

leading to the profits

$$\Pi^* = \frac{3}{4(2/\pi + \alpha/3)^2} \ .$$

Table 1 summarizes the results for some numerical values of the degree of consumer concentration α.

Table 1: Results for the location-price game with cone-shaped consumer concentration

	x_1^*	p^*	Π^*
$\alpha = 0$	1.178	3.701	1.851
$\alpha = 0.340$	1.000	2.667	1.333
$\alpha = 0.955$	0.785	1.645	0.822

In the limit case of $\alpha = 0$, the consumer distribution over the city is uniform, i.e., $f(r) = 1/\pi$. This implies the equilibrium locations $x_1^* = -x_2^* = (3/8)\pi \approx 1.178$ outside of the city, prices $p^* = (3/8)\pi^2 \approx 3.701$, and profits $\Pi^* = (3/16)\pi^2 \approx 1.851$.[3] The reason for locating outside of the city is the strategic effect that forces firms to sufficiently distance themselves in order to avoid a unilateral price deviation from the equilibrium. For intermediate degrees of concentration $\alpha \in (0, 3/\pi)$, the consumer distribution looks like a "marquee with open sides." The spatial distance of the firms, the prices, and the profits are all decreasing in the distribution parameter α. The centrifugal strategic effect of spatial differentiation is still at work, but the centripetal demand effect of a high consumer concentration around the city center induces firms to move closer to each other so that the intensity of competition increases, leading to a decline in profits. The concentration value of $\alpha = 3(3\pi - 8)/(4\pi) \approx 0.34$

[3]If the city border is binding, firms locate on the disk's perimeter at maximal distance. Substituting $x_1 = -x_2 = 1$, $D(\tilde{x} = 0) = 1/2$, and $D'(\tilde{x} = 0) = -2/\pi$ into the price equations (4) shows that the firms charge the prices $p = \pi < p^*$ and hence realize the profits $\Pi = (1/2)\pi < \Pi^*$. This solution is the constrained optimum as it was derived by Mazalov, Sakaguchi (2003) and Feldin (2012).

captures the setting where both firms locate exactly on the border of the city, i.e., at the ends of the horizontal diameter. When the degree of consumer concentration α is higher than this value, firms locate inside of the city. Finally, the limit case of $\alpha = 3/\pi \approx 0.955$ implies the density function $f(r) = 3(1-r)/\pi$. Consumer concentration is highest at the city center and declines to zero when approaching the outskirts of the city. Firms locate at $x_1^* = -x_2^* = (1/4)\pi \approx 0.785$, charge the prices $p^* = (1/6)\pi^2 \approx 1.645$ and realize the profits $\Pi^* = (1/12)\pi^2 \approx 0.822$.

Due to the upper limit $\alpha = 3/\pi$ of the distribution parameter, the cone-shaped density does not allow for an analysis of even more concentration, including the extreme case of a point distribution at the city center. This requires more flexible density functions. The desire to cover the whole range of consumer concentration motivated us to investigate another class of distributions by adapting the beta distribution to the two dimensions of the considered unit disk.[4] The polar-symmetric density functions that satisfy the feasibility condition (17) read

$$f(r) = (\beta+1)(1-r^2)^\beta/\pi \ ,$$

where $\beta \in [0, \infty]$ measures the degree of consumer concentration. For small parameter values $0 < \beta \leq 1$, the consumer distribution is concave and looks like a "dome" with its maximum at the city center. For higher values $1 < \beta < \infty$, the distribution has a concave-convex shape with a ring of inflection points at the distance $r = 1/\sqrt{2\beta - 1}$ from the city center and looks like a "bell."

We calculate $\int_0^1 f(r)\, dr = \Gamma(\beta{+}2)/[2\sqrt{\pi}\Gamma(\beta{+}3/2)]$, where Γ is the gamma function[5], and insert the expression into (13) to obtain

$$D'(\tilde{x}) = -\frac{\Gamma(\beta+2)}{\sqrt{\pi}\,\Gamma(\beta+3/2)} \ .$$

[4]This is a two-dimensional counterpart to the one-dimensional beta density function as used by Stadler (2012).

[5]Note that $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(n+1) = n\Gamma(n)$ for $n = \{1/2, 1\}$.

Then we derive from the equations (14) to (16) the firms' location decisions

$$x_1^* = -x_2^* = \frac{3\sqrt{\pi}}{4} \, \frac{\Gamma(\beta + 3/2)}{\Gamma(\beta + 2)} \ ,$$

the price decisions

$$p^* = \frac{3\pi}{2} \, \left(\frac{\Gamma(\beta + 3/2)}{\Gamma(\beta + 2)} \right)^2 \ ,$$

and hence the profits

$$\Pi^* = \frac{3\pi}{4} \, \left(\frac{\Gamma(\beta + 3/2)}{\Gamma(\beta + 2)} \right)^2 \ .$$

The limit case of $\beta = 0$ once again describes the uniform distribution with the consumer density $f(r) = 1/\pi$ as a benchmark. The spatial distance of the firms, the prices and profits are all decreasing in the degree of consumer concentration β. As before, the centripetal demand effect of a high consumer concentration around the city induces firms to move closer to each other so that the intensity of competition gets fiercer, leading to declining profits.

The intermediate case of $\beta = 1/2$ captures the setting where both firms locate exactly on the border of the city. Higher degrees of consumer concentration induce firms to locate inside of the city. When β approaches infinity, the consumer density converges to a mass point at the hub. The demand effect forces firms to locate back to back at the city center. Bertrand competition drives prices down to the unit production costs (which are normalized to zero) and no profits are left to the rivals.

Table 2: Results for the location-price game with dome-shaped and bell-shaped consumer concentration

	x_1^*	p^*	Π^*
$\beta = 0$	1.178	3.701	1.851
$\beta = 1/2$	1.000	2.667	1.333
$\beta = 1$	0.982	2.570	1.285
$\beta = 2$	0.859	1.968	0.984
$\beta = 3$	0.773	1.594	0.797
$\beta \to \infty$	0.000	0.000	0.000

The analysis of the centripetal force of demand concentration gives us the firms' unique equilibrium distances from the city center. The locations themselves, however, are not unique. Due to a rotation of the disk, infinitely many equivalent location equilibria exist. This means that the firms must somehow coordinate on one main axis through the city center. Unfortunately, this multiplicity problem cannot be resolved within the model. Instead, one would have to take into account additional considerations such as the location of adjacent cities in order to fix the main axis.

4 Socially Optimal Location

A regulator (such as the mayor and the city council), concerned with the surplus of the two firms as well as with that of all consumers in the market, aims to minimize the total transportation cost. In a symmetric configuration, each firm covers half of the market demand. To derive the socially optimal spatial differentiation, we consider the first quadrant of the unit disk. Here it proves useful to apply the polar coordinates $x = r\ cos\phi$ and $y = r\ sin\phi$ as shown in Figure 2. A consumer located at the distance r from the center with angle ϕ has to incur the transportation cost

$$t_1(r, \phi, \bar{x}_1) = (r\ sin\phi)^2 + (\bar{x}_1 - r\ cos\phi)^2 \ = \ \bar{x}_1^2 + r^2 - 2r\bar{x}_1 cos\phi \ ,$$

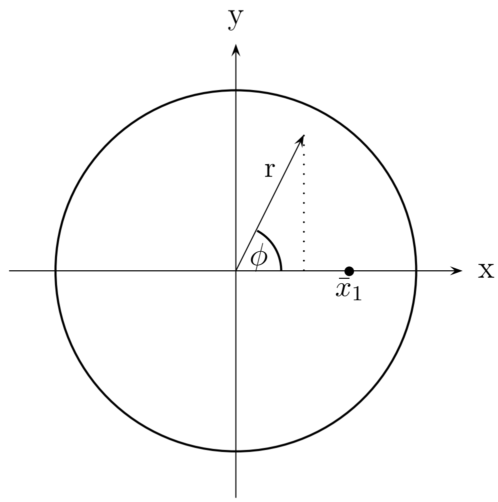

Figure 2: Consumer location and the transportation distance to firm 1

when buying from firm 1 located at $\bar{x}_1$. The total transportation cost of all consumers adds up to[6]

$$T = 4 \int_0^{\pi/2} \int_0^1 \left(\bar{x}_1^2 + r^2 - 2r\bar{x}_1 cos\phi \right) \; f(r) \; r \; dr \; d\phi \; .$$

In the case of the cone-shaped densities $f(r) = 1/\pi + \alpha(2/3 - r)$, we obtain

$$\begin{aligned} T &= \int_0^{\pi/2} \left(\frac{2}{\pi} \; \bar{x}_1^2 - \frac{24 - 2\alpha\pi}{9\pi} \; \bar{x}_1 cos\phi + \frac{15 - 2\alpha\pi}{15\pi} \right) \; d\phi \\ &= \bar{x}_1^2 - \frac{24 - 2\alpha\pi}{9\pi} \; \bar{x}_1 + \frac{15 - 2\alpha\pi}{30} \; . \end{aligned}$$

Minimizing of this cost with respect to the location $\bar{x}_1$ gives the first-order condition

$$\bar{x}_1 = -\bar{x}_2 = (12 - \alpha\pi)/(9\pi) \; .$$

[6]Note that the tiny area differential dA for the polar coordinates (r, ϕ) is $dA = r \; dr \; d\phi$.

Table 3: Comparison of equilibrium and socially optimal locations in the case of cone-shaped densities of consumer concentration

α	0	0.340	0.955
x_1^*	1.178	1.000	0.785
$\bar{x}_1$	0.424	0.387	0.318

As expected, the socially optimal spatial distance to the city center decreases in the concentration parameter α from $\bar{x}_1 = 4/(3\pi) \approx 0.424$ (when $\alpha = 0$) to $\bar{x}_1 = 1/\pi \approx 0.318$ (when $\alpha = 3/\pi \approx 0.955$). It can be shown that, for all feasible values of α, the socially optimal locations are nearer to the city center than the equilibrium locations are. Table 3 presents the results for some special values of α.

In case of the beta densities $f(r) = (\beta+1)(1-r^2)^\beta/\pi$, we are not able to derive a general solution. However, we can calculate explicit solutions for special values of the degree of concentration β. The limit case of $\beta = 0$ captures the uniform distribution and coincides with the solution $\bar{x}_1 = 4/(3\pi) \approx 0.424$, already derived in the case of the cone-shaped distribution with $\alpha = 0$.

The total transportation cost in the case of $\beta = 1$ is

$$T = (2/\pi)\int_0^{\pi/2} \left(\bar{x}_1^2 - (16/15)\,\bar{x}_1\; cos\phi + (1/3)\right)\, d\phi$$
$$= (1/2)\bar{x}_1^2 - (8/(15\pi))\bar{x}_1 + \pi/6\ .$$

Minimizing this cost with respect to the location $\bar{x}_1$ gives the first-order condition

$$\bar{x}_1 = -\bar{x}_2 = 16/(15\pi) \approx 0.340\ .$$

The total transportation cost in case of $\beta = 2$ is

$$T = (3/\pi)\int_0^{\pi/2} \left(\bar{x}_1^2 - (4/15)\bar{x}_1 cos\phi\right)\, d\phi$$
$$= (1/4)\bar{x}_1^2 - (16/(35\pi))\bar{x}_1 - 1/(8\pi)\ .$$

Minimization yields

$\bar{x}_1 = -\bar{x}_2 = 32/(35\pi) \approx 0.291$.

Of course, as the degree of consumer concentration approaches infinity, the socially optimal locations converge to the city center, i.e., $\bar{x}_1 = -\bar{x}_2 = 0$. Table 4 presents the results for some special values of β.

Table 4: Comparison of equilibrium and socially optimal locations in the case of beta densities of consumer concentration

	$\beta = 0$	$\beta = 1$	$\beta = 2$	...	$\beta \to \infty$
x_1^*	1.178	0.982	0.859	...	0.000
$\bar{x}_1$	0.424	0.340	0.291	...	0.000

The socially optimal spatial distance to the city center decreases in the concentration parameter from $\bar{x}_1 \approx 0.424$ (when $\beta = 0$) to $\bar{x}_1 = 0$ (when $\beta \to \infty$). The socially optimal locations are always nearer to the city center than the equilibrium locations are. The welfare implication is that firms should be forced by the regulator to locate nearer to the city center than they would do without such a regulation policy.

5 From Spatial to Product Differentiation

The location analysis is not restricted to the special case of a unit disk. Consider, for example, the uniform distribution. If the area of the disk is stretched to a radius of $R > 1$ or compressed to a radius $R < 1$, the equilibrium locations transform to

$\hat{x}_1^* = -\hat{x}_2^* = Rx_1^* = -Rx_2^*$

and the equilibrium prices and profits transform to $\hat{p}^* = R^2 p^*$ and $\hat{\Pi}^* = R^2 \Pi^*$. The larger the city area is, the greater the distance between the firms and the higher the prices and profits.[7] For example, in the case of a uniform consumer density of 1, the city radius takes the lower value $R = 1/\sqrt{\pi} \approx 0.564$, leading to the closer locations $\hat{x}_1^* = -\hat{x}_2^* = (3/8)\sqrt{\pi} \approx 0.665$ and the lower prices $\hat{p}^* = (3/8)\pi \approx 1.178$ and profits $\hat{\Pi}^* = (3/16)\sqrt{\pi} \approx 0.589$.

A radius approaching infinity allows for the application of further classes of distribution functions with infinite support. These distributions might serve as acceptable approximations of consumer densities in real cities. However, the actual advantage of such distributions is that they enable an appropriate analysis of firms' strategic product positioning in a two-dimensional product space.[8]

As is well known, models of spatial differentiation on the plane can equivalently be treated as models of horizontal differentiation in a two-dimensional product space. All that is necessary is to reinterpret the spatial consumer location in terms of their preferences for alternative product attributes and their transportation cost in terms of a loss of utility due to the fact that they are not able to buy the product with the most preferred attributes. In this modified interpretation of a positioning-price game, the firms $i = 1, 2$ produce the heterogeneous goods 1 and 2 such that product differentiation and the degree of heterogeneity is endogenized by the strategic (brand) positioning of the firms.

As an important example, let us consider the two-dimensional normal distribution with means zero and independent variances σ^2 which is described by the density function

$$f(r) = \frac{1}{2\pi\,\sigma^2} e^{-(1/2)r^2/\sigma^2} \ .$$

[7] Additionally, the population of the city - an alternative measure of the “city size” - does not have to be restricted to the mass of 1. An increasing mass would leave the location and price decisions of the firms unchanged but would proportionally increase their profits.

[8] In case of finite support, a rectangular (or a quadratic) product space would certainly be preferable to a disk space. A quadratic product space, however, is not tractable for an analysis of demand concentration since the property of polar-symmetry is not satisfied. In the case of infinite support, the geometrical difference between a disk and a square diminishes such that this problem is resolved.

This distribution implies polar-symmetric demand concentration around the modal consumer preference.[9] The marginal density of the demand distribution (13) can be calculated as

$$D'(\tilde{x}=0) = -\,2\int_0^\infty \frac{1}{2\pi\,\sigma^2} e^{-(1/2)r^2/\sigma^2} dr = -\int_{-\infty}^\infty \frac{1}{2\pi\,\sigma^2} e^{-(1/2)r^2/\sigma^2} dr = -\,\frac{1}{\sqrt{2\pi}\,\sigma}\;.$$

Then it follows from (14), (15), and (16), that the firms locate at

$$x_1^* = -x_2^* = \frac{3\sqrt{\pi}\,\sigma}{2\sqrt{2}} \;\approx\; 1.88\,\sigma\;,$$

charge the prices

$$p^* = 3\pi\sigma^2 \;\approx\; 9.425\,\sigma^2\;,$$

and realize the profits

$$\Pi^* = (3/2)\pi\sigma^2 \;\approx\; 4.712\,\sigma^2\;.$$

Due to the special properties of the normal distribution, this equilibrium solution of the positioning-price game coincides with that of a one-dimensional (product) space (see Meagher et al. 2008). The variance σ^2 clearly serves as an inverse measure of demand concentration. As in the case of a disk, demand concentration induces firms to position the attributes of their products nearer to each other even if this causes declining prices and profits. In the limit case of maximum demand concentration ($\sigma^2 \to 0$), product differentiation and hence market heterogeneity entirely vanish. As a consequence, Bertrand competition drives the prices down to the (zero) unit production costs and profits decline to zero.

[9] As pointed out e.g. by Ansari et al. (1994), the distribution of consumer preferences in many markets is unlikely to be uniform.

6 Conclusion

Consumer demand is usually concentrated around the centers of cities. Even if strategic effects of location induce firms to locate far away from each other, this demand effect stimulates firms to move towards the city center and thus closer to each other. The combination of a two-dimensional city area, perhaps best described by a disk model, and a non-uniform consumer distribution with concentration around the city center can explain the firms' location strategies in a rather satisfying way. In particular, it can be shown that in such an environment even duopolistic firms are likely to locate inside of the city when the degree of consumer concentration is sufficiently high. When the radius of the disk approaches infinity, the model of spatial differentiation can be reinterpreted in terms horizontal product differentiation. In this setting, it is the modal concentration of consumer preferences that induces firms to position the product attributes close to each other.

Of course, an interesting extension of the model would be the analysis of an oligopoly market with more than two firms. However, even in the case of uniform distributions, the theoretical analysis of oligopolistic competition in geographical or product space proves to be very complex. An alternative is to apply techniques of numerical simulation. On the one hand, such methods enable the search for unique or multiple location equilibria of oligopolies, even with non-uniform consumer densities. On the other hand, explicit solutions impose consistency on the theoretical thinking regarding the strategic interaction of firms. The present paper should be seen as a small piece in the mosaic of this broad framework.

7 Appendix

As described in the main text, the demand function of firm 1 reads

$$D(\tilde{x}) = 2\int_{\tilde{x}}^{1} f(r)\; r\; arccos(\tilde{x}/r)\; dr \tag{A.1}$$

and the demand of firm 2 is $1 - D(\tilde{x})$. By using Leibniz's rule and integrating by parts, we obtain the derivatives

$$\begin{aligned} D'(\tilde{x}) &= -2\int_{\tilde{x}}^{1} \frac{f(r)}{\sqrt{1-(\tilde{x}/r)^2}}\; dr \\ &= -2\left[f(1)\sqrt{1-\tilde{x}^2} - \int_{\tilde{x}}^{1} f'(r)\sqrt{r^2-\tilde{x}^2}\; dr\right] \end{aligned} \tag{A.2}$$

and

$$D''(\tilde{x}) = 2\tilde{x}\left[\frac{f(1)}{\sqrt{1-\tilde{x}^2}} - \int_{\tilde{x}}^{1} \frac{f'(r)}{\sqrt{r^2-\tilde{x}^2}}\; dr\right]\;. \tag{A.3}$$

In the case of $\tilde{x} = 0$ (symmetric firm location around the city center), one obtains[10]

$$D(\tilde{x} = 0) = 1/2\;. \tag{A.1'}$$

The first two derivatives amount to

$$D'(\tilde{x} = 0) = -2\int_{0}^{1} f(r)\; dr\;\; < 0 \tag{A.2'}$$

and

$$D''(\tilde{x} = 0) = 0\;. \tag{A.3'}$$

[10]Note that $arccos(0) = \pi/2$ and $2\pi\int_0^1 rf(r)\; dr\; = 1$ for all feasible polar-symmetric consumer densities.

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