Interior-Boundary Conditions as a Direct Description of QFT Hamiltonians

Dissertation

der Mathematisch-Naturwissenschaftlichen Fakultät

der Eberhard Karls Universität Tübingen

zur Erlangung des Grades eines

Doktors der Naturwissenschaften

(Dr. rer. nat.)

vorgelegt von

Julian Schmidt

aus Stuttgart

Tübingen

2019

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	01.04.2019
Dekan:	Prof. Dr. Wolfgang Rosenstiel
1. Berichterstatter:	Prof. Dr. Stefan Teufel
2. Berichterstatter:	Prof. Dr. Marcello Porta

Dissertation

Interior-Boundary Conditions as a Direct Description of QFT Hamiltonians

Julian Schmidt

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Contents

Summary **iv**
1 German . . . iv
2 English . . . v

List of publications **vi**

Personal contribution **vii**

1 Introduction **1**
1.1 A picture . . . 1
1.2 Brief history of self-energy renormalisation . . . 1

2 Objectives **6**

3 Results and Discussion **7**
3.1 Introduction to interior-boundary conditions . . . 7
3.1.1 A simple example . . . 7
3.1.1.1 Tunneling effects on the line . . . 7
3.1.1.2 Interior-boundary conditions on the line . . . 8
3.1.2 Three dimensional IBC . . . 9
3.1.2.1 Enlarging the domain . . . 9
3.1.2.2 Computing the non-symmetry . . . 10
3.1.2.3 The operators A and T_μ . . . 11
3.1.2.4 Renormalisation . . . 11
3.1.3 The abstract approach . . . 12
3.1.3.1 Vector notation . . . 12
3.1.3.2 Abstract reasoning for IBC's . . . 13
3.1.3.3 Action of the Hamiltonian H . . . 14
3.1.3.4 Self-adjointness of H . . . 14
3.1.3.5 The domain $\mathfrak{D}$ . . . 15
3.2 Evolution of the IBC method . . . 16
3.3 Conclusion . . . 18
3.4 Outlook: Towards more singular interactions . . . 18

Bibliography **21**

Appendix **24**
a) Accepted publications . . . 24
1) Article [LSTT18] . . . 24
2) Article [LS18] . . . 62
b) Submitted manuscripts . . . 105
3) Article [Sch18] . . . 105

4) Article [Sch19] 140

Summary

1 German

In dieser Dissertation werden quantenfeldtheoretische Modelle betrachtet, bei denen eine feste Anzahl Teilchen mit einem bosonischen Feld welchselwirkt. Die Teilchen sind dabei *linear* and das Feld gekoppelt. Das bedeutet, dass die Wechselwirkungsterme im Modell eine Summe aus Erzeugungs- und Vernichtungsoperatoren sind. Die Wechselwirkung zwischen den Teilchen, zum Beispiel Elektronen, und dem Feld stellt sich daher so dar, dass ein bosonisches Feldquant durch die Teilchen erzeugt und vernichtet werden kann, sobald beide sich treffen. Da elementare Teilchen punkt förmig sind, führt dies zu Schwierigkeiten, die allgemein als *UV-Problem* bezeichnet werden.

Hier wird eine neue Methode entwickelt, die es erlaubt, den Hamiltonoperator und seinen Definitionsbereich für eine Klasse dieser Modelle direkt hin zu schreiben. Die Methode ist insbesondere auf das in der Literatur häufig betrachtete Nelson-Modell anwendbar. Bisher war ein *Renormierungsverfahren* erforderlich, um einen selbstadjungierten Hamiltonoperator für dieses Modell zu konstruieren, und damit die Lösbarkeit der Schrödingergleichung für alle Zeiten zu garantieren. In einem Renormierungsverfahren wird der Operator als Grenzwert einer Operatorenfolge definiert. Für jedes der Folgenglieder ist die Wechselwirkung nicht punktförmig, sondern wird ad hoc regularisiert, indem die Teilchen als ausgedehnt betrachtet werden. Der Grenzwert dieser Folge wird als *renormierter Operator* bezeichnet. Der Definitionsbereich und die Wirkung der Operatoren lässt sich für die regularisierten Wechselwirkungen leicht angeben, für den Grenzwert gilt dies jedoch nicht. Erst im Jahr 2017 wurde von Marcel Griesemer und Andreas Wünsch eine von Edward Nelson bereits 1964 geäußerte Vermutung über den Definitionsbereich des renormierten Operators bestätigt. Über seine Wirkung auf Wellenfunktionen aus dem Definitionsbereich war weiterhin nichts bekannt.

Wir geben hier erstmals eine direkte Beschreibung des Operators und seines Definitionsbereichs an. Außer auf das (massive) Nelson-Modell lässt sich die Methode auch auf das Fröhlich Polaron, das Nelson-Modell mit masselosen Bosonen, sowie auf relativistische Varianten des Modells anwenden.

Es wird gezeigt, dass sich die Hamiltonoperatoren für derartige Modelle mittels so genannter *interior-boundary conditions (IBC's)* definieren lassen. Das sind Bedingungen, die Wellenfunktionen erfüllen müssen, um im Definitionsbereich des Operators zu liegen. Dabei werden die Werte der Wellenfunktion von n Bosonen, mit einem der Bosonen in der Nähe eines Teilchens, in Beziehung gesetzt zu den Werten der Wellenfunktion ohne dieses eine Boson. Eine solche Bedingung führt im Allgemeinen zu einem singulären Verhalten der Wellenfunktionen an Punkten wo sich Bosonen und Teilchen treffen. Auf diesen singulären Funktionen wird anschließend die Wirkung des Hamiltonoperators direkt definiert, ohne Verwendung eines Renormierungsverfahrens. Allerdings ist es a posteriori möglich, den direkt definierten Hamiltonoperator mit dem renormierten Operator zu identifizieren. Wegen der spezifischen Form des Definitionsbereichs lässt sich auch Nelsons Vermutung leicht bestätigen.

2 English

In this doctoral thesis models of quantum field theory are considered, where a fixed number of particles interacts with a bosonic quantum field. The particles and the field are coupled *linearly*. That means that the interaction terms in the model are the sum of creation and annihilation operators. Therefore, the quanta of the bosonic field can be emitted and absorbed by the particles, e.g. electrons, when both meet. Because elementary particles are point-like, this leads to difficulties, which are called an UV-problem.

Here a novel method is developed which allows to directly write down the Hamiltonian and its domain for a class of such models. In particular, the method can be applied to the Nelson model, which is well known in the literature. In order to construct a self-adjoint Hamiltonian for this model, and in this way prove the existence of a solution to the Schrödinger equation for all times, a *renormalisation procedure* used to be necessary. The operator is defined as the limit of a sequence of operators. For each of the elements of this sequence, the interaction is not point-like, but regularised. This ad hoc regularisation means that the particles are considered to be extended. The limit of this sequence is called the *renormalised operator*. For the regularised interactions, the domain and the action of the operators can be characterised easily. For the limit however, this is not true. It was only in 2017 when Marcel Griesemer and Andreas Wünsch confirmed a conjecture about the domain of the renormalised operator by Edward Nelson, which had been published already in 1964. Nothing was known about its action on wave functions in the domain.

Here we give, for the first time, a direct description of the operator and its domain. In addition to the (massive) Nelson model, the method can be applied to the Fröhlich Polaron, the Nelson model with massless bosons, as well as to relativistic variants of the model.

It is shown, that the Hamiltonians for such models can be defined via so called *interior-boundary conditions (IBC's)*. These are conditions, which wave functions have to fulfil in order to be an element of the domain of the operator. The values of the n-boson wave function, with one boson in the vicinity of the particle, are linked to the values of the wave function without this one boson. Such a condition in general leads to a singular behaviour of the wave functions at points where bosons and particles meet. On these singular functions, the action of the Hamiltonian is defined directly, without using a renormalisation procedure. It is however possible to identify the directly defined operator with the renormalised Hamiltonian. Due to the specific form of the domain, Nelson's conjecture can easily be confirmed.

List of publications

a) Accepted publications

1) **Particle Creation at a Point Source by Means of Interior-Boundary Conditions**, Jonas Lampart, Julian Schmidt, Stefan Teufel and Roderich Tumulka. Published in *Mathematical Physics, Analysis and Geometry*, 2018, Volume 21, Number 2. Cited in this thesis as [LSTT18] and included in the Appendix a).

2) **On the domain of Nelson-type Hamiltonians and abstract boundary conditions**, Jonas Lampart and Julian Schmidt. Originally, the title of the manuscript was "On Nelson-type Hamiltonians and abstract boundary conditions". Accepted in August 2018 for publication in *Communications in Mathematical Physics*. Cited in this thesis as [LS18] and included in the Appendix a).

b) Submitted manuscripts

3) **On a Direct Description of Pseudorelativistic Nelson Hamiltonians**, Julian Schmidt. In November 2018 submitted for publication in *Letters in Mathematical Physics*. Cited in this thesis as [Sch18] and included in the Appendix b).

4) **The Massless Nelson Hamiltonian and its Domain**, Julian Schmidt. Submitted in October 2018 as a contribution to the *Springer-INdAM volume "Mathematical Challenges of Zero-Range Physics"*. Cited in this thesis as [Sch19] and included in the Appendix b).

Personal contribution

In mathematics, the notion of a lead author of a publication is highly uncommon. Instead, authors of scientific publications are simply ordered alphabetically.

The categories “Data generation” and “Analysis and interpretation” do not apply.

a) Accepted publications

1) The writing of the article “Particle Creation at a Point Source by Means of Interior-Boundary Conditions” ([LSTT18]) was done in close collaboration by all four authors. Most of the ideas that are used in the more technical part are due to Jonas Lampart and Stefan Teufel, in particular the Propositions 5.2, 5.3 and 5.4 and their proofs in Appendix A. I contributed to these parts by providing clarifications and by writing them up. The proof of Proposition 6.1 is due to me. Stefan Keppeler contributed to an earlier version of this proof, which can already be found in my Diplomarbeit [Sch14]. I succeeded in setting up the framework of the sections 4 and 7. In particular, I provided the proofs of Theorems 4.1 and 4.4.

Scientific ideas by the candidate are estimated as 25%, paper writing as 40%.

2) The concept of the article “On Nelson-type Hamiltonians and abstract boundary conditions” ([LS18]) was developed by Jonas Lampart and myself while I was staying with him at the Laboratoire Interdisciplinaire Carnot de Bourgogne in Dijon. Afterwards, the writing was done in collaboration, while each of us contributed equal parts. The final version of the important Lemma 3.10, which improved an earlier version, is due to me. While I observed that there is a need for Lemma 2.2, its proof, as well as the one of Lemma A.1, was done entirely by Jonas Lampart. It was me who came up with an idea how to prove Proposition 4.2.

Scientific ideas by the candidate are estimated as 50%, paper writing as 50%.

b) Submitted manuscripts

I am the single author of the article **3)** “On a Direct Description of Pseudorelativistic Nelson Hamiltonians” ([Sch18]) as well as of the contribution **4)** “The Massless Nelson Hamiltonian and its Domain”([Sch19]). Jonas Lampart and Stefan Teufel provided me with a lot of helpful advice and feedback.

Scientific ideas and paper writing done by the candidate are both at 100%.

1 Introduction

1.1 A picture

What one has in mind, when talking about quantum field theory, is a picture that is actually quite mundane. We think of elementary particles as little balls, coloured according to their species, flying around freely. From time to time, two or more of these balls meet, and then one of them[1]disappears. This is called *annihilation of particles.* Some of the balls have the ability to spontaneously emit balls of another colour, then one speaks about particle *creation.* The particles, that can be emitted or absorbed, are the quanta of the field and particles of the type whose number is fixed are occasionally called nucleons. It is imagined that the emission and absoption of field quanta leads to an effective interaction between the nucleons, one that does not arise from an interaction potential but is caused solely by the exchange of particles.

While the process described above may seem fairly easy, it is actually very hard to make sense of this in a mathematically rigorous way. This is because one thing is surely wrong about this picture: elementary particles are not extended in space but in fact point-like. Consequently, the emission and absorption happens at exactly one position. To each of these two processes, there is an operator associated, the annihilation and creation operators. If one tries to put a delta distribution in these operators, which would be necessary for point-shaped particles, a so called *ultraviolet divergence* occurs.

1.2 Brief history of self-energy renormalisation

The first succesfull attempt to define a Hamiltonian for the elementary interaction process described above, was done by Edward Nelson in 1964. In the field theoretic model, which is nowadays called the Nelson model, a fixed number of nonrelativistic particles interacts with a relativistic, massive, bosonic quantum field.

There are also effective models in condensed matter physics that can be described in a field theoretic way. A well known case is the so called large or Fröhlich Polaron. In this model, some of the lattice modes of a crystal are considered as bosonic quasiparticles, called (optical) phonons, which can be created and annihilated in a non-local way by the electrons that move in the lattice, [FPZ50]. The combined system, or, more precisely, the ground state of the corresponding Hamiltonian, is then called a *Polaron.* The term Polaron was widely used in the following years to also label genuinely field theoretic models of creation and annihilation of elementary particles, [Gro73], or to describe effective models with different bosonic quasiparticles, such as, for example, Bogoliubov modes in a BEC, see e.g. [TCO$^+$09].

[1]Or both, but this case is not considered in this thesis.

The basic notion that is important for these kind of models is the *Fock space*, which is the natural Hilbert space for systems where the number of particles can be arbitrary. If $\mathfrak{h}$ is the one-particle Hilbert space, the bosonic (symmetric) Fock space that belongs to $\mathfrak{h}$ is given by

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathrm{Sym}_n \left(\bigotimes_{k=1}^{n} \mathfrak{h}_k \right)$$

The first term in this direct sum, where $n = 0$, is simply the complex numbers $\mathbb{C}$ and is called the *vacuum*. When $\mathfrak{h} = L^2(\mathbb{R}^d)$, we have $\mathrm{Sym}_n(\bigotimes_{k=1}^n \mathfrak{h}_k) = L^2_{\mathrm{sym}}(\mathbb{R}^{dn})$, the L^2-functions, which are symmetric under exchange of particles. Here $d \in \mathbb{N}$ is the dimension of the physical space. The configuration spaces of fixed particle number are $Q_n = \mathbb{R}^{dn}$, but if the number n of particles can be any number in $\mathbb{N}$, the full configuration space is actually the (disjoint) union, $Q = \dot{\bigcup}_{n=0}^{\infty} Q_n$. The space of square integrable functions on this non-connected space is the (infinite) sum of L^2-spaces:

$$L^2(Q) = L^2\left(\dot{\bigcup}_{n=0}^{\infty} Q_n\right) = \bigoplus_{n=0}^{\infty} L^2(Q_n) = \mathcal{F} \tag{1.1}$$

That is, the Fock space can, for $\mathfrak{h} = L^2(\mathbb{R}^d)$, be regarded as the space of square integrable functions on configuration space.

The spaces with fixed particle number n are the *sectors* of $\mathcal{F}$. On Fock spaces, there are some special operators that are widely used, the symbol $\mathrm{d}\Gamma(\omega)$ for example is called the *second quantisation* of ω. For multiplication operators ω, it acts on the sectors as the sum $\sum_{k=1}^n \omega(p_k)$. If ω is real, then its second quantisation is self-adjoint (on some domain). The operator $N := \mathrm{d}\Gamma(1)$ is the *number operator*. These operators still conserve the number of particles in the sense that their value in the n-th sector does only depend on the vectors in this sector. To model creation and annihilation of particles, one needs the ladder operators. In the case of L^2 spaces, the action of the annihilation operator is given by

$$(a(v)\psi)_n(x_1, \dots, x_n) := \sqrt{n+1} \int_{\mathbb{R}^d} \overline{v(x)} \psi_{n+1}(x_1, x_2, \dots, x_n, x) \, \mathrm{d}x$$

Here $v \in L^2(\mathbb{R}^d)$ is called the *form factor*. The value of $a(v)$ depends on the wave function ψ_{n+1} one sector above. The creation operator $a^*(v)$ is defined in such a way that it is the adjoint of $a(v)$ whenever the adjoint is well defined. More concretely

$$(a^*(v)\psi)_n(x_1, \dots, x_n) := n^{-1/2} \sum_{j=1}^{n} v(x_j) \psi_{n-1}(x_1, \dots, \hat{x}_j, \dots, x_n) \,.$$

The hat on top of k_j here means that this variable is omitted. These operators obviously do not conserve the number of particles. Note the most important feature of the creation operator: For $v \in \mathfrak{h}$, it is bounded with norm equal to the norm of v. If however $v \notin \mathfrak{h}$, then, no matter how regular ψ_{n-1} is[2], the expression $(a^*(v)\psi)_n$ is not an element of the n-th sector of Fock space. In the cases we are interested in, the form factors $\hat{v}$ in momentum space are distributions at least in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ but not square integrable on the

[2] As long as it is nonzero, $\psi_{n-1} \neq 0$.

whole space. That is, the integral of $|\hat{v}|^2$ over a ball of radius Λ in $\mathbb{R}^d$ diverges as Λ tends to infinity. The square integrability fails at infinity, this is called the *ultraviolet problem.*

The formal expression (that is, without specifying the domain) for a Hamiltonian that describes a fixed number M of nonrelativistic particles with mass one half is given by $-\sum_{i=1}^M \Delta_{x_i}$ and acts on the Hilbert space $L^2(\mathbb{R}^{dM})$.

The bosonic field is in momentum representation described by the second quantisation $\mathrm{d}\Gamma(\omega)$ of the field dispersion ω, acting on the bosonic Fock space $\mathcal{F}$. A non-interacting system of particles and field is modeled by the *free operator* $L = -\sum_{i=1}^M \Delta_{x_i} + \mathrm{d}\Gamma(\omega)$. Note that this operator acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{dM}) \otimes \mathcal{F}$ and, as used here, is a mixture between position representation (for the particles) and momentum representation (for the field). Given a non-negative function ω, there is a domain $D(L) \subset \mathcal{H}$ in this Hilbert space, where L is self-adjoint.

The formal expression of a Hamiltonian for an interacting system with interaction modeled by the form factor $\hat{v}$ in momentum space is

$$L + \sum_{i=1}^{M} a^*(\mathrm{e}^{-\mathrm{i}\langle x_i, \cdot\rangle}\hat{v}) + a(\mathrm{e}^{-\mathrm{i}\langle x_i, \cdot\rangle}\hat{v}) \tag{1.2}$$

Note that, via the phase factors, the interaction depends on the positions of the particles, as it is to be expected if annihilation and creation of bosons occurs at the position of the particles.

In [Nel64], Nelson considered a model where the dispersion of the bosons is $\omega(k) = \sqrt{k^2 + m^2}$ for some $m > 0$, which is called the boson mass. The interaction between particles (also called nucleons) and the bosons (or mesons) happens at the position of the particles. Due to the relativistic nature of the meson field, this does not lead to a constant form factor $\hat{v}$ (which is the Fourier transform of a delta distribution), but instead one has $\hat{v}(k) = \omega(k)^{-1/2}$. While this form factor is vanishing in the limit $k \to \infty$, it is not square integrable on $\mathbb{R}^3$. In order to make sense of the formal expression (1.2), a momentum cutoff is introduced. This means that $\hat{v}(k)$ is replaced by $\chi_\Lambda(k)\hat{v}(k)$, where χ_Λ is the characteristic function of a ball of radius Λ in $\mathbb{R}^d$. For finite Λ, the formal expression does in fact define a self-adjoint operator on the domain of the free operator $D(L)$, it is called the *cutoff Hamiltonian* H_Λ. The interaction part of the operator, consisting of the creation and annihilation operators, is for $\Lambda < \infty$ an operator perturbation of L; their sum can be defined via the Kato-Rellich theorem. While it was known that H_∞ can not be defined by all standard techniques, in the form above it was not clear what happens to the cutoff operator in the limit $\Lambda \to \infty$. In his article, Nelson applied a unitary, cutoff dependent *dressing transformation* U_Λ to the cutoff Hamiltonian[3]. It turns out that the transformed Hamiltonian is the sum of several terms, which can be grouped in the following way:

$$U_\Lambda H_\Lambda U_\Lambda^* = L + W_\Lambda - E_\Lambda \,.$$

Here L is the free operator, E_Λ is just multiplication by a constant that diverges as $\Lambda \to \infty$. The family W_Λ is an operator perturbation of L for $\Lambda < \infty$, but not in the limit as the cutoff tends to infinity. However, the quadratic form of W_∞ is form bounded

[3]Nelson attributed the tranformation to E. P. Gross. It is basically the Weyl operator on the Fock space for the field for a certain x-dependent function, made up of $\chi_\Lambda\hat{v}$ and ω.

by the form of L. Thus the sum of these forms can be defined and this gives rise to a unique self-adjoint operator, the *form sum* of L and W_∞, which we will denote by $L\dot{+}W_\infty$[4]. Because U_Λ has a strong limit U_∞, Nelson could prove that in strong resolvent sense

$$H_\Lambda + E_\Lambda \xrightarrow{\Lambda\to\infty} U_\infty^*(L\dot{+}W_\infty)U_\infty =: H_\infty\,.$$

This procedure is called (self-)energy *renormalisation* and E_Λ is the *renormalisation constant*, which, in this case, diverges logarithmically. Then H_∞ is the renormalised Hamiltonian, considered to be the correct Hamiltonian for the system. This can be understood as follows: the cutoff operator in some sense goes to $-\infty$ and this is compensated by adding a positive diverging constant E_Λ.Which part of H_Λ exactly causes the scalar divergence, is obscured by the use of the dressing transformation.

What can be said about H_∞, which is also called the Nelson Hamiltonian? By construction the operator is bounded from below. Concerning the operator domain however, one only gets $D(H_\infty) = U_\infty^* D(L\dot{+}W_\infty)$, and neither the domain of the form sum nor the mapping properties of U_Λ^* were known to Nelson. This is why, at the very end of his article [Nel64], he asked the following questions:

> It would be interesting to have a direct description of the operator H_∞. Is $D(H_\infty) \cap D(L^{1/2}) = \{0\}$?

While Nelson proved strong convergence of the resolvents of the cutoff operators H_Λ to the resolvent of H_∞, in [Amm00] it was observed that the cutoff operator in the Nelson model converges in fact in norm resolvent sense to the Nelson Hamiltonian. In [GHL14], the renormalisation procedure was carried out by using functional integration.

For the Fröhlich Hamiltonian, one can also employ a dressing transformation U_Λ. Because the form factor $\hat{v}(k) = |k|^{-1}$ is more regular, the operator that is the analogoue of W_Λ is actually operator bounded by L[5]. By investigating in detail the mapping properties of U_∞, in [GW16] Griesemer and Wünsch were able to give a characterisation of the operator domain $D(H_\infty)$ for the Fröhlich Hamiltonian. In particular, they showed that $D(H_\infty) \subset D(L^s)$ for all $s < 3/4$ but $D(H_\infty) \cap D(L^{3/4}) = \{0\}$. The latter is a variant of Nelson's second question in the quote above. Here it is proved that H_∞ is not an operator perturbation of L, proving the analogous result for $D(L^{1/2})$ for the Nelson Hamiltonian would prove that it is not even a form perturbation of the free operator. In [GW18], Griesemer and Wünsch extended their earlier work on the Fröhlich Hamiltonian. They refined the result on the mapping properties of U_∞ and used the fact that for the form domain of the Nelson Hamiltonian it still holds that $D(H_\infty^{1/2}) = U_\infty^* D(L^{1/2})$. In this way, they gave the answer to Nelson's second question: Yes, in fact $D(H_\infty) \cap D(L^{1/2}) = \{0\}$ while $D(H_\infty^{1/2}) \subset D(L^s)$ for all $s < 1/2$.

When computing the effect of the Gross transformation on the cutoff Hamiltonian for the Nelson model, one first computes the effect on p and then concludes what the effect on p^2 is. This method does not extend directly to more general dispersion relations.

[4]Nelson applied in his article this (by then not well known) theorem; it is today called the KLMN-Theorem (Kato, Lions, Lax and Milgram, Nelson)

[5]Although $v \notin L^2$ for the Fröhlich Hamiltonian, the limiting operator H_∞ could still be defined as a form perturbation. This does however not give as much information on $D(H_\infty)$ as the renormalisation procedure does.

Apparently, this was the reason why Eckmann chose another approach, when he aimed at renormalising a model with a relativistic dispersion $\sqrt{p^2+1}$ in [Eck70]. The approach is based on lecture notes of Hepp, [Hep69], and it employs the powerful tool of setting up self-adjoint operators via their resolvent, see also [Kat80, VIII §1.1]. It essentially means that, instead of defining a self-adjoint operator directly on some domain, one can also define an operator valued function on $\mathbb{C}$ and then show that it indeed is the resolvent of an operator. Eckmann first expanded the resolvent of the cutoff operator as an infinite series of certain combinations of creation and annihilation operators in the standard way. The domain in $\mathbb{C}$, where this series converges, does however depend on the value of Λ. Now he reordered the series in a a convenient way[6]and showed that this series extends the former but converges uniformly in Λ. Thus in the limit there is a domain in $\mathbb{C}$, where the operator valued function is defined and is the resolvent of a self-adjoint operator. This method gives a bit more insight into where the scalar divergence of the operator occurs. Eckmann discovered that the renormalisation constant E_Λ gets in the resolvent expansion compensated by terms of the form $a(\mathrm{e}^{-\mathrm{i}\langle x_i,\cdot\rangle}\hat{v})L^{-1}a^*(\mathrm{e}^{-\mathrm{i}\langle x_i,\cdot\rangle}\hat{v})$[7].

In [Wue17], Andreas Wünsch and Jacob Schach Møller used this method to treat a slightly different pseudorelativistic variant of the Nelson model in two dimensions. It had previously been renormalised by Sloan in [Slo74] using a different method. In addition to that, they were able to obtain some information on the domain, namely they gave supersets of $D(H_\infty)$ of the form $D(L^s)$. With these methods however, they could not investigate if and when $D(H_\infty)\cap D(L^s)$ is only the zero vector.

After Nelson's original work, the results have soon be extended to a variant of the model, the so called *massless* Nelson model, where $\omega(k)=|k|$, see e.g. [Fro74]. This poses technical difficulties, which can be overcome by imposing rather modest conditions on $\hat{v}$, that are met when taking $\hat{v}=|k|^{-1/2}$. The results of Griesemer and Wünsch are valid for the massless Nelson Hamiltonian as well.

The term *interior-boundary condition* was introduced by Teufel and Tumulka in [TT15] and [TT16], where they proposed IBC's as a way to solve the UV-difficulties that arise when point-like particles interact with a quantum field. Conditions very similar to IBC's appeared in the literature several times before, most notably in [Mos51], [Tho84], [Pav84] and [Yaf92].

[6] In most articles starting with Nelson, in fact a coupling constant is used. That is, $v=g\,w$ for some real constant g. Eckmann grouped terms together which are of the same power in the coupling constant.

[7] I mainly learned about Eckmann's approach reading the PhD thesis of Andreas Wünsch, who, together with Jacob Schach Møller, provided an understandable summary of the method.

2 Objectives

We began working on the current research project when Griesemer and Wünsch's result on the domain of the Fröhlich Hamiltonian was already known. But at the beginning, we were very much focused on local models, where the dispersion of the bosons (and those of the nucleons) is nonrelativistic and given by $k^2 + E_0$ for some real constant E_0. We did not believe, that interior-boundary conditions could be employed to obtain a direct description for the Nelson and the Fröhlich Hamiltonian. The common belief somehow was, that the concept of IBC's is too closely linked to the position representation, where the form factor is a delta distribution.

The main goal was to define the nonrelativistic model in three space dimensions for *dynamical sources*, i.e., particles.

However, we had to begin with the more easy case of one or several static sources. That is, we considered a model which is just the extension of the model of section 3.1.2 to an arbitrary number of bosons. For the results on this model, see [LSTT18].

Soon it turned out, that the case of nonrelativistic dynamical sources in three space dimensions was even more complicated than initially thought. Instead, we were looking at the analogous model in two space dimensions, which we expected to be less singular. The results on this model are one particular case of the models considered in [LS18].

3 Results and Discussion

Let us begin by introducing the concept of (abstract) interior-boundary conditions as we look at it today. That is, we will use a simple example, but with a focus on the tools and the notation, that were developed in the course of this project. For a chronological approach, see section 3.2.

3.1 Introduction to interior-boundary conditions

3.1.1 A simple example

Let us introduce the concept of interior-boundary conditions in a way that would be suitable for an introductory course on quantum mechanics. Firstly, we will look at a common example, which is discussed in order to explain tunneling effects for one particle. Then we will modify it to model annihilation and creation of the particle.

3.1.1.1 Tunneling effects on the line

In an introductory course, typically one of the first examples of quantum mechanical systems is the particle on the real line with a finite rectangular potential barrier. This system is used to demonstrate so called tunneling effects. The configuration space of the system is $Q = \mathbb{R}$ and the corresponding Hamiltonian (with unit square potential) is given by $H = -\partial_x^2 + \chi_{[0,1]}(x)$. Usually, in a physics course, one uses plane waves and invokes the continuity equation for the quantum flux at the point 0 to demonstrate that the waves can "reach the classically forbidden region". That is, that there is a nonzero flux between the left and the right half line, no matter what the "energy" of the waves is.

There are two subsystems that make up the system: one is the "starting region" on the negative half-line and one is the "forbidden region" on the right, the region where the potential is located, as well as the region behind it. The configuration space of the system is the (disjoint) union of two connection components $Q = (-\infty, 0) \cup [0, \infty) =: Q_1 \cup Q_0$. The corresponding Hilbert space is the sum of L^2-spaces of the components, $L^2(Q) = L^2(Q_1) \oplus L^2(Q_0)$. The operator H is self-adjoint with domain equal to the second Sobolev space, $D(H) = H^2(\mathbb{R})$. The Hamiltonian of the subsystem 1 on Q_1 is given by $H_1 = (-\partial_x^2, H^2(Q_1))$, the particle seems to "start freely moving". The Hamiltonian of the whole system is self-adjoint, but the Hamiltonians of the subsystems are not. This is a slightly more mathematical version of the fact that there are states initially located in Q_1 that reach the classically forbidden region Q_0. The probability flux out of Q_1 is compensated by one from Q_0 into Q_1. Put differently, the lack of self-adjointness of H_1, expressed as $\langle \varphi, H_1\psi\rangle_{L^2(Q_1)} - \langle H_1\varphi, \psi\rangle_{L^2(Q_1)}$ is compensated by the one of H_0. In this way one observes that, while the action of H_1 is equal $-\partial_x^2$, it is not self-adjoint, because the boundary conditions, that are imposed on wave functions $\psi_1 \in H^2(Q_1)$ at the right endpoint of Q_1, do not depend on ψ_1 alone. Instead, the wave functions and their first derivatives are chosen to be continuous at this point, $\psi_1(0) = \psi_0(0)$ and $\partial_x\psi_1(0) = \partial_x\psi_0(0)$ with $\psi_0 \in H^2(Q_0)$.

3.1.1.2 Interior-boundary conditions on the line

We would like to set up a different system in an analogous way, where the forbidden region is replaced by the vacuum. That is, we define $Q_0 := \{\text{vac}\}$ and consider this to be the region of configuration space, where the particle does not exist. Let $Q_1 = (-\infty, 0)$ be as above. If we are successful at finding a self-adjoint Hamiltonian H on the Hilbert space

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 := L^2(Q_0 \cup Q_1) = L^2(Q_0) \oplus L^2(Q_1) = \mathbb{C} \oplus L^2(Q_1)\,,$$

such that H_1 acts as $-\partial_x^2$ but is not symmetric on $\mathcal{H}_1$, we can safely say that the system models the creation and annihilation of a particle at the point 0.

Using integration by parts, one computes the non-symmetry of H_1 explicitly:

$$\langle \varphi_1, H_1\psi_1\rangle_{\mathcal{H}_1} - \langle H_1\varphi_1, \psi_1\rangle_{\mathcal{H}_1} = \overline{\varphi_1(0)}\partial_x\psi_1(0) - \overline{\partial_x\varphi_1(0)}\psi_1(0)\,.$$

In order to compensate this non-symmetry, we look for an operator H_0 on $\mathbb{C}$, which is not symmetric as well. Because we want to connect these non-symmetric parts, we have to identify the vacuum with the endpoint, vac $\simeq 0$. Then we make the following ansatz: $H_0\psi_0 = g\partial_x\psi_1(0)$ for some $g \in \mathbb{R}$. Note the peculiar feature, that the action of H_0 on any element in $\mathbb{C}$ does depend on ψ_1 but not on ψ_0, the element itself. It holds that

$$\langle \varphi_0, H_0\psi_0\rangle_{\mathcal{H}_0} - \langle H_0\varphi_0, \psi_0\rangle_{\mathcal{H}_0} = \overline{\varphi_0} g\partial_x\psi_1(0) - \overline{g\partial_x\varphi_1(0)}\psi_0\,.$$

We immediately observe that setting $\psi_0 = g\psi_1(0)$ and doing the same for φ turns the operator H symmetric. It can actually be proven, that it is also self-adjoint. The condition $\psi_0 = g\psi_1(0)$ is what we call an *interior-boundary condition*, abbreviated as *IBC*. The real g is the *coupling constant*.

We would like to stress that this construction should work not only for the situation of either one or no particle but also when the alternative is n particles or $n-1$ particles. As remarked in (1.1), the configuration space in the latter case would a priori be the disjoint union of the configuration space of n particles $\mathbb{R}^n_{\leq 0}$, and the configuration space of $n-1$ particles $\mathbb{R}^{n-1}_{\leq 0}$. The particles can however not reach all of Q_n. Something happens if one of them reaches zero: they get annihilated. So analogously to Q_1, we have to exclude the *collision configurations* and define $Q_n := (-\infty, 0)^n$. This space has a codimension-1 boundary, ∂Q_n consists exactly of those configurations where at least one particle reaches zero. The IBC relates the values of ψ_n on the collision configurations, i.e. on the *boundary* of $Q = Q_{n-1} \cup Q_n$ – for example at a point $(x_1, \ldots, x_{n-1}, 0)$ – to the values of ψ_{n-1} at the point $(x_1, \ldots, x_{n-1})$, in the *interior* of Q. This is where the term interior-boundary condition comes from.

For a more detailed investigation of interior-boundary conditions on the (whole) real line with the Laplacian as the free operator, see [KS16]. A model where the free evolution is governed by a Dirac operator on the line is considered in [LN18].

3.1.2 Three dimensional IBC

In the next step, we will repeat the construction for a particle moving in three dimensional physical space that can be created and annihilated at the origin, so we will this time keep Q_0, but set $Q_1 = \mathbb{R}^3 \setminus \{0\}$. For this simple example we will derive the interior-boundary condition and develop the tools and the notation, that is also used to treat the more complicated examples later on.

3.1.2.1 Enlarging the domain

We have to think about the domain for the operator H_1. The second Sobolev space is of course the domain of self-adjointness of $H_{\text{free}} = (-\Delta, H^2(\mathbb{R}^3))$ and therefore no flux into or out of the origin can occur for this operator. In the previous example, we have emphasised that the particle moves freely on Q_1. More precisely, we have $H\psi_1 = -\partial_x^2\psi_1$ for any $\psi_1 \in D(H)$ with $\text{supp}(\psi_1) \subset Q_1 = (-\infty, 0)$. We demand this now also for the present three dimensional example where $Q_1 = \mathbb{R}^3 \setminus \{0\}$. What is the maximal extension of this minimal operator $H_{\min} = (-\Delta, C_0^\infty(Q_1))$ as an operator on $\mathcal{H}_1 = L^2(Q_1)$? It turns out that this maximal distributional operator with domain $\{\psi \in L^2(Q_1) \subset \mathcal{D}'(Q_1) | -\Delta\psi \in L^2(Q_1)\}$ coincides[1]with the adjoint of $H_{\min}$, see, e.g. [Mic13, Lem. 4.2] for the reasoning. Consequently, our ansatz will be $H_1 := H_{\min}^*$. In order to compute the lack of symmetry of this operator, we need more detailed information on the adjoint domain $D(H_1)$. Recall that we have the following chain of operator inclusions: $H_{\min} \subset H_{\text{free}} \subset H_1$. Because H_{free} is self-adjoint, the domain of H_1 has, for any complex μ in the resolvent set $\rho(-H_{\text{free}})$, a direct sum decomposition

$$D(H_1) = D(H_{\text{free}}) \oplus \ker(H_1 + \mu)\,. \tag{3.1}$$

For the (easy) proof, see [Mic13, Lem. 3.12]. In the case at hand, where we consider only one particle, the dimension of $\ker(H_1 + \mu)$ is actually finite. What is more, the explicit form of the kernel for this model is well known. In [AGKH88], various variants of so called *point interactions* for models of one particle – in one, two or three space dimensions and with one or several points (centres) removed form configuration space – are considered in high detail. For point interactions, one looks for an operator, that acts freely on functions with support away from the centres, exactly as we do. Then one computes the non-symmetry of H_1 and looks for boundary conditions at these centres, that make the operator self-adjoint. It turns out that there are several possibilities to make H_1 self-adjoint on $\mathcal{H}_1$, of which H_{free} is only one. These self-adjoint restrictions of H_1 are called point interaction Hamiltonians.

Although we will also consider the lack of symmetry of H_1, our approach is very different from that. For we look for conditions relating the non-symmetry of H_1 to that of some, yet to be determined, operator H_0 acting on $\mathbb{C}$.

[1] The Laplacian here has to be understood in the distributional sense, it acts on distributions in $\mathcal{D}'(Q_1)$, the latter space being different from $\mathcal{D}'(\mathbb{R}^3)$.

3.1.2.2 Computing the non-symmetry

It holds that $\ker(H_1 + \mu) = \{(-\Delta + \mu)^{-1}\xi | \xi \in \mathbb{C}\}$ is a one-dimensional space. For convenience, we choose a real and positive μ. The non-symmetry of any operator is a skew-hermitean sesquilinear form. On the diagonal this form is given by the limit $r \to 0$ of the probability flux through a sphere of radius r. In general, we have

$$\begin{aligned}\langle \varphi_1, H_1\psi_1\rangle_{\mathcal{H}_1} - \langle H_1\varphi_1, \psi_1\rangle_{\mathcal{H}_1} &= \lim_{r\to 0}(1-\mathcal{K})r^2 \int_{\mathbb{S}^2} \overline{\varphi_1(r\omega)}\partial_r\psi_1(r\omega)\mathrm{d}\omega \\ = \lim_{r\to 0} r^2 \int_{\mathbb{S}^2} \overline{\varphi_1(r\omega)}\partial_r\psi_1(r\omega) - \overline{\partial_r\varphi_1(r\omega)}\psi_1(r\omega)\mathrm{d}\omega \,. \end{aligned} \tag{3.2}$$

Here $\mathcal{K} = CP_\varphi^\psi$ denotes the operation that interchanges φ and ψ and subsequently applies complex conjugation. We decompose elements $\psi_1 \in D(H_1)$ according to (3.1) for some $\xi_\psi \in \mathbb{C}$ as

$$\psi_1 = \psi_1^{\mathrm{fr}} - (-\Delta + \mu)^{-1}(2\pi)^{-3/2}\xi_\psi \,.$$

The map $\xi \mapsto -(-\Delta + \mu)^{-1}(2\pi)^{-3/2}\xi$ will become very important in the course of this introduction and for the whole method, we will call it $G_\mu\xi$. It is well known that $G_\mu\xi(x) = -\frac{\mathrm{e}^{-\sqrt{\mu}|x|}}{4\pi|x|}\xi$. Using this explicit form of $G_\mu\xi(x)$ and properties of $H^2(\mathbb{R}^3)$, one can show that $r\varphi_1^{\mathrm{fr}}(r\omega)$ and also $r\partial_r\varphi_1^{\mathrm{fr}}(r\omega)$ vanish in the limit $r \to 0$, and so

$$\lim_{r\to 0}(1-\mathcal{K})r^2 \int_{\mathbb{S}^2} \overline{\varphi_1(r\omega)}\partial_r\psi_1(r\omega)\mathrm{d}\omega = \lim_{r\to 0}(1-\mathcal{K}) \left(\overline{\varphi_1^{\mathrm{fr}}(0)} + \frac{\sqrt{\mu}\overline{\xi_\varphi}}{4\pi} + \frac{1}{r}\frac{\overline{\xi_\varphi}}{4\pi} \right) \xi_\psi.$$

Observe that the terms $\sqrt{\mu}(4\pi)^{-1}\sqrt{\mu}\overline{\xi_\varphi}\xi_\psi$ and $r^{-1}(4\pi)^{-1}\overline{\xi_\varphi}\xi_\psi$ are both invariant under exchange of ψ and φ composed with complex conjugation. That means that applying $1-\mathcal{K}$ would annihilate both terms. We will use this to get rid of the second, the diverging term. However, we choose to not use this for the other term $\frac{\sqrt{\mu}}{4\pi}\overline{\xi_\varphi}\xi_\psi$. This is possible, because the term has a limit as $r \to 0$. As a result, taking the limit, the non-symmetry of H_1 is given by

$$\langle \varphi_1, H_1\psi_1\rangle_{\mathcal{H}_1} - \langle H_1\varphi_1, \psi_1\rangle_{\mathcal{H}_1} = -\overline{\xi_\varphi}\left(\psi_1^{\mathrm{fr}}(0) + \frac{\sqrt{\mu}\xi_\psi}{4\pi}\right) + \overline{\left(\varphi_1^{\mathrm{fr}}(0) + \frac{\sqrt{\mu}\xi_\varphi}{4\pi}\right)}\xi_\psi.$$

Analogously to the one-dimensional example in 3.1.1.2, this can be compensated by the non-symmetric part of an operator H_0 on $\mathcal{H}_0 = \mathbb{C}$, if we set

$$H_0\psi_0 := \psi_1^{\mathrm{fr}}(0) + \frac{\sqrt{\mu}\xi_\psi}{4\pi} \tag{3.3}$$

$$\text{and} \quad \xi_\psi = \psi_0 \,. \tag{3.4}$$

The equation (3.4) is the interior-boundary condition for this system. In view of the decomposition (3.1), and using $r\psi_1^{\mathrm{fr}}(r\omega) \to 0$, this condition can be expressed as

$$4\pi \lim_{r\to 0} r\psi_1(r\omega) = \xi_\psi = \psi_0 \,. \tag{3.5}$$

Elements ψ_1 diverge as $r \to 0$ and the rate of divergence is given by ψ_0. To compute ψ_0, only ψ_1 near the origin has to be known.

3.1.2.3 The operators A and T_μ

The Hamiltonian H_0 acts differently on the two components of the decomposition (3.1). On ψ_1^{fr}, it is simply given by evaluation at the origin. This can not be the case for $G_\mu \xi_\psi$, because it diverges. Expanding near the origin, we can write it as

$$G_\mu \xi_\psi(r) = -\frac{\xi_\psi}{4\pi r} + \frac{\sqrt{\mu}\xi_\psi}{4\pi} + o(1) \qquad \text{for} \quad r \to 0\,. \tag{3.6}$$

Thus we can regard the map $G_\mu \xi_\psi \mapsto \frac{\sqrt{\mu}\xi_\psi}{4\pi}$ as a generalisation of the evaluation, namely taking the constant term in the asymptotic expansion (3.6). So the choice we made above, when we dropped the diverging term, can be supported by asking for a *local* extension of the evaluation operator to functions in $\ker(H_1 + \mu)$. For the same reason, we can not just drop $\frac{\sqrt{\mu}\xi_\psi}{4\pi}$ as well, for the decomposition (3.1) is evidently *non-local*. Taking the constant term in the asymptotic expansion however, is a local operation, and we can sum this up in defining a generalised evaluation A as

$$A\psi_1 := \psi_1^{\mathrm{fr}}(0) + T_\mu \xi_\psi = \psi_1^{\mathrm{fr}}(0) + \frac{\sqrt{\mu}}{4\pi}\xi_\psi\,. \tag{3.7}$$

Here we have defined the operator T_μ, which, in general, acts on the elements ξ that parametrise the kernel of the adjoint. In the theory of point interactions, the kernel is sometimes called the *charge space* or the boundary space, which makes sense because of the first equality in (3.5). Note however that, by imposing the IBC (3.4), we actually identify the charge space with the vacuum $\mathcal{H}_0$. So for us T_μ is always just an operator on this space. By comparing (3.3) and (3.7), we see that $H_0\psi_0 = A\psi_1$.

3.1.2.4 Renormalisation

The operator T_μ, which arises from regularising the otherwise ill-defined evaluation, can be related to what is called renormalisation. To see how this works, we will change from position to momentum representation. Let a hat denote the Fourier transformed elements, then $\widehat{G_\mu \xi}(k) = -\frac{(2\pi)^{-3/2}\xi}{k^2+\mu}$. Evaluation at a submanifold $\{x = 0\}$ corresponds in Fourier space to integrating over the coordinate k, which is conjugate to x. As expected, $\widehat{G_\mu \xi}$ is not integrable on $\mathbb{R}^3$. When dealing with such problems for point interactions and field theories, a cutoff is introduced. That is, the integration is restricted to a ball of radius $\Lambda > 0$. Then the terms can be modified in such a way that the limit $\Lambda \to \infty$ can be taken.

First observe that using spherical coordinates we obtain

$$\frac{1}{(2\pi)^{3/2}} \int_{B_\Lambda(0)} \widehat{G_\mu \xi}(k)\mathrm{d}k = -\frac{\xi}{(2\pi)^3} \int_{B_\Lambda(0)} \frac{1}{k^2 + \mu}\mathrm{d}k = -\frac{\xi}{2\pi^2} \int_0^\Lambda \frac{|k|^2}{|k|^2 + \mu}\mathrm{d}|k|\,.$$

Setting $u = |k|$, we rewrite the remaining integral in the following form:

$$\begin{aligned}\int_0^\Lambda \frac{u^2}{u^2+\mu}\mathrm{d}u &= \Lambda + \int_0^\Lambda \left(\frac{u^2}{u^2+\mu} - 1\right)\mathrm{d}u \\ &= \Lambda - \mu \int_0^\Lambda \frac{1}{u^2+\mu}\mathrm{d}u\,. \end{aligned} \tag{3.8}$$

The integral on the right hand side of (3.8) is convergent for $\Lambda \to \infty$ and yields the value $\frac{\pi}{2\sqrt{\mu}}$. This large-Λ-expansion is the Fourier version of the small-r-asymptotics in (3.6). Equation (3.8) implies that

$$\frac{1}{(2\pi)^{3/2}} \int_{B_\Lambda(0)} \widehat{G_\mu \xi}(k) \mathrm{d}k + \frac{\Lambda}{2\pi^2}\xi \to \frac{\sqrt{\mu}}{4\pi}\xi = T_\mu \xi \,.$$

We see that adding the so called renormalisation constant $E_\Lambda := \frac{\Lambda}{2\pi^2}$ to the cutoff evaluation of G_μ – and only then taking the limit Λ to infinity – yields a well defined operator, namely the already discovered expression for T_μ. This is what is called renormalisation.

3.1.3 The abstract approach

The goal of this subsection is to give an abstract reasoning for the IBC and the form of H. The tools, that will be developed in this section, will also be useful for the proof of self-adjointness of the IBC Hamiltonian H later.

The fact that evaluation of H^2-elements corresponds in Fourier space to multiplication by $(2\pi)^{-3/2}$ and integrating, can be cast like this: The Fourier transform of the delta distribution is regular, i.e. there is a locally integrable function $\hat{\delta} = (2\pi)^{-3/2}$, such that $\varphi(0) = \widehat{\delta\varphi} = \int \hat{\delta}\varphi$ for all $\varphi \in H^2(\mathbb{R}^3)$. So the delta distribution is a map from a subspace of $\mathcal{H}_1$ into $\mathcal{H}_0$. The dual map takes elements of $\mathcal{H}_0$ and multiplies them: $\hat{\psi}_0 \mapsto \hat{\delta} \cdot \hat{\psi}_0$. However, only the zero vector in $\mathbb{C}$ is mapped into $\mathcal{H}_1$, for nonzero elements it holds that $\hat{\delta} \cdot \hat{\psi}_0 \notin \mathcal{H}_1$, because no constant function is square integrable.

3.1.3.1 Vector notation

Let us introduce some vector notation for elements in and operators on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. If we impose the interior-boundary condition (3.5), and thus identify the boundary space with $\mathcal{H}_0$, we can regard G_μ as an operator on $\mathcal{H}$, which we denote by the same symbol. On vectors[2] $\hat{\psi} = (\hat{\psi}_0, \hat{\psi}_1) \in \mathcal{H}$ it acts in the following way:

$$\begin{aligned}\widehat{G_\mu \psi}(k) = \begin{pmatrix} \widehat{G_\mu \psi_0}(k) \\ 0 \end{pmatrix} &= \begin{pmatrix} -(k^2+\mu)^{-1}(2\pi)^{-3/2}\hat{\psi}_0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -(k^2+\mu)^{-1}\hat{\delta} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\psi}_1(k) \\ \hat{\psi}_0 \end{pmatrix} \,.\end{aligned}$$

We define an operator L_μ by setting

$$L_\mu \hat{\psi}(k) := \begin{pmatrix} (k^2+\mu)\hat{\psi}_1(k) \\ \mu\hat{\psi}_0 \end{pmatrix} = \begin{pmatrix} k^2+\mu & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \hat{\psi}_1(k) \\ \hat{\psi}_0 \end{pmatrix} \,.$$

Because $G_\mu \psi_0 \in \ker(H_1 + \mu)$, we have

$$\begin{pmatrix} H_1+\mu & 0 \\ 0 & \mu \end{pmatrix} G_\mu \psi = 0 \tag{3.9}$$

[2]We will write vectors sometimes as a row or as a column vector. We would like to be consistent with the notion of an upper, respectively lower, sector. This means however, that the mapping from row to column vectors is *not* given by transposition.

but if $\psi^{\mathrm{fr}} := (\psi_0, \psi_1^{\mathrm{fr}}) \in \mathbb{C} \oplus H^2(\mathbb{R}^3)$, it holds that

$$(H_1 + \mu)\psi^{\mathrm{fr}} = \begin{pmatrix} H_1 + \mu & 0 \\ 0 & \mu \end{pmatrix} \psi^{\mathrm{fr}} = \widetilde{L_\mu \psi^{\mathrm{fr}}} \,. \tag{3.10}$$

Note that we have defined a matrix version of the adjoint operator of the form $\mathrm{diag}(H_1, 0)$. We will denote it by the same symbol.

3.1.3.2 Abstract reasoning for IBC's

So far we have not mentioned any annihilation and creation operators in this introduction. The projection of the creation operator on the truncated Fock space, that we are considering here, is given by

$$a^*(\hat{\delta}) \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_0 \end{pmatrix} = \begin{pmatrix} \hat{\delta} \cdot \hat{\psi}_0 \\ 0 \end{pmatrix} .$$

Observe that the map $\hat{\psi} \mapsto (0, \hat{\delta} \cdot \psi_0)$ is only a map from $\mathcal{H}$ to $\mathbb{C} \oplus H^{-2}(\mathbb{R}^3)$. We can be sure about this, because the dual map $\hat{\psi} \mapsto (\widehat{\delta \psi_1}, 0)$ is well defined as a map from $\mathbb{C} \oplus H^2(\mathbb{R}^3)$ into the Hilbert space[3].

There is however a chance of repairing this, if we take the creation operator together with another operator that individually maps out of the Hilbert space as well. Then the singularities that arise could compensate each other, resulting in a square integrable element. Let L_μ, defined as a map from $\mathbb{C} \oplus L^2(\mathbb{R}^3)$ to $\mathbb{C} \oplus H^{-2}(\mathbb{R}^3)$, be this operator. Then we demand

$$\begin{aligned} L_\mu \hat{\psi} + a^*(\hat{\delta})\hat{\psi} \in \mathcal{H} \quad &\Longleftrightarrow \quad L_\mu \hat{\psi} + (0, (k^2+\mu)(k^2+\mu)^{-1} \hat{\delta} \cdot \hat{\psi}_0) \in \mathcal{H} \\ &\Longleftrightarrow \quad L_\mu \hat{\psi} + L_\mu (0, (k^2+\mu)^{-1} \hat{\delta} \cdot \hat{\psi}_0) \in \mathcal{H} \\ &\Longleftrightarrow \quad L_\mu \left(\hat{\psi} - \widehat{G_\mu \psi} \right) \in \mathcal{H} \\ &\Longleftrightarrow \quad \widehat{(1 - G_\mu)}\, \psi \in \{ \varphi \in \mathcal{H} | L_\mu \varphi \in \mathcal{H} \} \qquad (3.11) \\ &\Longleftrightarrow \quad (1 - G_\mu)\, \psi \in \widetilde{D(L_\mu)} = H^2(\mathbb{R}^3) \qquad (3.12) \end{aligned}$$

For the last implication, we have used that the maximal domain (3.11) is equal to the adjoint of the minimal realisation on, say, $\mathcal{S}(\mathbb{R}^3)$, see 3.1.2.1. But since this minimal operator is known to be essentially self-adjoint, the claim follows. Note now first that the domain $D(L_\mu)$ is evidently independent of μ. Consequently, we will from now on refer to it simply as $D(L)$ and omit the check. Secondly, note that for any $\psi \in \mathcal{H}$ it holds that $\psi = (1 - G_\mu)\psi + G_\mu \psi$. For ψ obeying condition (3.12), this is just a vector version of the decomposition (3.1) for $\xi = \psi_0$, because we can set $\psi^{\mathrm{fr}} = (\psi_0, \psi_1^{\mathrm{fr}}) := (1 - G_\mu)\psi$. That is, the interior-boundary condition already follows from this simple abstract reasoning that lead to (3.12) and we have

$$\{ \psi \in \mathbb{C} \oplus D(H_1) | \xi_\psi = \psi_0 \} = \{ \psi \in \mathcal{H} | (1 - G_\mu)\psi \in D(L) \} =: \mathfrak{D} \tag{3.13}$$

Here we have defined a subspace[4] $\mathfrak{D}$ that will be the domain of the final IBC operator.

[3] It also follows from $G\psi_0 \in L^2$ that $\hat{\delta} \cdot \psi_0 \in H^{-2}$.

[4] The resolvent identity yields that the difference $G_\mu \psi - G_\nu \psi$ is in $D(L)$. Therefore, the definition of the domain is in fact independet of μ. It also follows from the form (3.5) of the IBC.

3.1.3.3 Action of the Hamiltonian H

We have seen in the previous paragraph that it is exactly the domain $\mathfrak{D}$, which forces the sum of the operators L_μ and $a^*(\hat{\delta})$ to map back into the Hilbert space. In fact the above reasoning shows that $L_\mu(1-G_\mu)\hat{\psi} = L_\mu\hat{\psi}+a^*(\hat{\delta})\hat{\psi}$ as distributions in $H^{-2}(\mathbb{R}^3) = D(L_\mu)'$. Thus, we have already encoded the action of both the free operator and the creation operator in $L_\mu(1-G_\mu)$. Now use (3.9) and (3.10) to conclude that this is actually equal to $H_1+\mu$. What is missing is only the annihilation operator $a(\hat{\delta})$. We have seen in 3.1.2.3 that it is a priori not defined on the range of G_μ, which is equal to $\ker(H_1+\mu)$. Still, we can define a suitable extension of this evaluation operator. This is the operator A. The distributional action of $H_1 + A$ is – up to the regularisation that alters $a(\hat{\delta})$ and leads to A – the action of the formal expression we want to have. This operator is in fact equal to the one we have derived by different considerations in subsection 3.1.2. Consequently, it is reasonable to choose $H := H_1 + H_0 = H_1 + A$ to be the Hamiltonian of the system. Thus, as described in 3.1.2.4, it can be obtained through a renormalisation procedure.

3.1.3.4 Self-adjointness of H

We have concluded that $D(H) := \mathfrak{D} = \{\psi \in \mathcal{H} | (1-G_\mu)\psi \in D(L)\}$ is the domain of H.

Using what we know about the evaluation, we can also rewrite

$$\begin{aligned}\widehat{A\psi} &= \begin{pmatrix} 0 \\ \int_{\mathbb{R}^3} \hat{\delta}\hat{\psi}_1^{\mathrm{fr}} + T_\mu\psi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \langle L_\mu^{-1}\hat{\delta}, L_\mu\hat{\psi}_1^{\mathrm{fr}}\rangle_{\mathcal{H}_1} + T_\mu\psi_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \langle L_\mu^{-1}\hat{\delta},\ \cdot\rangle_{\mathcal{H}_1} & 0 \end{pmatrix} L_\mu\hat{\psi}^{\mathrm{fr}} + T_\mu\hat{\psi} \\ &= -G_\mu^* L_\mu\hat{\psi}^{\mathrm{fr}} + T_\mu\hat{\psi} = -G_\mu^* L_\mu(1-G_\mu)\hat{\psi} + T_\mu\hat{\psi}\,.\end{aligned}$$

We have defined matrix versions of A and T_μ and denoted them with the same symbol. Most importantly, we have identified the adjoint operator to G_μ, see also the introduction in 3.1.3. Note that in the last line we have also missed the fact that G_μ was defined as a position space operator. Instead, we will from now on drop any hats and write, for example, G_μ both for the position and momentum space representations.

With this at hand we can finally prove the self-adjointness of $H = H_1+A$ on the domain $D(H) = \mathfrak{D}$. In a first step, we rewrite its action to obtain a more symmetric form:

$$\begin{aligned}H\psi &= H_1\psi + A\psi = (H_1+\mu)\psi - G_\mu^* L_\mu(1-G_\mu)\psi + T_\mu\psi - \mu\psi \\ &= (H_1+\mu)(1-G_\mu)\psi + (H_1+\mu)G_\mu\psi - G_\mu^* L_\mu(1-G_\mu)\psi + T_\mu\psi - \mu\psi \\ &= L_\mu(1-G_\mu)\psi - G_\mu^* L_\mu(1-G_\mu)\psi + T_\mu\psi - \mu\psi \\ &= (1-G_\mu)^* L_\mu(1-G_\mu)\psi + T_\mu\psi - \mu\psi\,. \qquad (3.14)\end{aligned}$$

Because L_μ is symmetric on $D(L)$, the operator $Z_\mu := (1-G_\mu)^* L_\mu(1-G_\mu)$ is symmetric on $\mathfrak{D}$ and bounded from below by μ. Here we use that, by construction, $1-G_\mu$ maps $\mathfrak{D}$

into $D(L)$, i.e. $(1-G_\mu)\mathfrak{D} \subset D(L)$. To conclude the self-adjointness of H, the opposite inclusion $D(L) \subset (1-G_\mu)\mathfrak{D}$ has to be invoked as well:

$$\begin{aligned} D(L) \subset (1-G_\mu)\mathfrak{D} &\implies (L_\mu(1-G_\mu), \mathfrak{D})^* \subset (L_\mu, D(L))^* \\ &\implies (1-G_\mu)D(Z_\mu^*) \subset D(L^*) \\ &\implies (1-G_\mu)D(Z_\mu^*) \subset D(L) \subset (1-G_\mu)\mathfrak{D} \\ &\implies D(Z_\mu^*) \subset \mathfrak{D} \end{aligned}$$

This shows that the domain of Z_μ^* is exactly the domain $\mathfrak{D}$, the operator is self-adjoint. In most applications, it will be convenient to prove that $(1-G_\mu)$ is actually even continuously invertible. This would follow if we can give meaning to the Neumann series $\sum_{k=0}^\infty G_\mu^k$. In particular, this is true for any nilpotent G_μ, which also completes the proof for the present case. Another approach is to play with the spectral parameter μ: Show that the operator norm of G_μ is going to zero for $\mu \to \infty$. Then, for μ large enough, this implies that the norm of G_μ is less than one and thus the series converges in $\mathcal{H}$.

To prove self-adjointness of $H = Z_\mu + T_\mu - \mu$, we only have to observe that, in this particular example, T_μ is actually a bounded and symmetric operator. In more general settings, this final step in the proof is in fact the major obstacle. In order to prove self-adjointness and below boundedness of H, the Kato-Rellich theorem can be employed if T_μ is relatively bounded by Z_μ with relative bound smaller than one. This bound can be quite difficult to obtain and its proof constitutes the major part of the articles [LSTT18] and [Sch18].

3.1.3.5 The domain $\mathfrak{D}$

To this point, the domain $\mathfrak{D}$ has been characterised as the domain, which fulfils $(1-G_\mu)\mathfrak{D} = D(L)$. This form is very convenient to prove self-adjointness of the operator. Sometimes however, one is interested in a different kind of characterisation. Recall Nelson's second question in section 1.2: Is $D(H_\infty) \cap D(L^{1/2}) = \{0\}$? With the domain $\mathfrak{D}$ explicitly given, it is fairly easy to check, whether H is a form or operator perturbation of L, or none of both.

To do so, decompose again $\psi = (1-G_\mu)\psi + G_\mu\psi$ and let $\sigma \in [0, 1/2]$. Then $(1-G_\mu)\psi \in D(L^\sigma)$, as long as[5] $\psi \in \mathfrak{D}^{1/2}$. That means that $\psi \in D(L^\sigma)$, if and only if $G_\mu\psi \in D(L^\sigma)$. In Fourier space, the latter is equivalent to

$$\begin{aligned} L_\mu^\sigma G_\mu \psi \in \mathcal{H} \iff (k^2+\mu)^{-(1-\sigma)}\hat{\delta}\cdot\hat{\psi}_0 \in L^2(Q_1) \iff \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{(k^2+\mu)^{2(1-\sigma)}} < \infty \\ \iff 4(1-\sigma) > 3 \iff \sigma < 1/4 \end{aligned}$$

What we obtain is even a sharp result: For any $\sigma \in [0, 1/2]$, it holds that

$$\begin{aligned} &D(L^\sigma) \subset \mathfrak{D} \subset \mathfrak{D}^{1/2} \quad \sigma \in [0, 1/4) \quad \text{but} \\ &(D(L^\sigma) \cap \mathfrak{D}) \subset (D(L^\sigma) \cap \mathfrak{D}^{1/2}) = \{0\} \quad \sigma \in (1/4, 1/2]\,. \end{aligned}$$

In our particular example, we can exclude the possibility that H is a form perturbation of L.

[5] For this to be true in general, we should choose $\mu \geq 1$.

3.2 Evolution of the IBC method

At the beginning, we applied the method of renormalisation via the unitary Gross tranformation to the three dimensional nonrelativistic model with static sources. Then we tried to make a connection to the IBC appraoch. Because it is a Weyl operator, the dressing transformation of Gross acts naturally on coherent states. Already in my Diplomarbeit [Sch14], it was shown that this transforms the coherent domain over the one-particle IBC domain into the coherent domain over the free domain $H^2(\mathbb{R}^3)$. The observation made us confident that the IBC approach was in fact able to capture the form of the renormalised domain, which was known to be $U_\infty D(L)$ in the case of a fixed static source. This alone was however merely a curious observation. After all, the approach had promised to allow for a complete characterisation of the domain and the action of the Hamiltonian and in this respect coherent states are far from general enough. So the main goal of the first publication was then to show that the IBC domain is in fact the domain of self-adjointness.

I carried out a more detailed investigation of the variants of the static model in [LSTT18], where the free operator is replaced by a point interaction Hamiltonian at the origin. It was observed that they can have peculiar time asymmetry properties, which lead to the article [ST18].

Afterwards, I applied the theory of boundary forms on Krein spaces to interior-boundary conditions in order to classify all possible IBC's for a given model with a static source. The main theorem about boundary forms is sometimes called the Glazman-Krein-Naimark theorem, see [GTV12] and [EM05]. I was able to reproduce an earlier result by Yafaev [Yaf92]. The approach via boundary forms was significantly extended by Roderich Tumulka and used in the article [STT18]. For the treatment of moving sources however, Kreis spaces turned out to be not very promising.

The case of dynamical sources can not just be reduced to the one-boson cases for static sources. So we tried to understand how the domains of the Hamiltonians for *many-body point interactions* look like and also considered the method of *quasi boundary triples*, see [BL07] and [Mic13]. In the latter theory, the focus is on those elements in the adjoint domain on which a generalised Green's identity holds, which is very much in the spirit of the IBC approach as presented in subsection 3.1.2. We understood the relationship between the theory of boundary triples and the standard approach to many-body point interactions. What is called the Weyl function for example in triple theory is the Skornyakov–Ter-Martyrosyan (STM) operator in the theory of many-body point interactions. There the approach of triple theory is turned upside down, quasi boundary triples are effectively constructed using the gamma function and the Weyl operator, see [DFT94] and [CDF$^+$15]. This point of view was even adapted by proponents of triple theory in the article [BEHL18]. However, for an extension theoretic treatment (in contrast to an approach via quadratic forms) of many-body point interactions, it has to be proved that the whole adjoint domain may be parametrised in this way, see [MO17]. It was a relief, when we discovered that, in contrast, for IBC models it is irrelevant whether this parametrisation actually is equal to the whole adjoint domain.

Following the earlier work [Tho84], we focused on the possibility of setting up Hamiltonians for IBC models by constructing a resolvent. The first attempt was more constructive and needed explicit calculations, but worked for finitely many bosons. Together with

Jonas Lampart, we were able to simplify the procedure by defining the resolvent inductively. However the limit, where the particle number goes to infinity, did not work out as hoped.

Besides the more conceptual progress, at that time we also understood how to prove the more technical parts, that is, the regularity properties of the STM operator. There we profitted from the work of Moser and Seiringer, who showed in [MS17], that the quadratic form of the STM operator for n fermions and one other particle in three space dimensions can be bounded from below independently of n. While the proof for the lower bound makes essential use of the antisymmetry of the wave function, they also obtained an n-independent bound of the part of the operator, that is commonly called the off-diagonal part. We adapted some of the ideas of this proof and generalised it first to space dimension $d = 2$ and afterwards also to dispersion relations $\omega(k)$ other than k^2.

This came along with a shift from position space considerations to momentum space, thus leaving the ground where the idea was developed and where it has a very convenient interpretation and transform it more into a mathematical tool, as in 3.1.3. Of course for local operators, the original interpretation is still valid and can easily be recovered.

While Nelson's approach could not locate the divergence of the cutoff Hamiltonians, Eckmann's resolvent series already provided some more information, as described in 1.2. Looking at the STM operator in momentum space, we understood the connection between the IBC approach and the renormalisiation procedure. The action of the creation operator can be included in that of an extended free operator L. Renormalisation is necessary in order to extend the annihilation operator in the most natural way to the domain of the extended free operator. The formal action of the annihilation operator on the singular functions can, very loosely speaking, be thought of as consisting of three qualitatively different terms. To illustrate this, consider a situation with two nucleons and think of the possible processes in which one boson is created and one annihilated. If those two bosons are different, they contribute to the so called *off-diagonal* part of the STM operator. Then there are the situations, where the same boson that got created is annihilated again. If this happens at two different nucleons, we would speak of *θ-terms* as in [LS18]. If the boson gets created at one nucleon and annihilated again at the same nucleon, this is called the *diagonal* part. It is only this term that can be infinite regardless how regular the wave function is and it is exactly this term, that has to be regularised via renormalisation.

In this way, we were able to define renormalisation constants also for the nonrelativistic model of [LSTT18] in dimension two. If the cutoff is removed, this constant diverges logarithmically, exactly as in the Nelson model. Suddenly, we were very confident that also the better known Nelson model was in reach for the IBC method.

The above considerations also explain, in which cases no renormalisation is necessary. Yet, the IBC method can be applied to these cases as well and still simplifies the description of the Hamiltonians considerably. We decided to include those so called *form bounded* models in the presentation, of which the Fröhlich Hamiltonain is the best known example. In [LS18], they also served as an introduction to the more singular Nelson-type models.

Being a fellow member of the GRK 1838, Andreas Wünsch told me about his project with Jacob Schach Møller, where they considered a pseudorelativistic Nelson model in two space dimensions. His PhD-thesis [Wue17] helped me a lot to understand Eckmann's method. It became clear that the IBC method could be extended to treat a class of models, which includes in particular the model of [Wue17] as well as Eckmann's original model in [Eck70]. For reasons not known to me, the latter model had not been considered by Wünsch and Møller. In [Alb73], another variant of this model had been investigated.

At the conference "Mathematical Challenges of Zero-Range Physics", where I gave a talk on the connection between point interactions and IBC's, the question came up, whether the massless Nelson model could be treated with our method as well. I claimed that this was indeed the case. In a contribution to the Springer-INdAM volume of the conference, this was carried out, again for a larger class of models with nonnegative boson dispersion.

3.3 Conclusion

In the articles that make up this thesis, a method was developed, that allows to set up Hamiltonians for a certain class of otherwise ill-defined models of quantum field theory. In these models, a bosonic quantum field is linearly coupled to a finite and conserved number of particles. In addition, certain regularity assumptions are necessary. These depend on space dimension, interaction form factor and the dispersions of particles and field. The class of models contains in particular the Fröhlich Hamiltonian which describes Polarons, a model of nonrelativistic bosons and particles in two dimensions and last but not least the massive and the massless Nelson model as well as pseudorelativistic variants thereof. For the first time, a direct description of the Hamiltonians of those models was given. That means, that there is an explicit characterisation of the operator domain in terms of abstract boundary conditions on which the action of the operator is defined by an explicit formula. This is a significant improvement when compared to previous results, where in general only form domains could be characterised, and only in a more complicated form. For the more singular models, such as the Nelson model, an expression for the action of the Hamiltonians was lacking completely.

In general, we expect that the method of abstract interior-boundary conditions will lead to progress in the investigation of various aspects of the models, such as the effective mass of the particles, see e.g. [HO17] and [Spo87], and effective dynamics, see [Teu02]. We are confident that the results obtained in the articles of this thesis will pave the way to a rigorous definition of more realistic models of quantum field theory.

3.4 Outlook: Towards more singular interactions

After we had proved that the three dimensional model of [LSTT18] with static sources can be treated by using IBC's, there was of course the desire to apply the method also to the more realistic case of one or several moving sources. As was noted in [LS18], where the corresponding dynamical model in two dimensions was defined, the three dimensional version of T_μ fails to be well defined on the range of G_μ. In the framework of the publications of this thesis, this poses a serious problem. Shortly after we concluded the work on the article [LS18] however, in [Lam18] Jonas Lampart succeeded in setting up an enhanced version of the method. This method, in particular, makes it possible to define an IBC Hamiltonian also for the dynamical model of [LSTT18] in three dimensions. I will give the main idea and a short introduction below, see [Lam18] for all the details.

We will use the notation introduced in 3.1.3 and define L_μ, G_μ etc. for some spectral parameter $\mu > 0$. At first, consider again the case where T_μ is well defined on $\mathfrak{D}$. The latter is in particular the case, if T_μ is symmetric on $D(L)$ and $T_\mu G_\mu(\mathfrak{D}) \subset \mathcal{H}$. If this holds, the operator sum $L_\mu + T_\mu$ can be defined by the Kato-Rellich theorem. Recall, that the domain of the IBC Hamiltonian is defined by the equality $(1 - G_\mu)\mathfrak{D} = D(L)$. Now note that as long as $T_\mu G_\mu(\mathfrak{D}) \subset \mathcal{H}$, the domain $\mathfrak{D}$ can equivalently be characterised as $(1 - G_\mu^T)\mathfrak{D} = D(L)$, where

$$G_\mu^T := -(a(V)(L_\mu + T_\mu)^{-1})^* = G_\mu + (L_\mu + T_\mu)^{-1} T_\mu G_\mu$$

is a modified version of G_μ. On the right hand side of this equation it becomes clear why the condition $T_\mu G_\mu(\mathfrak{D}) \subset \mathcal{H}$ is important for the two definitions to be equivalent. Using G_μ^T, the expression for the Hamiltonian (3.14) can be rewritten in the following form:

$$\begin{aligned} H + \mu &= (1 - G_\mu)^* L_\mu (1 - G_\mu) + T_\mu \\ &= (1 - G_\mu^T)^* (L_\mu + T_\mu)(1 - G_\mu^T) + G_\mu^* T_\mu G_\mu^T \\ &=: (K_\mu^T + G_\mu^* T_\mu G_\mu^T \,. \end{aligned} \tag{3.15}$$

Furthermore, one expands $G_\mu^* T_\mu G_\mu^T = -a(V)(L_\mu + T_\mu)^{-1} T_\mu G_\mu$ and observes that this expression is again perfectly well defined as long as $T_\mu G_\mu$ maps into $\mathcal{H}$.

Now recall what happened, when the form factors are not square integrable. As described in detail in [LS18], although G_μ does not map into $D(L)$, it may be that $a(V)G_\mu = -(a(V)L_\mu^{-1/2})(a(V)L_\mu^{-1/2})^*$ is still well defined. This was called the form-bounded case. If $a(V)G_\mu$ is not defined, one had to regularise it and obtained the operator T_μ. We will refer to these steps as the first stage of the IBC approach.

In the same way we can expect that, when $(L_\mu + T_\mu)^{-1} T_\mu G_\mu$ fails to map into $D(L)$, the annihilation operator could still send $(L_\mu + T_\mu)^{-1} T_\mu G_\mu$ back into $\mathcal{H}$, in the very same way as it was the case for the term $G_\mu = L_\mu^{-1} a^*(V)$ before[6].

If T_μ is symmetric on $D(L)$, then the condition $(1 - G_\mu^T)\psi \in D(L)$ makes still sense, although it is not equivalent to $\psi \in \mathfrak{D}$ anymore if $T_\mu G_\mu \psi \notin \mathcal{H}$. So defining an alternative domain by $(1 - G_\mu^T)\mathfrak{D}^T = D(L)$, the operator K_μ^T becomes a self-adjoint operator again. This does not solve the problem of properly defining the action of $a(V)$ on $(L_\mu + T_\mu)^{-1} T_\mu G_\mu$ yet.

If the singularity of $T_\mu G_\mu$ is sufficiently mild, then it should be still possible to apply $a(V)$ as it stands, as it was the case with the form bounded models in [LS18]. To make the similarities with the first stage even more apparent, expand once more

$$G_\mu^* T_\mu G_\mu^T = G_\mu^* T_\mu G_\mu + G_\mu^* T_\mu (L_\mu + T_\mu)^{-1} T_\mu G_\mu \,.$$

Clearly, the second term is relatively bounded by L^δ for some $\delta < 1$, if the first term is. The first term is given by

$$G_\mu^* T_\mu G_\mu = (G_\mu^* T_\mu^{1/2})(G_\mu^* T_\mu^{1/2})^* = (a(V) L_\mu^{-1} T_\mu^{1/2})(a(V) L_\mu^{-1} T_\mu^{1/2})^* \,.$$

[6] Formally $T_\mu G_\mu$ replaces $a^*(V)$ and L_μ^{-1} is replaced by $(L_\mu + T_\mu)^{-1}$.

It is bounded as long as T_μ is form bounded by L_μ. This shows that, by defining a new domain $\mathfrak{D}^T$, we could cover not only cases where T_μ is relatively operator bounded – as in [LS18], [Sch19] and [Sch18] – but also cases where this operator is only form bounded. The definition of a different domain $\mathfrak{D}^T$ ensures that we can go to form bounded T-operators but at the same time do not have to define the full operator as a form perturbation. Instead, for form bounded T_μ, the perturbation $G_\mu^* T_\mu G_\mu^T$ is in fact bounded. Probably this helps to explain the missing region that can not be covered by IBC methods (using only the first stage), and the result of [GW18], where a larger region was covered by renormalisation methods.

If the formal action of $a(V)$ on the singular functions $(L_\mu + T_\mu)^{-1} T_\mu G_\mu \psi$ contains terms, that are formally infinite, then possibly it can be regularised and yields an operator S_μ. In the examples discussed in [Lam18], no renormalisation procedure was known before. Thus it was not an option to choose S_μ analogously to T_μ in such a way that the whole operator matches the one that has been obtained from the renormalisation. However, in these particular models, it is possible to go back to position space and define S_μ as the constant term in an asymptotic expansion near the collision configurations, see 3.1.2.3.

Of course there is no reason why we could not repeat the whole procedure yet again for the operator S_μ (if it is defined on $D(L)$). It is conjectured, that an IBC Hamiltonian can be defined in this way in finitely many steps as long as the annihilation operator is defined on $D(L^\sigma)$ for some $\sigma < 1$.

In [LN18], interior-boundary conditions were used to implement annihilation and creation of particles in a multi-time formulation. They considered a fixed number of Dirac-particles on the real line that interact with each other. A tentative analysis suggests that the Hamiltonian of a single time version of this model would be exactly at the boundary point of the first stage of the IBC method. Consequently, it could be an example of a setting with form bounded T-operator.

Bibliography

Articles of this thesis

[LS18] J. Lampart and J. Schmidt, *On Nelson-type Hamiltonians and abstract boundary conditions*, to appear in Comm. Math. Phys., arXiv:1803.00872v3 (2018).

[LSTT18] J. Lampart, J. Schmidt, S. Teufel and R. Tumulka, *Particle creation at a point source by means of interior-boundary conditions*, Math. Phys. Anal. Geom. **21** (2018) no. 2, 12.

[Sch18] J. Schmidt, *On a Direct Description of Pseudorelativistic Nelson Hamiltonians*, arXiv preprint arXiv:1810.03313 (2018).

[Sch19] J. Schmidt, *The Massless Nelson Hamiltonian and its Domain*, arxiv preprint arXiv:1901.05751 (2019).

References

[Alb73] S. Albeverio, *Scattering theory in a model of quantum fields. I*, J. Math. Phys. **14** (1973), 1800.

[AGKH88] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Texts and Monographs in Physics, Springer, 1988.

[Amm00] Z. Ammari, *Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: the Nelson model*, Math. Phys. Anal. Geom.**3** (2000), 217–285.

[BEHL18] J. Behrndt, P. Exner, M. Holzmann and V. Lotoreichik. *On the spectral properties of Dirac operators with electrostatic δ-shell interactions* J. Math. Pures Appl. **111** (2018), 47–78.

[BL07] J. Behrndt and M. Langer, *Boundary value problems for elliptic partial differential operators on bounded domains* J. Funct. Anal. **243** (2007), no. 2, 536.-565.

[CDF$^+$15] M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli, and A. Teta, *A class of Hamiltonians for a three-particle fermionic system at unitarity*, Math. Phys. Anal. Geom. **18** (2015), no. 1, 32.

[DFT94] G. Dell'Antonio, R. Figari and A. Teta, *Hamiltonians for systems of N particles interacting through point interactions*, Ann. Inst. H. Poincaré Phys. Théor. **60** (1994), no. 3, 253–290.

[Eck70] J. P. Eckmann, *A model with persistent vacuum*, Comm. Math. Phys. **18** (1970), 247–264.

[EM05] W. N. Everitt and L. Markus, *Complex symplectic spaces and boundary value problems*, Bull. Amer. Math. Soc. **42** (2005), 461–500.

[Fro74] J. Fröhlich, *Existence of dressed one electron states in a class of persistent models*, Fortschritte der Physik **2** (1974), 159–198.

[FPZ50] H. Fröhlich, H. Pelzer and S. Zienau. *Properties of slow electrons in polar materials*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **41** (1950), 221–242.

[GTV12] D. M. Gitman, I. V. Tyutin and B. L. Voronov, *Self-adjoint Extensions in Quantum Mechanics*, Progress in Mathematical Physics, Birkhäuser Basel, 2012.

[GW16] M. Griesemer and A. Wünsch, *Self-adjointness and domain of the Fröhlich Hamiltonian*, J. Math. Phys. **57** (2016), no. 2, 021902.

[GW18] M. Griesemer and A. Wünsch, *On the domain of the Nelson Hamiltonian*, J. Math. Phys. **59** (2018), no. 4, 042111.

[Gro73] L. Gross, *The Relativistic Polaron without Cutoffs*, Comm. Math. Phys. **31** (1973), 25–73.

[GHL14] M. Gubinelli, F. Hiroshima and J. Lőrinczi, *Ultraviolet renormalization of the Nelson Hamiltonian through functional integration*, J. Funct. Anal. **267** (2014), no. 9, 3125–3153.

[Hep69] K. Hepp, *Théorie de la renormalisation*, Lecture Notes in Physics Vol. 2., Springer (1969).

[HO17] F. Hiroshima and S. Osawa, *Mass Renormalization in the Nelson Model*, International Journal of Mathematics and Mathematical Sciences (2017).

[Kat80] T. Kato, *Perturbation theory for linear operators*, Grundlehren der Mathematischen Wissenschaften, Springer, 1980.

[KS16] S. Keppeler and M. Sieber, *Particle creation and annihilation at interior boundaries: one-dimensional models*, J. Phys. A. **49** (2016), no. 12, 125204.

[Lam18] J. Lampart, *A nonrelativistic quantum field theory with point interactions in three dimensions*, arXiv preprint arXiv:1804.08295 (2018).

[LN18] M. Lienert and L. Nickel, *Multi-time formulation of particle creation and annihilation via interior-boundary conditions* arXiv preprint arXiv:1808.04192 (2018).

[MO17] A. Michelangeli and A. Ottolini, *On point interactions realised as Ter-Martyrosyan–Skornyakov operators*, Rep. Math. Phys. **79** (2017), no. 2, 215–260.

[Mic13] T. Micheler, *Erweiterungstheorie symmetrischer Operatoren und elliptische Randwertprobleme auf Lipschitzgebieten*, PhD-Thesis, Technische Universität Berlin (2013). http://dx.doi.org/10.14279/depositonce-3747.

[MS17] T. Moser and R. Seiringer, *Stability of a fermionic N+1 particle system with point interactions*, Comm. Math. Phys. **356** (2017), no. 1, 329–355.

[Mos51] M. Moshinsky, *Boundary Conditions for the Description of Nuclear Reactions*, Physical Review **81** (1951), 347–352.

[Nel64] E. Nelson, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), no. 9, 1190–1197.

[Pav84] B. S. Pavlov, *Model of a zero-range potential with internal structure*, Theor. Math. Phys. **59** (1984), 544.

[Sch14] J. Schmidt, *Eine neue Methode zur Lösung des UV-Problems in einfachen Quantenfeldtheorien*, Diploma thesis, Universität Tübingen (2014).

[STT18] J. Schmidt, S. Teufel and R. Tumulka, *Interior-Boundary Conditions for Many-Body Dirac Operators and Codimension-1 Boundaries*, arxiv preprint arXiv:1811.02947 (2018).

[ST18] J. Schmidt, R. Tumulka, *Complex Charges, Time Reversal Asymmetry, and Interior-Boundary Conditions in Quantum Field Theory*, J. Phys. A. https://doi.org/10.1088/1751-8121/ab034c (2019).

[Slo74] A. D. Sloan, *The polaron without cutoffs in two space dimensions*, J. Math. Phys. **15** (1974), 190.

[Spo87] H. Spohn, *Effective mass of the polaron: A functional integral approach*, Annals of Physics **175** (1987), no. 2, 278–318.

[TCO+09] J. Tempere, W. Casteels, M. K. Oberthaler, S. Knoop, E. Timmermans and J. T. Devreese, *Feynman path-integral treatment of the BEC-impurity polaron* Phys. Rev. B **80** (2009).

[Teu02] S. Teufel, *Effective N-body dynamics for the massless Nelson model and adiabatic decoupling without spectral gap*, Annales Henri Poincare **3** (2002), 939–965.

[TT15] S. Teufel and R. Tumulka, *New type of Hamiltonians without ultraviolet divergence for quantum field theories*, arXiv preprint arXiv:1505.04847 (2015).

[TT16] S. Teufel and R. Tumulka, *Avoiding ultraviolet divergence by means of interior–boundary conditions*, Quantum Mathematical Physics (F. Finster, J. Kleiner, C. Röken, and J. Tolksdorf, eds.), Birkhäuser, 2016, 293–311.

[Tho84] L. E. Thomas, *Multiparticle Schrödinger Hamiltonians with point interactions*, Phys. Rev. D **30** (1984), 1233–1237.

[Wue17] A. Wünsch, *Self-adjointness and domain of a class of generalized Nelson models*, PhD thesis, Universität Stuttgart (2017). http://dx.doi.org/10.18419/opus-9384.

[Yaf92] D. R. Yafaev, *On a zero-range interaction of a quantum particle with the vacuum*, J. Phys. A: Math. Gen. **25** (1992), no. 4, 963.

Appendix

a) Accepted publications

1) Article [LSTT18]

Math Phys Anal Geom (2018) 21: 12
https://doi.org/10.1007/s11040-018-9270-8

Particle Creation at a Point Source by Means of Interior-Boundary Conditions

Jonas Lampart[1] · Julian Schmidt[2] · Stefan Teufel[2] · Roderich Tumulka[2]

Received: 18 August 2017 / Accepted: 25 April 2018 / Published online: 10 May 2018

Abstract We consider a way of defining quantum Hamiltonians involving particle creation and annihilation based on an *interior-boundary condition* (IBC) on the wave function, where the wave function is the particle-position representation of a vector in Fock space, and the IBC relates (essentially) the values of the wave function at any two configurations that differ only by the creation of a particle. Here we prove, for a model of particle creation at one or more point sources using the Laplace operator as the free Hamiltonian, that a Hamiltonian can indeed be rigorously defined in this way without the need for any ultraviolet regularization, and that it is self-adjoint. We prove further that introducing an ultraviolet cut-off (thus smearing out particles over a positive radius) and applying a certain known renormalization procedure (taking the limit of removing the cut-off while subtracting a constant that tends to infinity) yields, up to addition of a finite constant, the Hamiltonian defined by the IBC.

✉ Jonas Lampart
jonas.lampart@u-bourgogne.fr

Julian Schmidt
julian.schmidt@uni-tuebingen.de

Stefan Teufel
stefan.teufel@uni-tuebingen.de

Roderich Tumulka
roderich.tumulka@uni-tuebingen.de

1 CNRS & Laboratoire Interdisciplinaire Carnot de Bourgogne (UMR 6303), 9 Av. Alain Savary, 21078 Dijon, France

2 Fachbereich Mathematik, Eberhard-Karls-Universität, Auf der Morgenstelle 10, 72076 Tübingen, Germany

Keywords Ultraviolet divergence problem · Renormalization in quantum field theory · Self-adjoint Hamiltonian · Self-adjoint extensions of the Laplace operator · Point interactions · Particle-position representation · Ultraviolet cut-off

1 Introduction

Interior-boundary conditions (IBCs) provide a method of defining Hamiltonian operators with particle creation and annihilation that has received little attention so far. These Hamiltonians are related to extensions of differential operators to singular functions, somewhat like Hamiltonians describing point interactions. At least for some models, the IBC approach provides an alternative solution to the problem of ultraviolet (UV) divergences. In this paper, we consider a specific non-relativistic model of quantum field theory with point-shaped sources for creation and annihilation of bosonic particles, for which the UV problem has been solved by renormalization. For this model we prove that the IBC Hamiltonian H_{IBC} is a well-defined self-adjoint operator and agrees, up to addition of a constant, with the renormalized Hamiltonian H_∞.

The UV problem, in the form relevant to us, is the following. In the Fock space formulation of quantum field theories, the Hamiltonian involves annihilation and creation operators $a(\chi)$ and $a^*(\chi)$ that annihilate or create particles with wave function χ. For square-integrable functions χ these operators are densely defined operators on Fock space. However, in most physically relevant field theories the particles are created and annihilated at points in space, and χ is a distribution that is not square-integrable. For our model, χ will be the Dirac δ-distribution. While $a(\delta)$ can still be given mathematical sense as a densely defined operator, this is no longer possible for $a^*(\delta)$. Renormalization then amounts to making sense of the limit $\chi \to \delta$.

The IBC approach allows for the direct definition of Hamiltonians H_{IBC}, without a renormalization procedure, also in cases where χ is not square-integrable. It starts out from the particle-position representation of a vector in Fock space as a wave function on a configuration space of a variable number of particles. In this representation, the absorption of particle 1 by particle 2 corresponds to a jump from a configuration with 1 at the same location as 2 to the configuration without 1, while the emission of a particle corresponds to the opposite jump. These processes are therefore related to the flux of probability into (or out of) the set $\mathcal{C}$ of collision configurations in configuration space (i.e., the configurations with two particles at the same location). As we will show, a non-trivial such flux is possible for wave functions satisfying a suitable boundary condition, with $\mathcal{C}$ regarded as the boundary of configuration space; the relevant boundary condition is a relation between the values of the wave function at the two configurations connected by the jump just mentioned; since it relates a boundary point to an interior point of another sector, we call this condition an *interior-boundary condition* (IBC). Since wave functions in the domain of the Hamiltonian satisfy the IBC, the domain is not the same as that of a free field Hamiltonian. As we will show, the only common element of these domains is the zero vector. As a consequence, IBC Hamiltonians cannot be obtained as perturbations of free field Hamiltonians in any simple way.

While we discuss more general situations in [16, 17, 36], we focus in our present rigorous study on the simple model of a single non-relativistic bosonic scalar field whose quanta are created or annihilated at one or more point sources at fixed locations. For a single source at the origin, the formal expression for the Hamiltonian reads

$$H_\chi := \mathrm{d}\Gamma(h) + g\big(a(\chi) + a^*(\chi)\big), \tag{1.1}$$

with $\chi = \delta$. The free Hamiltonian $\mathrm{d}\Gamma(h)$ is the second quantization of the non-relativistic 1-particle Hamiltonian $h = -\Delta + E_0$, E_0 is a real constant called the rest energy, and g is a real coupling constant. Operators of the form (1.1) belong to the class of *van Hove Hamiltonians* [7, 34, 40]. Our model can be regarded as a non-relativistic variant of the Lee model [19] or Schweber's scalar field model [34, Sec. 12a].

A rigorous definition of van Hove Hamiltonians is discussed by Dereziński [7] for general h and χ. For our case of $h = -\Delta + E_0$ and $\chi = \delta$, the Hamiltonian H_∞ of [7] can be obtained through the following renormalization procedure. Consider a sequence of square-integrable functions χ_n approaching the δ distribution, $\chi_n \to \delta$. Then the sequence H_{χ_n} of Hamiltonians defined by (1.1) converges, after subtraction of a suitable divergent sequence of constants E_n, to H_∞. As described in more detail in Section 3, for $E_0 > 0$ there exists a unitary Weyl operator W_∞ such that $H_\infty = W_\infty^* \,\mathrm{d}\Gamma(h)\, W_\infty$ with domain $D(H_\infty) = W_\infty^* D(\mathrm{d}\Gamma(h))$. For $E_0 \in \mathbb{R}$, [7] provides an explicit formula for $\mathrm{e}^{-\mathrm{i}H_\infty t}$ and defines H_∞ as its generator. For a broader discussion of the UV problem, see, e.g., [10, 11, 19, 34, 40] and also Section 3.

Here we show instead that the IBC Hamiltonian H_{IBC} corresponding to the formal expression

$$H_\delta := \mathrm{d}\Gamma(h) + g\big(a(\delta) + a^*(\delta)\big), \tag{1.2}$$

is rigorously defined, self-adjoint, and (if $E_0 \geqslant 0$) bounded from below. The domain of H_{IBC} is explicitly characterized in terms of interior-boundary conditions. The action of H_{IBC} involves extensions of the Laplacian to functions singular on the set $\mathcal{C}$ where one particle hits the origin. Moreover, we show that H_{IBC} is equal, up to addition of a finite constant, to the Hamiltonian H_∞ obtained through renormalization. This yields a new explicit characterization of the domain of H_∞ and its action thereon that is not easily available otherwise. Thus, one conclusion from our results is that quantum field Hamiltonians obtained through renormalization can have a simple and explicit form when expressed in the particle-position representation, albeit not in terms of creation and annihilation operators but in terms of IBCs. And, as mentioned already, they are no longer defined on the domain of the free operator $\mathrm{d}\Gamma(h)$, but $D(\mathrm{d}\Gamma(h)) \cap D(H_\infty) = \{0\}$.

As a mathematical problem we have to study an infinite system of inhomogeneous boundary value problems. Here, the configuration space is the disjoint union of n-particle configuration spaces called sectors, and the boundary on each sector is a union of codimension-three planes. A particular difficulty arises from the fact that, in sectors with more than one particle, these planes intersect. This makes the regularity issues more complicated, and general approaches to elliptic problems with boundaries of higher codimension (e.g., [21]) cannot be applied directly. The intersections of these planes play an important role in the theory of point interactions involving more than two particles, see [3, 4, 23–25]. See also Remark 5.7 at the end of Section 5 for the relation of our results to the theory of abstract boundary value problems (e.g., [2]). In our case, some of the technical difficulties associated with the boundary value problem could be circumvented if we contented ourselves with proving merely *essential* self-adjointness, as we do for the generalized models of Section 4. However, in that case we do not obtain an explicit characterization of the domain of self-adjointness. Moreover, the enhanced understanding of these boundary value problems provided by our direct approach proves useful when dealing with further variants of the IBC approach and point interactions. In particular, in [16] the IBC approach is applied to a large class of models with dynamical sources, including the Nelson model [30] and the Fröhlich polaron. In [17] the model of the present paper is generalized to dynamical sources, a case that is of particular interest as no rigorous definition of the corresponding Hamiltonian was known before.

The plan of the paper is as follows: In Section 2 we motivate and define the IBC Hamiltonian $(H_{\text{IBC}}, D_{\text{IBC}})$ and state the main theorem about its self-adjointness for a single point source at the origin. In Section 3 we discuss the relation of the IBC Hamiltonian to a Hamiltonian obtained from a standard renormalization procedure. In Section 4 we explain that our results also apply to the situation of *several* (finitely many) point sources that can emit and absorb particles, located at fixed points in $\mathbb{R}^3$. Furthermore, we also provide in Section 4 a discussion of a 4-parameter family of IBCs and possibilities for further generalizations. In Sections 5–7 and the Appendix, we provide the proofs: In Section 5 we prove symmetry of H_{IBC} based on the regularity results provided in the Appendix. In Section 6 (essential) self-adjointness is proved by combining the symmetry result with the explicit knowledge of a core of the renormalized Hamiltonian H_∞. In Section 7 we prove the statements on generalized IBC Hamiltonians from Section 4.

Let us end the introduction with remarks on related literature. IBCs have been considered in the past, in some form or another, in [18, 26–29, 38, 39, 41]. Recent and upcoming works exploring various aspects of IBCs include [8, 9, 15–17, 36, 37]. Introductory presentations of the kind of models considered here can be found in [36, 37], and the physical motivation is discussed in [36]. Landau and Peierls [18] obtained conditions similar to IBCs when trying to formulate quantum electrodynamics in the particle-position representation, although their Hamiltonian was still ultraviolet divergent (and thus mathematically ill defined). Moshinsky [26, Sec. III] considered (as an effective description of nuclear reactions) a model with IBCs that is essentially equivalent to ours (including the 4-parameter family of IBCs discussed in Section 4), except that he considered only the sectors with $n = 0$ and $n = 1$ particles; he did not provide rigorous results about the Hamiltonian. Yafaev [41] independently considered the same model (again only the sectors with $n = 0$ and $n = 1$ particles), proved that the Hamiltonian is well defined and self-adjoint, and showed that the 4-parameter family mentioned above exhausts all possible IBC Hamiltonians in this case. Thomas [38] considered a model analogous to ours with moving sources, but only (what corresponds to) the sectors with $n = 2$ and $n = 1$ particles [38, Sec. III], respectively [38, Sec. II] with $n = 1$ and $n = 0$ particles, proving self-adjointness of the corresponding Hamiltonian. Moshinsky and Lopez [29] proposed a non-local kind of IBC for the Dirac and Klein-Gordon equations.

Tumulka and Georgii [39, Sec. 6] considered IBCs for boundaries of codimension 1 (whereas the boundary relevant here has codimension 3) and did not provide rigorous results. Keppeler and Sieber [15] described a physical reasoning leading to IBCs and discussed IBCs in 1 space dimension (though not rigorously). Galvan [9] suggested another approach towards a well defined Hamiltonian that has strong parallels to the IBC approach.

The mathematical study of Hamiltonians with IBCs is closely related to that of point interactions, a field that has recently received renewed attention. Hamiltonians for N-particle systems with point interactions were constructed rigorously using quadratic forms by Correggi, Dell'Antonio, Finco, Michelangeli, Teta [3, 4] and by Moser, Seiringer [25]. The problem was approached from the point of view of self-adjoint extensions by Minlos [24] and more recently by Michelangeli and Ottolini [23] (see also references therein for a more complete bibliography). Note that most of the literature on point interactions concerns fermionic systems. We expect that the IBC approach can also be applied to creation and annihilation of fermions.

2 The IBC Hamiltonian

We model the emission and absorption of non-relativistic particles at a point in $\mathbb{R}^3$, which we choose to be the origin. We thus call the origin the "source" and may think of it as a different kind of particle (which however remains at a fixed location).

Let $\mathfrak{H} := \mathrm{L}^2(\mathbb{R}^3) = \mathrm{L}^2(\mathbb{R}^3, \mathbb{C})$ be the one-particle Hilbert space, $\mathfrak{H}^n := \operatorname{Sym}\mathfrak{H}^{\otimes n}$ its n-fold symmetric tensor product, and $\mathfrak{F} := \Gamma(\mathfrak{H}) = \bigoplus_{n\in\mathbb{N}_0}\mathfrak{H}^n$ with $\mathfrak{H}^0 := \mathbb{C}$ the symmetric Fock space over $\mathfrak{H}$. An element ψ of $\mathfrak{F}$ has the form $\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \ldots)$ with

$$\psi^{(n)} = \psi^{(n)}(x_1, \ldots, x_n) \in \mathrm{L}^2(\mathbb{R}^{3n}), \tag{2.1}$$

symmetric under permutations of its arguments and $\sum_{n=0}^{\infty} \|\psi^{(n)}\|^2_{\mathfrak{H}^n} < \infty$. For a bounded operator T on $\mathfrak{H}$, an operator $\Gamma(T)$ on $\mathfrak{F}$ is defined by $(\Gamma(T)\psi)^{(n)} = T^{\otimes n}\psi^{(n)}$, and for a self-adjoint operator h (possibly unbounded), we define $\mathrm{d}\Gamma(h)$ as the generator of $\Gamma(\mathrm{e}^{-\mathrm{i}th})$. Its action is given by

$$(\mathrm{d}\Gamma(h)\psi)^{(n)} = \sum_{j=1}^{n} h_j \psi^{(n)}, \tag{2.2}$$

where $h_j = \mathbf{1} \otimes \ldots \otimes h \otimes \ldots \otimes \mathbf{1}$ is h acting on the jth factor. From now on we reserve the symbol h for the free one-particle Hamiltonian,

$$(h, D(h)) = (-\Delta + E_0, \mathrm{H}^2(\mathbb{R}^3)). \tag{2.3}$$

As a little digression, we point out how to set up a Hamiltonian with ultraviolet cut-off. We write $\overline{z}$ for the complex conjugate of $z \in \mathbb{C}$. For $\chi \in \mathfrak{H}$, the annihilation operator

$$(a(\chi)\psi)^{(n)}(x_1, \ldots, x_n) := \sqrt{n+1}\int_{\mathbb{R}^3} \mathrm{d}x\, \overline{\chi(x)}\, \psi^{(n+1)}(x, x_1, \ldots, x_n), \tag{2.4}$$

and its adjoint, the creation operator

$$(a^*(\chi)\psi)^{(n)}(x_1,\ldots,x_n) := \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\chi(x_j)\,\psi^{(n-1)}(x_1,\ldots,\hat{x}_j,\ldots,x_n), \tag{2.5}$$

(where $\hat{}$ denotes omission) are densely defined, closed operators on $\mathfrak{F}$ that are infinitesimally $\mathrm{d}\Gamma(h)$-bounded when $E_0 > 0$. Thus, for $E_0 > 0$ and any coupling constant $g \in \mathbb{R}$, the total Hamiltonian H_χ defined in (1.1) is self-adjoint on the domain of $\mathrm{d}\Gamma(h)$ by the Kato-Rellich theorem.

We now explain how to construct explicitly an operator H_{IBC} that captures, as we believe, the physical meaning of "H_δ" and agrees, as we will show, with the renormalized Hamiltonian up to addition of a finite constant. Recall that with the free Schrödinger evolution generated by the Laplacian on $\mathrm{L}^2(\mathbb{R}^3)$ there is associated a probability current

$$j^\psi(x) = 2\,\mathrm{Im}\,\overline{\psi(x)}\,\nabla\psi(x)\,. \tag{2.6}$$

In order to allow for annihilation or creation of particles at the origin, a non-vanishing probability current into or out of the origin must be possible. Using spherical coordinates $r = |x|$ and $\omega = \frac{x}{|x|} \in S^2 = \{v \in \mathbb{R}^3 : |v| = 1\}$, this current is

$$\begin{aligned} j_0^\psi &:= 2\lim_{r\to 0}\int_{S^2}\mathrm{d}\omega\, r^2\,\omega\cdot\mathrm{Im}\,\overline{\psi(r\omega)}\,\nabla\,\psi(r\omega), \\ &= 2\lim_{r\to 0}\int_{S^2}\mathrm{d}\omega\, r^2\,\mathrm{Im}\,\overline{\psi(r\omega)}\,\partial_r\,\psi(r\omega)\,. \end{aligned} \tag{2.7}$$

However, for j_0^ψ to be non-vanishing, ψ or $\partial_r\psi$ must be sufficiently singular at the origin. Since such singular functions are not in the standard domain $\mathrm{H}^2(\mathbb{R}^3)$ of the Laplacian, we need to consider the one-particle Laplace operator on a domain that includes singular functions that allow for non-vanishing currents into and out of the origin. Of course, such operators cannot be self-adjoint, since they cannot generate unitary groups.[1] In order to obtain a self-adjoint Hamiltonian and a unitary evolution on Fock space one thus needs to compensate the loss of probability in one sector by a corresponding gain in another sector. This is achieved by connecting different sectors with boundary conditions. Here, the configuration space is $\cup_{n=0}^{\infty}\mathbb{R}^{3n}$, and the "boundary" of its n-particle sector is the set

$$\mathcal{C}^n := \left\{x \in \mathbb{R}^{3n}\;\middle|\;\prod_{j=1}^{n}|x_j| = 0\right\}, \tag{2.8}$$

of those n-particle configurations with at least one particle at the origin. (This is the relevant set of collision configurations here; at these configurations, one of the moving particles collides with the source.) The "interior-boundary condition" connects the wave function $\psi^{(n)}$ on $\mathcal{C}^n$ with the wave function $\psi^{(n-1)}$ one sector below.

[1] Note that operators with δ-like potentials are defined in a similar way by enlarging the domain of the Laplacian, cf. [5]. However, in order to obtain a self-adjoint operator, an additional condition of the form $\lim_{r\to 0}(\partial_r r\psi(r\omega) - \alpha\, r\,\psi(r\omega)) = 0$ with $\alpha \in \mathbb{R}$ is imposed, precisely to ensure $j_0^\psi = 0$.

We now prepare for the precise definition of H_{IBC}. Define the operator Δ_n to be the Laplacian with domain $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^n) \subset \mathrm{L}^2(\mathbb{R}^{3n})$, which is defined as the closure of $C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^n)$ in the H^2-norm. We then set

$$\left(\Delta_n^*, D(\Delta_n^*)\right) \text{ is the adjoint of } \left(\Delta_n, \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^n)\right). \tag{2.9}$$

Since Δ_n is densely defined, closed and symmetric, the adjoint Δ_n^* extends Δ_n and its domain is given by (cf. [31, Sect. X.1])

$$D(\Delta_n^*) = D(\Delta_n) \oplus \ker(\Delta_n^* - \mathrm{i}) \oplus \ker(\Delta_n^* + \mathrm{i}). \tag{2.10}$$

We will always regard $D(\Delta_n^*)$ as a Banach space with the graph norm of Δ_n^*. Combining the Δ_n^* yields an operator $\Delta_{\mathfrak{F}}^*$ on Fock space, whose action is given by

$$(\Delta_{\mathfrak{F}}^* \psi)^{(n)} := \Delta_n^* \psi^{(n)}, \tag{2.11}$$

for those $\psi \in \mathfrak{F}$ such that $\psi^{(n)} \in D(\Delta_n^*)$.

The role of the annihilation operator $a(\delta)$ will be played by an operator A on Fock space that we define sector-wise by[2]

$$(A\psi)^{(n)}(x_1, \dots, x_n) := \frac{\sqrt{n+1}}{4\pi} \lim_{r \to 0} \partial_r\, r \int_{S^2} \mathrm{d}\omega\, \psi^{(n+1)}(r\omega, x_1, \dots, x_n). \tag{2.12}$$

Its dense domain will be specified later. It is not difficult to see that for $\psi^{(n+1)} \in \mathrm{H}^2(\mathbb{R}^{3(n+1)}) \cap \mathfrak{H}^{n+1}$,

$$(A\psi)^{(n)}(x_1, \dots, x_n) = \sqrt{n+1}\, \psi^{(n+1)}(0, x_1, \dots, x_n) = (a(\delta)\psi)^{(n)}(x_1, \dots, x_n). \tag{2.13}$$

However, some $\psi^{(n+1)}(r\omega, \dots)$ in the domain of H_{IBC} diverge like $1/r$ as $r \to 0$, and A is an extension of $a(\delta)$ to such functions.

The boundary conditions are formulated in terms of an operator B on Fock space that can again be defined sector-wise by

$$(B\psi)^{(n)}(x_1, \dots, x_n) := -4\pi \sqrt{n+1} \lim_{r \to 0} r\, \psi^{(n+1)}(r\omega, x_1, \dots, x_n). \tag{2.14}$$

We will define B on a dense domain where, in particular, the right hand side does not depend on ω. Again it is easy to see that for $\psi^{(n+1)} \in \mathrm{H}^2(\mathbb{R}^{3(n+1)}) \cap \mathfrak{H}^{n+1}$ we have $(B\psi)^{(n)} = 0$.

In the one-particle sector, $n = 1$, the domain $D(\Delta_1^*)$ is explicitly known and it is straightforward to prove that A and B are well defined functionals on $D(\Delta_1^*)$. For $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$ define the function

$$f_\gamma(x) := -\frac{1}{4\pi} \frac{\mathrm{e}^{-\gamma|x|}}{|x|}. \tag{2.15}$$

[2] Here and throughout the paper, we follow the convention, in order to write fewer brackets, that a derivative operator acts on all factors to the right of it, not just the one immediately to the right, unless otherwise indicated by brackets. Thus, in (2.12), ∂_r acts also on ψ.

Clearly, $f_\gamma \in \mathrm{L}^2(\mathbb{R}^3)$ but $f_\gamma \notin \mathrm{H}^2(\mathbb{R}^3)$. Moreover, $\Delta_1^* f_\gamma = \gamma^2 f_\gamma$ and f_γ is the unique L^2-solution to this equation. Consequently, with (2.10) it follows that

$$D(\Delta_1^*) = D(\Delta_1) \oplus V \qquad V = \mathrm{span}\Big\{ f_\gamma \Big| \gamma \in \{(1 \pm \mathrm{i})/\sqrt{2}\}\Big\}\,. \tag{2.16}$$

Then, writing $\psi \in D(\Delta_1^*)$ as $\psi_0 + \phi$ with $\psi_0 \in D(\Delta_1)$ and $\phi \in V$ and integrating by parts in spherical coordinates, one finds that that the degree of asymmetry of Δ_1^* can be expressed by A and B, that is

$$\langle \varphi, \Delta_1^* \psi\rangle_{\mathfrak{H}} - \langle \Delta_1^* \varphi, \psi\rangle_{\mathfrak{H}} = \langle B\varphi, A\psi\rangle_{\mathbb{C}} - \langle A\varphi, B\psi\rangle_{\mathbb{C}}\,. \tag{2.17}$$

We will give a rigorous proof of this equation and generalize it to the case $n \geqslant 2$ in Propositions 5.1 and 5.4 in Section 5. We remark that this implies that $-\Delta_1$ has a one-parameter family of self-adjoint extensions, known as point interactions (cf. [1]). Their domains correspond to one-dimensional subspaces of V on which the right hand side of (2.17) vanishes.

To illustrate the importance of (2.17), we define the simplest possible IBC Hamiltonian on the truncated Fock space $\mathfrak{F}^{(1)} := \mathbb{C} \oplus \mathrm{L}^2(\mathbb{R}^3)$ by

$$H_{\mathrm{IBC}}^{(1)} := \begin{pmatrix} 0 & gA \\ 0 & -\Delta_1^* + E_0 \end{pmatrix}, \tag{2.18}$$

on the domain

$$D_{\mathrm{IBC}}^{(1)} := \Big\{ (\psi^{(0)}, \psi^{(1)}) \in \mathfrak{F}^{(1)} \,\Big|\, \psi^{(1)} \in D(\Delta_1^*),\ B\psi^{(1)} = g\psi^{(0)} \Big\}\,. \tag{2.19}$$

Here $B\psi^{(1)} = g\psi^{(0)}$ is the interior-boundary condition (IBC). Equation (2.17) now implies that, contrary to what it may seem like, $H_{\mathrm{IBC}}^{(1)}$ is symmetric: for $\varphi, \psi \in D_{\mathrm{IBC}}^{(1)}$

$$\begin{aligned} & \langle \varphi, H_{\mathrm{IBC}}^{(1)} \psi\rangle_{\mathfrak{F}^{(1)}} - \langle H_{\mathrm{IBC}}^{(1)} \varphi, \psi\rangle_{\mathfrak{F}^{(1)}} \\ = \;& -\langle \varphi^{(1)}, \Delta_1^* \psi^{(1)}\rangle_{\mathfrak{H}} + \langle \Delta_1^* \varphi^{(1)}, \psi^{(1)}\rangle_{\mathfrak{H}} + \langle \varphi^{(0)}, gA\psi^1\rangle_{\mathbb{C}} - \langle gA\varphi^{(1)}, \psi^{(0)}\rangle_{\mathbb{C}} \\ \overset{(2.17)}{=} \;& \langle A\varphi^{(1)}, B\psi^{(1)}\rangle_{\mathbb{C}} - \langle B\varphi^{(1)}, A\psi^{(1)}\rangle_{\mathbb{C}} + g\langle \varphi^{(0)}, A\psi^1\rangle_{\mathbb{C}} - g\langle A\varphi^{(1)}, \psi^{(0)}\rangle_{\mathbb{C}} \\ \overset{\mathrm{IBC}}{=} \;& g\langle A\varphi^{(1)}, \psi^{(0)}\rangle_{\mathbb{C}} - g\langle \varphi^{(0)}, A\psi^1\rangle_{\mathbb{C}} + g\langle \varphi^{(0)}, A\psi^1\rangle_{\mathbb{C}} - g\langle A\varphi^{(1)}, \psi^{(0)}\rangle_{\mathbb{C}} \\ = \;& 0\,. \end{aligned} \tag{2.20}$$

It is not difficult to see (and was also shown in [41]) that $H_{\mathrm{IBC}}^{(1)}$ is even self-adjoint.

Our main result states that also the natural extension of $H_{\mathrm{IBC}}^{(1)}$ to the whole Fock space is (essentially) self-adjoint.

Theorem 2.1 *For every $g, E_0 \in \mathbb{R}$ the operator*

$$H_{\mathrm{IBC}} := -\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0) + gA, \tag{2.21}$$

is essentially self-adjoint on the domain

$$D_{\mathrm{IBC}} := \left\{ \psi \in \mathfrak{F} \;\middle|\; \begin{array}{l} \psi^{(n)} \in D(\Delta_n^*) \cap \mathfrak{H}^n \text{ for all } n \in \mathbb{N}\,, \\ H\psi \in \mathfrak{F}\,,\ A\psi \in \mathfrak{F}\,, \text{ and } B\psi = g\psi \end{array} \right\}. \tag{2.22}$$

If $g \neq 0$ then $D_{\mathrm{IBC}} \cap D(\mathrm{d}\Gamma(-\Delta)^{1/2}) = \{0\}$. For $E_0 \geqslant 0$, the Hamiltonian H_{IBC} is bounded from below.

If moreover $E_0 > 0$*, the domain of self-adjointness* $D(\overline{H}_{\mathrm{IBC}})$ *equals* D_{IBC} *and* $D(\overline{H}_{\mathrm{IBC}}) \cap D(\mathrm{d}\Gamma(h)^{1/2}) = \{0\}$*. In this case, the spectrum of* H_{IBC} *is given by* $\{E_{min}\} \cup [E_{min} + E_0, \infty)$*, and* $E_{min} = g^2\sqrt{E_0}/4\pi$ *is a simple eigenvalue.*

Note that the first two conditions in (2.22) just ensure that H maps the domain D_{IBC} back into Fock space. The third condition, $A\psi \in \mathfrak{F}$, might be redundant and follow from the second one, but we cannot show that. The last condition,

$$B\psi = g\psi\,, \tag{2.23}$$

is the interior-boundary condition, which connects the limiting behavior of $\psi^{(n)}$ at the boundary of the n-particle sector (where one particle reaches the origin) with the wave function $\psi^{(n-1)}$ one sector below. In particular, the IBC (2.23) immediately yields that if $\psi^{(n)} \neq 0$, then $\psi^{(k)} \neq 0$ for all $k > n$, and hence the Fock vacuum does not belong to D_{IBC}.

Formally, an analogous computation to the one for $H^{(1)}_{\mathrm{IBC}}$ shows that H_{IBC} is symmetric (see the proof of Corollary 5.5). However, in order to establish (2.17) for $n \geqslant 2$, we need to first investigate the regularity of functions in the adjoint domain $D(\Delta_n^*)$. This will be carried out in Section 5, with the main result given by Proposition 5.4. The proof of (essential) self-adjointness in Section 6 uses the symmetry established in Section 5, a dense domain of coherent states contained in (2.22) and a Weyl operator to be discussed in Section 6.2.

3 The Connection to Renormalization

As mentioned already, the formal expression H_δ as in (1.2) can be regularized by means of an ultraviolet cut-off, then the cut-off can be removed (while constants E_n tending to $\pm\infty$ get subtracted) in order to obtain a renormalized Hamiltonian H_∞. Our main result in this section, Theorem 3.1, asserts that H_{IBC} agrees with H_∞ (up to addition of a finite constant relative to the standard choice of E_n). We state Theorem 3.1 in Section 3.1 and then put it into perspective in Section 3.2 by connecting it to known facts, techniques, and hitherto open questions about H_∞.

3.1 Definition of H_∞ and Relation to H_{IBC}

We approximate the formal Hamiltonian H_δ with regularized (cut-off) Hamiltonians

$$H_n = \mathrm{d}\Gamma(h) + g\Big(a(\chi_n) + a^*(\chi_n)\Big) = H_0 + H_{I_n}, \tag{3.1}$$

with any choice of $\chi_n \in \mathrm{L}^2(\mathbb{R}^3)$ such that $\chi_n \to \delta$ as $n \to \infty$ in the sense that $\hat{\chi}_n \to \hat{\chi}_\infty := \hat{\delta} = (2\pi)^{-3/2}$ pointwise with $\|\hat{\chi}_n\|_\infty$ uniformly bounded. Here $\mathcal{F}\chi = \hat{\chi}$ denotes the Fourier transform of $\chi \in \mathrm{L}^2(\mathbb{R}^d)$. It is easy to see using standard arguments (and will be explained below) that if $E_0 > 0$ then $H_n - E_n$ converges in the strong resolvent sense for

$$E_n := -g^2\langle\chi_n, h^{-1}\chi_n\rangle_{\mathrm{L}^2}\,. \tag{3.2}$$

Note that for $E_0 > 0$ the free one-particle operator $h = -\Delta + E_0 \geqslant E_0 > 0$ is invertible. The limit is called the renormalized Hamiltonian,

$$H_\infty := \lim_{n\to\infty} (H_n - E_n)\,. \tag{3.3}$$

For $E_0 \leqslant 0$, a modification of the same procedure (or alternatively a formula for the unitary group $\mathrm{e}^{-\mathrm{i}H_\infty t}$) allows to define a Hamiltonian H_∞ as well [7]. However, we will compare H_∞ to H_{IBC} only for $E_0 > 0$.

Theorem 3.1 *For $E_0 > 0$, the renormalized operator $(H_\infty, D(H_\infty))$ agrees with $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ up to an additive constant:*

$$D_{\mathrm{IBC}} = D(H_\infty) \quad \textit{and} \quad H_{\mathrm{IBC}} = H_\infty + \frac{g^2\sqrt{E_0}}{4\pi}\mathbf{1}_{\mathfrak{F}}\,. \tag{3.4}$$

Theorem 3.1 is established in Section 6. Together with Theorem 2.1, it provides a new characterization of $D(H_\infty) = D_{\mathrm{IBC}}$, and of the action of H_∞ thereon (2.21), and shows that $D(H_\infty) \cap D(H_0^{1/2}) = \{0\}$.

3.2 Remarks on the Renormalization Procedure

The above described renormalization scheme is a particularly simple case of a somewhat more general renormalization procedure that can be applied to a wider class of UV divergent Hamiltonians with the following common structure. There is a self-adjoint operator $(H_0, D(H_0))$ and a sequence of operators H_{I_n} that are small perturbations of H_0 in the sense that

$$H_n := H_0 + H_{I_n}, \tag{3.5}$$

is self-adjoint on $D(H_0)$. If the interaction operator H_{I_n} converged as $n \to \infty$ to an operator that is relatively (form-)bounded by H_0 with relative bound smaller than one, then no renormalization would be necessary. In a typical manifestation of the UV problem, however, H_{I_n} does not converge. But in the cases of interest, there is a sequence of numbers $E_n \to \pm\infty$ such that $H_\infty = \lim_{n\to\infty}(H_n - E_n)$ exists in the strong resolvent sense.

In the examples we have in mind, the essential steps in finding this sequence E_n and proving the convergence of $H_n - E_n$ are, first, to construct a certain sequence of unitary operators W_n on Fock space, called dressing transformations, such that $W_n H_n W_n^*$ assumes a manageable form; second, to split $W_n H_n W_n^*$ into

$$W_n H_n W_n^* = H_n' + E_n, \tag{3.6}$$

such that H_n' converges in the strong resolvent sense to a well defined operator H_∞'. Third, one shows that W_n has a strong limit W_∞ (which is automatically unitary). Then it follows that

$$H_n - E_n = W_n^* H_n' W_n \xrightarrow{n\to\infty} W_\infty^* H_\infty' W_\infty = H_\infty, \tag{3.7}$$

in the strong resolvent sense.

Depending on the concrete model, the determination of the limiting Hamiltonian $H'_\infty = \lim_{n\to\infty} H'_n$ can be more or less tricky and, as a consequence, its domain can be more or less explicit. In all examples discussed in the following, W_n leaves invariant the domain $D(H_0)$, but this is no longer true for W_∞.

In his seminal paper [30], Nelson showed that the model nowadays named after him can be renormalized according to the general scheme just sketched. He used the so-called Gross transformation for W_n and was able to characterize $(H'_\infty, D(H'_\infty))$ as a form perturbation of H_0. Hence, he could not explicitly determine $D(H'_\infty)$ but merely conclude that $D(H'_\infty) \subset D(H_0^{1/2})$.

Whenever H'_∞ is an operator-bounded perturbation of H_0, one has $D(H'_\infty) = D(H_0)$ and $D(H_\infty) = W^*_\infty D(H_0)$ can be determined through the mapping properties of W^*_∞. Recently, Griesemer and Wünsch [12] proved that the Fröhlich Hamiltonian, which describes polarons, is of that type. In this case, one can define H_∞ also directly via its quadratic form without the detour via the dressing transformation. However, then the domain of H_∞ remains unknown, while the result of [12] provides an explicit characterization of it. In our model (1.2), the situation is even simpler, since it turns out that $H'_n = H'_\infty = H_0$.

After the existence of a self-adjoint renormalized Hamiltonian H_∞ is established, two questions remain in general open. First, is there a direct characterization of the domain $D(H_\infty) = W^*_\infty D(H'_\infty)$? And second, how does H_∞ act explicitly? As Nelson [30] put it:

> It would be interesting to have a direct description of the operator H_∞. Is $D(H_\infty) \cap D(H_0^{1/2}) = 0$?

The answer to the second question has been given by Griesemer and Wünsch for the Fröhlich Hamiltonian in [12] and for the Nelson model in [13] by studying the mapping properties of W^*_∞. A direct description in terms of IBCs, and thus a complete answer to both questions, is provided for our model in Theorem 3.1, and for the Fröhlich and Nelson Hamiltonians in [16].

Here is what the dressing transformation W_n looks like for our model (1.2). Since $h^{-1}\chi_n \in \mathrm{L}^2(\mathbb{R}^3)$ for $n \leqslant \infty$, the field operator

$$\Phi(h^{-1}\chi_n) := a(h^{-1}\chi_n) + a^*(h^{-1}\chi_n), \tag{3.8}$$

is self-adjoint. Therefore,

$$W_n := \mathrm{e}^{-\mathrm{i}\Phi(\mathrm{i}gh^{-1}\chi_n)}, \tag{3.9}$$

is unitary for all $n \leqslant \infty$. It is straightforward to show that (3.6) now holds with E_n as in (3.2) and $H'_n := \mathrm{d}\Gamma(h)$. The proof can be found in Section 6.3, or, for example, also in [6, 7]. Then $\lim_{n\to\infty} E_n = -\infty$, and $H'_\infty = \lim_{n\to\infty} H'_n = \mathrm{d}\Gamma(h)$ clearly exists. As a consequence,

$$H_\infty = W^*_\infty\, \mathrm{d}\Gamma(h)\, W_\infty \qquad \text{on} \qquad D(H_\infty) = W^*_\infty D(\mathrm{d}\Gamma(h))\,. \tag{3.10}$$

4 Variants of the IBC Hamiltonian

4.1 General Interior-Boundary Conditions

The IBC $B\psi = g\psi$ discussed in the previous sections is not the only possibility of implementing interior-boundary conditions for the Laplacian. In this section we present a four-parameter family of different interior-boundary conditions that all lead to a self-adjoint Hamiltonian on Fock space. In a certain sense, this family covers all possible types of IBCs.

The wider class of IBCs involves, instead of the values of the wave function on the boundary (like a Dirichlet boundary condition), a linear combination of the values and the derivative of the wave function on the boundary (like a Robin boundary condition); such IBCs were formulated in [36, 37] for boundaries of codimension 1 (and are also considered in [33] for particle creation, where the boundary has codimension 3). Specifically, in this wider class, we replace

$$B \to \mathrm{e}^{\mathrm{i}\theta}(\alpha B + \beta A), \quad A \to \mathrm{e}^{\mathrm{i}\theta}(\gamma B + \delta A), \tag{4.1}$$

where $\theta \in [0, 2\pi)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are such that

$$\alpha\delta - \beta\gamma = 1, \tag{4.2}$$

so that four of the five parameters can be chosen independently. We absorb the coupling constant g into the constants $\alpha, \beta, \gamma, \delta$. That is, we replace the IBC $B\psi = g\psi$ by

$$\mathrm{e}^{\mathrm{i}\theta}(\alpha B + \beta A)\psi = \psi \tag{4.3}$$

and the Hamiltonian $H_{\mathrm{IBC}} = -\Delta^*_{\mathfrak{F}} + \mathrm{d}\Gamma(E_0) + gA$ by

$$\tilde{H}_{\mathrm{IBC}} = -\Delta^*_{\mathfrak{F}} + \mathrm{d}\Gamma(E_0) + \mathrm{e}^{\mathrm{i}\theta}(\gamma B + \delta A). \tag{4.4}$$

The previous IBC (2.23) and Hamiltonian (2.21) are obviously contained in this scheme by chosing $\theta = 0 = \beta = \gamma$ and $\alpha^{-1} = g = \delta$. As discussed in detail in [33], the phase θ can be removed by means of the gauge transformation $\psi^{(n)} \to \mathrm{e}^{-\mathrm{i}\theta n}\psi^{(n)}$ if there is a single source, but not if there are several sources with different θ's, a situation that we consider in the next section. We refrain from stating and proving the analogue to Theorem 2.1 also for $\tilde{H}_{\mathrm{IBC}}$, although it could be proved along the same lines as for H_{IBC}. Instead, Theorem 4.1 below implies already a statement that is merely slightly weaker, namely that, for $E_0 > 0$, $\tilde{H}_{\mathrm{IBC}}$ is essentially self-adjoint on a dense domain satisfying the IBC (4.3).

To which extent does the family $\tilde{H}_{\mathrm{IBC}}$ cover all possible Hamiltonians with IBCs? Yafaev [41] showed that for the model on the truncated Fock space $\mathbb{C} \oplus \mathrm{L}^2(\mathbb{R}^3)$ with either zero or one particle all possible extensions of the (not densely defined) operator

$$H^\circ = (0, -\Delta) \quad \text{on} \quad D(H^\circ) = \{0\} \oplus C_0^\infty(\mathbb{R}^3 \setminus \{0\}), \tag{4.5}$$

are of the above type. On Fock space, however, one has in principle much more freedom. We could connect different sectors by different IBCs, i.e., make $\theta, \alpha, \beta, \gamma, \delta$ all depend on n, or even let them depend on the configuration of the other particles. But if we exclude such a dependence, then Yafaev's result shows that the family $\tilde{H}_{\mathrm{IBC}}$ is complete.

4.2 IBCs for Multiple Sources

We now consider a finite number N of sources fixed at (pairwise distinct) locations $\xi_1, \dots, \xi_N \in \mathbb{R}^3$. To keep things simple, we assume $E_0 > 0$ for the remainder of this section. For each source ξ_i, $1 \leqslant i \leqslant N$, we choose parameters

$$v_i := (\theta_i, \alpha_i, \beta_i, \gamma_i, \delta_i) \in [0, 2\pi) \times \mathbb{R}^4, \tag{4.6}$$

which fullfill separately

$$\alpha_i \delta_i - \beta_i \gamma_i = 1 \qquad 1 \leqslant i \leqslant N\,. \tag{4.7}$$

We write v for $(v_1, \dots, v_N)$. For suitable $\psi \in \mathfrak{H}$, define

$$A_i \psi := \lim_{x \to \xi_i} \partial_{r_i}(r_i \psi(x))\,, \quad B_i \psi := -4\pi \lim_{x \to \xi_i} (r_i \psi(x))\,, \quad \text{where } r_i := |x - \xi_i|\,, \tag{4.8}$$

and

$$X_i := \mathrm{e}^{\mathrm{i}\theta_i}(\alpha_i B_i + \beta_i A_i)\,, \qquad Y_i := \mathrm{e}^{\mathrm{i}\theta_i}(\gamma_i B_i + \delta_i A_i)\,, \qquad 1 \leqslant i \leqslant N\,. \tag{4.9}$$

The corresponding Fock space operators

$$X_i^{\mathfrak{F}}\big|_{\mathfrak{H}^{n+1}} := \sqrt{n+1}\, X_i \otimes \mathbf{1}_{\mathfrak{H}^n}\,, \qquad Y_i^{\mathfrak{F}}\big|_{\mathfrak{H}^{n+1}} := \sqrt{n+1}\, Y_i \otimes \mathbf{1}_{\mathfrak{H}^n}, \tag{4.10}$$

are densely defined in $\mathfrak{F}$. Then $(\Delta_1^*, D(\Delta_1^*)) := (\Delta_1, C_0^\infty(\mathbb{R}^3 \setminus \{\xi_1, \xi_2, \dots, \xi_N\}))^*$ is a closed but non-symmetric operator on $\mathfrak{H}$. Nevertheless, we will use the symbol $\mathrm{d}\Gamma(-\Delta_1^*)$ to denote the operator which acts as $-\sum_{j=1}^n \mathbf{1}_{1,\dots,j-1} \otimes \Delta_1^* \otimes \mathbf{1}_{j+1,\dots,n}$ on the n-th sector of Fock space. It is well known [1, 5] that

$$\tilde{h} := -\Delta_1^* + E_0 \quad \text{on} \quad U(v) := \bigcap_{i=1}^N \ker X_i \subset D(\Delta_1^*), \tag{4.11}$$

is a self-adjoint operator that is bounded from below. It is called the N-center point interaction with energy offset E_0 and parameters $a_i := \frac{\alpha_i}{\beta_i}$, where $\beta_i = 0$ corresponds to $a_i = +\infty$.

Theorem 4.1 *Let $E_0 > 0$ and v be any set of parameters obeying the condition (4.7) given above. There exists a dense subspace $\tilde{D}_{\mathrm{IBC}} \subset \mathfrak{F}$ such that for $\psi \in \tilde{D}_{\mathrm{IBC}}$ the IBCs*

$$X_i^{\mathfrak{F}} \psi = \psi \qquad \forall\, 1 \leqslant i \leqslant N, \tag{4.12}$$

hold and such that

$$\tilde{H}_{\mathrm{IBC}} := \mathrm{d}\Gamma(-\Delta_1^* + E_0) + \sum_{i=1}^N Y_i^{\mathfrak{F}}, \tag{4.13}$$

is essentially self-adjoint on $\tilde{D}_{\mathrm{IBC}}$. If $\tilde{h}$ is strictly positive,[3] *then $\tilde{H}_{\mathrm{IBC}}$ is bounded from below and possesses a unique ground state.*

[3] i.e., there is a positive constant c such that $\tilde{h} \geqslant c$.

Remark 4.2 Suppose that $\beta_i = 0$ for all $1 \leqslant i \leqslant N$. Then

$$\tilde{h} = h = (-\Delta_1^* + E_0, \mathrm{H}^2(\mathbb{R}^3)), \tag{4.14}$$

is the free one-particle operator, which is strictly positive. In this case $\tilde{H}_{\mathrm{IBC}}$ is bounded from below for any choice of distinct points $\xi_1, \dots, \xi_N$.

Remark 4.3 Let $N = 1$. In this case, for all values of $a_1 = \frac{\alpha_1}{\beta_1} \in (-\infty, \infty]$, the essential spectrum of the point-interaction operator is $\sigma_{ess}(\tilde{h}) = [E_0, \infty)$, cf. [1]. If $a_1 \geqslant 0$, then $\tilde{h}$ has no point spectrum. If $a_1 < 0$, then there is exactly one eigenvalue λ_0 of $\tilde{h}$. It is explicitly given as $\lambda_0 = E_0 - 16\pi^2 a_1^2$. Therefore $\tilde{H}_{\mathrm{IBC}}$ is bounded from below if $a_1 > 0$ or if $a_1 \leqslant 0$ but still $a_1 > -\frac{\sqrt{E_0}}{4\pi}$.

Under certain assumptions on v and E_0, we are able to further characterize $\tilde{H}_{\mathrm{IBC}}$. In order to state the theorem, we have to introduce some abbreviations:

For any $\lambda > 0$ let

$$w_i^\lambda(x) := f_{\sqrt{\lambda}}(x - \xi_i) = -\frac{\mathrm{e}^{-\sqrt{\lambda}|x-\xi_i|}}{4\pi|x - \xi_i|} \in \mathrm{L}^2(\mathbb{R}^3)\,, \tag{4.15}$$

and define the matrices

$$G_{ij}^\lambda := w_i^\lambda(\xi_j) = w_j^\lambda(\xi_i)\,, \tag{4.16}$$

and

$$S_{ij}(\lambda) := \delta_{ij}\mathrm{e}^{\mathrm{i}\theta_i}\left(\alpha_i + \frac{\sqrt{\lambda}}{4\pi}\beta_i\right) + (1 - \delta_{ij})\mathrm{e}^{\mathrm{i}\theta_i}\beta_i G_{ij}^\lambda\,, \tag{4.17}$$

where δ_{ij} denotes the Kronecker symbol. Note that S depends on all of $\lambda, \xi_1, \dots, \xi_N, v_1, \dots, v_N$.

Theorem 4.4 *Let* $(\tilde{H}_{\mathrm{IBC}}, \tilde{D}_{\mathrm{IBC}})$ *also denote the unique self-adjoint extension that has been constructed in Theorem 4.1. If the vector* $(1, 1, \dots, 1)^T$ *lies in the range of* $S(E_0)$*, then there exists* $\phi \in D(\Delta_1^*) \subset \mathfrak{H}$ *such that we have the equality*

$$\mathrm{e}^{\mathrm{i}\Phi(\mathrm{i}\phi)}\, \tilde{H}_{\mathrm{IBC}}\, \mathrm{e}^{-\mathrm{i}\Phi(\mathrm{i}\phi)} = \mathrm{d}\Gamma(\tilde{h}) + C(\phi)\mathbf{1}_{\mathfrak{F}}, \tag{4.18}$$

as self-adjoint operators on Fock space $\mathfrak{F}$*. Here* $C(\phi) \in \mathbb{R}$ *is a constant,* Φ *has been defined in* (3.8) *and* $\mathrm{d}\Gamma(\tilde{h})$ *denotes the second quantization of* $\tilde{h} = (-\Delta_1^* + E_0, U)$.

The definition of $\tilde{D}_{\mathrm{IBC}}$ in terms of coherent states obtained from vectors in $D(\Delta_1^*)$, as well as the proof of Theorems 4.1 and 4.4 and the explicit form of the ground state, of ϕ and of $C(\phi)$ are given in Section 7. As discussed in detail in [33], $\tilde{H}_{\mathrm{IBC}}$ is time reversal invariant if and only if all θ_i coincide up to addition of an integer multiple of π.

5 Symmetry of H_{IBC}

In this section we prove symmetry of $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$. The main ingredient is (2.17), which will be proved in Proposition 5.1 below, and its generalization to $n \geqslant 2$.

Proposition 5.1 *For $n = 1$ the maps A and B are well-defined continuous linear functionals on $D(\Delta_1^*)$ and for any $\varphi, \psi \in D(\Delta_1^*)$ we have*

$$\langle \varphi, \Delta_1^* \psi \rangle_{\mathfrak{H}} - \langle \Delta_1^* \varphi, \psi \rangle_{\mathfrak{H}} = \langle B\varphi, A\psi \rangle_{\mathbb{C}} - \langle A\varphi, B\psi \rangle_{\mathbb{C}} . \tag{5.1}$$

Proof Recall that $D(\Delta_1^*) = D(\Delta_1) \oplus V$ with $V = \operatorname{span}\left\{ f_\gamma \,\middle|\, \gamma \in \{(1 \pm \mathrm{i})/\sqrt{2}\} \right\}$. On the functions f_γ one easily evaluates

$$A f_\gamma = \frac{\gamma}{4\pi} \quad \text{and} \quad B f_\gamma = 1 . \tag{5.2}$$

On $D(\Delta_1)$ we have $A = 0$, since for $\psi \in C^1(\mathbb{R}^3)$

$$A\psi = \frac{1}{4\pi} \lim_{r \to 0} \int_{S^2} (\psi(r\omega) + r\omega \cdot \nabla\psi(r\omega)) \, \mathrm{d}\omega = \psi(0) , \tag{5.3}$$

and the point evaluation is continuous on $D(\Delta_1) = \mathrm{H}_0^2(\mathbb{R}^3 \setminus \{0\})$. Clearly also $B = 0$ on $D(\Delta_1)$. Now since H_0^2 is a closed subspace of $D(\Delta_1^*)$, the projection $p : D(\Delta_1^*) \to D(\Delta_1^*)/\mathrm{H}_0^2 \cong V$ is continuous. Thus $A, B : D(\Delta_1^*) \to \mathbb{C}$ are continuous as they can be written as the composition of p with a linear functional on a finite dimensional space.

The difference on the left hand side of (2.17) vanishes if either φ or ψ are elements of $\mathrm{H}_0^2(\mathbb{R}^3 \setminus \{0\})$, and so does the right hand side by the considerations above. Thus, it is sufficient to verify the claim for $\varphi = f_{\gamma_1}$, $\psi = f_{\gamma_2}$. As noted before we have $\Delta_1^* f_\gamma = \gamma^2 f_\gamma$ and

$$\langle f_{\gamma_1}, f_{\gamma_2} \rangle = \frac{1}{4\pi} \int_0^\infty \mathrm{d}r \, \mathrm{e}^{-(\overline{\gamma}_1 + \gamma_2) r} = \frac{1}{4\pi(\overline{\gamma}_1 + \gamma_2)} . \tag{5.4}$$

Thus

$$\begin{aligned} \langle f_{\gamma_1}, \Delta_1^* f_{\gamma_2} \rangle - \langle \Delta_1^* f_{\gamma_1}, f_{\gamma_2} \rangle &= \frac{\gamma_2^2 - \overline{\gamma}_1^2}{4\pi(\overline{\gamma}_1 + \gamma_2)} = \frac{\gamma_2 - \overline{\gamma}_1}{4\pi} \\ &= \overline{B f_{\gamma_1}} A f_{\gamma_2} - \overline{A f_{\gamma_1}} B f_{\gamma_2} . \end{aligned} \tag{5.5}$$

□

Proposition 5.1 can be understood as a generalized integration-by-parts formula for the singular functions in $D(\Delta_1^*)$. Its generalization to the case $n \geqslant 2$, given in Proposition 5.4 below, requires knowledge of the regularity properties of functions in $D(\Delta_n^*)$. These are rather subtle, as the following example shows:

Let $f \in \mathrm{H}^{-1/2}(\mathbb{R}^3)$, and set

$$\psi(x, y) = -\frac{\mathrm{e}^{T|x|}}{4\pi|x|} f(y) , \tag{5.6}$$

where $\mathrm{e}^{T|x|}$ denotes the contraction semi-group with generator $T = -\sqrt{-\Delta_y + 1}$, $D(T) = \mathrm{H}^1(\mathbb{R}^3)$, acting on $\mathrm{L}^2(\mathbb{R}^3_y)$. One easily checks that $\psi \in \mathrm{L}^2(\mathbb{R}^6)$ with norm proportional to $\|f\|_{\mathrm{H}^{-1/2}}$. By the smoothing properties of the semi-group, ψ is a smooth function on $\mathbb{R}^6 \setminus \{x = 0\} \supset \mathbb{R}^6 \setminus \mathcal{C}^2$. The action of Δ_2^* on ψ is thus given by differentiating on $\mathbb{R}^6 \setminus \mathcal{C}^2$ and yields

$$\Delta_2^* \psi = \psi \,, \tag{5.7}$$

so $\psi \in D(\Delta_2^*)$ is an eigenfunction of Δ_2^* with eigenvalue one. However, applying only the differential expression Δ_x gives $\Delta_x \psi = T^2 \psi$, which is not an element of $\psi \in \mathrm{L}^2(\mathbb{R}^6)$ unless $f \in \mathrm{H}^{3/2}(\mathbb{R}^3)$. Thus we have $\psi \in D(\Delta_2^*)$, but applying the Laplacian in only one of the variables does not give a square-integrable function, i.e. $\psi \notin D(\Delta_1^* \otimes 1)$. Furthermore, the formula for ψ suggests that $B\psi = \sqrt{2} f \in \mathrm{H}^{-1/2}(\mathbb{R}^3)$ is a distribution, so the "boundary values" of ψ on the collision configurations $\mathcal{C}^2$ will be of low regularity.

We now state our results concerning the definition of the operators A and B on $D(\Delta_n^*)$, which we prove in Appendix A. To allow for a lighter notation, we will use the symbol Ω_n to denote the configuration space of n particles, that is $\Omega_n := \mathbb{R}^{3n} \setminus \mathcal{C}^n$.

Lemma 5.2 *For any $n \in \mathbb{N}$, every $\varphi \in D(\Delta_n^*)$ has a representative for which the limits*

$$(A^{(n)}\varphi)(x_1, \dots, x_{n-1}) := \frac{\sqrt{n}}{4\pi} \lim_{r \to 0} \partial_r \int_{S^2} r\varphi(r\omega, x_1, \dots, x_{n-1}) \,\mathrm{d}\omega, \tag{5.8}$$

and

$$(B^{(n)}\varphi)(x_1, \dots, x_{n-1}) := -4\pi\sqrt{n} \lim_{r \to 0} r\varphi(r\omega, x_1, \dots, x_{n-1}), \tag{5.9}$$

exist in $\mathrm{H}^{-2}(\Omega_{n-1})$ and this defines continuous linear maps

$$A^{(n)}, B^{(n)} : D(\Delta_n^*) \to \mathrm{H}^{-2}(\Omega_{n-1}) \,. \tag{5.10}$$

Furthermore, $B^{(n)}$ vanishes on $\mathrm{H}^1(\mathbb{R}^{3n}) \cap D(\Delta_n^*)$ and the restriction of $A^{(n)}$ to $\mathrm{H}^2(\mathbb{R}^{3n})$ is given by the Sobolev-trace on $\{x_1 = 0\}$.

In the following we will drop the superscript from $A^{(n)}$ and $B^{(n)}$ for better readability. Let

$$D_n^* := \left\{ \psi \in D(\Delta_n^*) \cap \mathfrak{H}^n \,\middle|\, A\psi \in \mathrm{L}^2(\mathbb{R}^{3n-3}) \,,\ B\psi \in \mathrm{L}^2(\mathbb{R}^{3n-3}) \right\} \subset \mathfrak{H}^n \,. \tag{5.11}$$

and equip this space with the norm $\|\psi\|_{\mathfrak{H}^n} + \|\Delta_n^*\psi\|_{\mathfrak{H}^n} + \|A\psi\|_{\mathfrak{H}^{n-1}} + \|B\psi\|_{\mathfrak{H}^{n-1}}$. The following Proposition characterizes $\mathrm{H}^2 \subset D_n^*$ in terms of boundary values.

Proposition 5.3 *Let $\varphi \in D_n^*$. Then $B\varphi = 0$ if and only if $\varphi \in \mathrm{H}^2(\mathbb{R}^{3n})$.*

With this a priori information on the functions in D_n^* we can now characterize the asymmetry of Δ_n^* in terms on the operators A and B.

Proposition 5.4 *For all $\psi, \varphi \in D_n^*$ we have that*

$$\langle \Delta_n^*\psi, \varphi \rangle_{\mathfrak{H}^n} - \langle \psi, \Delta_n^*\varphi \rangle_{\mathfrak{H}^n} = \langle A\psi, B\varphi \rangle_{\mathfrak{H}^{n-1}} - \langle B\psi, A\varphi \rangle_{\mathfrak{H}^{n-1}} \,. \tag{5.12}$$

Proof By definition of the norm on D_n^*, the maps $A, B : D_n^* \to \mathfrak{H}^{n-1}$ are continuous, and so is the map

$$\mathfrak{B} : D_n^* \to \mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}, \qquad \psi \mapsto (B\psi, A\psi). \tag{5.13}$$

The skew-hermitean sesquilinear form

$$\beta(\psi, \varphi) := \langle \Delta_n^* \psi, \varphi \rangle - \langle \psi, \Delta_n^* \varphi \rangle, \tag{5.14}$$

is also continuous on D_n^*. Suppose for the moment that there exists a continuous, skew-hermitean sesquilinear form α on $\operatorname{ran}\mathfrak{B} \subset \mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}$ such that $\beta = \alpha \circ \mathfrak{B}$. Any continuous sesquilinear form on $\operatorname{ran}\mathfrak{B}$ is already determined by its values on any subspace of $\operatorname{ran}\mathfrak{B}$ which is dense in the $\|\cdot\|_{n-1} + \|\cdot\|_{n-1}$-norm. Therefore, β is already determined by its values on a subspace D_0 whose image $\mathfrak{B}(D_0)$ is dense in $\mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}$. That is, it suffices to verify (5.12) on D_0. Such a subspace is given by

$$D_0 := \{\psi \in D_n^* | \psi = \psi_A + \psi_B,\ \psi_A \in D_A^n,\ \psi_B \in D_B^n\} \qquad D_{A/B} := \ker A/B \subset D(\Delta_1^*). \tag{5.15}$$

Here D_A^n and D_B^n are the spans of symmetric n-fold tensor products of elements of $\ker A$ and $\ker B$ on $D(\Delta_1^*)$. These kernels are the domains of self-adjoint extensions of Δ_1; in fact $\ker B = \mathrm{H}^2(\mathbb{R}^3)$, and $\ker A$ is the domain of a point source with infinite scattering length. We have, by Proposition 5.1,

$$\mathfrak{B}(D_0) = (B(D_A^n), A(D_B^n)) = \left(D_A^{n-1}, D_B^{n-1}\right) \subset \mathfrak{H}^{n-1} \oplus \mathfrak{H}^{n-1}, \tag{5.16}$$

so $\mathfrak{B}(D_0)$ is in fact dense.

Now let $\psi = \psi_A + \psi_B, \varphi = \varphi_A + \varphi_B \in D_0$. The action of Δ_n^* on ψ, φ is given by the action of Δ_1^* on every factor. Because (Δ_1^*, D_A) and (Δ_1^*, D_B) are symmetric operators and β is skew-Hermitean, $\beta(\psi_A, \varphi_A) = \beta(\psi_B, \varphi_B) = 0$ and we only need to compute one cross-term $\beta(\psi_A, \varphi_B)$. Applying Proposition 5.1 yields

$$\begin{aligned}
\beta(\psi_A, \varphi_B) &= \sum_{i=1}^{n} \langle (\Delta_1^*)_{x_i} \psi_A, \varphi_B \rangle_{\mathfrak{H}^n} - \langle \psi_A, (\Delta_1^*)_{x_i} \varphi_B \rangle_{\mathfrak{H}^n} \\
&= n \left(\langle (\Delta_1^*)_{x_1} \psi_A, \varphi_B \rangle_{\mathfrak{H}^n} - \langle \psi_A, (\Delta_1^*)_{x_1} \varphi_B \rangle_{\mathfrak{H}^n} \right) \\
&= \langle A\psi_A, B\varphi_B \rangle_{\mathfrak{H}^{n-1}} - \langle B\psi_A, A\varphi_B \rangle_{\mathfrak{H}^{n-1}} \\
&= -\langle B\psi_A, A\varphi_B \rangle_{\mathfrak{H}^{n-1}}.
\end{aligned} \tag{5.17}$$

We still have to construct an α with $\beta = \alpha \circ \mathfrak{B}$. Here Proposition 5.3 enters as the key ingredient: we have that

$$\ker \mathfrak{B} = \ker B \cap \ker A = \{\psi \in \mathrm{H}^2(\mathbb{R}^{3n}) \cap \mathfrak{H}^n | A\psi = \psi|_{\mathcal{C}^n} = 0\} = \mathrm{H}_0^2(\Omega_n). \tag{5.18}$$

As a consequence $\beta(\psi, \varphi) = 0$ for all $\varphi \in D_n^*$ if $\psi \in \ker \mathfrak{B}$. Thus we can define on the quotient the sesquilinear form

$$\tilde{\alpha} : D_n^*/\ker\mathfrak{B} \times D_n^*/\ker\mathfrak{B} \to \mathbb{C}, \quad ([\psi], [\varphi]) \mapsto \beta(\psi, \varphi), \tag{5.19}$$

and (5.18) guarantees that this is well defined. Let π denote the quotient map. Then $\beta = \tilde{\alpha} \circ \pi$, which means that $\tilde{\alpha}$ is continuous in the quotient topology. There exists a unique continuous isomorphism $\mathfrak{B}' : D_n^*/\ker\mathfrak{B} \to \operatorname{ran}\mathfrak{B}$ such that $\mathfrak{B} = \mathfrak{B}' \circ \pi$.

Inserting the identity we get

$$\beta = \tilde{\alpha} \circ \pi = \tilde{\alpha} \circ (\mathfrak{B}')^{-1} \circ \mathfrak{B}' \circ \pi = \tilde{\alpha} \circ (\mathfrak{B}')^{-1} \circ \mathfrak{B} \,. \tag{5.20}$$

If we define $\alpha := \tilde{\alpha} \circ (\mathfrak{B}')^{-1}$, it is obviously continuous. This proves the claim. □

Corollary 5.5 *$(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ is symmetric for all $E_0 \in \mathbb{R}$.*

Proof Recall the definition of the domain

$$D_{\mathrm{IBC}} := \left\{ \psi \in \mathfrak{F} \;\middle|\; \begin{array}{l} \psi^{(n)} \in D(\Delta_n^*) \cap \mathfrak{H}^n \text{ for all } n \in \mathbb{N}\,, \\ H\psi \in \mathfrak{F}\,,\ A\psi \in \mathfrak{F}\,,\ \text{and } B\psi = g\psi \end{array} \right\} . \tag{5.21}$$

Now $H\psi \in \mathfrak{F}$ together with $A\psi \in \mathfrak{F}$ clearly implies $(-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\psi \in \mathfrak{F}$, so we may split the operator and compute with the help of Proposition 5.4:

$$\begin{aligned}
\langle \varphi, H\psi \rangle_{\mathfrak{F}} &= \langle \varphi, (-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&= \sum_{n\in\mathbb{N}} \langle \varphi^{(n)}, -\Delta_n^* \psi^{(n)} \rangle_n + \langle \varphi, \mathrm{d}\Gamma(E_0)\psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&\overset{(5.12)}{=} \sum_{n\in\mathbb{N}} \langle -\Delta_n^* \varphi^{(n)}, \psi^{(n)} \rangle_n + \langle A\varphi^{(n)}, B\psi^{(n)} \rangle_{n-1} - \langle B\varphi^{(n)}, A\psi^{(n)} \rangle_{n-1} \\
&\qquad + \langle \varphi, \mathrm{d}\Gamma(E_0)\psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&\overset{\mathrm{IBC}}{=} \langle (-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\varphi, \psi \rangle_{\mathfrak{F}} + \langle \varphi, gA\psi \rangle_{\mathfrak{F}} \\
&\qquad + \sum_{n\in\mathbb{N}} \langle A\varphi^{(n)}, g\psi^{(n-1)} \rangle_{n-1} - \langle g\varphi^{(n-1)}, A\psi^{(n)} \rangle_{n-1} \\
&= \langle (-\Delta_{\mathfrak{F}}^* + \mathrm{d}\Gamma(E_0))\varphi, \psi \rangle_{\mathfrak{F}} + \langle gA\varphi, \psi \rangle_{\mathfrak{F}} = \langle H\varphi, \psi \rangle_{\mathfrak{F}} \,.
\end{aligned} \tag{5.22}$$

□

Another simple corollary of our results in this section is the fact that, if $g \neq 0$, the intersection of D_{IBC} and the form-domain of $\mathrm{d}\Gamma(-\Delta)$ contains only the zero vector. For $E_0 \geqslant 0$, the form-domain of the free operator $\mathrm{d}\Gamma(h)$ is of course contained in that of $\mathrm{d}\Gamma(-\Delta)$.

Corollary 5.6 *If $g \neq 0$ we have for any $E_0 \in \mathbb{R}$*

$$D_{\mathrm{IBC}} \cap D\left(\mathrm{d}\Gamma(-\Delta)^{1/2}\right) = \{0\} \,. \tag{5.23}$$

Proof Take $\psi \neq 0 \in D_{\mathrm{IBC}}$. Then $\psi^{(n)} \neq 0$ for some $n \in \mathbb{N}$. This implies that $B\psi^{(n+1)} = g\psi^{(n)} \neq 0$. But $D(\mathrm{d}\Gamma(-\Delta)^{1/2})|_{\mathfrak{H}^{n+1}} = \mathrm{H}^1(\mathbb{R}^{3(n+1)}) \cap \mathfrak{H}^{n+1}$, and, by Lemma 5.2, B vanishes on this set. Hence $\psi \notin D(\mathrm{d}\Gamma(-\Delta)^{1/2})$. □

Remark 5.7 Propositions 5.3 and 5.4 prove that $(\mathfrak{H}^{n-1}, B, A)$ is a quasi boundary triple (in the sense of [2]) for the operator $(-\Delta_n^*, D_n^*)$. This allows for a complete characterization of the adjoint domain $D(\Delta_n^*)$ and the self-adjoint extensions of Δ_n (restricted to symmetric functions $\mathfrak{H}^n$). The following statements are consequences of the general theory [2, Prop. 2.9, 2.10], but can also be concluded directly in our setting from Propositions 5.3 and 5.4.

For any $\lambda > 0$ we have that

$$D(\Delta_n^*) \cap \mathfrak{H}^n = \mathrm{H}^2(\mathbb{R}^{3n}) \cap \mathfrak{H}^n \oplus K_\lambda \,, \tag{5.24}$$

with $K_\lambda = \ker(-\Delta_n^* + \lambda) \cap \mathfrak{H}^n$. The map

$$B : K_\lambda \to \left(\mathrm{H}^{1/2}(\mathbb{R}^{3(n-1)}) \cap \mathfrak{H}^{n-1}\right)' \subset \mathrm{H}^{-1/2}(\mathbb{R}^{3(n-1)}), \tag{5.25}$$

is continuous, as can easily be seen from the proof of Lemma 5.2. By Proposition 5.3 it is one-to-one. It is also surjective, with inverse given, as in (5.6), by

$$\begin{aligned} f \mapsto \; & \mathrm{Sym}_n \left(\frac{\mathrm{e}^{-\sqrt{-\Delta+1}|x_n|}}{4\pi |x_n|} f(x_1, \ldots, x_{n-1}) \right) \\ = \; & \mathrm{Sym}_n \left((-\Delta + 1)^{-1} f(x_1, \ldots, x_{n-1}) \delta(x_n) \right) . \end{aligned} \tag{5.26}$$

Such formulas for functions in $D(\Delta_n^*)$ have been widely used in the literature on point interactions, see e.g. [24]. An alternative proof that, for a similar problem with $n = 2$, the whole adjoint domain can be obtained in this way has been indicated recently in [23, Prop. 4].

6 Essential Self-Adjointness of H_{IBC}

6.1 Coherent Vectors and Denseness

The aim of this subsection is to introduce a set of coherent vectors in the domain D_{IBC} on which we can perform many computations explicitly. A standard choice of a dense set in Fock space is the space $\mathfrak{F}_0$ containing the vectors with a bounded number of particles, i.e., $\psi \in \mathfrak{F}_0$ iff there exists $N \in \mathbb{N}$ such that $\psi^{(n)} = 0$ for $n > N$. However, $\mathfrak{F}_0 \cap D_{\mathrm{IBC}} = \{0\}$ since the IBC $B\psi = g\psi$ immediately yields that if $\psi^{(n)} \neq 0$, then $\psi^{(k)} \neq 0$ for all $k > n$.

For $u \in \mathfrak{H}$ the associated *coherent vector* $\varepsilon(u) \in \mathfrak{F}$ is defined by

$$\varepsilon(u)^{(n)} := \frac{u^{\otimes n}}{\sqrt{n!}} \,. \tag{6.1}$$

It holds that $\langle \varepsilon(v), \varepsilon(u) \rangle_{\mathfrak{F}} = \exp(\langle v, u \rangle_{\mathfrak{H}})$; thus, the nonlinear map $\varepsilon : \mathfrak{H} \to \mathfrak{F}$, $u \mapsto \varepsilon(u)$, is continuous,

$$\begin{aligned} \|\varepsilon(v) - \varepsilon(u)\|^2 &= \langle \varepsilon(v), \varepsilon(v) \rangle_{\mathfrak{F}} + \langle \varepsilon(u), \varepsilon(u) \rangle_{\mathfrak{F}} - 2\mathrm{Re} \left(\langle \varepsilon(v), \varepsilon(u) \rangle_{\mathfrak{F}} \right) \\ &= \mathrm{e}^{\|v\|_{\mathfrak{H}}^2} + \mathrm{e}^{\|u\|_{\mathfrak{H}}^2} - 2\mathrm{Re}\, \mathrm{e}^{\langle v, u \rangle_{\mathfrak{H}}} \xrightarrow{v \to u} 0 \,. \end{aligned} \tag{6.2}$$

For a subset $D \subseteq \mathfrak{H}$, consider the subspace spanned by coherent vectors of elements of D, that is

$$E(D) := \mathrm{span}\{\varepsilon(u) | u \in D\} \subset \mathfrak{F} \,. \tag{6.3}$$

We will refer to this subspace as the *coherent domain over D*. When working with coherent vectors, we will need the following generalized polarization identity.

Proposition 6.1 *Let V be a complex vector space and $v_1, \dots, v_n \in V$. Then there exist vectors $u_1, \dots, u_m \in V$ and coefficients $d_1, \dots, d_m \in \mathbb{C}$ such that*

$$\mathrm{Sym}\,(v_1 \otimes \cdots \otimes v_n) = \sum_{k=1}^{m} d_k\, u_k^{\otimes n}\,. \tag{6.4}$$

See Appendix B for the proof, including an explicit formula for u_k and d_k. For a densely defined, non-self-adjoint operator (T, D), we use the expression $\mathrm{d}\Gamma(T)$ to denote the operator which acts as $\sum_{j=1}^{n} \mathbf{1}_{1,\dots,j-1} \otimes T \otimes \mathbf{1}_{j+1,\dots,n}$ on the n-th sector of Fock space. This expression obviously has meaning on $E(D)$.

Proposition 6.2 *If $D \subset \mathfrak{H}$ is dense, then $E(D)$ is a dense subspace of $\mathfrak{F}$. Moreover, let (T, D) be a densely defined operator on $\mathfrak{H}$. Then for $f \in \mathfrak{H}$ we have*

$$a(f)\,\varepsilon(u) = \langle f, u\rangle_{\mathfrak{H}}\,\varepsilon(u) \qquad \text{for all } u \in \mathfrak{H}\,, \tag{6.5}$$

$$a^*(f)\,\varepsilon(u) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varepsilon(u + tf) \qquad \text{for all } u \in \mathfrak{H}\,, \tag{6.6}$$

$$\mathrm{d}\Gamma(T)\,\varepsilon(u) = a^*(Tu)\,\varepsilon(u) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varepsilon(u + tTu) \qquad \text{for all } u \in D\,. \tag{6.7}$$

Proof For $u \in \mathfrak{H}$ the map $\mathbb{R} \to \mathfrak{F}$, $t \mapsto \varepsilon(tu)$, has derivatives of any order at $t = 0$ with

$$\left(\left.\frac{\mathrm{d}^n}{\mathrm{d}t^n}\right|_{t=0} \varepsilon(tu)\right)^{(m)} = \begin{cases} 0 & m \neq n \\ \sqrt{n!}\, u^{\otimes n} & m = n\,. \end{cases} \tag{6.8}$$

Thus, $E(\mathfrak{H})$ is dense in the span of all vectors of the form $(0, \dots, u^{\otimes n}, 0 \dots)$. Then, by the generalized polarization identity (Proposition 6.1) and standard approximation arguments, $E(\mathfrak{H})$ is also dense in $\mathfrak{F}$. The continuity of the map $u \mapsto \varepsilon(u)$ finally implies that $E(D)$ is dense in $E(\mathfrak{H})$ whenever D is dense in $\mathfrak{H}$. The formulas (6.5)–(6.7) follow directly from the definitions of the corresponding operators. □

The natural candidate for the set D is of course $D(\Delta_1^*)$. However, we still need to make sure that the coherent vectors generated by D satisfy the boundary condition. Let

$$D_g^\gamma := \left\{\varphi \in \mathfrak{H} \,\middle|\, \varphi = g f_\gamma + \phi,\ \phi \in \mathrm{H}^2(\mathbb{R}^3)\right\}, \tag{6.9}$$

for some γ with $\mathrm{Re}\,\gamma > 0$. The affine subspace D_g^γ is dense in $\mathfrak{H}$ because $\mathrm{H}^2(\mathbb{R}^3)$ is dense. Then, according to Proposition 6.2, the coherent domain $E(D_g^\gamma)$ over D_g^γ is a dense subspace of $\mathfrak{F}$; in fact, it is included in D_{IBC}:

Corollary 6.3 *We have that $E(D_g^\gamma) \subset D_{\mathrm{IBC}}$ for the value of g used in D_{IBC} and any $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma > 0$. As a consequence, D_{IBC} is dense in $\mathfrak{F}$.*

Proof Let $\varphi \in D_g^\gamma \subset D(\Delta_1^*)$. Then obviously $\varepsilon(\varphi)^{(n)} \in D_n^*$ as in (5.11), and

$$(B\varepsilon(\varphi))^{(n)} = \sqrt{n+1}(B\varphi)\frac{\varphi^{\otimes n}}{\sqrt{(n+1)!}} = g\frac{\varphi^{\otimes n}}{\sqrt{(n)!}} = g\varepsilon(\varphi)^{(n)}\,, \tag{6.10}$$

so $\varepsilon(\varphi)$ satisfies the interior-boundary condition. Additionally,

$$(A\varepsilon(\varphi))^{(n)} = \sqrt{n+1}(A\varphi)\frac{\varphi^{\otimes n}}{\sqrt{(n+1)!}} = (A\varphi)\varepsilon(\varphi)^{(n)}\,, \tag{6.11}$$

which defines an element of $\mathfrak{F}$ since A is bounded on $D(\Delta_1^*)$ by Proposition 5.1. Observe that $(\Delta_1^*)_{x_j}\varepsilon(\varphi)^{(n)} \in \mathrm{L}^2(\mathbb{R}^3_{x_j}, \mathrm{L}^2(\mathbb{R}^{3n-3}))$. Therefore the action of Δ_n^* coincides on $E(D_g^\gamma)$ with that of $\sum_{j=1}^n(-\Delta_1^*)_{x_j}$. It is also straightforward to check that $\Delta_{\mathfrak{F}}^*\varepsilon(\varphi) \in \mathfrak{F}$, and this completes the proof. □

6.2 Unitary Equivalence

To avoid unnecessary technicalities, we define the dressing transformation $\mathrm{e}^{-\mathrm{i}\Phi}$ directly for coherent states and not in terms of its generator $\Phi = a + a^*$. That is, we write $W(\varphi)$ for $\mathrm{e}^{-\mathrm{i}\Phi(\mathrm{i}\varphi)}$ and construct $W(\varphi)$ as follows. For $\varphi, u \in \mathfrak{H}$, let

$$W(\varphi)\,\varepsilon(u) := \mathrm{e}^{-\langle\varphi,u\rangle_{\mathfrak{H}} - \frac{\|\varphi\|^2_{\mathfrak{H}}}{2}}\,\varepsilon(u+\varphi)\,. \tag{6.12}$$

Lemma 6.4 *For every $\varphi \in \mathfrak{H}$, the map $W(\varphi)$ can be extended uniquely to a unitary transformation on Fock space; its inverse is given by $W(-\varphi)$.*

See, e.g., Section IV.1.9 in [22] for the rather elementary proof.

Proposition 6.5 *Let (T, D) be a self-adjoint operator on $\mathfrak{H}$. Then its second quantization $\mathrm{d}\Gamma(T)$ is essentially self-adjoint on the coherent domain $E(D)$.*

Proof The coherent domain $E(D)$ is a subspace of $D(\mathrm{d}\Gamma(T))$ and the associated unitary group of $\mathrm{d}\Gamma(T)$ is given by $\Gamma(\mathrm{e}^{-\mathrm{i}Tt})$. Since its action on coherent vectors is extremely simple, $\Gamma(\mathrm{e}^{-\mathrm{i}Tt})\varepsilon(u) = \varepsilon(\mathrm{e}^{-\mathrm{i}Tt}u)$, the coherent domain over D is *invariant* under $\Gamma(\mathrm{e}^{-\mathrm{i}Tt})$ because D is. Now the statement follows from Nelson's invariant domain theorem [32, Thm. VIII.11]. □

Lemma 6.6 *Let (T, D) be a densely defined operator on $\mathfrak{H}$. Suppose that $\varphi, u \in D$, and let $W(\varphi)$ be the corresponding unitary dressing transformation defined by (6.12). Then*

$$W(-\varphi)\mathrm{d}\Gamma(T)W(\varphi)\big|_{E(D)} = \mathrm{d}\Gamma(T) + a^*(T\varphi) + a(T\varphi) + G(T,\varphi)\,\big|_{E(D)}\,, \tag{6.13}$$

where $G(T,\varphi)$ is an operator on $E(D)$ whose action is given by

$$G(T,\varphi)\varepsilon(u) = \left(\langle\varphi, Tu\rangle_{\mathfrak{H}} - \langle T\varphi, u\rangle_{\mathfrak{H}} + \langle\varphi, T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u)\,. \tag{6.14}$$

Proof This is a consequence of Proposition 6.2 and the following straightforward computation:

$$\begin{aligned}
&W(-\varphi)\mathrm{d}\Gamma(T)W(\varphi)\varepsilon(u) \qquad (6.15)\\
&\overset{(6.7)}{=} W(-\varphi)\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\varepsilon(u+\varphi+tT(u+\varphi))\mathrm{e}^{-\langle\varphi,u\rangle-\frac{\|\varphi\|}{2}}\\
&\overset{(6.12)}{=} \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\varepsilon(u+tT(u+\varphi))\mathrm{e}^{t\langle\varphi,T(u+\varphi)\rangle}\\
&\overset{(6.6)}{=} \left(a^*(T(u+\varphi)) + \langle\varphi, Tu\rangle_{\mathfrak{H}} + \langle\varphi, T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u)\\
&\overset{(6.7)}{=} \left(\mathrm{d}\Gamma(T) + a^*(T\varphi) + \langle\varphi, Tu\rangle_{\mathfrak{H}} + \langle\varphi, T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u)\\
&\overset{(6.5)}{=} \left(\mathrm{d}\Gamma(T) + a^*(T\varphi) + a(T\varphi) + \langle\varphi, Tu\rangle_{\mathfrak{H}} - \langle T\varphi, u\rangle_{\mathfrak{H}} + \langle\varphi, T\varphi\rangle_{\mathfrak{H}}\right)\varepsilon(u)\,. \qquad \square
\end{aligned}$$

Corollary 6.7 *Let (T, D) be a self-adjoint operator on $\mathfrak{H}$ which is invertible, i.e. $0 \in \rho(T)$. Then for $\psi \in \mathfrak{H}$ and $u \in D$ it holds that*

$$\begin{aligned}
&W(-T^{-1}\psi)\mathrm{d}\Gamma(T)W(T^{-1}\psi)\big|_{E(D)}\\
&= \mathrm{d}\Gamma(T) + a^*(\psi) + a(\psi) + \langle\psi, T^{-1}\psi\rangle_{\mathfrak{H}}\mathbf{1}_{\mathfrak{F}}\,\big|_{E(D)}.
\end{aligned} \tag{6.16}$$

Proof Apply Lemma 6.6 with $\varphi = T^{-1}\psi$ and observe that, because T is symmetric, it holds that $\langle\varphi, Tu\rangle_{\mathfrak{H}} - \langle T\varphi, u\rangle_{\mathfrak{H}} = 0$. So the operator $G(T,\varphi)$ reduces to multiplication with the constant $\langle T^{-1}\psi, \psi\rangle_{\mathfrak{H}} = \langle\psi, T^{-1}\psi\rangle_{\mathfrak{H}}$. $\square$

Corollary 6.8 *Let $E_0 \in \mathbb{R}$, $\gamma > 0$, f_γ be given by (2.15) and let $h = -\Delta + E_0$ with domain $\mathrm{H}^2(\mathbb{R}^3)$. Then on the coherent domain $E(\mathrm{H}^2(\mathbb{R}^3))$ we have*

$$\begin{aligned}
&W(-gf_\gamma)\,H_{\mathrm{IBC}}\,W(gf_\gamma)\big|_{E(\mathrm{H}^2(\mathbb{R}^3))}\\
&= \mathrm{d}\Gamma(h) + (-\gamma^2 + E_0)\left(a^*(gf_\gamma) + a(gf_\gamma)\right) + C(g,\gamma,E_0)\mathbf{1}_{\mathfrak{F}}\,\big|_{E(\mathrm{H}^2(\mathbb{R}^3))},
\end{aligned} \tag{6.17}$$

where the constant reads

$$C(g,\gamma,E_0) = (-\gamma^2 + E_0)\|gf_\gamma\|^2_{\mathfrak{H}} + g^2\frac{\gamma}{4\pi}\,. \tag{6.18}$$

Proof We start by noting that (6.11) gives for $u \in \mathrm{H}^2(\mathbb{R}^3)$

$$gAW(gf_\gamma)\varepsilon(u) = g(A(gf_\gamma + u))W(gf_\gamma)\varepsilon(u) = \left(\frac{g^2\gamma}{4\pi} + gu(0)\right)W(gf_\gamma)\varepsilon(u)\,. \tag{6.19}$$

Now set $(T, D) = (-\Delta_1^* + E_0, D(\Delta_1^*))$ and $\varphi = gf_\gamma$ in Lemma 6.6. Then

$$\begin{aligned} & W(-gf_\gamma)\, H_{\mathrm{IBC}}\, W(gf_\gamma)\varepsilon(u) \\ &= W(-gf_\gamma)\mathrm{d}\Gamma(-\Delta_1^* + E_0)W(gf_\gamma)\varepsilon(u) + \left(\frac{g^2\gamma}{4\pi} + gu(0)\right)\varepsilon(u) \\ &= \left(\mathrm{d}\Gamma(h) + (E_0 - \gamma^2)\big(a^*(gf_\gamma) + a(gf_\gamma)\big) + G(T, \varphi) + \left(\frac{g^2\gamma}{4\pi} + gu(0)\right)\right)\varepsilon(u). \quad (6.20) \end{aligned}$$

It remains to show that for $u \in \mathrm{H}^2(\mathbb{R}^3)$

$$\left(G(T, \varphi) + \frac{g^2\gamma}{4\pi} + gu(0)\right)\varepsilon(u) = C(g, \gamma, E_0)\varepsilon(u)\,. \quad (6.21)$$

It follows from Proposition 5.1 that

$$\begin{aligned} G(T, \varphi) + gu(0) &= g\langle f_\gamma, Tu\rangle - g\langle T(f_\gamma), u\rangle + g^2\langle f_\gamma, Tf_\gamma\rangle + gAu \\ &= g\langle f_\gamma, -\Delta_1^* u\rangle - g\langle -\Delta_1^* f_\gamma, u\rangle + gAu + (-\gamma^2 + E_0)\|gf_\gamma\|_{\mathfrak{H}}^2 \\ &= gAf_\gamma Bu - gBf_\gamma Au + gAu + (-\gamma^2 + E_0)\|gf_\gamma\|_{\mathfrak{H}}^2 \\ &= (-\gamma^2 + E_0)\|gf_\gamma\|_{\mathfrak{H}}^2\,, \quad (6.22) \end{aligned}$$

since $Bu = 0$ and $Bf_\gamma = 1$. □

Proposition 6.9 *For all $E_0 \in \mathbb{R}$ the operator $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ is essentially self-adjoint and for any $\gamma > 0$ the space $W(gf_\gamma)E(\mathrm{H}^2(\mathbb{R}^3)) \subset D_{\mathrm{IBC}}$ is a core. If $E_0 \geqslant 0$, then the Hamiltonian H_{IBC} is bounded from below.*

Proof According to Corollary 6.8 and by symmetry of $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ it suffices to show that

$$\mathrm{d}\Gamma(h) + (-\gamma^2 + E_0)\left(a^*(gf_\gamma) + a(gf_\gamma)\right), \quad (6.23)$$

is essentially self-adjoint on $E(\mathrm{H}^2(\mathbb{R}^3))$. By Proposition 6.5, the operator $(\mathrm{d}\Gamma(h), E(\mathrm{H}^2(\mathbb{R}^3)))$ is essentially self-adjoint.

For $E_0 \geqslant 0$ the perturbation $a^*(gf_\gamma) + a(gf_\gamma)$ is infinitesimally bounded with respect to $\mathrm{d}\Gamma(h)$ (see Proposition 3.8 in [7]) and thus, by Kato-Rellich, essential self-adjointness of (6.23) on $E(\mathrm{H}^2(\mathbb{R}^3))$ holds. Here one uses the fact that

$$\hat{f}_\gamma(k) = -(2\pi)^{-\frac{3}{2}}(|k|^2 + \gamma^2)^{-1} = -\hat{\delta}(k)\cdot(|k|^2 + \gamma^2)^{-1} \qquad \mathrm{Re}(\gamma) > 0\,, \quad (6.24)$$

and therefore $\langle \hat{f}_\gamma, \hat{h}^{-1}\hat{f}_\gamma\rangle < \infty$ even for $E_0 = 0$.

If $E_0 < 0$, essential self-adjointness of (6.23) is shown using Nelson's Commutator Theorem (Theorem X.36 in [31]) with comparison operator $N = \mathbf{1}_{\mathfrak{F}} + \mathrm{d}\Gamma(h - E_0 + 1)$, cf. Proposition 3.11 in [7]. □

Proposition 6.10 *If $E_0 > 0$, then the operator $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$ is self-adjoint and*

$$H_{\mathrm{IBC}} = W(gf_{\sqrt{E_0}})\left[\mathrm{d}\Gamma(h) + \tfrac{g^2\sqrt{E_0}}{4\pi}\right]W(-gf_{\sqrt{E_0}})\,. \quad (6.25)$$

Proof As $E_0 > 0$, we may choose $\gamma = \sqrt{E_0}$ in Corollary 6.8 and set $\phi := g f_{\gamma=\sqrt{E_0}}$. The constant $C(g, \sqrt{E_0}, E_0)$ then reduces to $\frac{g^2\sqrt{E_0}}{4\pi}$ and the equality (6.25) holds on the common core $W(\phi)E(\mathrm{H}^2(\mathbb{R}^3))$. This extends to the common domain of self-adjointness $W(\phi)D(\mathrm{d}\Gamma(h))$.

The inclusion $D_{\mathrm{IBC}} \subseteq W(\phi)D(\mathrm{d}\Gamma(h))$ follows from the symmetry of $(H_{\mathrm{IBC}}, D_{\mathrm{IBC}})$, Proposition 5.5. To show that also $W(\phi)D(\mathrm{d}\Gamma(h)) \subseteq D_{\mathrm{IBC}}$, we use that $W(\phi)\, D(\mathrm{d}\Gamma(h))$ is the closure of $W(\phi)E(\mathrm{H}^2(\mathbb{R}^3))$ in the graph norm of $W(\phi)\mathrm{d}\Gamma(h)W(-\phi)$. We need to show that for $\psi \in W(\phi)\, D(\mathrm{d}\Gamma(h))$ we have $\psi^{(n)} \in D(\Delta_n^*)$ and $A\psi \in \mathfrak{F}$. Let $u \in \mathrm{H}^2(\mathbb{R}^3)$, then we have the estimate

$$\begin{aligned}
\|u(0)W(\phi)\varepsilon(u)\|^2_{\mathfrak{F}} &= \sum_{n\geqslant 0} \frac{1}{n!}\|u(0)u^{\otimes n}\|^2_{\mathfrak{H}^n} \\
&\leqslant \sum_{n\geqslant 0} \frac{C}{(n+1)!}(n+1)\|(-\Delta_{x_{n+1}} + E_0)u^{\otimes(n+1)}\|^2_{\mathrm{L}^2(\mathbb{R}^{3(n+1)})} \\
&\leqslant C\|\mathrm{d}\Gamma(h)\varepsilon(u)\|^2_{\mathfrak{F}}\,,
\end{aligned} \tag{6.26}$$

where we have used that $|u(0)| \leqslant C\|u\|_{\mathrm{H}^2}$ and that $\langle \Delta_{x_j} u^{\otimes(n+1)}, \Delta_{x_i} u^{\otimes(n+1)}\rangle \geqslant 0$. In view of (6.19) this implies that

$$\|AW(\phi)\varepsilon(u)\|_{\mathfrak{F}} \leqslant C\|\mathrm{d}\Gamma(h)\varepsilon(u)\|_{\mathfrak{F}}, \tag{6.27}$$

for some constant $C > 0$. This clearly implies that for any $n \in \mathbb{N}$

$$\|(-\Delta_n^* + nE_0)\,(W(\phi)\varepsilon(u))^{(n)}\,\|_{\mathfrak{H}^n} \leqslant \|(H - gA)W(\phi)\varepsilon(u)\|_{\mathfrak{F}} \leqslant C\|\mathrm{d}\Gamma(h)\varepsilon(u)\|_{\mathfrak{F}}\,. \tag{6.28}$$

As Δ_n^* is closed, it follows that $W(\phi)\, D(\mathrm{d}\Gamma(h))|_{\mathfrak{H}^n} \subset D(\Delta_n^*)$.

Consequently by Lemma 5.2 the expressions for A and B are well defined (as distributions) and continuous on each sector of $W(\phi)\, D(\mathrm{d}\Gamma(h))$. Now (6.27) implies that A maps $W(\phi)\, D(\mathrm{d}\Gamma(h))$ to $\mathfrak{F}$, so in particular $A\psi^{(n)} \in \mathrm{L}^2(\mathbb{R}^{3n-3})$. Since $B\psi = g\psi$ on the dense set $W(\phi)\, E(\mathrm{H}^2)$, this also holds on $W(\phi)\, D(\mathrm{d}\Gamma(h))$ by continuity, and we have proved $W(\phi)\, D(\mathrm{d}\Gamma(h)) \subset D_{\mathrm{IBC}}$. □

We remark that the expressions A and B defined on some natural domain $D \subset \bigoplus_n D(\Delta_n^*)$ are not necessarily closable, e.g., B vanishes on the dense (in $\mathfrak{F}$) subspace $D(\mathrm{d}\Gamma(h))$, so we cannot directly conclude from an estimate such as (6.27) that these expressions are well defined on the closure of $W(\phi)E(\mathrm{H}^2)$.

By virtue of the unitary equivalence, we can compute the ground state of H_{IBC} explicitly, provided $E_0 > 0$. The unique ground state of the free field $\mathrm{d}\Gamma\,(h)$ is the vector $\Omega_0 := (1, 0, 0, \ldots) \in \mathfrak{F}$, which is called the Fock vacuum. With $\phi = g f_{\gamma=\sqrt{E_0}}$ we conclude that $\psi_{min} := W(\phi)\Omega_0$ is the unique ground state of H_{IBC} with ground state energy $\frac{g^2\sqrt{E_0}}{4\pi}$, i.e.

$$H_{\mathrm{IBC}}\, \psi_{min} = \frac{g^2\sqrt{E_0}}{4\pi}\, \psi_{min}\,. \tag{6.29}$$

Note that because of $\Omega_0 = \varepsilon(0)$ we can calculate ψ_{min} explicitly by using (6.12),

$$\psi_{min} = W(\phi)\Omega_0 = W(\phi)\varepsilon(0) = \mathrm{e}^{-\frac{\|\phi\|^2}{2}}\ \varepsilon(\phi)\,. \tag{6.30}$$

Taken together, Corollary 5.6 and Propositions 6.9 and 6.10 prove Theorem 2.1.

6.3 Renormalization: Proof of Theorem 3.1

Let $h = (-\Delta + E_0, \mathrm{H}^2(\mathbb{R}^3))$, where we now assume that $E_0 > 0$. This operator is self-adjoint and invertible. In Section 3 we defined $W_n := W(gh^{-1}\chi_n)$ where χ_n is any sequence of elements of $\mathrm{L}^2(\mathbb{R}^3)$ such that $\chi_n \to \delta$ as $n \to \infty$ in the sense that $\hat{\chi}_n \to \hat{\chi}_\infty := \hat{\delta} = (2\pi)^{-3/2}$ pointwise with $\|\hat{\chi}_n\|_\infty$ uniformly bounded.
We first use Corollary 6.7 with $\psi = g\chi_n$ and $T = h$ to establish that, in the notation of Section 3,

$$\begin{aligned} W_n H_n W_n^* &= W_n \left(\mathrm{d}\Gamma(h) + a^*(g\chi_n) + a(g\chi_n)\right) W_n^* = \mathrm{d}\Gamma(h) - g^2\langle \chi_n, h^{-1}\chi_n\rangle_{\mathfrak{H}} \\ &= \mathrm{d}\Gamma(h) + E_n\,. \end{aligned} \tag{6.31}$$

The assumptions we made on the sequence χ_n imply that $\mathcal{F}(gh^{-1}\chi_n)$ converges in L^2 to the function $g(2\pi)^{-3/2}\hat{h}^{-1}$. Therefore, according to (6.24), $gh^{-1}\chi_n$ converges to $-gf_{\sqrt{E_0}}$. We have defined the family of unitary operators $W(\varphi)$ in (6.12) via coherent vectors. From this definition it follows that the mapping $\varphi \mapsto W(\varphi)\psi$ is continuous because the mapping $\varphi \mapsto \varepsilon(\varphi)$ is. As a consequence, the W_n converge strongly, and the limiting operator is

$$W_\infty = \lim_{n\to\infty} W_n = \lim_{n\to\infty} W(gh^{-1}\chi_n) = W(\lim_{n\to\infty} gh^{-1}\chi_n) = W(-gf_{\sqrt{E_0}})\,. \tag{6.32}$$

Moreover, for any $z \in \mathbb{C}\setminus\mathbb{R}$ also

$$\begin{aligned} \lim_{n\to\infty}(H_n - E_{n-z})^{-1} &= \lim_{n\to\infty} W_n^*(\mathrm{d}\Gamma(h) - z)^{-1}W_n = W_\infty^*(\mathrm{d}\Gamma(h) - z)^{-1}W_\infty \\ &= (W_\infty^*\mathrm{d}\Gamma(h)W_\infty - z)^{-1}, \end{aligned} \tag{6.33}$$

converges strongly because $\sup_n \|W_n^*\| = 1$. Recalling the definition (3.3) of H_∞, we find that

$$\begin{aligned} H_\infty &:= \lim_{n\to\infty}(H_n - E_n) = W_\infty^*\mathrm{d}\Gamma(h)W_\infty = W(gf_{\sqrt{E_0}})\,\mathrm{d}\Gamma(h)\,W(-gf_{\sqrt{E_0}}) \\ &\overset{(6.25)}{=} H_{\mathrm{IBC}} - \frac{g^2\sqrt{E_0}}{4\pi}, \end{aligned} \tag{6.34}$$

on $W(gf_{\sqrt{E_0}})D(\mathrm{d}\Gamma(h)) = D_{\mathrm{IBC}}$. We have proven Theorem 3.1.

7 Variants of the Model

Throughout this section, let $E_0 > 0$ and $N \in \mathbb{N}$ be fixed. We will use the notation that has been introduced in Section 4 and in particular assume the condition (4.7). Here we will properly define $\tilde{D}_{\mathrm{IBC}}$ and prove Theorems 4.1 and 4.4.

Observe that $w_i^\lambda \in D(\Delta_1^*)$ and that $\Delta_1^* w_i^\lambda = \lambda w_i^\lambda$ for $1 \leqslant i \leqslant N$, cf. [1]. It is known that that the maps $\psi \mapsto A_i\psi$ and $\psi \mapsto B_i\psi$ define continuous linear functionals on $D(\Delta_1^*)$. Furthermore, using a partition of unity, the degree of non-symmetry of Δ_1^* may be expressed with their help:

$$\langle\varphi, -\Delta_1^*\psi\rangle_{\mathfrak{H}} - \langle -\Delta_1^*\varphi, \psi\rangle_{\mathfrak{H}} = \sum_{i=1}^{N} \langle B_i\varphi, A_i\psi\rangle_{\mathbb{C}} - \langle A_i\varphi, B_i\psi\rangle_{\mathbb{C}}\,. \tag{7.1}$$

Note the following: The set $U(v) := \bigcap_{i=1}^N \ker X_i$ is a subspace of $D(\Delta_1^*)$, which is L^2-dense. By further inspection $X_i(\psi) = 0$ for all $1 \leqslant i \leqslant N$ is identified with the conditions that specify the domain of point interactions centered in $\xi_1, \ldots, \xi_N$ with parameters $a_i = \frac{\alpha_i}{\beta_i}$, where $\beta_i = 0$ formally corresponds to $a_i = \infty$, see [5].

The matrix $S(\lambda)$ is invertible if and only if $-\lambda$ is *not* an eigenvalue of the point-interaction operator $(-\Delta_1^*, U(v))$, see Theorem II.1.1.4 in [1]. The number of eigenvalues of this operator is finite, and all its eigenvalues are negative and situated below the essential spectrum, which covers the non-negative real axis. That implies, in particular, that for all $E_0 > 0$ and for all admissible choices of v there exists $\lambda > 0$ such that $S(\lambda)$ is invertible.

Lemma 7.1 *Let v obey the condition (4.7) and let $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(\lambda)$. Then there exists $\phi = \phi(\lambda) \in D(\Delta_1^*)$ with the properties*

$$\Delta_1^*\phi = \lambda\phi \tag{7.2}$$

$$X_k(\phi) = 1 \qquad 1 \leqslant k \leqslant N\,. \tag{7.3}$$

Proof For every choice of $c_1, \ldots, c_N \in \mathbb{C}$ the sum $\sum_{l=1}^N c_l w_l^\lambda$ is an eigenvector of Δ_1^* with eigenvalue λ. To obtain (7.3), we first compute

$$\begin{aligned} X_k\left(\textstyle\sum_{l=1}^N c_l w_l^\lambda\right) &= \sum_{l=1}^{N} c_l X_k(w_l^\lambda) = \sum_{l=1}^{N} c_l\alpha_k e^{i\theta_k} B_k(w_l^\lambda) + c_l\beta_k e^{i\theta_k} A_k(w_l^\lambda) \\ &= \sum_{l=1}^{N} c_l\alpha_k e^{i\theta_k}\delta_{kl} + c_l\beta_k e^{i\theta_k}\Big(\delta_{kl}\frac{\sqrt{\lambda}}{4\pi} + (1-\delta_{kl})G_{kl}^\lambda\Big) = \sum_{l=1}^{N} S_{kl}c_l\,. \end{aligned} \tag{7.4}$$

Since $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(\lambda)$, there are numbers $c_l \in \mathbb{C}$ such that $\sum_{l=1}^N S_{kl}c_l = 1$ for all $1 \leqslant k \leqslant N$. Then we set $\phi := \sum_{l=1}^N c_l w_l$. □

Lemma 7.2 *Let v obey the condition (4.7). Then the degree of non-symmetry of Δ_1^* can be expressed using X_i and Y_i: for $\varphi, \psi \in D(\Delta_1^*)$,*

$$\langle\varphi, -\Delta_1^*\psi\rangle_{\mathfrak{H}} - \langle -\Delta_1^*\varphi, \psi\rangle_{\mathfrak{H}} = \sum_{i=1}^{N} \langle X_i\varphi, Y_i\psi\rangle_{\mathbb{C}} - \langle Y_i\varphi, X_i\psi\rangle_{\mathbb{C}}\,. \tag{7.5}$$

Lemma 7.3 *Let* $\psi \in U(v) = \bigcap_{i=1}^{N} \ker X_i$ *and let* $\phi(\lambda) \in D(\Delta_1^*)$ *with the properties* (7.2) and (7.3). *Then*

$$\text{(a)} \quad \sum_{i=1}^{N} Y_i(\psi) = \langle \phi, (-\Delta_1^* + E_0)\psi \rangle_{\mathfrak{H}} - \langle (-\Delta_1^* + E_0)\phi, \psi \rangle_{\mathfrak{H}}$$

$$\text{(b)} \quad \sum_{i=1}^{N} Y_i(\phi) \in \mathbb{R}\,.$$

The proofs can be found in the Appendix B.

As mentioned above, the operator $\tilde{h} = (-\Delta_1^* + E_0, U)$ is self-adjoint and is called the N-center point-interaction with energy offset $E_0 > 0$. The coherent domain $E(U)$ is a core of $\mathrm{d}\Gamma(\tilde{h})$, see Proposition 6.5. Next we turn to another subset of $D(\Delta_1^*)$, which is an affine subspace. If $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(\lambda)$, define

$$M = M(\lambda) := \{\varphi \in D(\Delta_1^*) | \varphi = \phi(\lambda) + \psi\,,\, \psi \in U(v)\}\,. \tag{7.6}$$

Since $U(v)$ is L^2-dense, so is $M(\lambda)$ and therefore the coherent domain over $E(M)$ is a dense subspace of the symmetric Fock space $\mathfrak{F}$. Set $\tilde{D}_{\mathrm{IBC}} := E(M)$. Then on $\tilde{D}_{\mathrm{IBC}}$ we find

$$Y_i^{\mathfrak{F}}(\varepsilon(\varphi)) = Y_i(\phi + \psi)\varepsilon(\varphi) = (Y_i(\phi) + Y_i(\psi))\varepsilon(\varphi) \tag{7.7}$$

and

$$X_i^{\mathfrak{F}}(\varepsilon(\varphi)) = X_i(\phi + \psi)\varepsilon(\varphi) = X_i(\phi)\varepsilon(\varphi) = \varepsilon(\varphi)\,. \tag{7.8}$$

We are now in a position to define the operator $(\tilde{H}_{\mathrm{IBC}}, \tilde{D}_{\mathrm{IBC}})$ which depends on the set of parameters (v, E_0) where v obeys the relation (4.7):

$$\tilde{H}_{\mathrm{IBC}} := \mathrm{d}\Gamma(-\Delta_1^* + E_0) + \textstyle\sum_{i=1}^{N} Y_i^{\mathfrak{F}} \quad \text{on} \quad \tilde{D}_{\mathrm{IBC}} := E(M)\,. \tag{7.9}$$

Proof of Theorem 4.1 and Theorem 4.4 Let $\psi \in U$. Choose $\lambda > 0$ such that $S(\lambda)$ is invertible and use $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(\lambda)$ to construct $\phi(\lambda)$ with the properties (7.2) and (7.3). Due to property (7.2) of $\phi = \phi(\lambda)$, using Lemma 6.6 we get

$$\begin{aligned}
& W(-\phi)\mathrm{d}\Gamma(-\Delta_1^* + E_0)W(\phi)\varepsilon(\psi) + W(-\phi)\left[\textstyle\sum_{i=1}^{N} Y_i^{\mathfrak{F}}\right] W(\phi)\varepsilon(\psi) \\
&= \left(\mathrm{d}\Gamma(\tilde{h}) + (-\lambda + E_0)\left(a(\phi)^* + a(\phi)\right)\right)\varepsilon(\psi) \\
&\quad + \left(\langle \phi, (-\Delta_1^* + E_0)\psi \rangle_{\mathfrak{H}} - \langle (-\Delta_1^* + E_0)\phi, \psi \rangle_{\mathfrak{H}}\right)\varepsilon(\psi) \\
&\quad + \langle \phi, (-\Delta_1^* + E_0)\phi \rangle_{\mathfrak{H}}\varepsilon(\psi) + \left[\textstyle\sum_{i=1}^{N} Y_i(\phi) + Y_i(\psi)\right]\varepsilon(\psi) \\
&= \left(\mathrm{d}\Gamma(\tilde{h}) + (-\lambda + E_0)\left(a(\phi)^* + a(\phi)\right)\right)\varepsilon(\psi) \\
&\quad + \left[(-\lambda + E_0)\,\|\phi\|_{\mathfrak{H}} + \textstyle\sum_{i=1}^{N} Y_i(\phi)\right]\varepsilon(\psi)\,.
\end{aligned} \tag{7.10}$$

We have used statement (a) of Lemma 7.3. Due to statement (b) of this lemma, the constant in brackets is real. Because $\tilde{h}$ is bounded from below, we can use Nelson's Commutator Theorem to show essential self-adjointness of the operator on $E(U)$, cf.

Proposition 6.9 and [7]. Now essential self-adjointness of $\tilde{H}_{\mathrm{IBC}}$ on $W(\phi(\lambda))E(U) = E(M) = \tilde{D}_{\mathrm{IBC}}$ follows.

If $(1, 1, \ldots, 1)^T \in \operatorname{ran} S(E_0)$, set $\lambda = E_0$ to get (4.18). We have proven Theorem 4.4. In this case $\tilde{H}_{\mathrm{IBC}}$ may be unbounded from below.

If $\tilde{h}$ is strictly positive, then $-E_0$ is not an eigenvalue of $(-\Delta_1^*, U)$ and $S(E_0)$ is invertible. From the explicit form (4.18) we see that, because $\mathrm{d}\Gamma(\tilde{h})$ is strictly positive as well, Ω_0 is the unique ground state of $\mathrm{d}\Gamma(\tilde{h})$. As a consequence $\tilde{H}_{\mathrm{IBC}}$ is bounded from below by

$$C(\phi(E_0)) = \sum_{i=1}^{N} Y_i(\phi(E_0)), \tag{7.11}$$

and

$$\psi_{min} = \mathrm{e}^{-\frac{\|\phi(E_0)\|^2}{2}} \varepsilon(\phi(E_0)), \tag{7.12}$$

is the unique ground state of $\tilde{H}_{\mathrm{IBC}}$. □

Acknowledgments We thank Marcel Griesemer, Stefan Keppeler, and Andreas Wünsch for helpful discussions. This work was supported by the German Research Foundation (DFG) within the Research Training Group 1838 *Spectral Theory and Dynamics of Qauntum Systems*.

Appendix A: Regularity

Here, we give the details on the regularity questions regarding $D(\Delta_n^*)$, $A^{(n)}$, and $B^{(n)}$. We will need to work with Hilbert-space-valued distributions. Keep in mind for the following that for defining distributions the removal of a point $\{0\}$ from $\mathbb{R}^3$ or the sets $\mathcal{C}^n$ from $\mathbb{R}^{3n}$ matters, while $\mathrm{L}^2(\mathbb{R}^3 \setminus \{0\}, X) = \mathrm{L}^2(\mathbb{R}^3, X)$ and $\mathrm{L}^2(\mathbb{R}^{3n} \setminus \mathcal{C}^n, X) = \mathrm{L}^2(\Omega_n, X) = \mathrm{L}^2(\mathbb{R}^{3n}, X)$.

Lemma A.1 *Let $\varphi \in D(\Delta_n^*)$ and equip this space with the graph norm. Then for $j = 1, \ldots, n$*

$$\Delta_{x_j}\varphi \in \mathrm{L}^2\left(\mathbb{R}^3_{x_j}, \mathrm{H}^{-2}(\Omega_{n-1})\right), \tag{A.1}$$

where φ is regarded as a vector valued distribution on $\mathbb{R}^3_{x_j} \setminus \{0\}$ and Δ_{x_j} is the Laplacian of distributions on that domain taking values in $\mathrm{H}^{-2}(\Omega_{n-1})$. Moreover,

$$\|\Delta_{x_j}\varphi\|_{\mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2})} \leqslant \sqrt{2}\|\varphi\|_{D(\Delta_n^*)}. \tag{A.2}$$

Proof We will show the case $j = 1$. Recall that $D(\Delta_n) = \mathrm{H}_0^2(\Omega_n)$. For any $\varphi \in D(\Delta_n^*)$, the map

$$\Delta_{x_1}\varphi : \mathrm{H}_0^2(\Omega_n) \to \mathbb{C}, \quad \psi \mapsto \langle \varphi, \Delta_{x_1}\psi \rangle = \langle \Delta_n^*\varphi, \psi \rangle - \sum_{i=2}^{N} \langle \varphi, \Delta_{x_i}\psi \rangle, \tag{A.3}$$

extends by density to a bounded linear functional on the Bochner space $\mathrm{L}^2\left(\mathbb{R}^3_{x_1}, \mathrm{H}^2_0(\Omega_{n-1})\right)$, i.e.,

$$\Delta_{x_1}\varphi \ \in \ \mathrm{L}^2\left(\mathbb{R}^3_{x_1}, \mathrm{H}^2_0(\Omega_{n-1})\right)' . \tag{A.4}$$

Since $\mathrm{H}^{-2}(\Omega_{n-1}) := \mathrm{H}^2_0(\Omega_{n-1})'$ and this space is reflexive, we obtain that $\Delta_{x_1}\varphi \in \mathrm{L}^2\left(\mathbb{R}^3_{x_1}, \mathrm{H}^{-2}(\Omega_{n-1})\right)$. It remains to show that this $\Delta_{x_1}\varphi$ is in fact also the Laplacian of φ in the sense of $\mathrm{H}^{-2}(\Omega_{n-1})$-valued distributions, i.e. that for all $\phi \in C_0^\infty(\mathbb{R}^3\setminus\{0\})$ and $\xi \in \mathrm{H}^2_0(\Omega_{n-1})$ we have

$$(\Delta_{x_1}\varphi)(\phi\xi) = \int_{\mathbb{R}^3} \langle \varphi(x_1, \ldots, x_n)), \xi(x_2, \ldots, x_n)\rangle_{(\mathrm{H}^{-2},\mathrm{H}^2_0)} \Delta\phi(x_1)\, \mathrm{d}x_1 . \tag{A.5}$$

The left hand side is, by its definition (A.3),

$$(\Delta_{x_1}\varphi)(\phi\xi) = \langle \varphi, (\Delta\phi)\xi\rangle_{\mathrm{L}^2(\mathbb{R}^{3n})} , \tag{A.6}$$

and, since $\varphi \in \mathrm{L}^2(\mathbb{R}^3_{x_1}, \mathrm{L}^2(\mathbb{R}^{3n-3}))$, the right hand side equals

$$\int_{\mathbb{R}^3}\int_{\Omega_{n-1}} \overline{\varphi}(x_1, \ldots, x_n)\xi(x_2, \ldots, x_n)\Delta\phi(x_1)\, \mathrm{d}x = \langle \varphi, (\Delta_1\phi)\xi\rangle_{\mathrm{L}^2(\mathbb{R}^{3n})} . \tag{A.7}$$

Finally, the bound on the norm of $\Delta_{x_1}\varphi$ follows from (A.3) by

$$\begin{aligned}\|\Delta_{x_1}\varphi\|_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2}(\Omega_{n-1}))} &\leqslant \sup_{\|\psi\|_{\mathrm{L}^2\left(\mathbb{R}^3,\mathrm{H}^2_0\right)}=1} \Big(\|\Delta_n^*\varphi\|_{\mathrm{L}^2}\|\psi\|_{\mathrm{L}^2} + \|\varphi\|_{\mathrm{L}^2}\|\psi\|_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^2_0)}\Big)\\ &\leqslant \|\Delta_n^*\varphi\|_{\mathrm{L}^2(\mathbb{R}^{3n})} + \|\varphi\|_{\mathrm{L}^2(\mathbb{R}^{3n})} \leqslant \sqrt{2}\|\varphi\|_{D(\Delta_n^*)} . \quad \square\end{aligned}$$

Proof of Lemma 5.2 For clarity, we use the notation $A^{(n)}$ and $B^{(n)}$ in this proof for the operators on $D(\Delta_n^*) \subset \mathrm{L}^2(\mathbb{R}^{3n})$. The case $n = 1$ has been proved in Proposition 5.1 and we will use it here to show continuity of $A^{(n)}$ and $B^{(n)}$ for $n \geqslant 2$. Our proof basically follows ideas for the construction of distribution-valued trace maps on Sobolev spaces, as presented, e.g, in [20].

Define the space

$$D^*_{\mathrm{H}^{-2}} := \{\varphi \in \mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2}(\Omega_{n-1}))|\Delta_x\varphi \in \mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2}(\Omega_{n-1}))\} , \tag{A.8}$$

where Δ_x denotes the Laplacian on vector-valued distributions on $\mathbb{R}^3 \setminus \{0\}$, and

$$\|\varphi\|^2_{D^*_{\mathrm{H}^{-2}}} := \|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2})} + \|\Delta_x\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2})} . \tag{A.9}$$

Then, by Lemma 7.4, we have the continuous injection

$$D(\Delta_n^*) \hookrightarrow D^*_{\mathrm{H}^{-2}} . \tag{A.10}$$

We will show that $A^{(n)}$ is continuous on $D^*_{\mathrm{H}^{-2}}$, which of course implies continuity on $D(\Delta_n^*)$. To do so, we approximate any $\varphi \in D^*_{\mathrm{H}^{-2}}$ by a sequence φ_N in the following way: Let $(\eta_k)_{k\in\mathbb{N}}$ be a complete orthonormal set in $\mathrm{H}^{-2}(\Omega_{n-1})$ and set $\varphi_k(x) := \langle \eta_k, \varphi(x,\cdot)\rangle_{\mathrm{H}^{-2}}$. Because $\varphi \in \mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2}(\Omega_{n-1}))$, it holds that

$$\sum_{k=1}^{N} \varphi_k(x)\eta_k := \varphi_N(x) \overset{N\to\infty}{\to} \varphi(x), \tag{A.11}$$

pointwise in $\mathrm{H}^{-2}(\Omega_{n-1})$ and by dominated convergence in $\mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^{-2}(\Omega_{n-1}))$. Now let $\psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ and observe that, because $\langle \eta_k, \cdot\rangle_{\mathrm{H}^{-2}}$ is continuous on $\mathrm{H}^{-2}(\Omega_{n-1})$ and $\varphi(x,\cdot)\Delta\psi$ is integrable, we have that

$$\begin{aligned}\int_{\mathbb{R}^3} \varphi_k \Delta\psi \,\mathrm{d}x &= \int_{\mathbb{R}^3} \langle \eta_k, \varphi(x,\cdot)\Delta\psi\rangle_{\mathrm{H}^{-2}} \,\mathrm{d}x = \left\langle \eta_k, \int_{\mathbb{R}^3} \varphi(x,\cdot)\Delta\psi \,\mathrm{d}x \right\rangle_{\mathrm{H}^{-2}} \\ &= \left\langle \eta_k, \int_{\mathbb{R}^3} \psi \Delta_x \varphi(x,\cdot)\,\mathrm{d}x \right\rangle_{\mathrm{H}^{-2}} = \int_{\mathbb{R}^3} \langle \eta_k, \Delta_x\varphi(x,\cdot)\rangle_{\mathrm{H}^{-2}} \psi(x)\,\mathrm{d}x\,. \end{aligned} \tag{A.12}$$

Since $\varphi \in D^*_{\mathrm{H}^{-2}}$, $\langle \eta_k, \Delta_x\varphi(x,\cdot)\rangle_{\mathrm{H}^{-2}} \in \mathrm{L}^2(\mathbb{R}^3)$ and thus $\varphi_k \in D(\Delta_1^*)$ with $\Delta_1^*\varphi_k = \langle \eta_k, \Delta_x\varphi(x,\cdot)\rangle_{\mathrm{H}^{-2}}$.

To prove that the limit in the expression for $A^{(n)}$ exists, let

$$\tilde{\varphi}_k(r) := \frac{1}{4\pi}\int_{S^2} r\varphi_k(r\omega)\mathrm{d}\omega\,. \tag{A.13}$$

One easily sees that $\|\tilde{\varphi}_k\|_{\mathrm{H}^2((0,\infty))} = \|\varphi_k\|_{D(\Delta_1^*)}$, and thus $\tilde{\varphi}_k$ has a representative in $C^{1,\frac{1}{4}}([0,\infty))$ and there exists a constant K such that for $R, r \in [0,1]$

$$|\tilde{\varphi}_k'(R) - \tilde{\varphi}_k'(r)| \leqslant |R-r|^{1/4}\|\tilde{\varphi}_k\|_{C^{1,\frac{1}{4}}[0,1]} \leqslant K|R-r|^{1/4}\|\varphi_k\|_{D(\Delta_1^*)}. \tag{A.14}$$

where $\tilde{\varphi}_k'$ denotes the derivative of $\tilde{\varphi}_k$. Then we also have that

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}(\tilde{\varphi}_k'(R) - \tilde{\varphi}_k'(r))\eta_k\right\|^2_{\mathrm{H}^{-2}(\Omega_{n-1})} &= \sum_{k=1}^{\infty} |\tilde{\varphi}_k'(R) - \tilde{\varphi}_k'(r)|^2 \\ &\leqslant K^2|R-r|^{1/2}\sum_{k=1}^{\infty}\|\varphi_k\|^2_{D(\Delta_1^*)} \\ &= K^2|R-r|^{1/2}\sum_{k=1}^{\infty}\left(\|\varphi_k\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \|\Delta_1^*\varphi_k\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\right) \\ &= K^2|R-r|^{1/2})^2\left(\sum_{k=1}^{\infty}\|\varphi_k\eta_k\|^2_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2})} + \sum_{k=1}^{\infty}\|\Delta_x\varphi_k\eta_k\|^2_{\mathrm{L}^2(\mathbb{R}^3,\mathrm{H}^{-2})}\right) \\ &= K^2|R-r|^{1/2}\|\varphi\|^2_{D^*_{\mathrm{H}^{-2}}}\,. \end{aligned} \tag{A.15}$$

It follows that the limit $\lim_{r\to 0}\sum_{k=0}^{\infty}\tilde{\varphi}_k'(r)\eta_k$ exists for this representative of φ and yields the value of $A^{(n)}/\sqrt{n}$. In addition, we have that

$$\begin{aligned}\|A^{(n)}\varphi_N\|^2_{\mathrm{H}^{-2}} &= \left\|A^{(n)}\sum_{k=1}^{N}\varphi_k\eta_k\right\|^2_{\mathrm{H}^{-2}} = n\left\|\sum_{k=1}^{N}(A^{(1)}\varphi_k)\eta_k\right\|^2_{\mathrm{H}^{-2}}\\ &\leqslant n\|A^{(1)}\|^2_{D(\Delta_1^*)'}\sum_{k=1}^{N}\|\varphi_k\|^2_{D(\Delta_1^*)}\\ &= n\|A^{(1)}\|^2_{D(\Delta_1^*)'}\|\varphi_N\|^2_{D^*_{\mathrm{H}^{-2}}}\,. \end{aligned}\tag{A.16}$$

Thus, $A^{(n)}$ defines a bounded linear map. The proof for $B^{(n)}$ follows the same steps.

This proof shows that the action of $A^{(n)}$, $B^{(n)}$ is determined by the action of $A^{(1)}$, $B^{(1)}$ on the φ_k. If φ is an element of $\mathrm{H}^2(\mathbb{R}^{3n})$ or $\mathrm{H}^1(\mathbb{R}^{3n})$, then the φ_k are in the corresponding space over $\mathbb{R}^3$. In case $\varphi\in\mathrm{H}^1(\mathbb{R}^{3n})$ we thus have that $B^{(n)}\varphi=0$ since $B^{(1)}=0$ on $D(\Delta_1^*)\cap\mathrm{H}^1(\mathbb{R}^3)=\mathrm{H}^2(\mathbb{R}^3)$ because $f_\gamma\notin\mathrm{H}^1(\mathbb{R}^3)$. If $\varphi\in\mathrm{H}^2(\mathbb{R}^{(3n)})$, $A^{(n)}$ acts as the Sobolev-trace, because $A^{(1)}\varphi_k=\varphi_k(0)$. □

In order to establish regularity of the functions $\varphi\in D(\Delta_n^*)$ with $B^{(n)}\varphi=0$, we use a theorem of Hörmander, which is formulated using the following spaces:

$$H_{(2,s)} := \mathrm{L}^2([0,\infty),\mathrm{H}^{2+s}(\mathbb{R}^d))\cap\mathrm{H}^2((0,\infty),\mathrm{H}^s(\mathbb{R}^d))\,.\tag{A.17}$$

Theorem A.2 *Let $\frac{\mathrm{d}^2}{\mathrm{d}r^2}$ and $\Delta_{\mathbb{R}^d}$ denote the distributional Laplacians on $(0,\infty)$ and $\mathbb{R}^d$, respectively. The map*

$$\begin{aligned} H_{(2,s)} &\to \mathrm{L}^2([0,\infty),\mathrm{H}^s)\oplus\mathrm{H}^{s+\frac{3}{2}}(\mathbb{R}^d)\,,\\ \eta &\mapsto \left(\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\Delta_{\mathbb{R}^d}-1\right)\eta,\ \eta(0)\right), \end{aligned}\tag{A.18}$$

is an isomorphism of topological vector spaces.

This theorem is a direct consequence of [14, Corollary 10.4.1]. It gives rise to the following regularity lemma, where we denote by $P:\mathrm{L}^2(\mathbb{R}^3)\to\mathrm{L}^2(\mathbb{R}^3)$ the projection to the space of radial functions; for $j\in\{1,\ldots,n\}$, P_j is the projection P acting on the j-th factor of $\mathrm{L}^2(\mathbb{R}^{3n})=\mathrm{L}^2(\mathbb{R}^3)^{\otimes n}$; and $Q_j=1-P_j$.

Lemma A.3 *Let $\varphi\in D(\Delta_n^*)$ with $B\varphi=0$ and $\chi_\varepsilon\in C_{\mathrm{b}}^\infty(\mathbb{R}^{3n-3})$ such that, for some $\varepsilon>0$,*

$$\operatorname{supp}\chi_\varepsilon\subset\mathcal{U}_\varepsilon(\mathcal{C}^{n-1}) := \left\{(x_2,\ldots,x_n)\in\mathbb{R}^{3n-3}\,\middle|\,|x_i|>\varepsilon\ \textit{for all} i\right\}.\tag{A.19}$$

Then $\chi_\varepsilon P_1\varphi\in\mathrm{H}^2(\mathbb{R}^{3n})$.

Proof We assume without loss of generality that φ is radial in the first argument, i.e., $\varphi = P_1\varphi$. Let $\tilde\varphi(r, y) := r\,\chi_\varepsilon(y)\varphi(r, y) \in \mathrm{L}^2([0,\infty), \mathrm{L}^2(\mathbb{R}^{3n-3}))$. First note that

$$\begin{aligned}\Delta\tilde\varphi &= \chi_\varepsilon(\tfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \Delta_y)r\varphi + (\Delta_y\chi_\varepsilon)r\varphi + 2r\nabla_y\chi_\varepsilon\cdot\nabla_y\varphi \\ &= \chi_\varepsilon \underbrace{r\Delta_n^*\varphi}_{\in \mathrm{L}^2} + (\Delta_y\chi_\varepsilon)\underbrace{r\varphi}_{\in\mathrm{L}^2} + 2\nabla_y\chi_\varepsilon\cdot\underbrace{\nabla_y r\varphi}_{\in \mathrm{L}^2([0,\infty),\mathrm{H}^{-1})} ,\end{aligned} \tag{A.20}$$

and that $B\varphi = 0$ implies $\tilde\varphi(0) = 0 \in \mathrm{H}^{-2}$. This of course means that $\tilde\varphi(0) \in \mathrm{H}^{s+\frac{3}{2}}$ for any $s \in \mathbb{R}$. Thus, Theorem 7.5 implies that

$$\tilde\varphi \in H_{(2,-1)} \subset \mathrm{L}^2([0,\infty), \mathrm{H}^1(\mathbb{R}^{3n-3}))\,. \tag{A.21}$$

Plugging this information into (A.20), we conclude that $\Delta\tilde\varphi \in \mathrm{L}^2([0,\infty), \mathrm{L}^2)$. Another use of Theorem 7.5 then yields $\tilde\varphi \in H_{(2,0)}$ with $\tilde\varphi(0) = 0$. Hence

$$\frac{\tilde\varphi}{r} = \chi_\varepsilon P_1\varphi \in \mathrm{L}^2(\mathbb{R}^3, \mathrm{H}^2(\mathbb{R}^{3n-3})) \cap \mathrm{H}^2(\mathbb{R}^3, \mathrm{L}^2(\mathbb{R}^{3n-3})) = \mathrm{H}^2(\mathbb{R}^{3n})\,. \tag{A.22}$$

□

For $I \subset \{1, 2, \ldots, n\}$ define the following sets:

$$\mathcal{C}^I := \Big\{x \in \mathbb{R}^{3n} \;\Big|\; \prod_{j\in I}|x_j| = 0\Big\}\,. \tag{A.23}$$

Then we have $\mathcal{C}^I \subset \mathcal{C}^n = \mathcal{C}^{\{1,2,\ldots,n\}}$. We will also use the abbreviation $\mathcal{C}^k := \mathcal{C}^{\{n-k+1,n-k+2,\ldots,n\}}$.

Proof of Proposition 5.3 We will prove that $\varphi \in D(\Delta_n^*) \cap \mathfrak{H}^n$ together with $B\varphi = 0$ implies $\varphi \in \mathrm{H}^2(\mathbb{R}^{3n})$. This will prove the statement when combined with Lemma 5.2.

In this proof we write $D^*(X)$ for the adjoint domain of the Laplacian defined on $X \subset \mathrm{H}^2(\mathbb{R}^{3n})$. For $I \subset \{1, \ldots, n\}$ let $P_I := \prod_{i\in I} P_i$ and $Q_I := \prod_{i\in I}(1 - P_i)$. Then for $f \in \mathrm{L}^2(\mathbb{R}^{3n})$

$$f = \prod_{i=1}^{n}(P_i + Q_i)f = \sum_{I\subset\{1,\ldots,n\}} P_I Q_{I^c} f\,. \tag{A.24}$$

Now let $\psi \in \mathrm{H}^2(\mathbb{R}^{3n})$. Then

$$\langle\varphi, \Delta\psi\rangle = \sum_{I\subset\{1,\ldots,n\}} \langle P_I Q_{I^c}\varphi, \Delta\psi\rangle = \sum_{I\subset\{1,\ldots,n\}} \langle P_I\varphi, \Delta Q_{I^c}\psi\rangle\,. \tag{A.25}$$

Since $Q_j\psi|_{x_j=0} = 0$, we have that $Q_{I^c}\psi \in \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{I^c})$ (cf. [35]), and so it is sufficient to show that

$$P_I\varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{I^c})), \tag{A.26}$$

in order to conclude $\varphi \in D^*(\mathrm{H}^2(\mathbb{R}^{3n})) = \mathrm{H}^2(\mathbb{R}^{3n})$. By symmetry it suffices to consider the sets $I = \{1, \ldots, k\}$ for $k \leqslant n$, which will be done by induction over k.

For $k = 1$, $I = \{1\}$, (A.26) follows from Lemma 7.6 in the following way: Let $\psi \in \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{n-1})$ and let ψ_ε be a sequence in $C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^{n-1})$ with $\mathrm{supp}\psi_\varepsilon \subset \mathcal{U}_{2\varepsilon}$ converging to ψ in H^2. Then Lemma 7.6 implies

$$\langle P_1\varphi, \Delta_{x_1}\psi_\varepsilon\rangle = \langle \chi_\varepsilon(x_2, \ldots, x_n)P_1\varphi, \Delta_{x_1}\psi_\varepsilon\rangle = \langle \chi_\varepsilon \Delta_{x_1} P_1\varphi, \psi_\varepsilon\rangle = \langle \Delta_{x_1} P_1\varphi, \psi_\varepsilon\rangle\,, \tag{A.27}$$

where we have used a cutoff χ_ε with $\chi_\varepsilon \equiv 1$ on $\mathcal{U}_{2\varepsilon}$. Since $\psi_\varepsilon \in \mathrm{L}^2\left(\mathbb{R}^3_{x_1}, \mathrm{H}_0^2(\Omega_{n-1})\right)$, we find that

$$\begin{aligned}\langle P_1\varphi, \Delta\psi\rangle &= \lim_{\varepsilon\to 0}\langle P_1\varphi, \Delta\psi_\varepsilon\rangle \\ &\overset{(A.4)}{=} \lim_{\varepsilon\to 0}\left(\langle \Delta_{x_1} P_1\varphi, \psi_\varepsilon\rangle + \langle P_1\varphi, \sum_{j=2}^{n} \Delta_{x_j}\psi_\varepsilon\rangle\right) \\ &\overset{(A.3)}{=} \lim_{\varepsilon\to 0}\langle \Delta_n^* P_1\varphi, \psi_\varepsilon\rangle = \langle \Delta_n^* P_1\varphi, \psi\rangle\,.\end{aligned} \tag{A.28}$$

Hence, $P_1\varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{n-1}))$.

Now assume the induction hypothesis

$$P_{\{1,\ldots,k\}}\varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+1,\ldots,n\}}))\,. \tag{A.29}$$

By symmetry, the argument for $k = 1$ independently gives also

$$P_{\{k+1\}}\varphi \in D^*(\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{1,\ldots,k,k+2,\ldots,n\}}))\,. \tag{A.30}$$

Thus, $P_{\{1,\ldots,k+1\}}\varphi$ is in the intersection of these two domains (A.29) and (A.30). Clearly, for two dense domains D_1, D_2 it holds that $D^*(D_1) \cap D^*(D_2) \subset D^*(D_1 + D_2)$. We thus need to show that

$$\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+1,\ldots,n\}}) + \mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{1,\ldots,k,k+2,\ldots,n\}}), \tag{A.31}$$

is dense in $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+2,\ldots,n\}})$, as this implies that the adjoint domains are equal. The functions in this sum vanish on

$$\widetilde{\mathcal{C}} := \left(\mathcal{C}^{\{k+1\}} \cap \mathcal{C}^{\{1,\ldots,k\}}\right) \cup \mathcal{C}^{\{k+2,\ldots,n\}}\,. \tag{A.32}$$

Conversely, any function $f \in C_0^\infty(\mathbb{R}^{3n} \setminus \widetilde{\mathcal{C}})$ can be written as a sum $f = f_1 + f_2$ with $f_1 \in C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+1\}})$ and $f_2 \in C_0^\infty(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{1,\ldots,k\}})$. Thus the sum (A.31) is dense in $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \widetilde{\mathcal{C}})$, but the latter space is equal to $\mathrm{H}_0^2(\mathbb{R}^{3n} \setminus \mathcal{C}^{\{k+2,\ldots,n\}})$, as $\mathcal{C}^{\{k+1\}} \cap \mathcal{C}^{\{1,\ldots,k\}}$ has codimension six, see [35]. □

B Algebraic Identities

Proof of Proposition 6.1 We will prove the following formula:

$$\mathrm{Sym}(v_1 \otimes \cdots \otimes v_n) = \sum_{\mathbf{j}\in J} d_{\mathbf{j}} u_{\mathbf{j}}^{\otimes n}\,, \tag{A.33}$$

where $J = \{0, 1\}^n$ and

$$u_{\mathbf{j}} = \sum_{k=1}^{n} (-1)^{j_k} v_k\,, \qquad d_{\mathbf{j}} = \frac{(-1)^{j_1+\ldots+j_n}}{2^n n!}\,. \tag{A.34}$$

Note that we may rewrite $u_{\mathbf{j}}^{\otimes n}$ as a sum:

$$\begin{aligned} u_{\mathbf{j}}^{\otimes n} &= \left(\sum_{k=1}^{n} (-1)^{j_k} v_k\right)^{\otimes n} = \sum_{\mathbf{k}\in\{1,\ldots,n\}^n} (-1)^{j_{k_1}+\ldots+j_{k_n}}\; v_{k_1}\otimes\cdots\otimes v_{k_n} \\ &= \sum_{\mathbf{k}\in P} (-1)^{j_{k_1}+\ldots+j_{k_n}}\; v_{k_1}\otimes\cdots\otimes v_{k_n} \\ &\quad + \sum_{\mathbf{k}\in\{1,\ldots,n\}^n\setminus P} (-1)^{j_{k_1}+\ldots+j_{k_n}}\; v_{k_1}\otimes\cdots\otimes v_{k_n} \\ &=: (u_{\mathbf{j}})_P + (u_{\mathbf{j}})_{P^C}\,. \end{aligned} \tag{A.35}$$

Here we have introduced a set P of multi-indices:

$$P := \left\{x \in \mathbb{N}^n \,\middle|\, \exists\sigma\in S_n : x = \sigma(1,2,\ldots,n)\right\} \subset \{1,2,\ldots,n\}^n\,. \tag{A.36}$$

We will focus on $(u_{\mathbf{j}})_P$ first and insert it into our ansatz (A.33):

$$\begin{aligned} \sum_{\mathbf{j}\in J} d_{\mathbf{j}}(v_{\mathbf{j}})_P &= \frac{1}{2^n n!}\sum_{\mathbf{j}\in J}\sum_{\mathbf{k}\in P}(-1)^{j_1+\ldots+j_n}(-1)^{j_{k_1}+\ldots+j_{k_n}}\; v_{k_1}\otimes\cdots\otimes v_{k_n} \\ &= \frac{1}{2^n n!}\sum_{\mathbf{j}\in J}\sum_{\sigma\in S_n}(-1)^{j_1+\ldots+j_n}(-1)^{j_{\sigma(1)}+\ldots+j_{\sigma(n)}}\; v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)} \\ &= \frac{1}{2^n n!}\sum_{\mathbf{j}\in J}\sum_{\sigma\in S_n}(-1)^{j_1+\ldots+j_n}(-1)^{j_1+\ldots+j_n}\; v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)} \\ &= \frac{|\{0,1\}^n|}{2^n n!}\sum_{\sigma\in S_n} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)} = \mathrm{Sym}(v_1\otimes\cdots\otimes v_n)\,. \end{aligned} \tag{A.37}$$

It remains to show that $\sum_{\mathbf{j}} d_{\mathbf{j}}(u_{\mathbf{j}})_{P^C} = 0$:

$$\begin{aligned} &2^n n! \sum_{\mathbf{j}\in J} d_{\mathbf{j}}\,(u_{\mathbf{j}})_{P^C} \\ &= \sum_{\mathbf{j}\in J}\;\sum_{\mathbf{k}\in\{1,\ldots,n\}^n\setminus P}(-1)^{j_1+\ldots+j_n}(-1)^{j_{k_1}+\ldots+j_{k_n}}\; v_{k_1}\otimes\cdots\otimes v_{k_n} \\ &= \sum_{\mathbf{k}\in\{1,\ldots,n\}^n\setminus P}\left(\sum_{\mathbf{j}\in J}(-1)^{j_1+\ldots+j_n}(-1)^{j_{k_1}+\ldots+j_{k_n}}\right) v_{k_1}\otimes\cdots\otimes v_{k_n}\,. \end{aligned} \tag{A.38}$$

We will show that the expression in brackets vanishes. For every $\mathbf{k} \in \{1, \ldots, n\}^n \setminus P$ there is at least one $s \in \{1, \ldots, n\}$ such that none of the k_i is equal to s. Therefore, we can factor out

$$\begin{aligned} &\sum_{\mathbf{j}\in J}(-1)^{j_1+\ldots+j_n}(-1)^{j_{k_1}+\ldots+j_{k_n}} \\ =\; &\sum_{j_s=0}^{1}(-1)^{j_s}\sum_{\mathbf{j}\in\{0,1\}^{n-1}}(-1)^{j_1+\ldots+\widehat{j_s}+\ldots+j_n}(-1)^{j_{k_1}+\ldots+j_{k_n}}\,, \end{aligned} \tag{A.39}$$

because the remaining term on the right does not depend on j_s any more. Now $\sum_{j_s}(-1)^{j_s} = 0$. □

Proof of Lemma 7.2 For $\varphi, \psi \in D(\Delta_1^*)$,

$$\begin{aligned} &\langle X_i(\varphi), Y_i(\psi)\rangle_{\mathbb{C}} - \langle Y_i(\varphi), X_i(\psi)\rangle_{\mathbb{C}} \\ =\; &\left\langle e^{i\theta_i}(\alpha_i B_i + \beta_i A_i)(\varphi), e^{i\theta_i}(\gamma_i B_i + \delta_i A_i)(\psi)\right\rangle_{\mathbb{C}} \\ &-\left\langle e^{i\theta_i}(\gamma_i B_i + \delta_i A_i)(\varphi), e^{i\theta_i}(\alpha_i B_i + \beta_i A_i)(\psi)\right\rangle_{\mathbb{C}} \\ \overset{(4.7)}{=}\; &(\alpha_i\delta_i - \beta_i\gamma_i)\langle B_i\varphi, A_i\psi\rangle_{\mathbb{C}} - (\alpha_i\delta_i - \beta_i\gamma_i)\langle A_i\varphi, B_i\psi\rangle_{\mathbb{C}} \\ =\; &\langle B_i\varphi, A_i\psi\rangle_{\mathbb{C}} - \langle A_i\varphi, B_i\psi\rangle_{\mathbb{C}}, \end{aligned} \tag{A.40}$$

because the terms involving twice B_i or twice A_i cancel, and only the mixed terms survive. Summing the terms from all sources $i = 1, \ldots, N$ yields the claim. □

Proof of Lemma 7.3 By assumption, $X_i(\psi) = 0$ and $X_i(\phi) = 1$ for $i = 1, \ldots, N$. Thus, from Lemma 7.2 with $\varphi = \phi$,

$$\begin{aligned} \sum_{i=1}^{N} Y_i(\psi) &= \sum_{i=1}^{N}\langle X_i(\phi), Y_i(\psi)\rangle_{\mathbb{C}} = \sum_{i=1}^{N}\langle X_i(\phi), Y_i(\psi)\rangle_{\mathbb{C}} - \langle Y_i(\phi), X_i(\psi)\rangle_{\mathbb{C}} \\ &= \langle \phi, -\Delta_1^*\psi\rangle_{\mathfrak{H}} - \langle -\Delta_1^*\phi, \psi\rangle_{\mathfrak{H}} \\ &= \langle \phi, (-\Delta_1^* + E_0)\psi\rangle_{\mathfrak{H}} - \langle (-\Delta_1^* + E_0)\phi, \psi\rangle_{\mathfrak{H}}\,. \end{aligned} \tag{A.41}$$

This proves statement (a). To see why (b) is also true, observe that, since by assumption $\Delta_1^*\phi = \lambda\phi$,

$$\begin{aligned} 2\mathrm{i}\,\mathrm{Im}\left(\sum_{i=1}^{N} Y_i(\phi)\right) &= \sum_{i=1}^{N} Y_i(\phi) - \overline{Y_i(\phi)} = \sum_{i=1}^{N}\langle X_i(\phi), Y_i(\phi)\rangle_{\mathbb{C}} - \langle Y_i(\phi), X_i(\phi)\rangle_{\mathbb{C}} \\ &= \langle \phi, -\Delta_1^*\phi\rangle_{\mathfrak{H}} - \langle -\Delta_1^*\phi, \phi\rangle_{\mathfrak{H}} \\ &= \langle \phi, (-\lambda + E_0)\phi\rangle_{\mathfrak{H}} - \langle (-\lambda + E_0)\phi, \phi\rangle_{\mathfrak{H}} = 0\,, \end{aligned} \tag{A.42}$$

which completes the proof. □

References

1. Albeverio, S., Gesztesy, F., Hoegh-Krohn, H., Holden, H.: Solvable Models in Quantum Mechanics. American Mathematical Society, Providence (1988)
2. Behrndt, J., Micheler, T.: Elliptic differential operators on Lipschitz domains and abstract boundary value problems. J. Funct. Anal. **267**, 3657–3709 (2014)
3. Correggi, M., Dell'Antonio, G., Finco, D., Michelangeli, A., Teta, A.: Stability for a system of N fermions plus a different particle with zero-range interactions. Rev. Math. Phys. **24**, 1250017 (2012)
4. Correggi, M., Dell'Antonio, G., Finco, D., Michelangeli, A., Teta, A.: A class of hamiltonians for a three-particle fermionic system at unitarity. J. Math. Phys. Anal. Geom. **18**, 1–36 (2015)
5. Dell'Antonio, G., Figari, R., Teta, A.: A Brief Review on Point Interactions. Pages 171–189 in Inverse Problems and Imaging, Lecture Notes in Mathematics 1943. Springer, Berlin (2008)
6. Deckert, D.A.: Scalar Field Interaction Models. Diplom Thesis, Mathematisches Institut, Ludwig-Maximilians Universität, München (2004)
7. Dereziński, J.: Van hove Hamiltonians - exactly solvable models of the infrared and ultraviolet problem. Annales Henri Poincaré **4**, 713–738 (2003)
8. Dürr, D., Goldstein, S., Teufel, S., Tumulka, R., Zanghì, N.: Bohmian Trajectories for Hamiltonians with Interior-Boundary Conditions. In preparation
9. Galvan, B.: Quantum Field Theory Without Divergence: The Method of the Interaction Operators. Preprint. arXiv:1607.03876 (2016)
10. Glimm, J., Jaffe, A.: Quantum Field Theory and Statistical Mechanics. Birkhäuser (1985)
11. Glimm, J., Jaffe, A.: Quantum Physics—A Functional Integral Point of View. Springer, Berlin (1987)
12. Griesemer, M., Wünsch, A.: Self-adjointness and domain of the Fröhlich Hamiltonian. J. Math. Phys. **57**, 021902 (2016)
13. Griesemer, M., Wünsch, A.: On the domain of the Nelson Hamiltonian. J. Math. Phys. **59**, 042111 (2018)
14. Hörmander, L.: Linear Partial Differential Operators. Grundlehren der Mathematischen Wissenschaften 116. Springer, Berlin (1964)
15. Keppeler, S., Sieber, M.: Particle creation and annihilation at interior boundaries: one-dimensional models. J. Phys. A Math. Theor. **49**, 125204 (2016)
16. Lampart, J., Schmidt, J.: On the Domain of Nelson-Type Hamiltonians and Abstract Boundary Conditions. Preprint (2018). arXiv:1803.00872
17. Lampart, J.: A Nonrelativistic Quantum Field Theory with Point Interactions in Three Dimensions. Preprint (2018). arXiv:1804.08295
18. Landau, L., Peierls, R.: Quantenelektrodynamik im Konfigurationsraum. Zeitschrift für Physik **62**, 188–200 (1930)
19. Lee, T.D.: Some special examples in renormalizable field theory. Phys. Rev. **95**, 1329–1334 (1954)
20. Lions, J.L., Magenes, E.: Non-homogeneous Boundary Value Problems and Applications. Grundlehren der Mathematischen Wissenschaften 181. Springer, Berlin (1972)
21. Mazzeo, R.: Elliptic theory of differential edge operators I. Commun. Partial Differ. Equ. **16**, 1615–1664 (1991)
22. Meyer, P.-A.: Quantum Probability for Probabilists. Lecture Notes in Mathematics 1538. Springer, Berlin (1993)
23. Michelangeli, A., Ottolini, A.: On point interactions realised as Ter-Martirosyan-Skornyakov Hamiltonians. Rep. Math. Phys. **79**(2), 215–260 (2017)
24. Minlos, R.: On point-like interactions between n Fermions and another particle. Moscow Math. J. **11**(1), 113–127 (2011)
25. Moser, T., Seiringer, R.: Stability of a Fermionic $N + 1$-particle system with point interactions. Commun. Math. Phys. **356**, 329–355 (2017)
26. Moshinsky, M.: Boundary conditions for the description of nuclear reactions. Phys. Rev. **81**, 347–352 (1951)
27. Moshinsky, M.: Boundary conditions and time-dependent states. Phys. Rev. **84**, 525–532 (1951)
28. Moshinsky, M.: Quantum mechanics in Fock space. Phys. Rev. **84**, 533 (1951)
29. Moshinsky, M., López Laurrabaquio, G.: Relativistic interactions by means of boundary conditions: The Breit-Wigner formula. J. Math. Phys. **32**, 3519–3528 (1991)
30. Nelson, E.: Interaction of nonrelativistic particles with a quantized scalar field. J. Math. Phys. **5**, 1190–1197 (1964)

31. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness. Academic Press, Cambridge (1975)
32. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, I: Functional Analysis (revised and enlarged edition). Academic Press, Cambridge (1980)
33. Schmidt, J., Tumulka, R.: Interior-Boundary Conditions and Time Reversal Symmetry. In preparation
34. Schweber, S.S.: An Introduction to Relativistic Quantum Field Theory. Row, Peterson and Company, New York (1961)
35. Svendsen, E.: The effect of submanifolds upon essential self-adjointness and deficiency indices. Aust. J. Math. Anal. Appl. **80**, 551–565 (1981)
36. Teufel, S., Tumulka, R.: New Type of Hamiltonians Without Ultraviolet Divergence for Quantum Field Theories. Preprint (2015). arXiv:1505.04847
37. Teufel, S., Tumulka, R.: Avoiding ultraviolet divergence by means of interior-boundary conditions. In: Finster, F., Kleiner, J., Röken, C., Tolksdorf, J. (eds.) Quantum Mathematical Physics – A Bridge between Mathematics and Physics, pp. 293–311. Birkhäuser, Boston (2016). arXiv:1506.00497
38. Thomas, L.E.: Multiparticle Schrödinger Hamiltonians with point interactions. Phys. Rev. D **30**, 1233–1237 (1984)
39. Tumulka, R., Georgii, H.-O.: Some jump processes in quantum field theory. In: Deuschel, J.-D. and Greven, A. (edn), Interacting Stochastic Systems, pp. 55–73. Springer (2004)
40. van Hove, L.: Les difficultés de divergences pour un modèle particulier de champ quantifié. Physica **18**, 145–159 (1952)
41. Yafaev, D.R.: On a zero-range interaction of a quantum particle with the vacuum. J. Phys. A Math. Gen. **25**, 963–978 (1992)

2) Article [LS18]

On Nelson-type Hamiltonians and abstract boundary conditions

Jonas Lampart *, Julian Schmidt †

September 19, 2018

We construct Hamiltonians for systems of nonrelativistic particles linearly coupled to massive scalar bosons using abstract boundary conditions. The construction yields an explicit characterisation of the domain of self-adjointness in terms of boundary conditions that relate sectors with different numbers of bosons. We treat both models in which the Hamiltonian may be defined as a form perturbation of the free operator, such as Fröhlich's polaron, and renormalisable models, such as the massive Nelson model.

1 Introduction

We consider a system of nonrelativistic particles interacting with massive scalar bosons. For a linear coupling, the interaction between one particle and the bosons is (formally) given by $a(v(x-y))+a^*(v(x-y))$, where a, a^* are the bosonic annihilation and creation operators, v is the form factor of the interaction and x denotes the position of the particle, y that of a boson. Figuratively speaking, the particles act as sources that create and annihilate bosons with wavefunction v centred at their position x. We will discuss a class of ultraviolet-divergent models for which $v(y)$ is a singular function (or a distribution). In most examples $v(y)$ is singular at $y = 0$ but regular and decaying as $|y| \to \infty$. For example, for the Fröhlich polaron $v(y) \sim |y|^{-2}$, and in the Nelson model $v(y) \sim |y|^{-5/2}$ (both in three space-dimensions). The Hamiltonians for these models can be constructed using quadratic

*CNRS & ICB (UMR 6303), Université de Bourgogne Franche-Comté, 9 Av. A. Savary, 21078 Dijon Cedex, France. jonas.lampart@u-bourgogne.fr

†Fachbereich Mathematik, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. juls@maphy.uni-tuebingen.de

forms (for the Fröhlich model) or by a renormalisation procedure (for the Nelson model). However, these methods do not give detailed and explicit information on the domain of the operator (e.g. concerning regularity) or the action of the operator thereon. We will discuss a new method of construction that explicitly describes the domain in terms of abstract boundary conditions relating sectors with different numbers of bosons. More precisely, the elements of the domain will, for any given number $n \geq 1$ of bosons, be singular functions with singularities determined by the function with $n-1$ bosons. If the only singularity of v is at $y = 0$, these singularities are located on the planes in configuration space where the positions of (at least) a source and a boson coincide. The relation between the form of this singularity and the function with fewer bosons can be viewed as an inhomogeneous generalised boundary condition on the set of these planes.

Boundary conditions of this type were proposed as an approach to ultraviolet divergences by Teufel and Tumulka [TT15, TT16]. They were called interior-boundary conditions, as they concern points in the interior of the configuration space of the two species of particles. Similar boundary conditions had previously been investigated by Thomas [Tho84] in a specific model where the total number of particles is at most three. The emphasis of these works is on point interactions, where v is the δ-distribution and it is particularly natural to consider boundary conditions. A rigorous analysis of a model for nonrelativistic bosons, with $v = \delta$ and sources that are fixed at points in $\mathbb{R}^3$, was subsequently performed by Teufel, Tumulka, and the authors [LSTT18]. This extended a result of Yafaev [Yaf92], allowing only for the creation of a single particle. The one-dimensional variant of this model was studied by Keppeler and Sieber [KS16].

In the present article, we will explain how such an approach can be applied to models for nonrelativistic particles interacting with bosons, where the 'sources' are themselves dynamical objects. We also demonstrate that the method is sufficiently flexible to accommodate various interactions v and dispersion relations of the bosons, such as the relativistic dispersion of the Nelson model. Our class of models also contains a dynamical version of the model with nonrelativistic bosons and $v = \delta$ of [LSTT18] in two (instead of three) space-dimensions. Our method could also be applied to models that involve creation and annihilation of fermions, but we will restrict ourselves to bosons in this article. We obtain an explicit characterisation of the Hamiltonian and its domain of self-adjointness, which seems to be new for all of the cases under consideration. We also hope that this explicit characterisation will facilitate further research on the properties of these models, such as their energy-momentum spectrum and dynamics, which is an active area of investigation (see e.g. [AF14, BT17, GHL14, Miy18, MM17] for some recent results, and references therein).

1.1 Nonrelativistic particles interacting with scalar bosons

Let us now introduce some notation and discuss in more detail the models we will consider as well as our main results. We consider a fixed but arbitrary number M of nonrelativistic particles in $d \leq 3$ dimensions interacting with a variable number of scalar bosons. We do not impose any particular symmetry under permutations on the first type of particles. The Hilbert space on which we describe our system is given by

$$\mathscr{H} := L^2(\mathbb{R}^{dM}) \otimes \Gamma(L^2(\mathbb{R}^d)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^{dM}) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{dn}) = \bigoplus_{n\in\mathbb{N}} \mathscr{H}^{(n)} ,$$

where $\Gamma(L^2(\mathbb{R}^d))$ is the bosonic Fock space over $L^2(\mathbb{R}^d)$ and $\mathscr{H}^{(n)}$ the sector of $\mathscr{H}$ with n bosons. In the position representation, we will denote the positions of the first type of particles by $x_1, \dots, x_M$ and refer to these as the x-particles from now on. We will denote the positions of the bosons by $y_1, \dots$ and refer to them as the y-particles. In appropriate units, the formal expression for the linearly coupled Hamiltonian of this system reads

$$-\sum_{j=1}^{M} \Delta_{x_j} + \mathrm{d}\Gamma(\omega(-\mathrm{i}\nabla_y)) + g\sum_{j=1}^{M} \left(a^*(v(x_j - y)) + a(v(x_j - y))\right) , \tag{1}$$

where $\omega : \mathbb{R}^d \to \mathbb{R}_+$ is the dispersion relation of the bosons, $v \in \mathscr{S}'(\mathbb{R}^d)$ is the interaction, and $g \in \mathbb{R}$ is the coupling constant. When $v \in L^2$ and $\omega(k) \geq e_0 > 0$, then, by the Kato-Rellich theorem, this defines a self-adjoint operator on the domain

$$D(L) = \{\psi \in \mathscr{H} : L\psi \in \mathscr{H}\}$$

of the free operator (understood in the sense of tempered distributions)

$$L := -\sum_{j=1}^{M} \Delta_{x_j} + \mathrm{d}\Gamma(\omega(-\mathrm{i}\nabla_y)) . \tag{2}$$

Note that $D(L)$ is contained in the domain of the boson-number operator $N = \mathrm{d}\Gamma(1)$ if $\omega(k) \geq e_0 > 0$.

Our class of models concerns cases where the operator in Equation (1) above is not immediately well defined because $v \notin L^2(\mathbb{R}^d)$. We will only consider cases with an ultraviolet problem but no infrared problem, that is $\omega(k) \geq e_0 > 0$ and $\hat{v} \in L^2_{\mathrm{loc}}$. The problem in this case is that the creation operator $a^*(v(x-y))$ is not a densely defined operator on $\mathscr{H}$, so the expression (1) cannot be interpreted as a sum of unbounded operators on any dense domain. The annihilation operator $\sum_{j=1}^{M} a(v(x_j - y))$ is less problematic, as it is always densely defined, and under our assumptions it is defined on $D(L)$ (cf. Corollary 3.2 and the following remark). Depending on v and ω, this problem may be solvable by one of two well-known methods.

(1) If $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k < \infty$, the annihilation operator is continuous from $D(L^{1/2})$ to $D(N^{-1/2})$ and one can interpret the expression (1) as the quadratic form

$$\langle \psi, L\psi\rangle + \sum_{j=1}^{M} \langle \psi, a(v(x_j - y))\psi\rangle + \langle a(v(x_j - y))\psi, \psi\rangle, \tag{3}$$

on $D(L^{1/2}) \subset D(N^{1/2})$, since a^* is the formal adjoint of a. When this form is bounded below, one defines the Hamiltonian H to be the unique self-adjoint and semibounded operator associated with this form. This solves the problem of defining H, but yields only limited information, namely that $D(H) \subset D(L^{1/2})$ and that H is semibounded.

(2) When $a(v(x-y))$ is not defined on $D(L^{1/2})$ one can still hope to construct H using a renormalisation procedure due to Nelson [Nel64]. In this procedure, one first regularises v, for example by replacing it by v_Λ whose Fourier transform is $\hat{v}_\Lambda(k) = \hat{v}(k)\chi_\Lambda(k)$, where χ_Λ is the characteristic function of a ball of radius Λ. Then $v_\Lambda \in L^2$, so the operator H_Λ with this interaction is self-adjoint on $D(L)$ for every $\Lambda \in \mathbb{R}_+$ and v_Λ converges to v in $\mathscr{S}'(\mathbb{R}^d)$ as $\Lambda \to \infty$. Under appropriate conditions on v and ω, one can then find (explicit) numbers E_Λ, so that

$$H_\infty = \lim_{\Lambda\to\infty} H_\Lambda + E_\Lambda$$

exists in the norm resolvent sense and defines a self-adjoint and semibounded operator. This defines a Hamiltonian for the model up to a constant, since the numbers E_Λ can always be modified by adding a finite constant in this procedure. However, one retains virtually no information on the domain of H_∞, which led Nelson to pose in [Nel64] the following problem:

> It would be interesting to have a direct description of the operator H_∞. Is $D(H_\infty) \cap D(L^{1/2}) = \{0\}$?

The second question was answered, affirmatively, in a recent article by Griesemer and Wünsch [GW18]. We will provide a direct description of H_∞ and its domain in terms of abstract boundary conditions. From this description the answer to the second question will also be apparent.

The models we consider will fall into one of these two classes. They are form perturbations of L, as under point (1) above, if $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k < \infty$ and renormalisable in the sense of point (2) otherwise. The precise assumptions will be given in Condition 1.1 below. The class of v and ω we cover contains the following examples:

- The Fröhlich model ($d = 3$, $\omega = 1$, $\hat{v}(k) = |k|^{-1}$) describes the interaction of nonrelativistic electrons with phonons in a crystal. As noted above, this model falls into the class of form perturbations. A recent exposition of the construction and an investigation of its domain can be found in the article of Griesemer and Wünsch [GW16].

- The massive Nelson model ($d = 3$, $\omega(k) = \sqrt{k^2+1}$, $\hat{v}(k) = \omega(k)^{-1/2}$) describes the interaction of nonrelativistic particles with relativistic, massive, scalar bosons, whose mass we have chosen to be one. It was defined rigorously by Nelson [Nel64] and provides the blueprint for the renormalisation procedure described under point (2) above.

- Nonrelativistic point-particles in two dimensions ($d = 2$, $\omega(k) = k^2 + 1$, $v = \delta$). In this model, the nonrelativistic (x-) particles interact with nonrelativistic bosons (y-particles) by creation/annihilation at contact. This is a two-dimensional version of the model of [LSTT18] with dynamical sources. The renormalisation procedure can be applied to this model by following Nelson's proof line-by-line (see also [GW18]).

1.2 A Hamiltonian with abstract boundary conditions

Our approach to constructing a Hamiltonian for these models starts not from the quadratic form or a regularisation of the expression (1), but by considering extensions of L to singular functions, adapted to the singularity of v. This is analogous to the construction of Schrödinger operators with singular (pseudo-) potentials using the theory of self-adjoint extensions (see e.g. [AGHKH88, BFK$^+$17, MO17, Pos08]). In those problems, one considers a self-adjoint operator $(S, D(S))$ (e.g. $S = -\Delta$ on $H^2(\mathbb{R}^d)$) and restricts it to the kernel of a singular 'potential'. This could be the Sobolev trace on some lower dimensional set, the 'boundary', or some other linear functional on $D(S)$. The restriction of S then defines a closed, symmetric operator S_0, and one searches for self-adjoint extensions of S_0, or, equivalently, restrictions of S_0^*. These extensions incorporate interactions through (generalised) boundary conditions. We remark that, in many examples, such models can also be constructed using renormalisation techniques (see e.g. [DFT94, DR04, KS95]), giving the same operators. This is also true for our models, as we will show in Theorem 1.4 below.

Let L_0 be the restriction of L to the domain

$$D(L_0) = D(L) \cap \ker\left(\sum_{j=1}^{M} a(v(x_j - y))\right). \tag{4}$$

Then L_0^* is an extension of L whose domain contains, in particular, elements of the form

$$\psi = G\varphi := -g\left(\sum_{j=1}^{M} a(v(x_j - y))L^{-1}\right)^{*}\varphi = -gL^{-1}\sum_{j=1}^{M} a^{*}(v(x_j - y))\varphi, \qquad (5)$$

for $\varphi \in \mathscr{H}$. In this expression, $a^*(v(x_j - y))$ is to be understood as the adjoint of $a(v(x_j - y)) : D(L) \to \mathscr{H}$ that maps $\mathscr{H}$ to $D(L)' = D(L^{-1})$, the dual of $D(L)$. Note also that L is invertible on the sectors with at least one boson since we assume $\omega \geq e_0 > 0$.

We will define an extension A of $\sum_{j=1}^{M} a(v(x_j - y))$ to functions in the range of G. One can then consider the operator $L_0^* + gA$ on the domain

$$\{\psi \in \mathscr{H} | \exists \varphi \in \mathscr{H} : \psi - G\varphi \in D(L)\}\,.$$

Since $G\varphi \notin D(L)$ for $\varphi \neq 0$, the function φ in this decomposition is unique. The condition means that the singular part of $\psi^{(n)}$, i.e. the part not in $D(L)$, is determined by the 'boundary value' $\varphi^{(n-1)}$. Note that, since $\mathscr{H}$ is the sum over all sectors $\mathscr{H}^{(n)}$, the space on which the operator acts and the space of boundary values are both equal to $\mathscr{H}$. The operator $L_0^* + gA$ is not symmetric on this domain, but it has symmetric restrictions obtained by imposing boundary conditions, in the sense of linear relations between ψ and φ.

To find the boundary condition corresponding to the formal Hamiltonian (1), first observe that the range of G is contained in the kernel of L_0^*, because for all $\psi \in D(L_0)$

$$\langle L_0^* G\varphi, \psi\rangle = \langle \varphi, G^* L_0 \psi\rangle = -g\sum_{j=1}^{M}\langle \varphi, a(v(x_j - y))\psi\rangle = 0\,. \qquad (6)$$

For any ψ with $\psi - G\varphi \in D(L)$ we then have

$$L_0^*\psi = L_0^*(\psi - G\varphi) = L(\psi - G\varphi) = L\psi + g\sum_{j=1}^{M} a^*(v(x_j - y))\varphi. \qquad (7)$$

The final expression is a sum of vectors in $D(L)'$ that lies in $\mathscr{H}$, because it equals the left hand side. Imposing the relation $\varphi = \psi$, i.e. that $\psi - G\psi \in D(L)$, then gives the equality

$$L_0^*\psi + gA\psi = L\psi + g\sum_{j=1}^{M} a^*(v(x_j - y))\psi + gA\psi$$

in $D(L)'$. This is essentially the formal Hamiltonian (1), but on a domain different from $D(L)$ chosen in such a way that the singularities of the first two terms cancel each other, and with the annihilation operator suitably extended to this domain. Our main result is that the Hamiltonian $H = L_0^* + gA$ is self-adjoint and bounded from below on the domain with this boundary condition. For the appropriate choice of extension A, it equals the Hamiltonian defined as a quadratic form, or by renormalisation, respectively.

Our hypothesis on $\hat{v}$ and ω is that they have upper, respectively lower, bounds by appropriate powers of $|k|$ or $1+k^2$, which is the case in all of the relevant examples. For simplicity we also set the rest-mass e_0 of the y-particles to one.

Condition 1.1. Let $v \in \mathscr{S}'(\mathbb{R}^d)$, $v \notin L^2(\mathbb{R}^d)$ and $\omega : \mathbb{R}^d \to \mathbb{R}_+$. We have bounds $|\hat{v}(k)| \leq |k|^{-\alpha}$ and $\omega(k) \geq (1+k^2)^{\beta/2}$ with parameters $0 \leq \alpha < \frac{d}{2}$, $0 \leq \beta \leq 2$ satisfying additionally one of the following two conditions:

(1) $\alpha > \frac{d}{2} - 1$ and thus $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)} \mathrm{d}k < \infty$;

(2) $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)} \mathrm{d}k = \infty$ and

$$\alpha = 0 \text{ and } \beta > 0 \text{ if } d = 2$$
$$\alpha > \tfrac{1}{2} - \tfrac{\beta^2}{8+\beta^2} \text{ if } d = 3\,.$$

Note that the condition $\alpha < \frac{d}{2}$ implies $\hat{v} \in L^2_{\mathrm{loc}}$. Later on, we will often state our results in terms of the parameter

$$D := d - 2\alpha - 2,$$

which measures the (non)-integrability of $|\hat{v}(k)|^2(1+k^2)^{-1}$ and thus the singularity of the interaction. The first case of the condition corresponds to $D < 0$ and the second to $D \geq 0$.

Definition 1.2. Assume Condition 1.1 holds and $d \in \{1,2,3\}$. We define A with domain $D(A)$ as the extension of

$$\sum_{j=1}^{M} a(v(x_j - y)) : D(L) \to \mathscr{H}$$

given in

- Equation (13) if $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)} \mathrm{d}k < \infty$, or
- Equations (27) and (32) if $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)} \mathrm{d}k = \infty$.

The integrability condition determines which of the cases in Condition 1.1 applies. Our main result is:

Theorem 1.3. *Let $d \in \{1,2,3\}$ and assume that v and ω satisfy Condition 1.1. Then the operator $H = L_0^* + gA$ with domain*

$$D(H) = \{\psi \in \mathscr{H} | \psi - G\psi \in D(L)\}$$

is self-adjoint and bounded from below. Its domain is contained in the domain of the number operator N and for $\psi \in D(H)$ we have the equality

$$H\psi = L\psi + g\sum_{j=1}^{M} a^*(v(x_j - y))\psi + gA\psi. \tag{8}$$

in the dual of $D(L)$.

For the Fröhlich model we are in the first case of Condition 1.1 and have $\alpha = 1$. For the Nelson model we can choose $\beta = 1$, $\alpha = \frac{1}{2}$. For $\beta = 1$ the condition on α is $\alpha > \frac{7}{18}$, which also allows for slightly more singular cases. For our model of nonrelativistic point-particles in two dimensions the conditions are satisfied with $\beta = 2$ and $\alpha = 0$. The corresponding model in one dimension, which is an extension of the one treated in [KS16] with moving sources, is a form perturbation. In fact, in one dimension we always have $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)} \mathrm{d}k < \infty$ since we assume a bound with $0 \le \alpha < \frac{1}{2}$. For nonrelativistic bosons in three dimensions with $\beta = 2$ our condition is $\alpha > \frac{1}{6}$. This excludes $v = \delta$, corresponding to a model which is not known to be renormalisable (in sense of operators explained above). However, our methods can be adapted to construct a Hamiltonian also in this case. This will be the subject of an upcoming publication by the first author [Lam18].

Our result provides a self-adjoint operator H whose action is given by (1), if the separate terms are interpreted as elements of $D(L)'$ and $\sum_{j=1}^{M} a(v(x_j - y))$ is suitably extended. In the case of form perturbations, the annihilation operator is automatically well defined on $D(H) \subset D(L^{1/2})$. Our theorem then also implies that the quadratic form of H is indeed given by the usual expression (3), since in this case Equation (7) also holds in the sense of quadratic forms on $D(L^{1/2})$.

For the more singular models the extension of the annihilation operator involves an operation that can be interpreted as the addition of an 'infinite constant', and it is certainly not unique. These models can also be treated by a renormalisation technique, see [GW18]. We make a choice of the extension A for which H coincides with the operator H_∞ obtained by renormalisation (see also Remark 3.4). The following theorem, proved in Section 3.4, implies that $H = H_\infty$.

Theorem 1.4. *Let the conditions of Theorem 1.3 be satisfied and $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)} \mathrm{d}k = \infty$. For $\Lambda \in \mathbb{R}_+$ let H_Λ be the Hamiltonian with the regularised interaction defined by $\hat{v}_\Lambda(k) = \hat{v}(k)\chi_\Lambda(k)$, where χ_Λ is the characteristic function of a ball of radius Λ,* *and let*

$$E_\Lambda = g^2 M \int \frac{|\hat{v}_\Lambda(k)|^2}{k^2+\omega(k)} \mathrm{d}k\,.$$

Then $H_\Lambda + E_\Lambda$ converges to H in the strong resolvent sense.

The domain of H is explicit and for any given $\psi \in \mathscr{H}$ it is easy to check whether it belongs to $D(H)$ or not. In particular, the regularity properties of $\psi \in D(H)$ are easily deduced from the regularity of $G\psi$. This allows us to answer Nelson's second question.

Corollary 1.5. *Let the conditions of Theorem 1.3 be satisfied and additionally $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. Then $D(H) \subset D(L^{1/2})$ if and only if $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k < \infty$. Moreover, if $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k = \infty$, then $D(H) \cap D(L^{1/2}) = \{0\}$.*

This corollary follows from our more precise discussion of the regularity properties of $D(H)$ in Section 4. Essentially the same result for $M = 1$ was recently obtained [GW16, GW18] by different methods.

The structure of the proof of our main result, Theorem 1.3, is essentially the same for the cases of form perturbations ($\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k < \infty$) and renormalisable models ($\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k = \infty$). However, the technical difficulties are slightly different, and much greater in the second case. For this reason, we will give the proof of the first case separately, in Section 2. This may also serve as a less technical presentation of the general strategy. The proof for the second case will be given in Section 3. In both cases, the crucial technical ingredients of the proof are bounds on the operator $T = gAG$ that are sufficiently good regarding both regularity and particle number. This operator also appears in the theory of point interactions (with $v = \delta$), where it is known as the Ter-Martyrosyan–Skornyakov operator, see e.g. [CDF+15, DFT94, MS17, MS18]. We will build on some of the results obtained in this area, as we explain in Remark 3.9.

2 Form perturbations

In this section we will prove Theorem 1.3 under the assumptions of the first case in Condition 1.1. That is, we assume that $\omega(k) \geq 1$, $v \in \mathscr{S}'(\mathbb{R}^d)$, $v \notin L^2(\mathbb{R}^d)$, and that $|\hat{v}(k)| \leq |k|^{-\alpha}$ for some $\frac{d}{2} > \alpha > \frac{d}{2} - 1$, respectively $d - 2\alpha - 2 = D < 0$. We will use the notation

$$a(V) := \sum_{i=1}^{M} a(v(x_i - y)). \tag{9}$$

Under the assumptions of this section $a(V)$ is operator-bounded by L, as will be proved in Lemma 2.1 below.

In the following we will often work in the Fourier representation. We denote by $P = (p_1, \ldots, p_M)$, $K = (k_1, \ldots, k_n)$ the conjugate Fourier variables to $X = (x_1, \ldots x_M)$, $Y = (y_1, \ldots, y_n)$. The vector $\hat{Q}_j \in \mathbb{R}^{d\nu}$ is $Q \in \mathbb{R}^{d(\nu+1)}$ with the j-th entry deleted and e_i is the inclusion of the i-th summand in $\mathbb{R}^{d\mu} = \bigoplus_{i=1}^{\mu} \mathbb{R}^d$. We will denote the Fourier representation of the operator L (on the n-boson sector) as multiplication by the function

$$L(P, K) := P^2 + \sum_{j=1}^{n} \omega(k_j) =: P^2 + \Omega(K).$$

Lemma 2.1. *If Condition 1.1 holds with $D < 0$ then*

$$a(V) L^{-\frac{1}{2}} N^{-\frac{2+D}{4}}$$

is a bounded operator on $\mathscr{H}$.

Proof. Since we are not concerned with the dependence of the norm on M it is sufficient to estimate one term in the sum (9) and then bound the norm of the sum by the sum of the norms.

In Fourier representation, we have

$$\begin{aligned}
&\Big(\mathcal{F} a(v(x_1 - y)) L^{-\frac{1}{2}} N^{-\frac{2+D}{4}} \psi\Big)^{(n)} (P, \hat{K}_{n+1}) \\
&= \sqrt{n+1} \int_{\mathbb{R}^d} \frac{\overline{\hat{v}(k_{n+1})} \hat{\psi}^{(n+1)}(P - e_1 k_{n+1}, K)}{L(P - e_1 k_{n+1}, K)^{\frac{1}{2}} (n+1)^{\frac{2+D}{4}}} \, \mathrm{d}k_{n+1} \, .
\end{aligned}$$

To prove our claim it is sufficient to show that for some constant $C > 0$ it holds that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \frac{\overline{\hat{v}(k_{n+1})} \hat{\psi}^{(n+1)}(P - e_1 k_{n+1}, K)}{L(P - e_1 k_{n+1}, K)^{\frac{1}{2}}} \, \mathrm{d}k_{n+1} \right|^2 \\
&\leq C (n+1)^{\frac{D}{2}} \int_{\mathbb{R}^d} \left| \hat{\psi}^{(n+1)}(P - e_1 k_{n+1}, K) \right|^2 \mathrm{d}k_{n+1} \, ,
\end{aligned} \tag{10}$$

because we may afterwards integrate in P and $\hat{K}_{n+1}$ and perform a change of variables $P \to P - e_1 k_{n+1}$.

Using the Cauchy-Schwarz inequality, and our assumptions on $\hat{v}$ and ω, we can bound the integral from above by

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \frac{\overline{\hat{v}(k_{n+1})} \hat{\psi}^{(n+1)}(P - e_1 k_{n+1}, K)}{L(P - e_1 k_{n+1}, K)^{\frac{1}{2}}} \, \mathrm{d}k_{n+1} \right|^2 \\
&\leq \int_{\mathbb{R}^d} \frac{|q|^{-2\alpha}}{(p_1 - q)^2 + n + 1} \, \mathrm{d}q \int_{\mathbb{R}^d} \left| \hat{\psi}^{(n+1)}(P - e_1 k_{n+1}, K) \right|^2 \mathrm{d}k_{n+1} \, .
\end{aligned}$$

The integral in q takes its maximal value at $p_1 = 0$, by the Hardy-Littlewood inequality. Rescaling by $(n+1)^{-1/2}$ then yields the upper bound

$$(n+1)^{-1+\frac{d-2\alpha}{2}} \int_{\mathbb{R}^d} \frac{|q'|^{-2\alpha}}{q'^2+1} \mathrm{d}q' \int_{\mathbb{R}^d} \left|\hat{\psi}^{(n+1)}(P - e_1 k_{n+1}, K)\right|^2 \mathrm{d}k_{n+1} ,$$

and this proves the claim. □

2.1 The extended domain

Lemma 2.1 has several important consequences. First of all, $a(V) : D(L) \to \mathscr{H}$ is continuous in the graph norm of L. Thus $D(L_0)$, defined in (4) as the kernel of $a(V)$ in $D(L)$, is a closed subspace of $D(L)$ with this norm. Due to our assumption that $v \notin L^2$, this subspace is also dense in $\mathscr{H}$.

Lemma 2.2. *If Condition 1.1 is satisfied the space $D(L_0)$ is dense in $\mathscr{H}$.*

Proof. The Hilbert space $\mathscr{H}$ is equal to the direct integral $\mathscr{H} = \int_{\mathbb{R}^{Md}}^{\oplus} \Gamma(L^2(\mathbb{R}^d))\mathrm{d}X$. We start by proving that for almost every $X \in \mathbb{R}^{Md}$ the kernel of $a(V(X)) = \sum_{i=1}^{M} a(v(x_i - y))$ is dense in $\Gamma(L^2(\mathbb{R}^d))$.

The first step is to show that the kernel of the linear functional defined by $\hat{V}(X,k) = \sum_{i=1}^{M} \mathrm{e}^{\mathrm{i}kx_i}\hat{v}(k) \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ if $\hat{V}(X) \notin L^2(\mathbb{R}^d)$. The set of X where $\hat{V}(X) \in L^2$ has measure zero, see Lemma A.1 in the appendix. Let $\hat{v}_\Lambda(k) = \hat{v}(k)\chi_\Lambda(k)$ with the characteristic function χ_Λ of a ball of radius $\Lambda > 0$ and let $\hat{V}_\Lambda$ be defined like $\hat{V}$, with $\hat{v}_\Lambda$ replacing $\hat{v}$. Let $f \in H^2(\mathbb{R}^d)$ and set

$$\hat{f}_\Lambda(X,k) := \hat{f}(k) - \frac{\hat{V}(X,k)\chi_\Lambda(k)}{\int |\hat{V}_\Lambda(X,k')|^2 \mathrm{d}k'} \int \overline{\hat{V}(X,k')}\hat{f}(k')\mathrm{d}k'.$$

Now $\lim_{\Lambda\to\infty} \int |\hat{V}_\Lambda(X,k')|^2\mathrm{d}k' = \infty$, because $\hat{V}(X,k) \notin L^2(\mathbb{R}^d)$, so $\hat{f}_\Lambda(k)$ converges to $\hat{f}(k)$ in $L^2(\mathbb{R}^d)$. On the other hand $\int \overline{\hat{V}(X,k)}\hat{f}_\Lambda(k)\mathrm{d}k = 0$ so, after taking the inverse Fourier transform in k, f_Λ is in the kernel of $V(X)$.

This implies that coherent states generated by functions in $H^2(\mathbb{R}^d) \cap \ker(V(X))$ are dense in $\Gamma(L^2(\mathbb{R}^d))$, see e.g. [LSTT18, Prop.12]. Such states are in the kernel of $a(V(X))$ since for the coherent state $\Phi(f)$ generated by f we have $a(V(X))\Phi(f) = \langle V(X), f\rangle\Phi(f)$. Consequently, the kernel of $a(V(X))$ is dense in $\Gamma(L^2(\mathbb{R}^d))$ for almost every X.

To conclude the proof, notice that the approximants $f_\Lambda(X)$ above are in $H^2(\mathbb{R}^d)$ and depend smoothly on X. We can thus approximate any $\Gamma(L^2(\mathbb{R}^d))$-valued L^2-function of X by smooth functions taking values in the kernel of $a(V(X))$. Such functions are elements of $D(L)$ and this proves the claim. □

We have established that L_0, the restriction of L to the kernel of $a(V)$, is a densely defined, closed, symmetric operator. As explained in the introduction, we are going to extend L to a subspace of the domain of L_0^*. This space is spanned by functions of the form $\psi + G\varphi$ with $\psi \in D(L)$ and $\varphi \in \mathscr{H}$, where

$$G\varphi := \left(-ga(V)L^{-1}\right)^* \varphi = -gL^{-1}a^*(V)\varphi. \tag{11}$$

The operator G is bounded on $\mathscr{H}$ by Lemma 2.1. It maps $\mathscr{H}$ to the kernel of L_0^* by Equation (6). Application of G also improves regularity or decay in the particle number.

Lemma 2.3. *If Condition 1.1 holds with $D < 0$ the operator G is continuous from $\mathscr{H}$ to $D(N^{-D/4})$ and from $D(N^{\frac{2+D}{4}})$ to $D(L^{1/2})$.*

Proof. In view of Equation (11) and the fact that $L \geq N$, this is immediate from Lemma 2.1. □

The next lemma is concerned with the map $1 - G$ which is not only bounded but also invertible.

Lemma 2.4. *Assume Condition 1.1 holds with $D < 0$. Then $1 - G$ is invertible and there exists a constant $C > 0$ such that*

$$\|N\psi\|_{\mathscr{H}} \leq C(\|N(1-G)\psi\|_{\mathscr{H}} + \|\psi\|_{\mathscr{H}}) . \tag{12}$$

Proof. Due to Lemma 2.3 there is a constant $C > 0$ such that sector-wise

$$\|G\|^2_{\mathscr{H}^{(n-1)} \to \mathscr{H}^{(n)}} \leq Cn^{\frac{D}{2}} .$$

Using this we estimate the k-th power of G acting on $\psi \in \mathscr{H}$ by

$$\begin{aligned}\left\|G^k\psi\right\|^2_{\mathscr{H}} &= \sum_{n\geq 1} \left\|\left(G^k\psi\right)^{(n)}\right\|^2_{\mathscr{H}^{(n)}} \\ &\leq \sum_{n\geq k} \prod_{\ell=1}^{k} \|G\|^2_{\mathscr{H}^{(n-\ell)} \to \mathscr{H}^{(n-\ell+1)}} \left\|\psi^{(n-k)}\right\|^2_{\mathscr{H}^{(n-k)}} \\ &\leq C \|\psi\|^2_{\mathscr{H}} \sup_{m\geq 0} \prod_{r=1}^{k} (r+m)^{\frac{D}{2}} \leq C \|\psi\|^2_{\mathscr{H}} (k!)^{\frac{D}{2}} .\end{aligned}$$

This implies that the Neumann series $\sum_{k\geq 0} G^k$ converges in $\mathscr{H}$, hence $1 - G$ is invertible.

To prove (12), first note that G is a bounded operator from $D(N)$ to itself, because it maps $\mathscr{H}^{(n)}$ to $\mathscr{H}^{(n+1)}$. Define for any $\mu \geq 0$ a modified map by

$$G_\mu := -g \left(a(V)(L+\mu^2)^{-1}\right)^*$$

The norm $\|G_\mu\|_{D(N)\to D(N)} := c_\mu$ is decreasing in μ, so for sufficiently large μ we have $c_\mu < 1$. Then $(1-G_\mu)^{-1}$ is a bounded operator on $D(N)$ with norm at most $(1-c_\mu)^{-1}$. By the resolvent formula we then have

$$\begin{aligned}\|N\psi\| &\leq (1-c_\mu)^{-1}\left(\|N(1-G_\mu)\psi\| + \|(1-G_\mu)\psi\|\right)\\ &\leq (1-c_\mu)^{-1}\left(\|N(1-G)\psi\| + \left\|\mu^2 N(L+\mu^2)^{-1}G_\mu\psi\right\| + \|(1-G_\mu)\psi\|\right).\end{aligned}$$

Since $L \geq N$ this proves the claim. □

2.2 The annihilation operator A

So far we have considered $a(V)$ as an operator on $D(L)$. In view of Lemma 2.1 we may also define it sector-wise on $D(L^{1/2})$ in the case $D < 0$ of the current section. By Lemma 2.3 the annihilation operator thus makes sense on $G\varphi^{(n)}$, for any $n \in \mathbb{N}$.

Lemma 2.5. *Assume that Condition 1.1 holds with $D < 0$ and let $T = ga(V)G$. Then T defines a symmetric operator on the domain $D(T) = D(N^{1+D/2})$.*

Proof. Using Equation (11) we can write T as

$$T = -G^* L G = -g^2 \left(a(V)L^{-\frac{1}{2}}\right)\left(a(V)L^{-\frac{1}{2}}\right)^*.$$

This defines a continuous operator from $D(N^{1+D/2})$ to $\mathscr{H}$ by Lemma 2.1, and this operator is clearly symmetric. □

On the set $D(A) = D(L) \oplus GD(T)$, which contains $D(H)$, we now define the annihilation operator A by

$$gA(\psi + G\varphi) := ga(V)(\psi + G\varphi) = ga(V)\psi + T\varphi. \tag{13}$$

Remark 2.6. The objects we have discussed so far occur naturally in the context of abstract boundary conditions. Let K denote the restriction of L_0^* to $D(A) = D(L) \oplus GD(T)$ and denote by $B(\eta + G\varphi) = \varphi$ a left inverse of G. Then $(D(T), B, -gA)$ is a quasi boundary triple for K in the sense of Behrndt et al. [BFK+17]. In particular we have the identity

$$\langle K\varphi, \psi\rangle - \langle \varphi, K\psi\rangle = -\langle gA\varphi, B\psi\rangle + \langle B\varphi, gA\psi\rangle.$$

The family of operators $G(z) = -g(L+z)^{-1}a^*(V)$ are called the γ-field, and $T(z) = gAG(z)$ the Weyl-operators associated to this triple.

In specific cases the operators B and A can be expressed as local boundary value operators on the configurations where at least one x-particle (source) and one y-particle (boson) meet, see [TT15, LSTT18] and also Remark 3.4 for details.

2.3 Proof of Theorem 1.3 for $D < 0$

We will now prove that $H = L_0^* + gA$ is self-adjoint on the domain

$$D(H) = \{\psi \in \mathscr{H} | (1-G)\psi \in D(L)\} = (1-G)^{-1}D(L)$$

in the case of form perturbations, $D < 0$. The domain $D(H)$ is contained in $D(N)$ because $D(L) \subset D(N)$ and the domain of N is preserved by $(1-G)^{-1}$, see Lemma 2.4. We start the proof of self-adjointness by rewriting H in a more symmetric form. First, we use the fact that $L_0^* G = 0$, by Equation (6), to write for $\psi \in D(H)$

$$\begin{aligned} H\psi &= L_0^*(1-G)\psi + gA\psi \\ &= L(1-G)\psi + ga(V)(1-G)\psi + T\psi \,. \end{aligned} \tag{14}$$

Here, we have also used the 'boundary condition' that $(1-G)\psi \in D(L)$ for $\psi \in D(H)$. Since $G^* L = -ga(V)$ we can further rewrite this as

$$\begin{aligned} H\psi &= (1-G^*)L(1-G)\psi + G^*L(1-G)\psi + ga(V)(1-G)\psi + T\psi \\ &= (1-G)^*L(1-G)\psi + T\psi \,. \end{aligned} \tag{15}$$

We will prove that H is self-adjoint by showing that it is a perturbation of the self-adjoint operator $(1-G)^*L(1-G)$.

Lemma 2.7. *The operator* $H_0 := (1-G)^*L(1-G)$ *is self-adjoint on* $D(H)$ *and positive.*

Proof. The operator H_0 is clearly positive and symmetric on $D(H_0) = D(H)$, so it suffices to show that $D(H_0^*) \subset D(H)$. If $\varphi \in D(H_0^*)$, $\psi \in D(H_0) = (1-G)^{-1}D(L)$, we have

$$\langle \varphi, H_0\psi \rangle = \langle (1-G)\varphi, L(1-G)\psi \rangle,$$

and thus $(1-G)\varphi \in D(L^*) = D(L)$. This proves the claim. □

To prove self-adjointness of H we now show that T is infinitesimally H_0-bounded. By Lemma 2.5 and Young's inequality we have, keeping in mind that $D < 0$,

$$\|T\psi\|_{\mathscr{H}} \le C \left\| N^{1+D/2}\psi \right\|_{\mathscr{H}} \le \frac{C}{2}\left((2+D)\varepsilon \|N\psi\|_{\mathscr{H}} - D\varepsilon^{\frac{2+D}{D}} \|\psi\|_{\mathscr{H}} \right), \tag{16}$$

for any $\varepsilon > 0$. Now Lemma 2.4 together with $L \ge N$ yields the inequality

$$\begin{aligned} \|N\psi\|_{\mathscr{H}} &\le C(\|N(1-G)\psi\|_{\mathscr{H}} + \|\psi\|_{\mathscr{H}}) \\ &\le C\left(\left\| (1-G)^{-1} \right\| \|(1-G)^*L(1-G)\| + \|\psi\| \right). \end{aligned} \tag{17}$$

This proves an infinitesimal bound on T relative to H_0 and thus that $H = H_0 + T$ is self-adjoint on $D(H)$, by the Kato-Rellich theorem.

3 Renormalisable models

In this section we will deal with models falling into the second case of Condition 1.1. This means that $|\hat{v}(k)| \leq |k|^{-\alpha}$ for some $\alpha \geq 0$, $\int |\hat{v}(k)|^2 (k^2+\omega(k))^{-1}\mathrm{d}k = \infty$ (so necessarily $2\alpha \leq d-2$) and $\omega(k) \geq (1+k^2)^{\beta/2}$ for some $0<\beta\leq 2$. In dimension $d=2$ this leaves $\alpha = 0$ as the only case. In $d=3$ we assume

$$\frac{1}{2} \geq \alpha > \frac{1}{2} - \frac{\beta^2}{\beta^2+8}.$$

In terms of $D = d-2\alpha-2$ this means that

$$0 \leq D < \frac{2\beta^2}{\beta^2+8} \leq \frac{\beta}{2}. \tag{18}$$

Following the structure of Section 2, we start this section by discussing the extended domain. We then turn to the definition of the annihilation operator A and finally prove Theorem 1.3 and Theorem 1.4.

3.1 The extended domain

As in Section 2, we consider the extension of L (or the restriction of L_0^*) to vectors of the form $\psi + G\varphi$ with $\psi \in D(L)$, $\varphi \in \mathscr{H}$ and $G = -gL^{-1}a^*(V)$. We start by discussing the mapping properties of G, showing in particular that $a^*(V) : \mathscr{H} \to D(L)'$ and $a(V) : D(L) \to \mathscr{H}$ are continuous. In Section 2, where $D<0$, we showed that G maps into the form domain of L. For $D \geq 0$ however, G will not map into the form domain of L but instead into $D(L^\eta)$ for some $\eta < \frac{2-D}{4} \leq \frac{1}{2}$. We first prove a bound on G that will allow us to use the regularity and the decay in the particle number N in an optimal way later on.

Proposition 3.1. *Let Condition 1.1 be satisfied and define the affine transformation* $u(s) := \frac{\beta}{2}s - \frac{D}{2}$. *Then for any* $s \geq 0$ *such that* $u(s) < 1$ *and all* $0 \leq \eta < \frac{1+u(s)-s}{2}$ *there exists a constant* C *such that for all* $n \in \mathbb{N}$

$$\|L^\eta G\psi\|_{\mathscr{H}^{(n+1)}} \leq C\left(1+n^{\frac{\max(0,1-s)}{2}}\right)\left\|\psi^{(n)}\right\|_{\mathscr{H}^{(n)}}.$$

Proof. Note first that, since $\beta \leq 2$, the function $u(s)-s$ is non-increasing and thus $\eta < \frac{1+u(0)}{2} = \frac{2-D}{4}$. The expression for the Fourier transform of $G\psi^{(n)}$ is given by

$$\widehat{G\psi^{(n)}}(P,K) = \frac{-g}{\sqrt{n+1}} \sum_{i=1}^{M}\sum_{j=1}^{n+1} \frac{\hat{v}(k_j)\hat{\psi}^{(n)}(P+e_ik_j,\hat{K}_j)}{L(P,K)}. \tag{19}$$

As we are not interested in the dependence of the constant C on M or g it is sufficient to estimate the $\mathscr{H}^{(n+1)}$-norm of the expression

$$\gamma\hat{\psi}^{(n)}(P,K) = \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1}\frac{\hat{v}(k_j)\hat{\psi}^{(n)}(P+e_1k_j,\hat{K}_j)}{L(P,K)^{1-\eta}}\,. \tag{20}$$

We first multiply by $\omega(k_j)^{\frac{s}{2}}$ and its inverse, and then use the finite-dimensional Cauchy-Schwarz inequality to obtain

$$\left|\gamma\hat{\psi}^{(n)}(P,K)\right|^2 \leq \frac{1}{n+1}\sum_{i,j=1}^{n+1}\frac{|\hat{v}(k_j)|^2\left|\hat{\psi}^{(n)}(P+e_1k_j,\hat{K}_j)\right|^2}{L(P,K)^{2(1-\eta)}\omega(k_j)^s}\omega(k_i)^s\,. \tag{21}$$

Let $\left|\gamma^{(d)}\hat{\psi}^{(n)}\right|^2$ denote the sum of terms in this sum with $i=j$, and $\left|\gamma^{(od)}\hat{\psi}^{(n)}\right|^2$ the sum of the remaining terms. We have

$$\left|\gamma^{(d)}\hat{\psi}^{(n)}\right|^2 = \frac{1}{n+1}\sum_{j=1}^{n+1}\frac{|\hat{v}(k_j)|^2\left|\hat{\psi}^{(n)}(P+e_1k_j,\hat{K}_j)\right|^2}{L(P,K)^{2(1-\eta)}}\,, \tag{22}$$

$$\left|\gamma^{(od)}\hat{\psi}^{(n)}\right|^2 \leq \frac{n^{\max(0,1-s)}}{n+1}\sum_{j=1}^{n+1}\frac{|\hat{v}(k_j)|^2\left|\hat{\psi}^{(n)}(P+e_1k_j,\hat{K}_j)\right|^2}{L(P,K)^{2(1-\eta)}\omega(k_j)^s}\Omega(\hat{K}_j)^s\,. \tag{23}$$

In the second line we have used the bound (with the notation $\sum_{j\in J}\omega(q_j)=\Omega(Q)$)

$$\sum_{i=1}^{n}\omega(k_i)^s \leq n^{\max(0,1-s)}\Omega(K)^s\,, \tag{24}$$

which for $s<1$ follows from the Hölder inequality, while for $s\geq 1$ it holds by interpolation between the ℓ^1-norm and the ℓ^∞-norm.

Note that both sums in (22), (23) are just symmetrisations and every summand has the same integral. Integrating (22) and performing a change of variables thus yields

$$\int\left|\gamma^{(d)}\hat{\psi}^{(n)}(P,K)\right|^2\,\mathrm{d}P\mathrm{d}K = \int\frac{|\hat{v}(k_{n+1})|^2\left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2}{L(P-e_1k_{n+1},K)^{2(1-\eta)}}\,\mathrm{d}P\mathrm{d}K\,.$$

We notice that the square of $\hat{\psi}^{(n)}$ does not depend on k_{n+1} anymore. Using that $4\eta<2-D=4+2\alpha-d$, and the Hardy-Littlewood inequality, the integral over

k_{n+1} can be bounded by

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\hat{v}(k_{n+1})|^2}{L(P - e_1 k_{n+1}, K)^{2(1-\eta)}} \,\mathrm{d}k_{n+1} \\
&\quad \leq \int_{\mathbb{R}^d} \frac{|k_{n+1}|^{-2\alpha}}{((p_1 - k_{n+1})^2 + \Omega(\hat{K}_{n+1}) + 1)^{2(1-\eta)}} \,\mathrm{d}k_{n+1} \\
&\quad \leq C(\Omega(\hat{K}_{n+1}) + 1)^{-2(1-\eta)-\alpha+\frac{d}{2}}\,, \qquad (25)
\end{aligned}$$

The exponent here is negative, which proves the required bound for $\left|\gamma^{(d)}\psi^{(n)}\right|^2$.

The integration of (23) gives

$$\begin{aligned}
&\int \left|\gamma^{(od)}\hat{\psi}^{(n)}(P,K)\right|^2 \,\mathrm{d}P\mathrm{d}K \\
&\quad \leq n^{\max(0,1-s)} \int \frac{|\hat{v}(k_{n+1})|^2\, \Omega(\hat{K}_{n+1})^s \left|\hat{\psi}^{(n)}(P, \hat{K}_{n+1})\right|^2}{L(P - e_1 k_{n+1}, K)^{2(1-\eta)}\omega(k_{n+1})^s} \,\mathrm{d}P\mathrm{d}K\,.
\end{aligned}$$

The condition $u(s) < 1$ implies that $\beta s + 2\alpha < d$, so we can bound the integral in k_{n+1} by

$$\int_{\mathbb{R}^d} \frac{|k_{n+1}|^{-2\alpha-\beta s}}{((p_1 - k_{n+1})^2 + \Omega(\hat{K}_{n+1}))^{2(1-\eta)}} \,\mathrm{d}k_{n+1} \leq C\Omega(\hat{K}_{n+1})^{-2(1-\eta)-\frac{2\alpha+\beta s}{2}+\frac{d}{2}}\,.$$

It follows that

$$\begin{aligned}
&\int \left|\gamma^{(od)}\hat{\psi}^{(n)}(P,K)\right|^2 \,\mathrm{d}P\mathrm{d}K \\
&\quad \leq Cn^{\max(0,1-s)} \int \Omega(\hat{K}_{n+1})^{2\eta-1-u(s)+s} \left|\hat{\psi}^{(n)}(P, \hat{K}_{n+1})\right|^2 \,\mathrm{d}P\mathrm{d}\hat{K}_{n+1}\,.
\end{aligned}$$

The exponent of $\Omega(\hat{K}_{n+1})$ in this integral is negative by hypothesis, and this proves the claim. ☐

A simple consequence of this proposition is that G maps $\mathscr{H}$ into the domain of some power of L, and thus also of N.

Corollary 3.2. *Assume Condition 1.1 holds with $D \geq 0$. There exists an $\eta \in (0, 1/2)$ such that G is a continuous operator from $\mathscr{H}$ to $D(L^\eta)$.*

Proof. We apply Proposition 3.1, distinguishing two cases. First, if $D = 0$ and $\beta = 2$, then $u(s) = s$ and we choose, for some $1 > \varepsilon > 0$, $s_\varepsilon = 1 - \varepsilon$ and $\eta_\varepsilon = \frac{1-\varepsilon}{2}$. Proposition 3.1 then gives the bound

$$\left\|L^{\frac{1-\varepsilon}{2}} G\psi\right\|_{\mathscr{H}^{(n+1)}} \leq C(1 + n^{\varepsilon/2}) \left\|\psi^{(n)}\right\|_{\mathscr{H}^{(n)}}\,.$$

Dividing by $(1+n^{\varepsilon/2}) \leq cL^{\varepsilon/2}$ then shows that G maps $\mathscr{H}$ to $D(L^{1/2-\varepsilon})$ for all $0<\varepsilon\leq\frac{1}{2}$, in this case.

In all other cases, we have $u(1)=(\beta-D)/2<1$, by (18), and we may choose in Proposition 3.1 $s=1$ and any $0\leq\eta<\frac{\beta-D}{4}$. □

An important consequence of this is that $ga(V)L^{-1}=-G^*$ is a continuous operator on $\mathscr{H}$, so $a(V)$ is well defined on $D(L)$. We can thus define L_0 and its adjoint in the very same way as in Section 2. We can also prove the analogue of Lemma 2.4.

Lemma 3.3. *Let Condition 1.1 be satisfied. Then $1-G$ is invertible and there exists a constant $C>0$ such that*

$$\|N\psi\|_{\mathscr{H}} \leq C(\|N(1-G)\psi\|_{\mathscr{H}} + \|\psi\|_{\mathscr{H}})\,. \tag{26}$$

Proof. Using Corollary 3.2 and the fact that $N\leq L$ the proof for the case $D\geq 0$ is exactly the same as in Lemma 2.4 for $D<0$. □

3.2 Extending the annihilation operator for $D \geq 0$

In this section we will extend the annihilation operator $a(V)$ to certain vectors in the range of G, defining the operator A. To do so, for any symmetric operator $(T, D(T))$ we could define an extension gA on the set $D(A)=D(L)\oplus GD(T)$ by

$$gA(\psi+G\varphi) := ga(V)\psi + gAG\varphi = ga(V)\psi + T\varphi\,. \tag{27}$$

In the case of a form perturbation, where G maps sector-wise into $D(L^{1/2})$, the right extension of $a(V)$ to these elements is obviously $a(V)$ itself. As a result, we have simply chosen $T=ga(V)G$ in Section 2. However, this choice is not possible if the domain of $a(V)$ and the range of G do not match, as is the case if $D\geq 0$. We will define T by slightly modifying the expression for $ga(V)G$, in such a way that the operator H we obtain coincides with the one constructed by renormalisation. In Fourier representation, $ga(V)G$ is formally given by

$$g\sqrt{n+1}\sum_{\ell=1}^{M}\int_{\mathbb{R}^d}\overline{\hat{v}(k_{n+1})}\widehat{G\varphi^{(n)}}(P-e_\ell k_{n+1},K)\,\mathrm{d}k_{n+1}\,.$$

Expanding the formal action by spelling out $\widehat{G\varphi^{(n)}}$ as in (19) gives

$$-g^2\sum_{\ell=1}^{M}\sum_{i=1}^{M}\sum_{j=1}^{n+1}\int_{\mathbb{R}^d}\frac{\overline{\hat{v}(k_{n+1})}\hat{v}(k_j)\hat{\varphi}^{(n)}(P-e_\ell k_{n+1}+e_ik_j,\hat{K}_j)}{L(P-e_\ell k_{n+1},K)}\,\mathrm{d}k_{n+1}\,. \tag{28}$$

Have a look at the sum above. In the terms where $j=n+1$ and $i=\ell$, the function $\hat{\varphi}^{(n)}$ does not depend on the variable k_{n+1} anymore. Formally, these terms define a multiplication operator, with the multiplier given by a sum over integrals of the form

$$-g^2 \int_{\mathbb{R}^d} \frac{|\hat{v}(k_{n+1})|^2}{L(P - e_\ell k_{n+1}, K)} \, \mathrm{d}k_{n+1} \, .$$

This is what we will call the *diagonal* part in the following. However, this integral is divergent. In order to obtain a well-defined operator, we replace this integral by a regularised version. We set

$$I_\ell(P, \hat{K}_{n+1}) := \int_{\mathbb{R}^d} |\hat{v}(k_{n+1})|^2 \left(\frac{1}{L(P - e_\ell k_{n+1}, K)} - \frac{1}{k_{n+1}^2 + \omega(k_{n+1})} \right) \mathrm{d}k_{n+1} \tag{29}$$

and define T_d, the diagonal part of T, in Fourier representation as

$$\widehat{T_\mathrm{d}\varphi^{(n)}}(P, \hat{K}_{n+1}) := -g^2 \hat{\varphi}^{(n)}(P, \hat{K}_{n+1}) \sum_{\ell=1}^{M} I_\ell(P, \hat{K}_{n+1}) \, . \tag{30}$$

The remaining expressions in (28) constitute the *off-diagonal* part of T. It is a sum of integral operators and we will show that they are defined on suitable spaces, without modification. Spelled out, we have

$$\begin{aligned} &\widehat{T_\mathrm{od}\varphi^{(n)}}(P, \hat{K}_{n+1}) \qquad (31) \\ &\quad := -g^2 \sum_{\ell=1}^{M} \sum_{\substack{i=1 \\ i \neq \ell}}^{M} \int_{\mathbb{R}^d} \frac{|\hat{v}(k_{n+1})|^2 \, \hat{\varphi}^{(n)}(P - (e_\ell - e_i)k_{n+1}, \hat{K}_{n+1})}{L(P - e_\ell k_{n+1}, K)} \, \mathrm{d}k_{n+1} \\ &\qquad - g^2 \sum_{\ell=1}^{M} \sum_{i=1}^{M} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\overline{\hat{v}(k_{n+1})} \hat{v}(k_j) \hat{\varphi}^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)}{L(P - e_\ell k_{n+1}, K)} \, \mathrm{d}k_{n+1} \, . \end{aligned}$$

We define the operator

$$T\varphi^{(n)} := T_\mathrm{d}\varphi^{(n)} + T_\mathrm{od}\varphi^{(n)} \tag{32}$$

by the expressions above, on a domain (or rather a family of admissible domains) to be specified in Proposition 3.5 below.

Remark 3.4. As noted before, the choice of the operator T is not unique. In fact, any operator T that is symmetric on an appropriate domain will lead to a self-adjoint operator H. We have made the choice for which this operator coincides with the one constructed by renormalisation, with the usual choice of renormalisation constant E_Λ, cf. Theorem 1.4. Observe that the the regularised integral (29) is formally obtained by subtracting the 'constant' $E_\infty = \int |\hat{v}(k)|^2 (k^2 + \omega(k))^{-1} \mathrm{d}k$. In this sense, the operator A may be viewed as the 'renormalised' annihilation operator.

Another way to interpret the expression for T_d is that the distribution $v(x_\ell - y_{n+1})$ is not applied to the function $G\varphi^{(n)}$, but to the more regular function

$$G\varphi^{(n)} + g\varphi^{(n)}(X, \hat{Y}_{n+1}) f(x_\ell - y_{n+1}),$$

where $\hat{f}(k) = \hat{v}(k)(k^2 + \omega(k))^{-1}$. Here, the second term effectively cancels the local singularities of $G\varphi^{(n)}$ in the directions parametrised by $x_\ell - y_{n+1}$. This point of view is particularly natural if $v(y)$ is singular only at $y = 0$, and thus $G\varphi^{(n)}$ is singular on the planes $\{x_\ell = y_j\}$. In this case, the off-diagonal operator T_od comes from the application of $v(x_\ell - y_{n+1})$ to functions $L^{-1} v(x_i - y_j)\varphi^{(n)}(X, \hat{Y}_j)$ in directions where they are regular.

In concrete examples, there might be other criteria that single out a choice of T, respectively A. For example in the case of $v = \delta$, $d = 2$, $\omega(k) = k^2 + 1$, the annihilation operator $a(V)$ is (the sum of) evaluation operators on the planes where $x_\ell = y_j$. These are local boundary values and one would want the extension A to be local in this sense as well. In this example, the functions in the range of G are singular, with an asymptotic expansion

$$G\varphi^{(n)}(X, Y) = \frac{c \log |x_\ell - y_j| \varphi^{(n)}(X, \hat{Y}_j)}{\sqrt{n+1}} + F(X, Y)$$

as $|x_\ell - y_j| \to 0$, where c is a universal constant and F is a function that has a (suitable) limit almost-everywhere on $\{x_\ell = y_j\}$. One can view φ as a local boundary value of this function, since

$$\varphi(X, \hat{Y}_j) = \sqrt{n+1} \lim_{|x_\ell - y_j| \to 0} \frac{G\varphi^{(n)}(X, Y)}{c \log |x_\ell - y_j|}.$$

It is then natural to choose $AG\varphi^{(n)}$ as the evaluation of the regular part $F(X, Y)$ of $G\varphi^{(n)}$, more precisely

$$AG\varphi^{(n)}(X, \hat{Y}_{n+1}) = \lim_{r \to 0} \sum_{\ell=1}^{M} \int_{|x_\ell - y_{n+1}| = r} \Big(\sqrt{n+1} G\varphi^{(n)}(X, Y) - c \log(r) \varphi^{(n)}(X, \hat{Y}_{n+1}) \Big) \mathrm{d}\omega.$$

This is clearly a local boundary value, and one can check that this coincides with our choice of A up to the addition of a global constant. Such boundary values are discussed in [Lam18, LSTT18, TT15] for a variety of models involving creation and annihilation of particles. Boundary values for a two-dimensional model with point interactions were treated by Dell'Antonio, Figari, and Teta [DFT94, Sec.5].

The next proposition states the important mapping properties of T. For our model of non-relativistic point-particles in two dimensions ($d = 2$, $v = \delta$, $\omega(k) = k^2 + 1$), we show that T_{d} is defined on $D(L^\varepsilon)$ for any $\varepsilon > 0$ (in fact, it is a Fourier multiplier of logarithmic growth), and that T_{od} is a bounded operator on $\mathscr{H}^{(n)}$ whose norm grows at most like n^ε. For the Nelson model ($d = 3$, $\omega(k) = \sqrt{1 + k^2}$, $\hat{v}(k) = \omega(k)^{-1/2}$), T_{d} is also bounded by any power of L, and T_{od} is an operator $D(L^\varepsilon) \cap \mathscr{H}^{(n)} \to \mathscr{H}^{(n)}$ whose norm grows at most like $n^{1-2\varepsilon}$.

Proposition 3.5. *Assume Condition 1.1 holds with $D \geq 0$, set $u(s) = \frac{\beta}{2}s - \frac{D}{2}$ and define T for every $n \in \mathbb{N}$ by the expression* (32)*.*

- *If $D = 0$ and $\beta = 2$ then, for any $\varepsilon > 0$, T defines a symmetric operator on the domain $D(T) = D(L^\varepsilon)$.*
- *If either $D > 0$ or $\beta < 2$ then, for all $s > 0$ such that the following two conditions are satisfied*

$$\begin{aligned} u(s) &< 1 \\ 0 &< u(u(s)) \,, \end{aligned}$$

the operator T is symmetric on $D(T) = D((N+1)^{\max(0,1-s)} L^{s-u(s)})$.

Proof. The proof will be split into three lemmas. In Lemma 3.6 we deal with the diagonal operator T_{d}. We will show that T_{d} defines a symmetric operator on the domain $D(L^{\max(\varepsilon, D/2)})$ for any $\varepsilon > 0$. We further decompose the *off-diagonal* part in (31) as

$$\widehat{T_{\mathrm{od}}\varphi}^{(n)} := \sum_{\ell=1}^{M} \sum_{i=1, i\neq\ell}^{M} \theta_{i\ell}\hat{\varphi}^{(n)} + \sum_{\ell=1}^{M}\sum_{i=1}^{M} \tau_{i\ell}\hat{\varphi}^{(n)}$$

with

$$\theta_{i\ell}\hat{\varphi}^{(n)}(P, \hat{K}_{n+1}) := \int_{\mathbb{R}^d} \frac{|\hat{v}(k_{n+1})|^2\, \hat{\varphi}^{(n)}(P + (e_i - e_\ell)k_{n+1}, \hat{K}_{n+1})}{L(P - e_\ell k_{n+1}, K)}\, \mathrm{d}k_{n+1} \tag{33}$$

and

$$\tau_{i\ell}\hat{\varphi}^{(n)}(P, \hat{K}_{n+1}) := \sum_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\overline{\hat{v}(k_{n+1})}\hat{v}(k_j)\hat{\varphi}^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)}{L(P - e_\ell k_{n+1}, K)}\, \mathrm{d}k_{n+1}\,. \tag{34}$$

In Lemma 3.7 the properties of the θ-terms and in Lemma 3.8 those of the τ-terms are described. Both of these lemmas rely on modifications of the Schur test, but the second one will be more difficult due to the additional sum over n terms in $\tau_{i\ell}$.

If $D = 0$ and $\beta = 2$, Lemma 3.6 shows that T_d is defined on $D(L^\varepsilon)$ for any $\varepsilon > 0$. Regarding the terms $\theta_{i\ell}$, Lemma 3.7 shows that they are bounded and that their sum is symmetric. Now because $u(s) = s$, the conditions on the parameter s in Lemma 3.8 reduce to $s \in (0, 1)$. The lemma then states that the operators $\tau_{i\ell}$ are defined on $D(N^{1-s})$ and their sum is symmetric. Choosing $s_\varepsilon = 1-\varepsilon$ and estimating $N \le L$ yields the claim in this case.

If either $D > 0$ or $\beta < 2$, strictly, we have for sufficiently small $\varepsilon > 0$

$$s - u(s) = \frac{1}{2}(2-\beta)s + \frac{D}{2} \ge \max\left(\varepsilon, \frac{D}{2}\right).$$

This means that $D((N+1)^{\max(0,1-s)}L^{s-u(s)}) \subset D(L^{\max(\varepsilon,D/2)})$ for such an ε. Therefore, Lemmas 3.6 – 3.8 together prove the claim. □

Lemma 3.6. *Assume Condition 1.1 holds with $D \ge 0$. Then for any $\varepsilon > 0$ the expression T_d given by* (30) *defines a symmetric operator on the domain $D(T_\mathrm{d}) = D(L^{\max(\varepsilon,D/2)})$.*

Proof. The integral (29) defining T_d is real, so T_d is a real Fourier multiplier and it is sufficent to prove that it maps the domain $D(T_\mathrm{d})$ to $\mathscr{H}$. Specifying as usual to $\ell = 1$ we have to show that there exists a constant $C > 0$ such that the inequality

$$I_1(P, \hat{K}_{n+1}) \le C\left(L(P, \hat{K}_{n+1})^{\max(\varepsilon,D/2)} + 1\right) \tag{35}$$

holds pointwise on $\mathbb{R}^{Md} \times \mathbb{R}^{nd}$. We will use that

$$I_1(P, \hat{K}_{n+1}) = \int_{\mathbb{R}^d} |\hat{v}(k_{n+1})|^2 \frac{2p_1 \cdot k_{n+1} - p_1^2 - \left(\hat{P}_1^2 + \Omega(\hat{K}_{n+1})\right)}{L(P - e_1 k_{n+1}, K)(k_{n+1}^2 + \omega(k_{n+1}))} \mathrm{d}k_{n+1}$$

and distinguish between $d = 2$ and $d = 3$.

If $d = 2$ then necessarily $\alpha = 0$ and $D = 0$. We denote the integration variable by q instead of k_{n+1} and also write p for p_1. The absolute value of the integral I_1 can, for any $\varepsilon \in (0, 1)$, be bounded by

$$\int_{\mathbb{R}^2} \frac{2|p||q| + p^2 + \left(\hat{P}_1^2 + \Omega(\hat{K}_{n+1})\right)}{\left((p-q)^2 + \hat{P}_1^2 + \Omega(\hat{K}_{n+1})\right)(q^2+1)} \mathrm{d}q$$

$$\le \int_{\mathbb{R}^2} \frac{2|p|(q^2+1)^{\frac{1}{2}} + p^2}{((p-q)^2+1)(q^2+1)} \mathrm{d}q + \int_{\mathbb{R}^2} \frac{\hat{P}_1^2 + \Omega(\hat{K}_{n+1})}{\left((p-q)^2 + \hat{P}_1^2 + \Omega(\hat{K}_{n+1})\right)|q|^{2(1-\varepsilon)}} \mathrm{d}q.$$

The second term is bounded by some constant times $(\hat{P}_1^2 + \Omega(\hat{K}_{n+1}))^\varepsilon$. For the first term we use Lemma A.2 in the appendix, which yields

$$\int_{\mathbb{R}^2} \frac{2|p|(q^2+1)^{\frac{1}{2}} + p^2}{((p-q)^2+1)(q^2+1)} \mathrm{d}q \le 3C(\log(1+|p|)+1) \le \tilde{C}(|p|^\varepsilon + 1),$$

for some $\tilde{C} > 0$.

Now let $d = 3$ and $D > 0$. The absolute value of the integral I_1 is bounded by

$$\int_{\mathbb{R}^3} \frac{2|p||q| + p^2}{((p-q)^2+1)|q|^{2+2\alpha}} \mathrm{d}q + \int_{\mathbb{R}^3} \frac{\hat{P}_1^2 + \Omega(\hat{K}_{n+1})}{\Big((p-q)^2 + \hat{P}_1^2 + \Omega(\hat{K}_{n+1})\Big)|q|^{2+2\alpha}} \mathrm{d}q\,. \tag{36}$$

The integrals converge because $2 + 2\alpha = d - D < d$ and $\alpha > 0$. The second term is easily seen to be bounded by a constant times $(\hat{P}_1^2 + \Omega(\hat{K}_{n+1}))^{\frac{D}{2}}$. For the first term we can use Lemma A.2 in the appendix which gives

$$\int_{\mathbb{R}^3} \frac{2|p||q| + p^2}{((p-q)^2+1)|q|^{2+2\alpha}} \mathrm{d}q \leq \frac{2C|p|}{|p|^{2\alpha}} + \frac{Cp^2}{|p|^{1+2\alpha}} \leq \tilde{C}|p|^D \leq \tilde{C}L(P,\hat{K}_{n+1})^{\frac{D}{2}}\,.$$

If $D = 0$, the function $|q|^{-2-2\alpha} = |q|^{-d}$ is not locally integrable. We thus use the estimate $q^2 + 1 \geq q^{2(1-\varepsilon)}$, for any $\varepsilon \in (0,1)$. This yields a bound on $|I_1|$ as in Equation (36), but with $|q|^{-2-2\alpha}$ replaced by $|q|^{-d+2\varepsilon}$. Applying Lemma A.2 then gives a bound on $|I_1|$ by some constant times $L(P,\hat{K}_{n+1})^{\varepsilon}$. □

Lemma 3.7. *Assume Condition 1.1 holds with $D \geq 0$. Then, for any $i,\ell \in \{1,\dots,M\}$ with $i \neq \ell$, the operator $\theta_{i\ell}$ defined by* (33) *is continuous from $D(L^{D/2})$ to $\mathscr{H}$ and $\theta_{i\ell} + \theta_{\ell i}$ is symmectric on this domain.*

Proof. We will prove continuity for $\theta := \theta_{12}$. We multiply (33) by $|p_2 - k_{n+1}|^{D+\varepsilon}$ and its inverse for any $\varepsilon > 0$, and use the Cauchy-Schwarz inequality to obtain

$$\left|\theta\hat{\psi}^{(n)}\right|^2 \leq \int_{\mathbb{R}^d} \frac{|\hat{v}(q)|^2 \,\mathrm{d}q}{L(P - e_2 q, \hat{K}_{n+1}, q)\,|p_2 - q|^{2(D+\varepsilon)}}$$
$$\times \int_{\mathbb{R}^d} \frac{|\hat{v}(k_{n+1})|^2 \left|\hat{\psi}^{(n)}(P + (e_1 - e_2)k_{n+1}, \hat{K}_{n+1})\right|^2 |p_2 - k_{n+1}|^{2(D+\varepsilon)}}{L(P - e_2 k_{n+1}, K)} \mathrm{d}k_{n+1}\,.$$

Using the Hardy-Littlewood inequality and scaling, the integral in q can be bounded by

$$\int_{\mathbb{R}^d} \frac{|q|^{-2\alpha}}{(p_1^2 + q^2)\,|q|^{2(D+\varepsilon)}} \mathrm{d}q \leq C\,|p_1|^{-(D+2\varepsilon)}\,,$$

for $0 < \varepsilon < 1/2$. Integrating in the remaining variables $(P, \hat{K}_{n+1})$ and performing a change of variables $P \to P + (e_1 - e_2)k_{n+1}$ then gives

$$\int \left|\theta\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \mathrm{d}\hat{K}_{n+1}\mathrm{d}P$$
$$\leq C \int \frac{|\hat{v}(k_{n+1})|^2 \left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 |p_2|^{2(D+\varepsilon)}}{L(P - e_1 k_{n+1}, K)\,|p_1 - k_{n+1}|^{D+2\varepsilon}} \mathrm{d}k_{n+1}\mathrm{d}\hat{K}_{n+1}\mathrm{d}P\,.$$

Because $2+2\alpha+D+2\varepsilon = d+2\varepsilon$ the k_{n+1}-integral can, for $0<\varepsilon<1$, be bounded as above by some constant times $|p_1|^{-2\varepsilon}$. We thus obtain

$$\int \left|\theta\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \mathrm{d}\hat{K}_{n+1}\mathrm{d}P \le C\int \left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 |p_2|^{2D}\,\mathrm{d}\hat{K}_{n+1}\mathrm{d}P\,,$$

and this proves continuity.

To prove symmetry, we use the change of variables $Q = P+(e_i-e_\ell)k_{n+1}$ in

$$\begin{aligned}
&\langle\hat{\varphi}^{(n)},\theta_{i\ell}\hat{\psi}^{(n)}\rangle_{\mathscr{H}^{(n)}}\\
&\quad=\int\overline{\hat{\varphi}^{(n)}}(P,\hat{K}_{n+1})\frac{|\hat{v}(k_{n+1})|^2\,\hat{\psi}^{(n)}(P+(e_i-e_\ell)k_{n+1},\hat{K}_{n+1})}{L(P-e_\ell k_{n+1},K)}\,\mathrm{d}P\mathrm{d}K\\
&\quad=\int\overline{\hat{\varphi}^{(n)}}(Q+(e_\ell-e_i)k_{n+1},\hat{K}_{n+1}),\hat{K}_{n+1})\frac{|\hat{v}(k_{n+1})|^2\,\hat{\psi}^{(n)}(Q,\hat{K}_{n+1})}{L(Q-e_i k_{n+1},K)}\,\mathrm{d}P\mathrm{d}K.
\end{aligned}$$

Together with the bounds we have just proved, this implies that $\theta^*_{i\ell}$ extends $\theta_{\ell i}$ (defined on $D(L^{D/2})$), so the sum of the two is symmetric on this domain. □

Lemma 3.8. *Assume Condition 1.1 holds with $D\ge 0$ and let $u(s)=\frac{\beta}{2}s-\frac{D}{2}$. Then, for all $s>0$ such that the following two conditions are satisfied*

$$u(s) < 1 \tag{37}$$

$$0 < u(u(s))\,, \tag{38}$$

and for all $i,\ell\in\{1,\dots,M\}$, the operator $\tau_{i\ell}$, defined in (34)*, is bounded from $D(N^{\max(0,1-s)}L^{s-u(s)})$ to $\mathscr{H}$ and $\tau_{i\ell}+\tau_{\ell i}$ is symmetric on this domain.*

Proof. We start by proving the bound

$$\left\|\tau_{i\ell}\hat{\psi}^{(n)}\right\|_{\mathscr{H}^{(n)}} \le Cn^{\max(0,1-s)}\left\|L^{s-u(s)}\psi^{(n)}\right\|_{\mathscr{H}^{(n)}}$$

for any fixed i,ℓ and $n\ge 1$ (note that $\tau_{i\ell}=0$ for $n=0$). Note that, because $D\ge 0$ and $\beta\le 2$, it holds that $u(s)\le s$ and therefore the conditions (37) and (38) already imply that

$$u(s),u(u(s))\in(0,1)\,. \tag{39}$$

Now we denote $\tau=\tau_{i\ell}$ and write

$$\begin{aligned}
\tau\hat{\psi}^{(n)} = \sum_{j=1}^{n}\int_{\mathbb{R}^d}&\omega(k_{n+1})^{\frac{s}{2}}\frac{\hat{v}(k_j)\hat{\psi}^{(n)}(P-e_\ell k_{n+1}+e_i k_j,\hat{K}_j)}{L(P-e_\ell k_{n+1},K)^{\frac12}\omega(k_j)^{\frac{s}{2}}}\\
&\times\omega(k_j)^{\frac{s}{2}}\frac{\overline{\hat{v}(k_{n+1})}}{L(P-e_\ell k_{n+1},K)^{\frac12}\omega(k_{n+1})^{\frac{s}{2}}}\,\mathrm{d}k_{n+1}\,.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality on $L^2(\mathbb{R}^d \times \{1, \ldots, n\})$ and using the assumptions on $\hat{v}$ and ω, we obtain

$$\left|\tau\hat{\psi}^{(n)}\right|^2 \leq \sum_{j=1}^{n} \int_{\mathbb{R}^d} \omega(k_{n+1})^s \frac{|\hat{v}(k_j)|^2 \left|\hat{\psi}^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)\right|^2}{L(P - e_\ell k_{n+1}, K)\omega(k_j)^s} \, \mathrm{d}k_{n+1}$$
$$\times \sum_{\mu=1}^{n} \omega(k_\mu)^s \int_{\mathbb{R}^d} \frac{1}{((p_\ell - q)^2 + \Omega(\hat{K}_{n+1}))\, |q|^{\beta s + 2\alpha}} \, \mathrm{d}q\,.$$

Since $u(s) \in (0,1)$, the integral in the second line is bounded by a constant times $\Omega(\hat{K}_{n+1})^{-u(s)}$. In order to deal with the sum over $\mu = 1, \ldots, n$, we split the term $\mu = j$ from the rest and use (24). This gives

$$\begin{aligned}\sum_{\mu=1}^{n} \omega(k_\mu)^s \Omega(\hat{K}_{n+1})^{-u(s)} &\leq \omega(k_j)^{s-u(s)} + n^{\max(0,1-s)} \Omega(\hat{K}_{n+1,j})^s \Omega(\hat{K}_{n+1})^{-u(s)} \\ &\leq \omega(k_j)^{s-u(s)} + n^{\max(0,1-s)} \Omega(\hat{K}_{n+1,j})^{s-u(s)} \\ &\leq \omega(k_j)^{s-u(s)} + n^{\max(0,1-s)} \Omega(\hat{K}_j)^{s-u(s)}\,,\end{aligned}$$

where we have also used that $s \geq u(s) > 0$. Consequently, we have a bound of the form

$$\left|\tau\hat{\psi}^{(n)}\right|^2 \leq C \left|\tau^{(d)}\hat{\psi}^{(n)}\right|^2 + C \left|\tau^{(od)}\hat{\psi}^{(n)}\right|^2\,,$$

with

$$\left|\tau^{(d)}\hat{\psi}^{(n)}\right|^2 := \sum_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\omega(k_{n+1})^s\, |\hat{v}(k_j)|^2 \left|\hat{\psi}^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)\right|^2}{\omega(k_j)^{u(s)} L(P - e_\ell k_{n+1}, K)} \, \mathrm{d}k_{n+1} \tag{40}$$

and

$$\left|\tau^{(od)}\hat{\psi}^{(n)}\right|^2 := n^{\max(0,1-s)} \sum_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\omega(k_{n+1})^s \left|\hat{\psi}^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)\right|^2}{L(P - e_\ell k_{n+1}, K)} \times \frac{|\hat{v}(k_j)|^2\, \Omega(\hat{K}_j)^{s-u(s)}}{\omega(k_j)^s} \, \mathrm{d}k_{n+1}\,. \tag{41}$$

To treat the term (40), we integrate in $(P, \hat{K}_{n+1})$, perform a change of variables

$P \to P - e_\ell k_{n+1} + e_i k_j$, and then rename the variables $k_j \leftrightarrow k_{n+1}$. This yields

$$\begin{aligned}\int & \left|\tau^{(d)}\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \,\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \\ &= \sum_{j=1}^{n} \int \frac{\omega(k_{n+1})^s\,|\hat{v}(k_j)|^2 \left|\hat{\psi}^{(n)}(P,\hat{K}_j)\right|^2}{\omega(k_j)^{u(s)} L(P - e_i k_j, K)} \,\mathrm{d}P\mathrm{d}K \\ &= \sum_{j=1}^{n} \int \frac{\omega(k_j)^s\,|\hat{v}(k_{n+1})|^2 \left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2}{\omega(k_{n+1})^{u(s)} L(P - e_i k_{n+1}, K)} \,\mathrm{d}P\mathrm{d}K \,,\end{aligned}$$

where, in the last step, we have used the permutation symmetry. The k_{n+1}-integral can be estimated, using the assumptions on $\hat{v}$ and ω and the fact that $u(u(s)) \in (0,1)$, by

$$\int_{\mathbb{R}^d} \frac{|\hat{v}(k_{n+1})|^2}{L(P - e_i k_{n+1}, K)\omega(k_{n+1})^{u(s)}} \,\mathrm{d}k_{n+1} \leq C\Omega(\hat{K}_{n+1})^{-u(u(s))} \,.$$

Therefore, using again the bound (24), we conclude

$$\begin{aligned}\int & \left|\tau^{(d)}\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \,\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \\ &\leq C\sum_{j=1}^{n} \int \omega(k_j)^s \left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \Omega(\hat{K}_{n+1})^{-u(u(s))} \,\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \\ &\leq C n^{\max(0,1-s)} \int \left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \Omega(\hat{K}_{n+1})^{s-u(u(s))} \,\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \,.\end{aligned}$$

We proceed similarly with the second term (41) and obtain

$$\left|\tau^{(od)}\hat{\psi}^{(n)}\right|^2 \leq C n^{2\max(0,1-s)} \int \left|\hat{\psi}^{(n)}(P,\hat{K}_{n+1})\right|^2 \Omega(\hat{K}_{n+1})^{2(s-u(s))} \,\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \,.$$

This proves the desired bound, because $u(s) \leq s$ (as $D \geq 0$ and $\beta \leq 2$) and thus

$$s - u(u(s)) \leq s - u(s) + u(s - u(s)) \leq 2(s - u(s)) \,.$$

Symmetry follows from this as in Lemma 3.7. In this case, the change of variables one makes is $P \mapsto P - e_\ell k_{n+1} + e_i k_j$. Additionally, one also uses the symmetry of functions in $\mathscr{H}^{(n)}$, while renaming $k_j \leftrightarrow k_{n+1}$. □

Remark 3.9. An operator very similar to the operator T plays an important role in the context of point interactions of nonrelativistic particles, where $v = \delta$ and $\omega(k) = 1 + k^2$. This operator is known as the Ter-Martyrosyan–Skornyakov operator.

In two dimensions, this was studied in [DFT94, Lem.3.1], where estimates similar to ours (but with a linear growth in n) were proved. These bounds were refined by Griesemer and Linden [GL18].

The three-dimensional case has received more attention, see e.g. [DFT94, CDF$^+$15, MS17, MS18]. Recently, Moser and Seiringer [MS17] proved, in particular, an n-independent bound on T_{od} for this model, as an operator from $H^{1/2}(\mathbb{R}^{3+3n})$ to $H^{-1/2}(\mathbb{R}^{3+3n})$ (with $M = 1$). Our proof of Lemma 3.8 is inspired by their technique.

The lemmas above do provide bounds on T for the case $d = 3$, $v = \delta$, $\omega = k^2 + 1$ (for which $D = 1$), as an operator on $D(L^{1/2})$. In particular, an n-independent bound on T_{od} on $H^1(\mathbb{R}^{3(M+n)})$ is obtained from Lemma 3.8 by choosing $s = 1 + \varepsilon$. However, this model is not known to be renormalisable by Nelson's method and it does not satisfy the assumptions of Theorem 1.3. The reason is that, since G does not map into $D(L^{1/2})$, we do not have $D(T) \subset GD(L)$ and $D(H) \subset D(A)$. See [Lam18] for a modification of our method that works for this model.

3.3 Proof of Theorem 1.3 for $D \geq 0$

We are now ready to prove Theorem 1.3 under the assumptions of this section (Condition 1.1,(2)). As in the case of form perturbations treated in Section 2, we rewrite the Hamiltonian $H = L_0^* + gA$ as (cf. Equation (15))

$$H = (1 - G)^* L(1 - G) + T\,.$$

From Lemma 2.7 we already know that $H_0 := (1 - G)^* L(1 - G)$ is self-adjoint on $D(H_0) = D(H) = (1 - G)^{-1} D(L)$. It is thus sufficient to prove that T is symmetric and infinitesimally H_0-bounded on this domain. We will do this, distinguishing two cases.

The case $D = 0$ and $\beta = 2$. In this case, Proposition 3.5 states that T is symmetric on the domain $D(T) = D(L^\varepsilon)$, for any $\varepsilon > 0$. Writing any $\psi \in D(H)$ as $(1 - G)\psi + G\psi$, the first summand is an element of $D(L)$, and the second is in $D(L^\varepsilon)$ by Corollary 3.2. We thus have $D(H) \subset D(L^\varepsilon) = D(T)$ and T is symmetric on $D(H)$.

To prove the relative bound on T, we decompose its action on $D(H)$ as $T = T(1 - G) + TG$. Because G maps $\mathscr{H}$ to the domain of T, the operator TG is bounded on $\mathscr{H}$. To prove that $T(1 - G)$ is relatively bounded by H_0 we simply use Young's inequality as in Equation (16).

The general case. We will now cover the remaining cases, including the Nelson model. Given that D and β are within the bounds defined by Condition 1.1,(2) the condition that either $\beta < 2$ or $D > 0$ is equivalent to $\beta - 2 < D$. We also recall from Equation (18) that Condition 1.1,(2) implies

$$0 \leq D < \frac{2\beta^2}{\beta^2 + 8} \leq \frac{\beta}{2}$$

for the case at hand.

We will now use the flexibility of Proposition 3.5 that gives a family of domains on which T is symmetric, by choosing a parameter $s(\beta, D)$ such that this domain is contained in $D(H)$.

Lemma 3.10. *For any $s > 0$ let $D_s(T) = D((N+1)^{\max(0,1-s)} L^{s-u(s)})$, with $u(s) = \frac{\beta}{2}s - \frac{D}{2}$. If Condition 1.1 is satisfied with $D \geq 0$, there exists $s = s(\beta, D)$, satisfying the conditions of Proposition 3.5, and numbers $\delta_1(\beta, D), \delta_2(\beta, D) \in [0, 1)$ such that*

- *$D(L^{\delta_1}) \subset D_s(T)$, and*
- *G is a continuous operator from $D(N^{\delta_2})$ to $D_s(T)$.*

Proof. For $\beta = 2$, $D = 0$ this was already proved above, so we may restrict to $\beta - 2 < D$. We will find s, depending on β and D, such that the second statement holds. The first claim is then immediate, because

$$(N+1)^{\max(0,1-s)} L^{s-u(s)} \leq \begin{cases} (L+1)^{1-u(s)} & s \leq 1 \\ L^{s-u(s)} & s > 1, \end{cases}$$

and $u(s) > 0$ (by the hypothesis $u(u(s)) > 0$ of Proposition 3.5), as well as $s - u(s) < 1/2$ (this follows from Equation (43) below since $\sigma - u(\sigma) > 0$).

To prove the second claim, recall that, by Proposition 3.1, G maps $\mathscr{H}^{(n)}$ to $D(L^\eta) \cap \mathscr{H}^{(n+1)}$, for an appropriate $\eta > 0$ and any $n \in \mathbb{N}$. For G to map into $D_s(T)$, we need to apply this with $\eta = s - u(s)$. If the hypothesis of Proposition 3.1 are satisfied for some $\sigma \geq 0$, we then obtain the bound

$$\|G\psi\|_{D_s(T)} \leq C \left\| (N+1)^{\frac{\max(0,1-\sigma)}{2} + \max(0,1-s)} \psi \right\|_{\mathscr{H}}.$$

We will now prove the claim by showing that there is a possible choice of $(s, \sigma) \in (0, \infty) \times [0, \infty)$, satisfying the conditions of Proposition 3.5, respectively Proposition 3.1, such that $\delta_2 = \frac{1}{2}\max(0, 1-\sigma) + \max(0, 1-s)$ is less than one.

The parameter σ needs to satisfy the hypothesis of Proposition 3.1 with $\eta = s - u(s)$:

$$u(\sigma) < 1\,, \tag{42}$$

$$s - u(s) + \frac{\sigma - u(\sigma) - 1}{2} < 0\,. \tag{43}$$

For s, the hypothesis of Proposition 3.5 have to hold:

$$u(s) < 1\,, \tag{44}$$

$$u(u(s)) > 0\,. \tag{45}$$

Set for $\beta < 2$

$$S_1 := \frac{2+D}{\beta}\,, \qquad S_2 := \frac{1-\frac{3}{2}D}{2-\beta}\,,$$

and $S_1 = 1 + D/2$, $S_2 = \infty$ for $\beta = 2$. Note that $u(S_1) = 1$ and $S_1 > 1$, because $\beta < D + 2$. Furthermore, using that $D < \frac{2\beta^2}{\beta^2+8}$ and $0 < \beta \leq 2$, we also have that

$$S_2 = \frac{1-\frac{3}{2}D}{2-\beta} > \frac{1-\frac{3\beta^2}{\beta^2+8}}{2-\beta} = \frac{\frac{\beta^2+8-3\beta^2}{\beta^2+8}}{2-\beta} = 2\frac{\frac{4-\beta^2}{\beta^2+8}}{2-\beta} = 2\frac{2+\beta}{\beta^2+8} > \frac{1}{2}\,. \tag{46}$$

We now define a family of pairs $(s_\varepsilon, \sigma_\varepsilon)$ such that they fulfil the conditions (43) – (45) as long as ε is small enough. For any $\varepsilon > 0$, let

$$(s_\varepsilon, \sigma_\varepsilon) := \Big(\min\{S_1, S_2\} - \varepsilon\,,\ \max\,[0, \min\{2(S_2 - S_1), S_1\} - 2\varepsilon]\Big).$$

For ε small enough, we can determine $(s_\varepsilon, \sigma_\varepsilon)$ in all possible cases

$$\begin{aligned}
(s_\varepsilon, \sigma_\varepsilon) &= \begin{cases} (S_1 - \varepsilon, \min\{2(S_2 - S_1), S_1\} - 2\varepsilon) & S_1 < S_2 \\ (S_2 - \varepsilon, 0) & S_1 \geq S_2 \end{cases} \\
&= \begin{cases} (S_1 - \varepsilon, S_1 - 2\varepsilon) & \frac{3}{2}S_1 \leq S_2 \leq \infty \\ (S_1 - \varepsilon, 2(S_2 - S_1) - 2\varepsilon) & S_1 < S_2 < \frac{3}{2}S_1 \\ (S_2 - \varepsilon, 0) & \frac{1}{2} < S_2 \leq S_1\,. \end{cases}
\end{aligned} \tag{47}$$

In the last step we have used (46). Observe that s_ε and σ_ε are always finite, and $s_\varepsilon > 0$, $\sigma_\varepsilon \geq 0$ if ε is small enough.

It is clear from the definition that we have $\sigma_\varepsilon < S_1$. Since u is increasing for $\beta > 0$ and $u(S_1) = 1$, we conclude that $u(\sigma_\varepsilon) < u(S_1) = 1$, and (42) holds. As also $s_\varepsilon < S_1$, this equally shows that (44) is fulfilled.

To check (45), observe that, because $\beta > D$,

$$u(u(S_1)) = u(1) = \frac{\beta}{2} - \frac{D}{2} > 0\,.$$

This shows (45) if $S_1 \leq S_2$ and ε is small enough. If $S_2 < S_1$, then necessarily $\beta < 2$ and, using the hypothesis $D < \frac{2\beta^2}{\beta^2+8}$, we see that

$$u(u(S_2)) = \frac{\beta^2}{4}\frac{1-\frac{3}{2}D}{2-\beta} - \frac{(2+\beta)D}{4} > \frac{\beta^2}{4(2-\beta)}\left(1 - \frac{3\beta^2}{\beta^2+8} - \frac{2(4-\beta^2)}{\beta^2+8}\right) = 0\,.$$

This proves that (45) holds for any sufficiently small ε.

The last condition to prove is (43). By computing $2s_\varepsilon + \sigma_\varepsilon$ in the different cases of Equation (47), we find

$$2s_\varepsilon + \sigma_\varepsilon = \begin{cases} 3S_1 - 4\varepsilon & \frac{3}{2}S_1 \le S_2 \le \infty \\ 2S_2 - 4\varepsilon & S_1 < S_2 < \frac{3}{2}S_1 \\ 2S_2 - 2\varepsilon & \frac{1}{2} < S_2 \le S_1\,. \end{cases}$$

From this we see that $2s_\varepsilon + \sigma_\varepsilon < 2S_2$, and thus

$$\begin{aligned} s_\varepsilon - u(s_\varepsilon) - \frac{1}{2}(1 - \sigma_\varepsilon + u(\sigma_\varepsilon)) &= \frac{1}{2}\left(1 - \frac{\beta}{2}\right)(2s_\varepsilon + \sigma_\varepsilon) + \frac{3D}{4} - \frac{1}{2} \\ &< \frac{1}{2}\left(1 - \frac{3}{2}D\right) + \frac{3}{4}D - \frac{1}{2} = 0\,, \end{aligned}$$

which proves (43).

It remains to compute $\delta_2 = \max(0, 1 - s_\varepsilon) + \frac{1}{2}\max(0, 1 - \sigma_\varepsilon)$ and see that $\delta_2 < 1$. Since, for ε small enough, $S_1 - \varepsilon > 1$, we find for the different cases of Equation (47)

$$\delta_2 = \begin{cases} 0 & \frac{3}{2}S_1 \le S_2 \le \infty \\ \frac{1}{2}\max(0, 1 - 2(S_2 - S_1) + 2\varepsilon) & S_1 < S_2 < \frac{3}{2}S_1 \\ \max(0, 1 - S_2 + \varepsilon) + \frac{1}{2} & \frac{1}{2} < S_2 \le S_1\,. \end{cases}$$

In the first case, we are finished. In the second case, $S_2 - S_1 > 0$ and choosing ε smaller than this quantity proves the claim. For the last one, it is sufficient to choose $\varepsilon < S_2 - \frac{1}{2}$, which is positive by (46). This completes the proof. □

This lemma proves that $D(H)$ in $D_s(T)$, because $\psi = (1-G)\psi + G\psi$, with both of these terms in $D_s(T)$ since $D(H) \subset D(N)$ by Lemma 3.3. Since $\delta_1, \delta_2 < 1$, the lemma also implies that $(T, D_s(T))$ is infinitesimally H_0-bounded, because N is H_0-bounded by Equation (17). We have thus proven that H is self-adjoint and bounded from below under the assumptions of Theorem 1.3.

The expression (8), involving the creation operators, for H as an operator from $D(H)$ to the dual of $D(L)$ was already derived in Equation (7). Note that $A\psi \in \mathscr{H}$ for $\psi \in D(H)$, since T maps $(1-G)^{-1}D(L)$ to $\mathscr{H}$, as we have just shown.

3.4 Proof of Theorem 1.4

We will now prove that the operator H, whose self-adjointness was proved in the previous section, is equal to an operator H_∞ constructed by renormalisation.

Let us recall the definition of H_∞. Let, for $\Lambda > 0$, v_Λ be the interaction defined by $\hat{v}_\Lambda(k) = \chi_\Lambda(k)\hat{v}(k)$, where χ_Λ is the characteristic function of a ball with radius Λ. Then let

$$H_\Lambda = L + g\sum_{i=1}^{M} a(v_\Lambda(x_i - y)) + a^*(v_\Lambda(x_i - y)) .$$

Since $v_\Lambda \in L^2(\mathbb{R}^d)$, this operator is self-adjoint on the domain $D(H_\Lambda) = D(L)$. In order to consider the limit of H_Λ as $\Lambda \to \infty$ it is necessary to modify it by adding

$$E_\Lambda := g^2 M \int_{\mathbb{R}^d} \frac{|\hat{v}_\Lambda(k)|^2}{k^2 + \omega(k)} \,\mathrm{d}k .$$

Note that, since we are assuming that the second case of Condition 1.1 holds, the numbers E_Λ diverge as $\Lambda \to \infty$.

It is known that, under appropriate assumptions on $\hat{v}$ and ω, the limit as $\Lambda \to \infty$ of $H_\Lambda + E_\Lambda$ exists (see [GW18, Thm 3.3]).

Theorem ([Nel64, GW18]). *Let Condition 1.1 be satisfied with $D \geq 0$. Then $H_\Lambda + E_\Lambda$ converges in norm resolvent sense as $\Lambda \to \infty$ to an operator $(H_\infty, D(H_\infty))$ that is self-adjoint.*

We will now prove Theorem 1.4, which states that under the same hypothesis $H_\Lambda + E_\Lambda$ converges to H in the strong resolvent sense. This obviously implies $H = H_\infty$. With a more involved analysis one could certainly also prove convergence of the resolvents in norm. However, this seems unnecessary as the main point is to show that $H = H_\infty$, and this already implies norm resolvent convergence by [GW18, Thm 3.3].

In the following proof, an important role will be played by G and its regularised variant $G_\Lambda := -gL^{-1}a^*(V_\Lambda)$. The operators $(1 - G_\Lambda)$ are somewhat analogous to the Gross transformation U_Λ that is used in the renormalisation procedure. This is a family of unitary operators on $\mathscr{H}$ with the property that $H_\Lambda + E_\Lambda = U_\Lambda^*(L + R_\Lambda)U_\Lambda$, with operators R_Λ that have a limit as $\Lambda \to \infty$, in the sense of quadratic forms on $D(L^{1/2})$. The limit $\lim_{\Lambda\to\infty} U_\Lambda =: U_\infty$ also exists and one has

$$H_\infty = U_\infty^*(L \dotplus B_\infty)U_\infty ,$$

where $L \dotplus B_\infty$ denotes the self-adjoint operator defined by the sum of the quadratic forms. This implies that $D(|H_\infty|^{1/2}) = U_\infty^* D(L^{1/2})$. However, for an explicit characterisation of $D(H_\infty)$ one would need to know the domain of $L \dotplus B_\infty$ and an explicit description of the action of U_∞ on this domain. On the other hand, using the operators G_Λ and $G_\infty = G$, we will find that

$$H_\Lambda + E_\Lambda = (1 - G_\Lambda)^* L (1 - G_\Lambda) + T_\Lambda + E_\Lambda .$$

The operators $T_\Lambda + E_\Lambda$ will converge as $\Lambda \to \infty$ to T (strongly as operators $D(T) \to \mathscr{H}$). We have shown, in Section 3.3, that T is a perturbation of $(1-G)^*L(1-G)$ in the sense of operators, and thus $D(H) = (1-G)^{-1}D(L)$.

While these procedures look rather similar, there are some notable differences. The Gross transformation is constructed as a Weyl operator from the one-particle function $\hat{v}_\Lambda(k)/(k^2+\omega(k))$, it is unitary and maps the form domain of H_Λ, respectively H_∞, to $D(L^{1/2})$. On the other hand, the operator $1-G$ uses the resolvent of the multi-particle operator L and it is invertible, but not unitary. Like U_∞, this operator maps $D(|H|^{1/2})$ to $D(L^{1/2})$ (see also (53)), but additionally also $D(H)$ to $D(L)$. The action of $(1-G)$ on a generic element of $\mathscr{H}$ is also somewhat easier to analyse. This is because $((1-G)\psi)^{(n)}$ depends only on $\psi^{(n)}$ and $\psi^{(n-1)}$, whereas $(U_\infty \psi)^{(n)}$ will depend on all of the $\psi^{(j)}$, $j \in \mathbb{N}$.

Proof of Theorem 1.4. Let $a(V_\Lambda) = \sum_{i=1}^M a(v_\Lambda(x_i - y))$, define $G_\Lambda = -gL^{-1}a^*(V_\Lambda)$, and

$$T_\Lambda := -G_\Lambda^* L G_\Lambda = -g^2 a(V_\Lambda)L^{-1}a(V_\Lambda)^* \,.$$

Since $v_\Lambda \in L^2$ for $\Lambda < \infty$ and $L \geq N$, one easily sees that G_Λ and T_Λ are bounded operators on $\mathscr{H}$. We then have

$$\begin{aligned}
(1-G_\Lambda)^*L(1-G_\Lambda) + T_\Lambda &= L - G_\Lambda^* L - LG_\Lambda + G_\Lambda^* L G_\Lambda + T_\Lambda \\
&= L + g\left(a(V_\Lambda) + a^*(V_\Lambda)\right) \\
&= H_\Lambda \,.
\end{aligned}$$

Using this representation, we calculate the difference of resolvents

$$\begin{aligned}
&(H+\mathrm{i})^{-1} - (H_\Lambda + E_\Lambda + \mathrm{i})^{-1} \\
&\quad = (H+\mathrm{i})^{-1}\Big(H_\Lambda + E_\Lambda - H\Big)(H_\Lambda + E_\Lambda + \mathrm{i})^{-1} \\
&\quad = (H+\mathrm{i})^{-1}\Big((1-G)^*L(G-G_\Lambda)\Big)(H_\Lambda + E_\Lambda + \mathrm{i})^{-1} && (48) \\
&\qquad + (H+\mathrm{i})^{-1}\Big((G^* - G_\Lambda^*)L(1-G_\Lambda)\Big)(H_\Lambda + E_\Lambda + \mathrm{i})^{-1} && (49) \\
&\qquad + (H+\mathrm{i})^{-1}\Big(T_\Lambda + E_\Lambda - T\Big)(H_\Lambda + E_\Lambda + \mathrm{i})^{-1} \,. && (50)
\end{aligned}$$

We need to prove that this converges to zero, strongly on $\mathscr{H}$.

Consider first

$$G - G_\Lambda = gL^{-1}\left(a^*(V_\Lambda) - a^*(V)\right) = gL^{-1}\left(\sum_{i=1}^M a^*\Big((v_\Lambda - v)(x_i - y)\Big)\right).$$

Following the proof of Proposition 3.1, with $\hat{v}$ replaced by $\hat{v}(\chi_\Lambda - 1)$, one easily sees that this converges to zero, since integrals such as (25) tend to zero with the modified interaction. This proves the convergence of the term (48), because T is H-bounded, as shown in Section 3.3, and thus $(H+\mathrm{i})^{-1}(1-G)^*L$ is bounded. The proof of this statement, with v replaced by v_Λ can also be used to show that $T_\Lambda + E_\Lambda$ is bounded relative to $(1-G_\Lambda)^*L(1-G_\Lambda)$ with constants independent of Λ, because all of the estimates are given by certain integrals of $\hat{v}_\Lambda$ that are bounded by the integral with $\hat{v}$ (see also the discussion of T_Λ below). This implies that $L(1-G_\Lambda)(H_\Lambda + E_\Lambda + \mathrm{i})^{-1}$ is bounded uniformly in Λ and gives the desired result for (49).

We now turn to $T_\Lambda + E_\Lambda = T_{\mathrm{d},\Lambda} + E_\Lambda + T_{\mathrm{od},\Lambda}$, with $T_{\mathrm{d},\Lambda}$, $T_{\mathrm{od},\Lambda}$ defined in analogy with T_d, T_od (see Equations (30), (31)). In Fourier representation the action of $T_{\mathrm{d},\Lambda} + E_\Lambda$ is just multiplication by the function

$$-g^2 \sum_{\ell=1}^{M} \int_{|k_{n+1}|<\Lambda} |\hat{v}(k_{n+1})|^2 \left(\frac{1}{L(P - e_\ell k_{n+1}, K)} - \frac{1}{k_{n+1}^2 + \omega(k_{n+1})} \right) \mathrm{d}k_{n+1} \,.$$

As $\Lambda \to \infty$ this converges to the function defining T_d, given in (29), pointwise. Using the bound of Lemma 3.6 one then sees that $T_{\mathrm{d},\Lambda} + E_\Lambda \to T_\mathrm{d}$ in the strong topology of operators from $D(L^{\max(\varepsilon, D/2)})$ to $\mathscr{H}$.

Concerning $T_{\mathrm{od},\Lambda}$, we spell out the action of $ga(V_\Lambda)G_\Lambda$ in the same way as in (28) and decompose as in (31) to arrive at

$$T_{\mathrm{od},\Lambda} - T_\mathrm{od} := -g^2 \sum_{\ell=1}^{M} \sum_{i=1, i\neq\ell}^{M} (\theta_{i\ell,\Lambda} - \theta_{i\ell}) - g^2 \sum_{\ell=1}^{M} \sum_{i=1}^{M} (\tau_{i\ell,\Lambda} - \tau_{i\ell}) \,.$$

Explicitly, we have

$$\begin{aligned} &(\theta_{i\ell,\Lambda} - \theta_{i\ell})\hat{\varphi}^{(n)}(P, \hat{K}_{n+1}) \\ &\qquad = \int_{\mathbb{R}^d} \frac{(\chi_\Lambda(k_{n+1}) - 1)\, |\hat{v}(k_{n+1})|^2 \, \hat{\psi}^{(n)}(P + (e_i - e_\ell)k_{n+1}, \hat{K}_{n+1})}{L(P - e_\ell k_{n+1}, K)} \mathrm{d}k_{n+1} \,, \end{aligned} \tag{51}$$

and

$$\begin{aligned} (\tau_{i\ell,\Lambda} - \tau_{i\ell})\hat{\varphi}^{(n)}(P, \hat{K}_{n+1}) = \sum_{j=1}^{n} \int_{\mathbb{R}^d} & \frac{\overline{\hat{v}(k_j)}\hat{v}(k_{n+1})\hat{\psi}^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)}{L(P - e_\ell k_{n+1}, K)} \\ & \times (\chi_\Lambda(k_j)\chi_\Lambda(k_{n+1}) - 1)\ \mathrm{d}k_{n+1} \,. \end{aligned} \tag{52}$$

With the expression (51) at hand, going through the proof of Lemma 3.7 shows that $\sum_{\ell=1}^{M} \sum_{i=1, i\neq\ell}^{M} (\theta_{i\ell,\Lambda} - \theta_{i\ell})$ converges to zero strongly as an operator from $D(L^{D/2})$ to $\mathscr{H}$. To show the analogue for the τ-terms, one first inserts the equality

$$\chi_\Lambda(k_j)\chi_\Lambda(k_{n+1}) - 1 = \chi_\Lambda(k_j)(\chi_\Lambda(k_{n+1}) - 1) + (\chi_\Lambda(k_j) - 1)$$

into (52). Then, one observes that at least one of the the integrals in k_j or k_{n+1} performed in the proof of Lemma 3.8 converges to zero. This implies that (52) converges to zero strongly as an operator from $D(N^{\max(0,1-s)}L^{s-u(s)})$ to $\mathscr{H}$.

To summarise, we have found that $T_\Lambda+E_\Lambda-T$ tends to zero strongly as an operator from $D(T)$ to $\mathscr{H}$, for any domain $D(T)$ that can be chosen in Proposition 3.5. Combining this with the fact that T is bounded relative to H implies that for any $\psi \in \mathscr{H}$

$$\lim_{\Lambda\to\infty} \left\| (T_\Lambda + E_\Lambda - T)\,(H-\mathrm{i})^{-1}\psi \right\|_{\mathscr{H}} = 0\,.$$

This shows convergence of (50) and completes the proof. □

4 Regularity of domain vectors

In this section we will discuss the regularity of vectors in $D(H)$. These results apply both to the case of form perturbations of Section 2 and the renormalisable models treated in Section 3. Due to the boundary condition $(1-G)\psi \in D(L)$, a vector $\psi \in D(H)$ is exactly as regular as $G\psi = \psi - (1-G)\psi$ is. The same reasoning also applies on the form domain of H. Since we proved in Sections 2.3 and 3.3 that H is a perturbation of $H_0 = (1-G)^*L(1-G)$, the quadratic form of H is a perturbation of that of H_0 and its domain is

$$D(|H|^{1/2}) = (1-G)^{-1}D(L^{1/2}) \subset D(N^{1/2})\,. \tag{53}$$

This domain is characterised by the abstract boundary condition $\psi - G\psi \in D(L^{1/2})$, which is non-trivial if $G\psi \notin D(L^{1/2})$, i.e. for the models treated in Section 3. In this case, $\psi \in D(|H|^{1/2})$ has the same regularity (with respect to L) as $G\psi$.

We will prove sharp results on the regularity of $G\psi$ below. Together, these will imply the Corollary 1.5 stated in the introduction.

Proposition 3.1 establishes that if $|\hat{v}(k)| \leq |k|^{-\alpha}$, then the vectors in the domain of the operator H with interaction v have the regularity of those in $D(L^\eta)$ for all $\eta < \frac{2-D}{4} = 1 - \frac{d-2\alpha}{4}$. Note that if $\int \frac{|\hat{v}(k)|^2}{k^2+\omega(k)}\mathrm{d}k < \infty$ Condition 1.1 implies that we are in the case of form perturbations with $D<0$ treated in Section 2 and the following corollary holds for some $\eta > 1/2$.

Corollary 4.1. *Let the conditions of Theorem 1.3 be satisfied. Then for every* $0 \leq \eta < \frac{2-D}{4}$ *we have*

$$D(H) \subset D(L^\eta) \quad \textit{and} \quad D(|H|^{1/2}) \subset D(L^{\min(\eta,1/2)}).$$

Proof. Let $\psi \in D(H)$, respectively $\psi \in D(|H|^{1/2})$. To show that $G\psi \in D(L^\eta)$ we can apply Proposition 3.1 with $s=0$, since $\eta - \frac{2-D}{4} < 0$. This yields

$$\|L^\eta G\psi\|_{\mathscr{H}} \leq C\|\sqrt{N+1}\psi\|_{\mathscr{H}}\,.$$

Together with the fact that $D(|H|^{1/2}) \subset D(N^{1/2})$ this implies that $G\psi \in D(L^\eta)$. $\square$

For the Fröhlich model this means that $D(H) \subset D(L^\eta)$ for $\eta < 3/4$. For the Nelson model as well as our model for point-particles in two dimensions with $v = \delta$ we have $D(|H|^{1/2}) \subset D(L^\eta)$ for $\eta < 1/2$.

We will now show that these results are sharp, in the sense that $D(H) \cap D(L^\eta) = \{0\}$ for all larger η. The intuition behind this is that the (worst) singularities of $G\psi$ behave exactly like those of $(-\Delta + \omega(-\mathrm{i}\nabla))^{-1} v(x_\ell - y_j)$. Similar results for $M = 1$ were also proved in [GW16] and [GW18] using the Gross transform.

Proposition 4.2. *Assume the hypothesis of Theorem 1.3 hold and additionally that* $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. *Let* $0 < \eta < 1$ *be such that* $\int \frac{|\hat{v}(k)|^2}{(k^2+\omega(k))^{2(1-\eta)}} \mathrm{d}k = \infty$, *then*

$$D(H) \cap D(L^\eta) = \{0\},$$

and if $\eta \leq 1/2$ *we also have*

$$D(|H|^{1/2}) \cap D(L^\eta) = \{0\}.$$

Proof. We will show that G maps no $0 \neq \psi \in \mathscr{H}$ into $D(L^\eta)$, which implies our claim as discussed above.

Let $n \in \mathbb{N}$ be such that $\psi^{(n)} \neq 0$ and recall that

$$\widehat{G\psi^{(n)}}(P, K) = \frac{-g}{\sqrt{n+1}} \sum_{i=1}^{M} \sum_{j=1}^{n+1} L^{-1}(P, K) \hat{v}(k_j) \hat{\psi}^{(n)}(P + e_i k_j, \hat{K}_j).$$

Let $U \subset \mathbb{R}^{Md} \times \mathbb{R}^{(n+1)d}$ be the set

$$U = \{(P, K) : |p_j| < R \text{ and } |k_j| < R \text{ for all } j > 1\} = \mathbb{R}^d \times B_R(0)^{M-1} \times \mathbb{R}^d \times B_R(0)^n,$$

where $R > 0$ is a parameter, to be chosen later. We will prove that

$$\int_U \left| L^\eta \widehat{G\psi^{(n)}}(P, K) \right|^2 \mathrm{d}P \mathrm{d}K = \infty,$$

which implies that $G\psi^{(n)} \notin D(L^\eta)$. We first use that $(a+b)^2 \geq \frac{1}{2} a^2 - b^2$ and the Cauchy-Schwarz inequality to obtain the lower bound

$$\left| L^\eta \widehat{G\psi^{(n)}}(P, K) \right|^2 \geq \frac{g^2}{2(n+1)} \frac{|\hat{v}(k_1)|^2 \left| \hat{\psi}^{(n)}(P + e_1 k_1, \hat{K}_1) \right|^2}{L(P, K)^{2-2\eta}} \tag{54}$$

$$- g^2 M \sum_{(i,j) \neq (1,1)} \frac{|\hat{v}(k_j)|^2 \left| \hat{\psi}^{(n)}(P + e_i k_j, \hat{K}_j) \right|^2}{L(P, K)^{2-2\eta}}. \tag{55}$$

We will see that the terms of the second line have a finite integral over U, while the integral of the first is infinite. In the sum over $(i,j) \neq (1,1)$ in Equation (55) consider a term with $i=1, j>1$. By the change of variables $p_1 \mapsto q = p_1 - k_j$ (note that the domain of integration for p_1 is $\mathbb{R}^d$) we obtain the bound

$$\int\limits_U \frac{|\hat{v}(k_j)|^2 \left|\hat{\psi}^{(n)}(P+e_1k_j,\hat{K}_j)\right|^2}{L(P,K)^{2-2\eta}} \mathrm{d}P\mathrm{d}K \leq \left\|\psi^{(n)}\right\|^2 \sup_{q\in\mathbb{R}^d} \int\limits_{|k|<R} \frac{|\hat{v}(k)|^2}{((q-k)^2+1)^{2-2\eta}} \mathrm{d}k\,.$$

This is finite since $\hat{v} \in L^2_{\mathrm{loc}}$. The terms with $i,j>1$ can be bounded by enlarging the domain of integration in the variable p_i to $\mathbb{R}^d$ and proceeding as for $i=1$. The terms with $j=1, i>1$ are estimated in the same way, where the change of variables is performed in k_1 and the remaining integral is then over p_i.

To show that the integral over the term (54) is infinite, we perform a change of variables $p_1 \to p_1 - k_1$. Then we restrict the domain of integration to $\{|p_1|<R\} \cap U$ to bound it from below by

$$\int\limits_{B_R(0)^{M+n}} \left|\hat{\psi}^{(n)}(P,\hat{K}_1)\right|^2 \int\limits_{\mathbb{R}^d} \frac{|\hat{v}(k_1)|^2}{L(p_1-k_1,P,K)^{2\eta-2}} \mathrm{d}k_1\mathrm{d}P\mathrm{d}\hat{K}_1\,. \tag{56}$$

Since we have restricted to $(P,\hat{K}_1) \in B_R(0)^{M+n}$ and assumed that $\omega \in L^\infty_{\mathrm{loc}}$, it holds that $P^2 + \Omega(\hat{K}_1) \leq C$ for some $C>0$ that depends on R. Because, in particular $|p_1|<R$ and $1 \leq \omega(k_1)$, we can then estimate

$$L(p_1-k_1,\hat{P}_1,K) \leq (k_1-p_1)^2 + \omega(k_1) + C \leq C'(k_1^2+\omega(k_1))\,,$$

for some $C'>0$. Hence the integral (56) is bounded from below by some constant times

$$\int\limits_{B_R(0)^{M+n}} \left|\hat{\psi}^{(n)}(P,\hat{K}_1)\right|^2 \mathrm{d}P\mathrm{d}\hat{K}_1 \int_{\mathbb{R}^d} \frac{|\hat{v}(k_1)|^2}{(k_1^2+\omega(k_1))^{2\eta-2}} \mathrm{d}k_1\,.$$

Because $\psi^{(n)} \neq 0$, we can choose an $R>0$ such that

$$\int\limits_{B_R(0)^{M+n}} \left|\hat{\psi}^{(n)}(P,\hat{K}_1)\right|^2 \mathrm{d}P\mathrm{d}K > 0\,.$$

But since the integral in k_1 is infinite by hypothesis we have proved the claim. □

Acknowledgments. We thank Stefan Keppeler, Stefan Teufel and Roderich Tumulka for helpful discussions. J.S. was supported by the German Research Foundation (DFG) within the Research Training Group 1838 *Spectral Theory and Dynamics of Quantum Systems* and thanks the Laboratoire Interdisciplinaire Carnot de Bourgogne for its hospitality during his stay in Dijon.

A Appendix

Lemma A.1. *Let $v \in \mathscr{S}'(\mathbb{R}^d)$ with $\hat{v} \in L^\infty(\mathbb{R}^d) + L^2(\mathbb{R}^d)$. If $v \notin L^2(\mathbb{R}^d)$, then for any $M \geq 2$ the set of $X = (x_1, \ldots, x_M)$ such that*

$$V(X, y) = \sum_{j=1}^{M} v(x_j - y)$$

is an element of $L^2(\mathbb{R}^d, \mathrm{d}y)$ has Lebesgue measure zero in $\mathbb{R}^{Md}$.

Proof. Assume to the contrary that the set where $V(X, y) \in L^2(\mathbb{R}^d, \mathrm{d}y)$ has positive measure. This set is the union over all $R > 0$ of the sets

$$U_R := \{X \in \mathbb{R}^{Md} : |X| < R \text{ and } \|V(X, y)\|_{L^2(\mathbb{R}^d)} < R\},$$

and thus U_R has positive (and finite) measure $|U_R| > 0$ for some $R > 0$. Integrating over this set, we see that $\int_{U_R} V(X, x_1 - y)\mathrm{d}X \in L^2(\mathbb{R}^d)$, since

$$\left\| \int_{U_R} V(X, x_1 - y)\mathrm{d}X \right\|_{L^2(\mathbb{R}^d)} \leq \int_{U_R} \|V(X, y)\|_{L^2(\mathbb{R}^d)} \, \mathrm{d}X < R \, |U_R| \, .$$

On the other hand, denoting by χ_{U_R} the characteristic function of U_R, we have

$$\begin{aligned} &\int_{U_R} V(X, x_1 - y)\mathrm{d}X \\ &\quad = |U_R| \, v(y) + \sum_{j=2}^{M} \int_{U_R} v(x_j - x_1 + y)\mathrm{d}X \\ &\quad = |U_R| \, v(y) + \sum_{j=2}^{M} \int_{\mathbb{R}^d} v(y - x_1) \int_{\mathbb{R}^{(M-1)d}} \chi_{U_R}(X + e_1 x_j)\mathrm{d}X \, . \end{aligned}$$

Since U_R has finite measure, the functions

$$f_j(x) = \int_{\mathbb{R}^{(M-1)d}} \chi_{U_R}(X + e_1 x_j)\mathrm{d}x_2 \cdots \mathrm{d}x_M$$

are in $L^1 \cap L^2(\mathbb{R}^d)$. Since $\hat{v} \in L^2 + L^\infty$, the convolution $v * f_j$ is then in $L^2(\mathbb{R}^d)$. But this implies that

$$|U_R| \, v(y) = \int_{U_R} V(X, x_1 - y)\mathrm{d}X - \sum_{j=2}^{M} (v * f_j)(y) \in L^2(\mathbb{R}^d) \, ,$$

a contradiction. □

Lemma A.2. *Let $p \in \mathbb{R}^3$ and $\theta \in (1,3)$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}q}{((p-q)^2+1)\,|q|^\theta} \le \frac{C}{|p|^{\theta-1}}\,. \tag{57}$$

Let $p \in \mathbb{R}^2$ and $\theta \in \{1,2\}$. Then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}q}{((p-q)^2+1)(q^2+1)^{\frac{\theta}{2}}} \le \frac{C}{|p|^\theta}(\log(1+|p|)+1)\,. \tag{58}$$

Proof. For $d = 3$, we will use spherical coordinates and write p instead of $|p|$ when it is clear what is intended. Let $R > 0$ be any positive number. For $p \ge R$ we have:

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\mathrm{d}q}{((p-q)^2+1)\,|q|^\theta} &= 2\pi \int_0^\infty \frac{1}{r^{\theta-2}} \int_{-1}^1 \frac{1}{r^2-2rps+p^2+1}\,\mathrm{d}s\mathrm{d}r \\ &= \frac{\pi}{p}\int_0^\infty \frac{1}{r^{\theta-1}} \log\left(\frac{r^2+2rp+p^2+1}{r^2-2rp+p^2+1}\right)\mathrm{d}r\end{aligned}$$

We perform a change of variables $r \to \frac{r}{p} =: t$ which yields

$$\frac{\pi}{p^{\theta-1}}\int_0^\infty \frac{1}{t^{\theta-1}} \log\left(\frac{(t+1)^2+p^{-2}}{(t-1)^2+p^{-2}}\right)\mathrm{d}t\,.$$

Now we split the domain of integration into three parts. For $t \in [0,\frac{1}{2}]$ we will use that

$$\frac{(t+1)^2+p^{-2}}{(t-1)^2+p^{-2}} = 1 + \frac{4t}{(t-1)^2+p^{-2}}\,,$$

which implies

$$\log\left(\frac{(t+1)^2+p^{-2}}{(t-1)^2+p^{-2}}\right) \le \frac{4t}{(t-1)^2+p^{-2}}\,. \tag{59}$$

So we estimate the integral there using $\theta < 3$ by

$$\frac{4\pi}{p^{\theta-1}}\int_0^{\frac{1}{2}} \frac{1}{t^{\theta-2}}\frac{1}{(t-1)^2+p^{-2}}\,\mathrm{d}t \le \frac{4\pi}{p^{\theta-1}}\int_0^{\frac{1}{2}} \frac{1}{t^{\theta-2}}\frac{1}{(t-1)^2}\,\mathrm{d}t \le \frac{C}{p^{\theta-1}}\,.$$

We can do the exact same thing for $t \in [\frac{3}{2},\infty)$ and obtain the same bound. It remains to deal with the integrable singularity at $t = 1$:

$$\begin{aligned}&\frac{\pi}{p^{\theta-1}}\int_{\frac{1}{2}}^{\frac{3}{2}} \log\left(\frac{(t+1)^2+p^{-2}}{(t-1)^2+p^{-2}}\right)\mathrm{d}t \\ &\qquad \le \frac{\pi}{p^{\theta-1}}\int_{\frac{1}{2}}^{\frac{3}{2}} \log\Big((t+1)^2+R^{-2}\Big)\,\mathrm{d}t - \frac{\pi}{p^{\theta-1}}\int_{\frac{1}{2}}^{\frac{3}{2}} \log\Big((t-1)^2\Big)\,\mathrm{d}t \le \frac{C}{p^{\theta-1}}\,.\end{aligned}$$

If however $p < R$, then we simply estimate the integral by a constant. Since R was arbitrary, this yields the claim in the case $d = 3$.

If $d = 2$, we first observe that for $a \in (0,1)$ it holds that

$$\frac{\partial}{\partial x} \arctan\left(\sqrt{\frac{a+1}{a-1}} \tan(x)\right) = \frac{\sqrt{a^2-1}}{1 - a\cos(2x)} \,. \tag{60}$$

We will use this to integrate in the angular variable:

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{\mathrm{d}q}{((p-q)^2+1)(q^2+1)^{\frac{\theta}{2}}} &= \int_0^\infty \frac{r}{(r^2+1)^{\frac{\theta}{2}}} \int_{-\pi}^{\pi} \frac{1}{r^2 - 2rp\cos(s) + p^2 + 1} \,\mathrm{d}r\mathrm{d}s \\ &\leq \int_0^\infty \frac{1}{(r^2+1)^{\frac{\theta-1}{2}}(r^2+p^2+1)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1 - \frac{2rp}{r^2+p^2+1}\cos(2\xi)} \,\mathrm{d}r\mathrm{d}\xi\end{aligned}$$

Now set $a = \frac{2rp}{r^2+p^2+1}$ and use (60) to obtain

$$\begin{aligned}&\int_0^\infty \frac{a}{2rp(r^2+1)^{\frac{\theta-1}{2}}} \left[\frac{1}{\sqrt{a^2-1}} \arctan\left(\sqrt{\frac{a+1}{a-1}}\tan(\xi)\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}r \\ &= \int_0^\infty \frac{\pi}{(r^2+1)^{\frac{\theta-1}{2}}((r-p)^2+1)^{\frac{1}{2}}((r+p)^2+1)^{\frac{1}{2}}} \,\mathrm{d}r \,.\end{aligned}$$

We then perform a change of variables $r \to \frac{r}{p} =: x$. This yields

$$\frac{\pi}{p^\theta} \int_0^\infty \frac{1}{(x^2+p^{-2})^{\frac{\theta-1}{2}}((x-1)^2+p^{-2})^{\frac{1}{2}}((x+1)^2+p^{-2})^{\frac{1}{2}}} \,\mathrm{d}x \,.$$

For $\theta = 1$ the integral has one singularity at $x = 1$. For $\theta = 2$ there are two singularities remaining, one at zero and another one at $x = 1$. For that reason we split the integral at $r = \frac{1}{2}$ and $r = \frac{3}{2}$ (as in the case $d = 3$ above). The integral from $r = \frac{3}{2}$ to infinity is finite and bounded independent of p. For the other two terms we use the fact that $\operatorname{arsinh}(x)' = (x^2+1)^{-1/2}$ and conclude that

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+p^{-2})^{\frac{\theta-1}{2}}((x-1)^2+p^{-2})^{\frac{1}{2}}((x+1)^2+p^{-2})^{\frac{1}{2}}} \leq C_1 \operatorname{arsinh}\left(\frac{|p|}{2}\right) + C_2 \,,$$

for some constants $C_1, C_2 > 0$. Choose $C = \pi \max(C_1, C_2)$ and note that $\operatorname{arsinh}(x) = \log(x + \sqrt{x^2+1}) \leq \log(2x+1)$ for non-negative x. This yields the claim. □

References

[AF14] Z. Ammari, and M. Falconi, *Wigner Measures Approach to the Classical Limit of the Nelson Model: Convergence of Dynamics and Ground State Energy*, J. Stat. Phys. **157** (2014), no. 2, 330–362.

[AGHKH88] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Texts and Monographs in Physics, Springer, 1988.

[BFK+17] J. Behrndt, R. L. Frank, C. Kühn, V. Lotoreichik, and J. Rohleder, *Spectral theory for Schrödinger operators with δ-interactions supported on curves in $\mathbb{R}^3$*, Ann. Henri Poincaré **18** (2017), no. 4, 1305–1347.

[BT17] G. A. Bley, and L. E. Thomas, *Estimates on functional integrals of quantum mechanics and non-relativistic quantum field theory*, Comm. Math. Phys. **350** (2017), no. 1, 79–103.

[CDF+15] M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli, and A. Teta, *A class of Hamiltonians for a three-particle fermionic system at unitarity*, Math. Phys. Anal. Geom. **18** (2015), no. 1, 32.

[DFT94] G. Dell'Antonio, R. Figari, and A. Teta, *Hamiltonians for systems of N particles interacting through point interactions*, Ann. Inst. H. Poincaré Phys. Théor. **60** (1994), no. 3, 253–290.

[DR04] J. Dimock and S. G. Rajeev, *Multi-particle Schrödinger operators with point interactions in the plane*, J. Phys. A: Math. Gen. **37** (2004), no. 39, 9157.

[GHL14] M. Gubinelli, F. Hiroshima, and J. Lőrinczi, *Ultraviolet renormalization of the Nelson Hamiltonian through functional integration*, J. Funct. Anal. **267** (2014), no. 9, 3125–3153.

[GL18] M. Griesemer and U. Linden, *Stability of the two-dimensional Fermi polaron*, Lett. Math. Phys. **108** (2018), no. 8, 1837–1849.

[GW16] M. Griesemer and A. Wünsch, *Self-adjointness and domain of the Fröhlich Hamiltonian*, J. Math. Phys. **57** (2016), no. 2, 021902.

[GW18] M. Griesemer and A. Wünsch, *On the domain of the Nelson Hamiltonian*, J. Math. Phys. **59** (2018), no. 4, 042111.

[KS95] A. Kiselev and B. Simon, *Rank one perturbations with infinitesimal coupling*, J. Funct. Anal. **130** (1995), no. 2, 345–356.

[KS16] S. Keppeler and M. Sieber, *Particle creation and annihilation at interior boundaries: one-dimensional models*, J. Phys. A: Math. Gen. **49** (2016), no. 12, 125204.

[Lam18] J. Lampart, *A nonrelativistic quantum field theory with point interactions in three dimensions*, arXiv preprint arXiv:1804.08295 (2018).

[LSTT18] J. Lampart, J. Schmidt, S. Teufel, and R. Tumulka, *Particle creation at a point source by means of interior-boundary conditions*, Math. Phys. Anal. Geom. **21** (2018), no. 2, 12.

[Miy18] T. Miyao, *On the semigroup generated by the renormalized Nelson Hamiltonian*, arXiv preprint arXiv:1803.08659 (2018).

[MM17] O. Matte, and J. S. Møller, *Feynman-Kac formulas for the ultra-violet renormalized Nelson model*, arXiv preprint arXiv:1701.02600 (2017).

[MO17] A. Michelangeli and A. Ottolini, *On point interactions realised as Ter-Martyrosyan–Skornyakov operators*, Rep. Math. Phys. **79** (2017), no. 2, 215–260.

[MS17] T. Moser and R. Seiringer, *Stability of a fermionic N+1 particle system with point interactions*, Comm. Math. Phys. **356** (2017), no. 1, 329–355.

[MS18] T. Moser and R. Seiringer, *Stability of the 2 + 2 fermionic system with point interactions*, Math. Phys. Anal. Geom. **21** (2018), no. 3, 19.

[Nel64] E. Nelson, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), no. 9, 1190–1197.

[Pos08] A. Posilicano, *Self-adjoint extensions of restrictions*, Oper. Matrices **2** (2008), no. 4, 483–506.

[RS75] M. Reed and B. Simon, *Methods of modern mathematical physics:II Fourier analysis, self-adjointness*, Academic Press, 1975.

[Tho84] L. E. Thomas, *Multiparticle Schrödinger Hamiltonians with point interactions*, Phys. Rev. D **30** (1984), 1233–1237.

[TT15] S. Teufel and R. Tumulka, *New type of Hamiltonians without ultraviolet divergence for quantum field theories*, arXiv preprint arXiv:1505.04847 (2015).

[TT16] S. Teufel and R. Tumulka, *Avoiding ultraviolet divergence by means of interior–boundary conditions*, Quantum Mathematical Physics (F. Finster, J. Kleiner, C. Röken, and J. Tolksdorf, eds.), Birkhäuser, 2016, 293–311.

[Yaf92] D. R. Yafaev, *On a zero-range interaction of a quantum particle with the vacuum*, J. Phys. A: Math. Gen. **25** (1992), no. 4, 963.

b) Submitted manuscripts

3) Article [Sch18]

On a Direct Description of Pseudorelativistic Nelson Hamiltonians

Julian Schmidt *

Abstract interior-boundary conditions (IBC's) allow for the direct description of the domain and the action of Hamiltonians for a certain class of ultraviolet-divergent models in Quantum Field Theory. The method was recently applied to models where nonrelativistic scalar particles are linearly coupled to a quantised field, the best known of which is the Nelson model. Here, we extend the IBC method to pseudorelativistic scalar particles that interact with a real bosonic field. We construct the Hamiltonians for such models via abstract boundary conditions, describing their action explicitly. In addition, we obtain a detailed characterisation of their domain and make the connection to renormalisation techniques. As an example, we apply the method to two relativistic variants of Nelson's model, which have been renormalised for the first time by J. P. Eckmann and A. D. Sloan in 1970 and 1974, respectively.

MSC: 81T10, 81Q10, 81T16. Keywords: Mathematical Quantum Field Theory, Nelson model, Relativistic Polaron, Yukawa interaction, Renormalisation, Interior-boundary conditions.

1 Introduction

In the recent article [LS18], J. Lampart together with the author used abstract boundary conditions to characterise the domain and the action of certain otherwise ultraviolet-divergent Hamiltonians. Those Hamiltonians describe models where nonrelativistic scalar particles (often called nucleons) are linearly coupled to a field of

*Fachbereich Mathematik, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. juls@maphy.uni-tuebingen.de

massive scalar bosons, the most prominent of which is the so called Nelson model ([Nel64]). To characterise the domains and to set up the Hamiltonians, an abstract variant of interior-boundary conditions (IBC's) was used. These conditions relate the wave functions of different sectors of Fock space. The IBC method allows for the direct description of the Nelson Hamiltonian H_∞ without cutoff: no renormalisation procedure is needed. In this article, we will extend the method to also treat variants of Nelson's model where not only the kinematics of the field but also of the nucleons is relativistic. In such models, instead of a renormalisation constant, an operator valued counter term can arise, as will be explained in the following paragraphs.

The formal Hamiltonian of the original Nelson model is the sum of the free operator of nucleons and field and an interaction term. For one nucleon, the free operator in Fourier representation reads $L = p^2 + \mathrm{d}\Gamma(\omega)$, and acts as a self-adjoint operator on the Hilbert space

$$\mathscr{H} := L^2(\mathbb{R}^d) \otimes \Gamma(L^2(\mathbb{R}^d)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^d) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{dn}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^{(n)} .$$

Here p denotes the momentum of the nucleon and $\mathrm{d}\Gamma(\omega)$ is the second quantisation of the field dispersion $\omega(k) = \sqrt{k^2+1}$, which acts on the bosonic Fock space $\Gamma(L^2(\mathbb{R}^d))$. The sectors $\mathscr{H}^{(n)}$ are equal to $L^2(\mathbb{R}^d) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{dn})$, the subspaces of functions in L^2 that are symmetric under exchange of the k-variables. The interaction term of the Nelson model is formally given by $a(V)+a^*(V)$ where $V : L^2(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ is a (formal) operator (for more details on these generalised creation and annihilation operators see, e.g., [GW16, App. B]). The operator V acts as $(V\psi)(p,k) := v(k)\psi(p+k)$ with $v \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ called the *form factor*. In the Nelson model we have $v = \omega^{-1/2}$. That means that v is not square integrable at infinity and therefore $a^*(V)$ is ill-defined as an operator into $\mathscr{H}$.

The interaction in the Nelson model can be understood to be a coupling of the form $\int \Psi^+(x)(\varphi^+(x) + \varphi^-(x))\Psi^-(x)\,\mathrm{d}x$ where $\Psi^-(x)$ is the nonrelativistic complex scalar nucleon field, $\Psi^+(x)$ its adjoint and

$$\varphi^+(x) + \varphi^-(x) = \int \omega(k)^{-1/2}(\mathrm{e}^{\mathrm{i}k\cdot x}a(k) + \mathrm{e}^{-\mathrm{i}k\cdot x}a^*(k))\,\mathrm{d}k$$

is the real bosonic field operator with form factor $v(k) = \omega(k)^{-1/2} = (k^2+1)^{-1/4}$. In trying to adapt this expression to include nucleons with relativistic kinematics, two different choices have been made:

- Eckmann [Eck70] took $\Psi^\pm(x)$ to be, analogously to $\varphi^\pm(x)$, the annihilation and creation part of a relativistic scalar nucelon field. The nucleons are assumed to have dispersion relation $\Theta(p) = \sqrt{p^2+\mu^2}$, where $\mu \geq 0$ is the nucleon mass. With this choice, the operators $\Psi^\pm(x)$ feature an additional factor $\Theta(p)^{-1/2}$ when compared to the Nelson model. For this interaction operator, the number of nucleons is still conserved, and thus restricting the investigation to a fixed number of nucleons is convenient. For one particle, the interaction in Fourier representation is still of the form $a(V) + a^*(V)$ but the form factor v now becomes the function $v \in L^2_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ given by $v_p(k) = \Theta(p)^{-1/2}\Theta(p+k)^{-1/2}\omega(k)^{-1/2}$. This function is not in $L^2(\mathbb{R}^d_k)$ for any $p \in \mathbb{R}^d$ if $d \geq 2$. The dependence of the form factor on p is one major difference between Eckmann's model and the original Nelson model. However, the form factor of the relativistic model at hand is more regular in k for $\mu > 0$: it holds that $\Theta(p)^{-1/2}\Theta(p+k)^{-1/2} \leq (\mu\,|k|)^{-1/2}$ pointwise on $\mathbb{R}^d \times \mathbb{R}^d$.

- Gross [Gro73] also assumed relativistic kinematics of the form $\Theta(p) = \sqrt{p^2 + \mu^2}$ for the nucleons (resulting in $L = \Theta(p) + \mathrm{d}\Gamma(\omega)$) but kept the operators $\Psi^{\pm}(x)$ as they were in the Nelson model: just the creation and annihilation operators for the nucleons, without any additional factors. This implies that the form factor $v_p(k) = v(k) = \omega(k)^{-1/2}$ is independent of $p \in \mathbb{R}^d$. It is however more singular than the one chosen by Eckmann. For the IBC method to work, one needs at least that $a(V)L^{-1}$ is continuous. Therefore, in this model, one has to restrict to $d = 2$. On the other hand in Gross' model we can treat also the case $\mu = 0$.

Compared to a full Yukawa-type coupling of a complex and a real scalar field, the pair creation and pair annihilation terms have been dropped in both of these models. Models of the above type have been called *polarisation-free Yukawa interaction* ([Alb73]), *spinless Yukawa model* ([DP14]), or as having a *persistent vacuum* ([Fro74, Eck70]). Note that also the interaction of the Pauli-Fierz Hamiltonian is of the form $a(V) + a^*(V)$, when the pair creation and annihilation terms are dropped. In this case however, $v_p(k, \lambda) = e_\lambda(k) \cdot (p+k)\omega(k)^{-1/2}$ is not only singular in k but also even more so in p.

We will later assume that $v_p(k)$ is uniformly bounded by $|k|^{-\alpha}$ for some $\alpha \in [0, d/2)$, as in [LS18]. Such form factors do not exhibit infrared-problems, because they are in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$. There is however an ultraviolet-problem present due to the fact that these form factors are not necessarily square integrable at infinity and thus not in $L^2(\mathbb{R}^d)$. In order to make sense of the Hamiltonian

$$H = \Theta(p) + \mathrm{d}\Gamma(\omega) + a(V) + a^*(V) \tag{1}$$

by using a renormalisation procedure, one multiplies the form factor $v_p(k)$ by a momentum cutoff $\chi_\Lambda(k)$ for some $\Lambda < \infty$ where χ_Λ denotes the characteristic function of the ball of radius Λ in $\mathbb{R}^d$. The resulting operator H_Λ is self-adjoint on the domain of the free operator $L = \Theta(p) + \mathrm{d}\Gamma(\omega)$. Renormalisation amounts to finding a sequence E_Λ such that $H_\Lambda + E_\Lambda$ converges to a self-adjoint operator H_∞ in some generalised sense. Note that if v_p depends on p, then in general also E_Λ does. That is, $E_\Lambda(p)$ is an operator on $L^2(\mathbb{R}^d)$ that effectively alters the dispersion of the nucleons, already for finite Λ. We will refer to it as a (renormalisation) counter term. In 1970, Eckmann showed that the first model can be renormalised in this sense with $H_\Lambda + E_\Lambda$ converging in norm resolvent sense. He used a reordering of the resolvent of H_Λ which is originally due to Hepp [Hep69]. Sloan [Slo74] showed strong resolvent convergence for the model considered by Gross in $d = 2$. Fröhlich investigated the infrared behaviour of both models in [Fro74] and Albeverio [Alb73] worked on scattering theory for Eckmann's model and a related one where E_Λ is replaced by a different operator E'_Λ. In [Wue17], Wünsch, Schach Møller and Griesemer applied Eckmann's method to Gross' model in $d = 2$ in order to show that the domain of the renormalised operator $D(H_\infty)$ is contained in $D(L^\eta)$ for all $0 \leq \eta < 1/2$.

In this paper we will use a different approach based on interior-boundary conditions. They were introduced in [TT15] and it was suggested that they could be used to directly define otherwise UV divergent models of mathematical QFT. Similar boundary conditions relating different sectors of Fock space have been used several times in the past, see e.g. [Mos51], [Tho84] and [Yaf92]. However, they have never been applied to models on the full Fock space until [LSTT17], where a nonrelativistic model in three dimensions with a static source was investigated.

In the present article we will show that the abstract IBC method of [LS18] can be applied to Eckmann's (in $d = 3$) and to Gross' model (in $d = 2$). This will allow for the direct description of H_∞ as a self-adjoint operator on $\mathscr{H}$. The action of H_∞ and the characterisation of its domain $D(H_\infty)$ will be given in terms of abstract boundary conditions. As a Corollary, we will see that $D(|H_\infty|^{1/2}) \subset D(L^\eta)$ for all $\eta \in [0, 1/2)$ but $D(|H_\infty|^{1/2}) \cap D(L^{1/2}) = \{0\}$. In Section 5, we will also sketch the construction for the case of massless bosons in Eckmann's model.

In both models dicussed so far, the counter terms E_Λ diverges for fixed $p \in \mathbb{R}^d$ logarithmically when $\Lambda \to \infty$, exactly as in the original Nelson model. With the method applied in [LS18] and in the present note, slightly more singular interactions can be treated (depending on various parameters and in a way to be made precise below). Recently it was shown in [Lam18] that the IBC approach, if modified in a suitable way, also allows for the definition of a more singular model. In this nonrelativistic model, the divergence of the renormalisation constant is linear in Λ and most importantly, a renormalisation procedure has not been worked out before.

Let us briefly sketch the definition of the Hamiltonian. Under the assumptions we will make on V, Θ and ω, the annihilation operator $a(V)$ is an operator which maps $D(L)$ into the Hilbert space $\mathscr{H}$. This implies that the operator $G := -(a(V)L^{-1})^*$, which maps $\mathscr{H}^{(n)}$ into $\mathscr{H}^{(n+1)}$, is continuous on $\mathscr{H}$. Then we show that $(1 - G)$ is invertible and with its help define the domain of our Hamiltonian $D(H) = \{\psi \in \mathscr{H} | (1 - G)\psi \in D(L)\}$. The condition $(1 - G)\psi \in D(L)$ is the abstract variant of the interior-boundary condition. It states that elements in the domain of H consist of a regular part $(1-G)\psi$ and a singular part $G\psi$ which is completely determined by the wave function one sector below. On $D(H)$ one can define the self-adjoint and non-negative operator $(1-G)^*L(1-G)$. The main task in the construction is to extend the action of the annihilation operator in a suitable way to the domain $D(H)$, i.e., to define a properly regularised symmetric operator $(T, D(H))$ which can replace the ill-defined operator $a(V)G$. Then we define, using Kato-Rellich, the Hamiltonian $H := (1-G)^*L(1-G) + T$.

We will now explain why this well defined operator is, on a formal level, a version of the expression (1) and why we consider it to be the direct description, the correct Hamiltonian for the model. Recall the definition of G^*, which yields

$$H = L(1-G) - G^*L(1-G) + T = L(1-G) + a(V)(1-G) + T\,. \tag{2}$$

Consider the action of the term $L(1-G)$ on any $\psi \in D(H)$ first. By definition of the domain, this is an element of $\mathscr{H}$. Using the definition of G, we see that formally

$$L(1-G)\psi = L\psi + a^*(V)\psi\,.$$

Individually, each term maps ψ into $D(L)' \supset \mathscr{H}$, but the boundary condition $(1-G)\psi \in D(L)$ ensures that their sum is an element of the Hilbert space. Compared to (1), in the second term on the right hand side of (2), the expression $a(V)G$ has been replaced by T. This is necessary, because the singularity of $a(V)G$ is of a different nature: in fact it contains a contribution that is formally equal to multiplication by a (negative) infinite constant. The operator T accounts for the action of $a(V)G$ without this infinite constant. More concretely, introducing an UV-cutoff, thereby replacing G by $G_\Lambda = -L^{-1}a^*(V_\Lambda)$, the extension T is chosen in such a way that $a(V_\Lambda)G_\Lambda + E_\Lambda \to T$ as $\Lambda \to \infty$. Here E_Λ is the standard renormalisation counter term. Recall that the usual cutoff Hamiltonian is equal to $H_\Lambda = L + a^*(V_\Lambda) + a(V_\Lambda)$, which we can rewrite in the following way:

$$H_\Lambda + E_\Lambda = L(1-G_\Lambda) + a(V_\Lambda)(1-G_\Lambda) + a(V_\Lambda)G_\Lambda + E_\Lambda\,.$$

Comparing this to (2), we can see that $H_\Lambda + E_\Lambda$ converges to H, as we will show, in norm resolvent sense. This implies that H agrees with the renormalised Hamiltonian H_∞. Because we explicitly identified the limiting Hamiltonian, instead of having to deal with dressing transformations or resolvent series, we are left with the well-posed task of proving a relative bound of T with respect to $(1-G)^*L(1-G)$ in order to obtain a direct description of the desired operator.

The construction sketched above is in some respect analogous to the one used in setting up zero-range Hamiltonians and the technical tools employed here are in fact inspired by previous works on many-body point interactions, in particular [CDF$^+$15] and [MS17].

In the general case we consider a system of M nucleons such that the Hilbert space is given by $\mathscr{H} = L^2(\mathbb{R}^{dM}) \otimes \Gamma(L^2(\mathbb{R}^d))$ and the free operator becomes $L = \sum_{i=1}^{M} \Theta(p_i) + \mathrm{d}\Gamma(\omega)$. The general coupling operator V is of the form

$$V\varphi(P,k) = \sum_{i=1}^{M} V^i\varphi(P,k) = \sum_{i=1}^{M} v^i_{p_i}(k)\varphi(P+e_ik,k) \tag{3}$$

Here $P = (p_1,\dots,p_M)$ and e_i denotes the inclusion of the i-th component into $\mathbb{R}^{Md}$. We have absorbed the common coupling constant g of [LS18] into the form factors. Since we do not assume any statistics for the nucleons, different particles could couple differently to the field and consequently the form factors would not be the same. It may however be helpful to think of them as being of the form $v^i_p = g_iv_p$ with $g_i \in \mathbb{C}$. As will be discussed in the upcoming work [ST18], different phases of the coupling constants g_i can be interpreted as complex charges and the Hamiltonians then fail to be invariant under time reversal.

2 Assumptions and Theorems

Let $d \in \mathbb{N}$ denote the dimension of the physical space and let $M \in \mathbb{N}$ be the number of nucleons. Let $\alpha \in [0, d/2)$, $\gamma > 0$ and $0 < \beta \le \gamma$ be real constants. Set $\mathcal{D} := d - 2\alpha - \gamma$. In order to define the Hamiltonian, we will make the following three assumptions.

Condition 2.1.

a) Let $\Theta, \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}_{\ge 0})$ and $v^i_p \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ for all $p \in \mathbb{R}^d$ and all $1 \le i \le M$. Assume that $v^i_{p-k}(k) = v^i_p(-k)$, and that there is a constant $c > 0$ such that $\left|v^i_p(k)\right| \le c\,|k|^{-\alpha}$ for all $p \in \mathbb{R}^d$ and any $1 \le i \le M$. In addition assume the bounds $|\Theta(p)| \ge |p|^\gamma$ and $\omega(k) \ge (1+k^2)^{\beta/2}$.

b) For any $\varepsilon > 0$ there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^d} \frac{|v_p(-k)|^2\,|\Theta(k) - \Theta(p-k)|}{(\Theta(p-k)+\omega(k))(\Theta(k)+\omega(k))}\,\mathrm{d}k \le C\left(|p|^{\mathcal{D}+\gamma\varepsilon} + 1\right).$$

for all $p \in \mathbb{R}^d$.

c) We have $0 \le \mathcal{D} < \frac{\gamma\beta^2}{\beta^2+2\gamma^2}$.

Condition 2.1 a) is a global condition that will be assumed throughout the paper. When dealing with renormalisation, the parameter $\mathcal{D} := d - 2\alpha - \gamma$ will basically measure the dependence of E_Λ on Λ, with $\mathcal{D} = 0$ corresponding to $E_\Lambda \sim \log\Lambda$.

The Condition 2.1 b) is concerned with the only part of the method, for which scaling is not sufficient. This is the definition of the diagonal part of the T-operator. For the two models that have been discussed above, we have $\gamma = 1$.

Condition 2.1 c) is the generalisation of the Condition 1.1 (2) of [LS18] to the case $\gamma \neq 2$. The upper bound ensures that TG is a well defined operator while the lower bound implies that E_Λ diverges (pointwise). This excludes the more regular cases where $a(V)$ is defined on the form domain of the free operator, see [LS18, Sect. 2]

In the literature on renormalisation, two different choices for the sequences of renormalisation counter terms have been made, resulting in different limiting Hamiltonians H_∞. In our setting, this will be reflected in the fact that the extension of the annihilation operator, the T-operator, comes in one of two variants. They will be defined later in (16) below. One will be denoted as variant $T^{\nu=1}$ and the other one as $\nu = 2$. For this reason, we will state the main theorem also for two different operators H^ν.

Theorem 2.2. *Assume the Conditions 2.1. Then the operator* $G := -(a(V)L^{-1})^*$ *is continuous and the domain* $D(H) := \{\psi \in \mathscr{H} | (1-G)\psi \in D(L)\}$ *is dense in* $\mathscr{H}$*. The operator* T^ν *– defined in* (16) *for* $\nu = 1, 2$ *– is symmetric on* $D(H)$ *and*

$$H^\nu := (1-G)^* L (1-G) + T^\nu \tag{4}$$

is self-adjoint and bounded from below on $D(H)$*.*

We will prove that the models obtained by renormalisation techniques are in fact equal to our Hamiltonian H. As stated above, we will give two variants, in order to include both choices of the renormalisation counter term. For the convergence of the renormalised Hamiltonians to be uniform we need another assumption.

Condition 2.3. For any $\varepsilon > 0$ there is a positive function $F \in C_0[0, \infty)$ such that

$$\int_{\mathbb{R}^d} \frac{(1-\chi_\Lambda(k))\, |v_p(-k)|^2\, |\Theta(k) - \Theta(p-k)|}{(\Theta(p-k) + \omega(k))(\Theta(k) + \omega(k))} \mathrm{d}k \leq F(\Lambda) \left(|p|^{\mathcal{D}+\gamma\varepsilon} + 1 \right) .$$

Note that this Condition 2.3 is stronger than Condition 2.1 b), the latter follows from this one by setting $C := F(0)$.

Proposition 2.4. *Assume Conditions 2.1 and 2.3 and let the counter term be defined in one of two different ways:*

$$E^\nu_\Lambda(P) := \begin{cases} \sum_{i=1}^M \int_{B_\Lambda} \left| v^i_{p_i - k}(k) \right|^2 (\Theta(k) + \omega(k))^{-1} \mathrm{d}k & \nu = 1 \\ \sum_{i=1}^M \int_{B_\Lambda} \left| v^i_{p_i - k}(k) \right|^2 (\Theta(p_i - k) + \omega(k))^{-1} \mathrm{d}k & \nu = 2\,. \end{cases} \tag{5}$$

Let $(H_\Lambda, D(L))$ *be the Hamiltonian which is given by the formal expression* (1)*, where the form factors* v^i_p *are replaced by* $v^i_p \chi_\Lambda \in L^2(\mathbb{R}^d)$*. Then* $H_\Lambda + E^\nu_\Lambda \to H^\nu$ *in norm resolvent sense.*

For a discussion of how to choose T^ν and E^ν_Λ, see Remark 3.5. The Theorem 2.4 is a slight improvement when compared to [LS18, Thm. 1.4], where only strong resolvent convergence was proved. Note that the equality $H = H_\infty$ easily follows from the weaker result because the limit is unique. However, we find that it is more satisfactory to prove convergence in norm directly by using the IBC method.

The Condition 2.5 is necessary in order to prove that intersections of the form $D(|H|^{1/2}) \cap D(L^\eta)$ only contain the zero vector for suitable $\eta > 0$. If we assume Condition 2.5, we suppose that v_p, ω, and Θ behave essentially like powers of the distance, while in general we only assume an upper bound on v_p and lower bounds for Θ and ω.

Condition 2.5. For any $R > 0$ there exist constants $C', C > 0$ such that $\Theta(q-p) \leq C(|q|^\gamma + 1)$ and $v_p(k) \geq C'(|k|^\alpha + 1)^{-1}$ for all $p \in \mathbb{R}^d$ with $|p| < R$. Furthermore, there exists a constant $\tilde{C} > 0$, such that $\omega(k) \leq \tilde{C}(|k|^\gamma + 1)$.

The Proposition 2.6 gives quite strong results when compared to [LS18, Thm. 4.2] but only because the Condition 2.5 is more restrictive. All concrete examples we have in mind fulfill these conditions.

Proposition 2.6. *Assume Conditions 2.1 and 2.5. Then for any $\nu \in \{1, 2\}$ and all $\eta \in [0, \frac{\gamma - \mathcal{D}}{2\gamma})$ it holds that $D(|H^\nu|^{1/2}) \subset D(L^\eta)$. If however $\eta \geq \frac{\gamma-\mathcal{D}}{2\gamma}$ then it holds that $D(|H^\nu|^{1/2}) \cap D(L^\eta) = \{0\}$.*

In Section 3 we will construct the Hamiltonian in the general setting and prove Theorem 2.2 and the Proposition 2.4 for $\gamma = \beta$ and Proposition 2.6. The proof of Theorem 2.2 and Proposition 2.4 in the general case $\beta < \gamma$ will be given in the Appendix A.

In Section 4 we will apply the results we have obtained to the two models that have been discussed in the introduction. In the end we will prove the following Corollary:

Corollary 2.7. *Let $\omega(k) = \sqrt{k^2+1}$ and $\Theta(p) = \sqrt{p^2 + \mu^2}$.*

- *If $d = 3$, $v_p(k) = \Theta(p)^{-1/2}\Theta(p+k)^{-1/2}\omega(k)^{-1/2}$ and $\mu > 0$, then the renormalised operator of Eckmann [Eck70] is equal to $H^{\nu=2}$.*
- *If $d = 2$, $v_p(k) = \omega(k)^{-1/2}$ and $\mu \geq 0$ then the renormalised operator for Gross' model that has been obtained in [Wue17] is equal to $H^{\nu=1}$.*

It holds that $D(|H^\nu|^{1/2}) \subset D(L^\eta)$ for any $\eta < 1/2$. If $\eta \in [1/2, 1]$ then $D(|H^\nu|^{1/2}) \cap D(L^\eta) = \{0\}$ in both models.

In [Wue17], Wünsch proved that the *operator domain* $D(H^{\nu=1})$ is contained in $D(L^\eta)$ for all $\eta < 1/2$ in the renormalised model of Gross and Sloan. The corresponding statement for the form domain as well as its analogue for Eckmann's model seem to be new. For both models, this is apparently also the first proof of the converse – the fact that in both models $D(|H^\nu|^{1/2}) \cap D(L^{1/2}) = \{0\}$.

3 Construction of the Hamiltonian

In the whole Section, the global Condition 2.1 a) is assumed to hold. Because our goal is to apply the results of this section to models with $\Theta(p) = \sqrt{p^2+\mu^2}$ and $\omega(k) = \sqrt{k^2+1}$, we will pay special attention to the case of $\beta = \gamma$ where Condition 2.1 c) reduces to $0 \le \mathcal{D} < \gamma/3$. Some issues concerning the general case of $0 < \beta < \gamma$ will be treated only in the Appendix A.

3.1 The domain of the Hamiltonian

We start with a technical lemma that will turn out to be very useful later on. The proof can be found in the Appendix A. We will always denote the characteristic function of a ball of radius Λ in $\mathbb{R}^d$ by χ_Λ.

Lemma 3.1. *Let $\Lambda, \Omega \ge 0$. For any $\gamma, r, \beta > 0$ and $\nu, \sigma \ge 0$ such that $d \in (\nu+\sigma, \nu+\sigma+r\gamma)$ there exists a $\delta_0 > 0$ and a constant $C > 0$ such that for any $0 \le \delta < \delta_0$ it holds that*

$$\int_{\mathbb{R}^d} (1-\chi_\Lambda(k)) \frac{|k|^{-\nu}\,|p-k|^{-\sigma}}{(|p-k|^\gamma + |k|^\beta + \Omega)^r}\,\mathrm{d}k \le C\,\Omega^{-r+(d-\nu-\sigma)/\gamma+\delta_\Lambda}\,\Lambda^{-\beta\delta_\Lambda}$$

for all $p \in \mathbb{R}^d$. The function δ_Λ is defined as $\delta_\Lambda := \delta \cdot (1-\chi_{[0,1]}(\Lambda))$.

The action of the free operator on the n-boson sector is given by multiplication with the function

$$L(P,K) := \sum_{i=1}^{M} \Theta(p) + \sum_{j=1}^{n} \omega(k_j) := \Theta(P) + \Omega(K)\,, \tag{6}$$

where we make use of the notation $\sum_{j\in J}\omega(q_j) = \Omega(Q)$. We can now generalise [LS18, Prop. 3.1] and prove that, for $0 \le \mathcal{D} < \gamma$, the operator $G = -(a(V)L^{-1})^* = -L^{-1}a^*(V)$ maps into $D(L^\eta)$ for some $0 \le \eta < \frac{1}{2} - \frac{\mathcal{D}}{2\gamma} \le \frac{1}{2}$.

Proposition 3.2. *Define the affine transformation $u(s) := \frac{\beta}{\gamma}s - \frac{\mathcal{D}}{\gamma}$ and let $s \ge 0$ be such that $u(s) < 1$. Then for all $0 \le \eta < \frac{1+u(s)-s}{2}$ the operator G is bounded from $D(N^{\max(0,1-s)/2})$ to $D(L^\eta)$ and $G_\Lambda \to G$ in this norm of continuous operators.*

Proof. We will prove a bound of the form $\|L^\eta(G-G_\Lambda)\psi\| \le f(\Lambda)\left\|N^{\max(0,1-s)/2}\psi\right\|$ for a continuous function f on $[0,\infty)$ which tends to zero as $\Lambda \to \infty$. This proves convergence. Boundedness follows by setting $\Lambda = 0$. We write V also for the variant of the interaction operator that acts on the n-th sector, i.e. $V\psi^{(n)} = \sqrt{n+1}\mathrm{Sym}((V \otimes \mathbf{1}_n)\psi^{(n)})$, where V acts on $L^2(\mathbb{R}^{dM})$. Sector-wise, the action of $G - G_\Lambda$ is given by

$$(G - G_\Lambda)\psi^{(n)}(P,K) = -\sum_{i=1}^{M}(L^{-1}\mathrm{Sym}((V^i - V^i_\Lambda)\psi^{(n)}))(P,K)$$
$$= \frac{-1}{\sqrt{n+1}}\sum_{i=1}^{M}\sum_{j=1}^{n+1}\frac{(1-\chi_\Lambda(k_j))v^i_{p_i}(k_j)\psi^{(n)}(P+e_ik_j,\hat{K}_j)}{L(P,K)}.$$

Here $\hat{K}_j$ denotes the variables K with the j-th entry omitted. We will define $\xi_\Lambda(k_j) := 1 - \chi_\Lambda(k_j)$. Observe that it is sufficient to estimate the norm of $L^\eta(G - G_\Lambda)\psi^{(n)}$ by the sum over the norms of $\kappa_i\psi^{(n)} := L^{-(1-\eta)}\mathrm{Sym}((V^i - V^i_\Lambda)\psi^{(n)})$. To do so, we use the finite dimensional Cauchy-Schwarz inequality and obtain

$$\left|\kappa_i\psi^{(n)}(P,K)\right|^2 \le (n+1)^{-1}\sum_{j,\mu=1}^{n+1}\frac{\left|\xi_\Lambda(k_j)v^i_{p_i}(k_j)\right|^2\left|\psi^{(n)}(P+e_ik_j,\hat{K}_j)\right|^2}{L(P,K)^{2(1-\eta)}\omega(k_j)^s}\omega(k_\mu)^s\,.$$

Using the inequality

$$\sum_{i=1}^{n}\omega(k_i)^s \le n^{\max(0,1-s)}\Omega(K)^s\,, \tag{7}$$

we can bound the μ-sum by $\omega(k_j)^s + n^{\max(0,1-s)}\Omega(\hat{K}_j)^s$. Then we use the assumptions $\left|v^i_{p_i}(k)\right| \le c\,|k|^{-\alpha}$ and $\omega(k) \ge |k|^\beta$ as well as $v^i_{p_i-k}(k) = v^i_{p_i}(-k)$ and obtain for the translated $\left|\kappa_i\psi^{(n)}(P-e_ik_j,K)\right|^2$ the bound

$$(n+1)^{-1}\sum_{j=1}^{n+1}\frac{c^2\xi_\Lambda(k_j)\left|\psi^{(n)}(P,\hat{K}_j)\right|^2}{L(P-e_ik_j,K)^{2(1-\eta)}}\left(n^{\max(0,1-s)}\,|k_j|^{-2\alpha-\beta s}\,\Omega(\hat{K}_j)^s + |k_j|^{-2\alpha}\right)$$
$$= \mathrm{Sym}_k\left[\frac{c^2\xi_\Lambda(k_1)\left|\psi^{(n)}(P,\hat{K}_1)\right|^2}{L(P-e_ik_1,K)^{2(1-\eta)}}(n^{\max(0,1-s)}\,|k_1|^{-2\alpha-\beta s}\,\Omega(\hat{K}_1)^s + |k_1|^{-2\alpha})\right]. \tag{8}$$

Here we have used the symmetry of ψ and L. Now bound $L(P - e_ik_1, K)$ from below by $|p_i - k_1|^\gamma + |k_1|^\beta + \Omega(\hat{K}_1)$ and recall that Condition 2.1 c) implies in particular $\beta > 0$. This together with $u(s) < 1$ implies that the hypothesis of Lemma 3.1 is fulfilled for the first term in (8) and consequently

$$c^2\int_{\mathbb{R}^d}\frac{\xi_\Lambda(k_1)\Omega(\hat{K}_1)^s\,|k_1|^{-2\alpha-\beta s}}{L(P-e_ik_1,K)^{2(1-\eta)}}\,\mathrm{d}k_1 \le C\Omega(\hat{K}_1)^{2(\eta-1)+\frac{d-2\alpha-\beta s}{\gamma}+s+\delta_\Lambda}\Lambda^{-\beta\delta_\Lambda}\,, \tag{9}$$

where $\delta_\Lambda := \delta(1 - \chi_{[0,1]}(\Lambda))$. If $\delta > 0$ is small enough, then

$$2(\eta - 1) + \frac{d - 2\alpha - \beta s}{\gamma} + s + \delta_\Lambda = 2\left(\eta - \frac{1 + u(s) - s}{2}\right) + \delta_\Lambda < 0\,.$$

Because $\Omega \geq 1$, that means that we can simply estimate $\Omega^{2(\eta - \frac{1+u(s)-s}{2}) + \delta_\Lambda} \leq 1$ in (9).

The corresponding bound for the second term of (8) follows by setting $s = 0$. Because the function $u(s) - s$ is non-increasing it holds that $2(\eta - 1) + \frac{d-2\alpha}{\gamma} + \delta_\Lambda < 0$ for the same choice of $\delta > 0$. Integrating in the remaining variables $(P, \hat{K}_1)$ yields the claim. □

Corollary 3.3. *Assume $0 \leq \mathcal{D} < \beta$. There exists an $\eta \in (0, 1/2)$ such that G is a continuous operator from $\mathscr{H}$ to $D(L^\eta)$ and $G_\Lambda \to G$ in norm as operators in $\mathcal{L}(\mathscr{H}, D(L^\eta))$. In particular, if $\beta = \gamma$, for any $\varepsilon > 0$ small enough we can choose $\eta = \frac{1 - \mathcal{D}/\gamma}{2} - \varepsilon$.*

Proof. We apply Proposition 3.2, distinguishing two cases. First, if $\mathcal{D} = 0$ and $\beta = \gamma$, then $u(s) = s$ and we choose, for some $\varepsilon > 0$, $s_\varepsilon = 1 - \varepsilon$ and $\eta_\varepsilon = \frac{1-\varepsilon}{2}$. Proposition 3.2 then gives the bound

$$\left\| L^{\frac{1-\varepsilon}{2}} (G - G_\Lambda)\psi \right\|_{\mathscr{H}^{(n+1)}} \leq C(\Lambda)(1 + n^{\varepsilon/2}) \left\| \psi^{(n)} \right\|_{\mathscr{H}^{(n)}}\,,$$

with $C(\Lambda) \to 0$ as $\Lambda \to \infty$. This shows that G maps $\mathscr{H}$ to $D(L^{1/2-\varepsilon})$ for all $0 < \varepsilon \leq \frac{1}{2}$ in this case and that $G_\Lambda \to G$ in $\mathcal{L}(\mathscr{H}, D(L^\eta))$.

In all other cases, we have $u(1) = (\beta - \mathcal{D})/\gamma < 1$ and we may choose in Proposition 3.2 $s = 1$ and any $0 \leq \eta < \frac{\beta - \mathcal{D}}{2\gamma}$. □

Lemma 3.4. *Let $0 \leq \mathcal{D} < \beta$. Then $1 - G$ is invertible and there exists a constant $C > 0$ such that*

$$\|N\psi\|_{\mathscr{H}} \leq C(\|N(1 - G)\psi\|_{\mathscr{H}} + \|\psi\|_{\mathscr{H}})\,. \tag{10}$$

Proof. See [LS18, Lemma 2.4]. □

We can now define what will be the domain of our Hamiltonian. We choose $D(H) := \{\psi \in \mathscr{H} | (1 - G)\psi \in D(L)\} = (1 - G)^{-1}D(L)$. Since $a(V)L^{-1} = -G^*$ is a continuous operator on $\mathscr{H}$, the annihilation operator $a(V)$ is well defined on $D(L)$. It is however not defined on the range of G, because G does not map into $D(L^{1/2})$. In the next section we will extend the action of $a(V)$ in a suitable way to elements of the form $G\varphi$.

3.2 The extension of the annihilation operator

In this section we will extend the annihilation operator $a(V)$ to $D(H) = \{\psi \in \mathscr{H} | (1-G)\psi \in D(L)\}$. Decomposing elements $\varphi \in D(H)$ as $\varphi = (1-G)\varphi + G\varphi$, we observe that $a(V)$ is well defined on $(1-G)\varphi$ but not on $G\varphi$. For that reason, we have to define an operator T, which is a regularised version of the operator $a(V)G$. The formal expression for the latter is given by

$$\begin{aligned} &a(V)G\psi^{(n)}(P, \hat{K}_{n+1}) \\ &= \sqrt{n+1}\sum_{\ell=1}^{M} \int_{\mathbb{R}^d} \overline{v^{\ell}_{p_\ell - k_{n+1}}(k_{n+1})} G\psi^{(n)}(P - e_\ell k_{n+1}, K)\, \mathrm{d}k_{n+1} \\ &= -\sum_{i,\ell=1}^{M}\sum_{j=1}^{n+1} \int_{\mathbb{R}^d} \overline{v^{\ell}_{p_\ell - k_{n+1}}(k_{n+1})} v^{i}_{p_i - \delta_{\ell i} k_{n+1}}(k_j) \\ &\qquad\qquad \times \frac{\psi^{(n)}(P - e_\ell k_{n+1} + e_i k_j, \hat{K}_j)}{L(P - e_\ell k_{n+1}, K)}\, \mathrm{d}k_{n+1}\,. \end{aligned} \tag{11}$$

Here $\delta_{\ell i}$ denotes the usual Kronecker-delta. The integrals in the terms where $j = k_{n+1}$ and $\ell = i$ do not converge in general. In order to obtain a well defined operator, we have to replace the integrals in these so called *diagonal parts* of the sum by regularised ones. To do so we employ the assumption $v^{\ell}_{p-k}(k) = v^{\ell}_{p}(-k)$ for all ℓ and set

$$I_\ell(P, \hat{K}_{n+1}) := \int_{\mathbb{R}^d} \frac{\left|v^{\ell}_{p_\ell}(-k_{n+1})\right|^2}{L(P - e_\ell k_{n+1}, K)} - \frac{\left|v^{\ell}_{p_\ell}(-k_{n+1})\right|^2}{\Theta(k_{n+1}) + \omega(k_{n+1})}\, \mathrm{d}k_{n+1} \tag{12}$$

and

$$J(p_\ell) := \int_{\mathbb{R}^d} \frac{\left|v^{\ell}_{p_\ell}(-k_{n+1})\right|^2}{\Theta(k_{n+1}) + \omega(k_{n+1})} - \frac{\left|v^{\ell}_{p_\ell}(-k_{n+1})\right|^2}{\Theta(p_\ell - k_{n+1}) + \omega(k_{n+1})}\, \mathrm{d}k_{n+1}\,. \tag{13}$$

Then we define two variants of the *diagonal* part of the operator T:

$$T^{\nu}_{\mathrm{d}}\varphi^{(n)}(P, \hat{K}_{n+1}) := \begin{cases} -\sum_{\ell=1}^{M} I_\ell(P, \hat{K}_{n+1})\varphi^{(n)}(P, \hat{K}_{n+1}) & \nu = 1 \\ -\sum_{\ell=1}^{M} (I_\ell(P, \hat{K}_{n+1}) + J(p_\ell))\varphi^{(n)}(P, \hat{K}_{n+1}) & \nu = 2\,. \end{cases} \tag{14}$$

The remaining expressions in (11) constitute the *off-diagonal* part of T. There is no need to regularise these expressions; it can be shown that they are well defined on suitable spaces:

$$\begin{aligned}
&T_{\mathrm{od}}\varphi^{(n)}(P,\hat{K}_{n+1}) &(15)\\
&:= -\sum_{\substack{i,\ell=1\\ i\neq \ell}}^{M}\int_{\mathbb{R}^d}\overline{v^{\ell}_{p_\ell}(-k_{n+1})}\frac{v^i_{p_i}(k_{n+1})\psi^{(n)}(P+(e_i-e_\ell)k_{n+1},\hat{K}_{n+1})}{L(P-e_\ell k_{n+1},K)}\,\mathrm{d}k_{n+1}\\
&\quad -\sum_{i,\ell=1}^{M}\sum_{j=1}^{n}\int_{\mathbb{R}^d}\overline{v^{\ell}_{p_\ell}(-k_{n+1})}\frac{v^i_{p_i-\delta_{\ell i}k_{n+1}}(k_j)\psi^{(n)}(P-e_\ell k_{n+1}+e_ik_j,\hat{K}_j)}{L(P-e_\ell k_{n+1},K)}\,\mathrm{d}k_{n+1}.
\end{aligned}$$

We define for $\nu\in\{1,2\}$ the operator

$$T^{\nu}\varphi^{(n)} := T^{\nu}_{\mathrm{d}}\varphi^{(n)} + T_{\mathrm{od}}\varphi^{(n)} \tag{16}$$

sector-wise, by the expressions above, on a domain that will be specified in Proposition 3.9 below.

Remark 3.5. Clearly, the choice of T_{d} is not unique. There are, however, several possible criteria why to prefer one regularisation over the other. First of all, if $v=\hat{\delta}$ and Θ and ω are quadratic, then the theory allows for a convenient intepretation in the position representation. It is most natural to define $T\varphi$ as the constant part in an asymptotic expansion of $G\varphi$ as $y_{n+1}\to x_i$. For more details, see [LS18, Rem. 3.4]. In Fourier representation, this choice corresponds to $\nu=1$.

In general, observe that, formally, T^{ν}_{d} is equal to the unregularised diagonal part plus $E_{\Lambda=\infty}$, the counter term at infinity. This will be made rigorous in the proof of Proposition 2.4 below. If $v^i_p=v^i$ are independent of p, then choosing $\nu=1$ means that H^{ν} can be approximated by a cutoff operator where the sequence of counter terms does not depend on p, i.e., is in fact an actual constant. This is the choice that has been made by Nelson and also in [LS18]. If the form factors v^i_p do however depend on p, then choosing the variant $\nu=2$, as Eckmann did, seems a viable option because E_Λ will anyway be an operator. Albeverio has noted in [Alb73] that the counter term used by Eckmann has "the disadvantage of not having the correct relativistic spectrum of the physical one nucleon energies". We can make the following observation: On any sector, the operator $T^{\nu=2}_{\mathrm{d}}$ is given by a bounded function of P. In particular, for $M=1$ the full operator $T^{\nu=2}$ equals zero on the lowest sector (which corresponds to no bosons).

We will in the next Lemmas prove the main results about the various parts of T and how to approximate them. We remark that the notation for $T^{\nu}_{\mathrm{d},\Lambda}$ differs from the one that has been used in [LS18].

Lemma 3.6. *Assume Condition 2.1 b) and let $0\le\mathcal{D}<\gamma$. Then, for any $\nu\in\{1,2\}$ and any $\varepsilon>0$ small enough, the operators T^{ν}_{d} defined in (14) are symmetric* *operators on the domain $D(L^{\mathcal{D}/\gamma+\varepsilon})$. Let $T^\nu_{\mathrm{d},\Lambda}$ be the same operator with v^i_p replaced by $\chi_\Lambda v^i_p$ and assume Condition 2.3. Then $T^\nu_{\mathrm{d},\Lambda} \to T^\nu_{\mathrm{d}}$ in norm as operators on $\mathcal{L}(D(L^{\mathcal{D}/\gamma+\varepsilon}), \mathscr{H})$.*

Proof. We will prove a bound of the form $\left\|(T^\nu_{\mathrm{d},\Lambda} - T^\nu_{\mathrm{d}})\psi\right\| \leq f(\Lambda)\left\|L^{\mathcal{D}/\gamma+\varepsilon}\psi\right\|$ for a continuous function f on $[0,\infty)$ which tends to zero as $\Lambda \to \infty$. This proves convergence. Boundedness follows by setting $\Lambda = 0$. The integrals (12) and (13) defining T^ν_{d} are real, so T^ν_{d} is a real Fourier multiplier. First, let $\nu = 2$, define $\xi_\Lambda(q) := 1 - \chi_\Lambda(q)$ and observe that the action of $T^\nu_{\mathrm{d},\Lambda} - T^\nu_{\mathrm{d}}$ is given by a sum over M terms of the form

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\xi_\Lambda(q)\,|v_{p_i}(-q)|^2}{L(P - e_i q, K, q)} - \frac{\xi_\Lambda(q)\,|v_{p_i}(-q)|^2}{\Theta(p_i - q) + \omega(q)}\,\mathrm{d}q \\
&\qquad = \int_{\mathbb{R}^d} \frac{-\xi_\Lambda(q)\,|v_{p_i}(-q)|^2\,(\Theta(\hat{P}_i) + \Omega(K))}{L(P - e_i q, K, q)(\Theta(p_i - q) + \omega(q))}\,\mathrm{d}q\,.
\end{aligned}$$

Note that this vanishes for $M = 1$ and $n = 0$. If $\gamma > \mathcal{D} > 0$ the absolute value of the integral can, using Lemma 3.1, be bounded by

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\xi_\Lambda(q)\,|q|^{-2\alpha}\,(\Theta(\hat{P}_i) + \Omega(K))}{(|p_i - q|^\gamma + |q|^\beta + \Theta(\hat{P}_i) + \Omega(K))\,|p_i - q|^\gamma}\,\mathrm{d}q \\
&\qquad \leq C(\Theta(\hat{P}_i) + \Omega(K))^{\frac{\mathcal{D}}{\gamma} + \delta_\Lambda}\Lambda^{-\beta\delta_\Lambda}
\end{aligned}$$

with $\delta_\Lambda := \delta(1 - \chi_{[0,1]}(\Lambda))$ and $\delta > 0$ small enough. If $\mathcal{D} = 0$ however, we estimate the integral for any $\varepsilon \in (0, 2)$ by

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{\xi_\Lambda(q)\,|q|^{-2\alpha}\,(\Theta(\hat{P}_i) + \Omega(K))}{(|p_i - q|^\gamma + |q|^\beta + \Theta(\hat{P}_i) + \Omega(K))\,|p_i - q|^{\gamma(1-\varepsilon/2)}}\,\mathrm{d}q \\
&\qquad \leq C(\Theta(\hat{P}_i) + \Omega(K))^{\frac{\varepsilon}{2} + \delta_\Lambda}\Lambda^{-\beta\delta_\Lambda}\,.
\end{aligned}$$

Choosing $\delta = \varepsilon/2$ small enough, this shows (because $\Omega(K) \geq 1$) that $T^{\nu=2}_{\mathrm{d}}$ is symmetric on $D(L^{\mathcal{D}/\gamma+\varepsilon})$ and that $T^{\nu=2}_{\mathrm{d}} - T^{\nu=2}_{\mathrm{d},\Lambda} \to 0$ in norm. According to Condition 2.1 b), for any $\varepsilon > 0$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|v_{p_i}(-q)|^2\,(\Theta(q) - \Theta(p_i - q))}{(\Theta(p_i - q) + \omega(q))(\Theta(q) + \omega(q))}\,\mathrm{d}q &\leq C\left((|p_i|^\gamma)^{\mathcal{D}/\gamma+\varepsilon} + 1\right) \\
&\leq C\left(\Theta(p_i)^{\mathcal{D}/\gamma+\varepsilon} + 1\right)\,.
\end{aligned}$$

If we assume Condition 2.3, we even have

$$\int_{\mathbb{R}^d} \frac{\xi_\Lambda(q)\,|v_{p_i}(-q)|^2\,(\Theta(q) - \Theta(p_i - q))}{(\Theta(p_i - q) + \omega(q))(\Theta(q) + \omega(q))}\,\mathrm{d}q \leq F(\Lambda)\left(\Theta(p_i)^{\mathcal{D}/\gamma+\varepsilon} + 1\right)$$

for some function $F \in C_0[0,\infty)$. This shows the claims for $T^{\nu=1}_{\mathrm{d}}$ as well. □

We will now separate two different terms in T_{od}, see (15). First, define

$$\theta_{i\ell}\psi^{(n)} := \int_{\mathbb{R}^d} \overline{v^\ell_{p_\ell}(-k_{n+1})}\frac{v^i_{p_i}(k_{n+1})\psi^{(n)}(P+(e_i-e_\ell)k_{n+1},\hat{K}_{n+1})}{L(P-e_\ell k_{n+1},K)}\,\mathrm{d}k_{n+1}\,. \quad (17)$$

Without loss of generality, we will specify to $(i,\ell)=(1,2)$.

Lemma 3.7. *Assume $\mathcal{D}\geq 0$. For any $\varepsilon>0$ small enough the operator θ_{12} defined in* (17) *is continuous from $D(L^{\mathcal{D}/\gamma+\varepsilon})$ to $\mathscr{H}$ and $\theta_{12}+\theta_{21}$ is symmetric on this domain. Let $\theta_{12,\Lambda}$ be the same operator with v^i_p replaced by $\chi_\Lambda v^i_p$. Then $\theta_{12,\Lambda}\to\theta_{12}$ in norm as operators on $\mathcal{L}(D(L^{\mathcal{D}/\gamma+\varepsilon}),\mathscr{H})$.*

Proof. We will prove convergence and boundedness first by a bound of the form $\|(\theta_{12}-\theta_{12,\Lambda})\psi\|\leq f(\Lambda)\left\|L^{\mathcal{D}/\gamma+\varepsilon}\psi\right\|$ for a continuous function f on $[0,\infty)$ which tends to zero as $\Lambda\to\infty$. This proves convergence. Boundedness follows by setting $\Lambda=0$. Set $\xi_\Lambda(q):=1-\chi_\Lambda(q)$. Then multiply by $|p_2-k_{n+1}|^{2\frac{(\mathcal{D}+\epsilon)}{2}}$ and its inverse for any $\epsilon>0$, and estimate using the Cauchy-Schwarz inequality

$$\begin{aligned}&\left|(\theta_{12}-\theta_{12,\Lambda})\psi^{(n)}\right|^2\\ &\leq \int_{\mathbb{R}^d}\frac{\xi_\Lambda(k)\left|v^1_{p_1}(k)\right|^2\left|\psi^{(n)}(P+(e_1-e_2)k,\hat{K}_{n+1})\right|^2|p_2-k|^{2(\mathcal{D}+\epsilon)}}{L(P-e_2k,\hat{K}_{n+1},k)}\,\mathrm{d}k\\ &\quad\times\int_{\mathbb{R}^d}\frac{\xi_\Lambda(q)\left|v^2_{p_2}(-q)\right|^2}{L(P-e_2q,\hat{K}_{n+1},q)\,|p_2-q|^{2(\mathcal{D}+\epsilon)}}\,\mathrm{d}q\,.\end{aligned}$$

The integral in q can, for ϵ small enough, be bounded by

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{|q|^{-2\alpha}\,|p_2-q|^{-2(\mathcal{D}+\epsilon)}}{(|p_2-q|^\gamma+|q|^\beta+|p_1|^\gamma+1)}\,\mathrm{d}q&\leq C(|p_1|^\gamma+1)^{-(\mathcal{D}+2\epsilon)/\gamma+\delta_\Lambda}\Lambda^{-\beta\delta_\Lambda}\\ &\leq C\,|p_1|^{-(\mathcal{D}+2\epsilon)+\gamma\delta}\,\Lambda^{-\beta\delta_\Lambda}\,,\end{aligned}$$

where we have used Lemma 3.1, $|p_1|^\gamma+1\geq 1$ and the fact that $-(\mathcal{D}+2\epsilon)+\gamma\delta<0$ for δ small enough. Integrating in $(P,\hat{K}_{n+1})$ and performing a change of variables $P\to P+(e_1-e_2)k_{n+1}$ then gives

$$\begin{aligned}&\int\left|(\theta_{12}-\theta_{12,\Lambda})\psi^{(n)}\right|^2(P,\hat{K}_{n+1})\,\mathrm{d}\hat{K}_{n+1}\mathrm{d}P\\ &\leq C\Lambda^{-\beta\delta_\Lambda}\int\int_{\mathbb{R}^d}\frac{\xi_\Lambda(k)\left|v^1_{p_1-k}(k)\right|^2\left|\psi^{(n)}(P,\hat{K}_{n+1})\right|^2|p_2|^{2(\mathcal{D}+\epsilon)}}{L(P-e_1k,\hat{K}_{n+1},k)\,|p_1-k|^{\mathcal{D}+2\epsilon-\gamma\delta}}\,\mathrm{d}k\mathrm{d}\hat{K}_{n+1}\mathrm{d}P\,.\end{aligned}$$

In the next step we can safely bound $\xi_\Lambda(k)$ by one, apply Lemma 3.1 and obtain the upper bound

$$C'\Lambda^{-\beta\delta_\Lambda}\int \left|\psi^{(n)}(P,\hat{K}_{n+1})\right|^2 |p_2|^{2\mathcal{D}+\gamma\delta}\,\mathrm{d}\hat{K}_{n+1}\mathrm{d}P\,.$$

Choosing $\delta = 2\varepsilon$ proves continuity and convergence because $|p|^{\mathcal{D}} \leq \Theta(p)^{\mathcal{D}/\gamma}$. To prove symmetry, we use a change of variables:

$$\begin{aligned}
&-\langle\varphi,\theta_{12}\psi\rangle\\
&=\int \overline{\varphi(P,\hat{K}_{n+1})v^2_{p_2-k_{n+1}}(k_{n+1})}\frac{v^1_{p_1}(k_{n+1})\psi^{(n)}(P+(e_1-e_2)k_{n+1},\hat{K}_{n+1})}{L(P-e_2k_{n+1},K)}\mathrm{d}P\mathrm{d}K\\
&=\int \frac{\overline{\varphi(P-(e_1-e_2)k_{n+1},\hat{K}_{n+1})v^2_{p_2}(k_{n+1})}v^1_{p_1-k_{n+1}}(k_{n+1})}{L(P-e_1k_{n+1},K)}\psi^{(n)}(P,\hat{K}_{n+1})\mathrm{d}P\mathrm{d}K\,.
\end{aligned}$$

□

The remaining parts of T_{od} are sums over terms of the form

$$\tau_{i\ell}\psi^{(n)} := \sum_{j=1}^{n}\int_{\mathbb{R}^d}\overline{v^\ell_{p_\ell}(-k_{n+1})}\frac{v^i_{p_i-\delta_{\ell i}k_{n+1}}(k_j)\psi^{(n)}(P-e_\ell k_{n+1}+e_ik_j,\hat{K}_j)}{L(P-e_\ell k_{n+1},K)}\,\mathrm{d}k_{n+1}\,. \tag{18}$$

The domain of the operators $\tau_{i\ell}$ can be characterised in terms of the domain of powers of the operator $\Omega := \mathrm{d}\Gamma(\omega)$ alone.

Lemma 3.8. *Assume $\mathcal{D}\geq 0$ and let $u(s)=\frac{\beta}{\gamma}s-\frac{\mathcal{D}}{\gamma}$. Then, for all $s>0$ such that the following two conditions are satisfied*

$$u(s) < 1 \tag{19}$$

$$0 < u(u(s))\,, \tag{20}$$

the operators $\tau_{i\ell}+\tau_{\ell i}$ defined in (18) *are symmetric on $D(N^{\max(0,1-s)}\Omega^{s-u(s)+\varepsilon/2})$ for any $\varepsilon>0$ small enough. Let $\tau_{i\ell,\Lambda}$ be the same operator with v^i_p replaced by $\chi_\Lambda v^i_p$. Then $\tau_{i\ell,\Lambda}\to\tau_{i\ell}$ in norm as operators on $\mathcal{L}(D(N^{\max(0,1-s)}\Omega^{s-u(s)+\varepsilon/2}),\mathscr{H})$.*

Proof. We restrict to $n\geq 1$ because $\tau_{i\ell}=0$ for $n=0$. Denote $\tau=\tau_{i\ell}$ for some (i,ℓ). We will prove a bound of the form $\|(\tau-\tau_\Lambda)\psi\| \leq f(\Lambda)\left\|N^{\max(0,1-s)}\Omega^{s-u(s)+\varepsilon/2}\psi\right\|$ for a continuous function f on $[0,\infty)$ which tends to zero as $\Lambda\to\infty$. This proves convergence. Boundedness follows by setting $\Lambda=0$. Note that, because $\mathcal{D}\geq 0$ and $\beta\leq\gamma$, it holds that $u(s)\leq s$ and therefore the conditions (19) and (20) already imply that

$$u(s), u(u(s)) \in (0,1)\,. \tag{21}$$

We multiply by $\omega(k_j)^{\frac{s}{2}}\omega(k_{n+1})^{\frac{s}{2}}$ and its inverse, apply the Cauchy-Schwarz inequality on $L^2(\mathbb{R}^d \times \{1,\dots,n\})$ and use the assumptions on v_p^l and ω to obtain

$$\begin{aligned}
&\left|(\tau-\tau_\Lambda)\psi^{(n)}\right|^2 \\
&\le \sum_{j=1}^n \int_{\mathbb{R}^d} \omega(k_{n+1})^s \frac{\left|v^i_{p_i-\delta_{\ell i}k_{n+1}}(k_j)\right|^2 \left|\psi^{(n)}(P-e_\ell k_{n+1}+e_i k_j,\hat{K}_j)\right|^2}{L(P-e_\ell k_{n+1},K)\omega(k_j)^s}\,\mathrm{d}k_{n+1} \\
&\qquad \times c\sum_{\mu=1}^n \omega(k_\mu)^s \int_{\mathbb{R}^d} \frac{(1-\chi_\Lambda(k_j)\chi_\Lambda(q)))\,|q|^{-2\alpha-\beta s}}{L(P-e_\ell q,\hat{K}_{n+1},q)}\,\mathrm{d}q\,.
\end{aligned}$$

First of all, we have to estimate

$$\begin{aligned}
(1-\chi_\Lambda(k_j)\chi_\Lambda(q))^2 &= (1-\chi_\Lambda(k_j)+\chi_\Lambda(k_j)(1-\chi_\Lambda(q)))^2 \\
&\le (1-\chi_\Lambda(k_j)+1-\chi_\Lambda(q))^2 \\
&\le 2(\xi_\Lambda(k_j)+\xi_\Lambda(q))
\end{aligned}$$

Since $u(s)\in(0,1)$, we can apply Lemma 3.1 to the integral in the second line. We deal separately with the term that does involve a $\xi_\Lambda(q)$ and the one that does not, such that they are bounded by a constant times

$$\begin{aligned}
&\xi_\Lambda(k_j)\Omega(\hat{K}_{n+1})^{-u(s)} + \Lambda^{-\beta\delta_\Lambda}\Omega(\hat{K}_{n+1})^{-u(s)+\delta_\Lambda} \\
&\qquad \le (\xi_\Lambda(k_j)+\Lambda^{-\beta\delta_\Lambda})\Omega(\hat{K}_{n+1})^{-u(s)+\delta}\,.
\end{aligned}$$

Here we have used that $\Omega(\hat{K}_{n+1})\ge 1$. In order to deal with the sum over μ, we separate the term $\mu=j$ from the rest and use (7), giving

$$\sum_{\mu=1}^n \omega(k_\mu)^s\Omega(\hat{K}_{n+1})^{-u(s)+\delta} \le \omega(k_j)^{s-u(s)+\delta} + (n-1)^{\max(0,1-s)}\Omega(\hat{K}_j)^{s-u(s)+\delta}\,.$$

Consequently, we have a bound of the form

$$\left|(\tau-\tau_\Lambda)\psi^{(n)}\right|^2 \le C\left|(\tau-\tau_\Lambda)^{(d)}\psi^{(n)}\right|^2 + C\left|(\tau-\tau_\Lambda)^{(od)}\psi^{(n)}\right|^2\,,$$

with

$$\begin{aligned}
\left|(\tau-\tau_\Lambda)^{(d)}\psi^{(n)}\right|^2 &:= \sum_{j=1}^n \int_{\mathbb{R}^d} \frac{(\xi_\Lambda(k_j)+\Lambda^{-\beta\delta_\Lambda})\left|\psi^{(n)}(P-e_\ell k_{n+1}+e_i k_j,\hat{K}_j)\right|^2}{L(P-e_\ell k_{n+1},K)} \\
&\qquad\qquad \times \frac{\left|v^i_{p_i-\delta_{\ell i}k_{n+1}}(k_j)\right|^2}{\omega(k_{n+1})^{-s}\omega(k_j)^{u(s)-\delta}}\,\mathrm{d}k_{n+1}
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\left|(\tau-\tau_\Lambda)^{(od)}\psi^{(n)}\right|^2 \\
&:= n^{\max(0,1-s)}\sum_{j=1}^{n}\int_{\mathbb{R}^d}\frac{(\xi_\Lambda(k_j)+\Lambda^{-\beta\delta_\Lambda})\left|\psi^{(n)}(P-e_\ell k_{n+1}+e_ik_j,\hat{K}_j)\right|^2}{L(P-e_\ell k_{n+1},K)} \\
&\qquad\qquad\times\frac{\left|v^i_{p_i-\delta_{\ell i}k_{n+1}}(k_j)\right|^2\Omega(\hat{K}_j)^{s-u(s)+\delta}}{\omega(k_{n+1})^{-s}\omega(k_j)^s}\,\mathrm{d}k_{n+1}\,.
\end{aligned} \tag{23}$$

To treat the term (22), we integrate in $(P,\hat{K}_{n+1})$, perform a change of variables $P\to P-e_\ell k_{n+1}+e_ik_j$, and then rename the variables $k_j\leftrightarrow k_{n+1}$. This yields

$$\begin{aligned}
&\int\left|(\tau-\tau_\Lambda)^{(d)}\psi^{(n)}(P,\hat{K}_{n+1})\right|^2\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \\
&=\sum_{j=1}^{n}\int\frac{(\xi_\Lambda(k_j)+\Lambda^{-\beta\delta_\Lambda})\omega(k_{n+1})^s\left|v^i_{p_i-k_j+\delta_{\ell i}k_{n+1}}(k_j)\right|^2\left|\psi^{(n)}(P,\hat{K}_j)\right|^2}{\omega(k_j)^{u(s)-\delta}L(P-e_ik_j,K)}\,\mathrm{d}P\mathrm{d}K \\
&=\sum_{j=1}^{n}\int\frac{(\xi_\Lambda(k_{n+1})+\Lambda^{-\beta\delta_\Lambda})\omega(k_j)^s\left|v^i_{p_i+\delta_{\ell i}k_j}(-k_{n+1})\right|^2\left|\psi^{(n)}(P,\hat{K}_{n+1})\right|^2}{\omega(k_{n+1})^{u(s)-\delta}L(P-e_ik_{n+1},K)}\,\mathrm{d}P\mathrm{d}K,
\end{aligned}$$

where, in the last step, we have used the permutation symmetry and our assumption on v^i_p. Because we have $u(u(s))\in(0,1)$ we can choose δ so small such that also $u(u(s)-\delta)\in(0,1)$. This allows us to apply again Lemma 3.1 to the k_{n+1}-integral in the usual way and to bound it from above by a constant times

$$\begin{aligned}
&\Lambda^{-\beta\delta_\Lambda}(\Omega(\hat{K}_{n+1})^{-u(u(s)-\delta)+\delta}+\Omega(\hat{K}_{n+1})^{-u(u(s)-\delta)}) \\
&\qquad\le 2\Lambda^{-\beta\delta_\Lambda}\Omega(\hat{K}_{n+1})^{-u(u(s)-\delta)+\delta}\,.
\end{aligned}$$

Therefore, using again the bound (7), we conclude

$$\begin{aligned}
&\int\left|(\tau-\tau_\Lambda)^{(d)}\psi^{(n)}(P,\hat{K}_{n+1})\right|^2\mathrm{d}P\mathrm{d}\hat{K}_{n+1} \\
&\quad\le C\Lambda^{-\beta\delta_\Lambda}n^{\max(0,1-s)}\int\left|\Omega^{\frac{s-u(u(s)-\delta)+\delta}{2}}\psi^{(n)}(P,\hat{K}_{n+1})\right|^2\mathrm{d}P\mathrm{d}\hat{K}_{n+1}\,.
\end{aligned}$$

We proceed similarly with the second term (23) and obtain

$$\left|\tau^{(od)}\psi^{(n)}\right|^2\le C\Lambda^{-\beta\delta_\Lambda}n^{2\max(0,1-s)}\int\left|\Omega^{s-u(s)+\delta}\psi^{(n)}(P,\hat{K}_{n+1})\right|^2\mathrm{d}P\mathrm{d}\hat{K}_{n+1}\,.$$

This proves the desired bounds for $\delta=\varepsilon/2$, because u is subadditive, $u(s)\le s$ and thus

$$\begin{aligned}
s-u(u(s)-\delta)+\delta &= s-u(s)+u(s)-u(u(s)-\delta)+\delta \\
&\le s-u(s)+u(s-u(s)+\delta)+\delta\le 2(s-u(s)+\delta)\,.
\end{aligned}$$

Symmetry follows also by a change of variables as in Lemma 3.7 together with an additional renaming $k_j \leftrightarrow k_{n+1}$ similar to the one we used above. □

3.3 Proof of Theorem 2.2 for $\gamma = \beta$

The next proposition gives a domain for T as a whole in the case $\gamma = \beta$. For the general case $\beta < \gamma$, the result can be found in Proposition A.1.

Proposition 3.9. *Assume Conditions 2.1. If $\beta = \gamma$ and $\mathcal{D} < \gamma/2$ then, for any $\varepsilon > 0$ small enough and any $\nu \in \{1, 2\}$, the operators T^ν define symmetric operators on the domain $D(T) = D(L^{\mathcal{D}/\gamma+\varepsilon})$.*

Proof. We have to deal with T^ν_{d} and T_{od} separately. The Lemma 3.6 states that T^ν_{d} defines a symmetric operator on the domain $D(L^{\mathcal{D}/\gamma+\varepsilon})$ for any $\varepsilon > 0$ and $\nu \in \{1, 2\}$. If $\beta = \gamma$, the function $u(s)$ of Lemma 3.8 is equal to $s - \mathcal{D}/\gamma$. Therefore the conditions on the parameter s in this lemma reduce to $s \in (2\mathcal{D}/\gamma, 1 + \mathcal{D}/\gamma)$. The Lemmas 3.8 and 3.7 taken together combined with the estimate $\Omega \leq L$ then yield that T_{od} is symmetric on $D(N^{1-s}L^{\mathcal{D}/\gamma+\varepsilon/2})$ because

$$T_{\mathrm{od}}\varphi^{(n)} = -\sum_{\substack{i,\ell=1 \\ i\neq\ell}}^{M} \theta_{i\ell}\varphi^{(n)} - \sum_{i,\ell=1}^{M} \tau_{i\ell}\varphi^{(n)} .$$

We choose $s_\varepsilon = 1 - \varepsilon/2$, which is possible for ε small enough because $\mathcal{D} < \gamma/2$. Estimating $N \leq L$ yields the claim. □

Proof of Theorem 2.2 for $\gamma = \beta$. Recall that under the assumption $\gamma = \beta$, Condition 2.1 c) reduces to $0 \leq \mathcal{D} < \gamma/3$. Any $\psi \in D(H)$ can be decomposed into $\psi = (1 - G)\psi + G\psi$. The first term belongs to $D(L)$ by definition. Corollary 3.3 shows that G is bounded from $\mathscr{H}$ to $D(L^{(1-\mathcal{D}/\gamma)/2-\varepsilon})$ for any $\varepsilon > 0$, so clearly $D(H) \subset D(L^{(1-\mathcal{D}/\gamma)/2-\varepsilon})$.

Since by Proposition 3.9 the operator T is symmetric on $D(L^{\mathcal{D}/\gamma+\varepsilon})$, we conclude that it is symmetric on $D(H)$ as long as $\mathcal{D} < \gamma/3$ (and ε is chosen appropriately). To prove the self-adjointness, we decompose:

$$H^\nu = (1 - G)^* L(1 - G) + T = H_0 + T(1 - G) + TG . \tag{24}$$

From [LS18, Prop. 2.7] we know that $H_0 := (1 - G)^* L(1 - G)$ is self-adjoint and positive. Because the range of G and the domain of T match together we conclude that TG is a bounded operator on $\mathscr{H}$. To prove that $T(1-G)$ is relatively bounded by H_0, we simply use Young's inequality as in [LS18, Sect. 2.3]. □

The proof of the Theorem 2.2 in the general case is given in Proposition A.2.

3.4 Renormalisation

We will now prove that the operator H can be approximated by a sequence of cutoff Hamiltonians $H_\Lambda + E_\Lambda$. Let us first recall the definition of these cutoff Hamiltonians. Let V_Λ be the interaction operator with form factors v_p^i replaced by $v_p^i \chi_\Lambda$, where χ_Λ is the characteristic function of a ball with radius Λ (in the variable k only). Since $v_p^i \in L^2_{\mathrm{loc}}$, the operator V_Λ maps into $L^2(\mathbb{R}^{dM}) \otimes L^2(\mathbb{R}^d)$. Thus the operator

$$H_\Lambda = L + a(V_\Lambda) + a^*(V_\Lambda)$$

is self-adjoint on $D(H_\Lambda) = D(L)$. Define $G_\Lambda = -L^{-1}a^*(V_\Lambda)$. We can rewrite the cutoff Hamiltonian analogously to H and arrive at

$$H_\Lambda + E_\Lambda^\nu = (1 - G_\Lambda)^* L (1 - G_\Lambda) + T_\Lambda + E_\Lambda^\nu \,.$$

Because V_Λ is regular, here T_Λ is simply the bounded and in particular self-adjoint operator

$$T_\Lambda := a(V_\Lambda) G_\Lambda = -G_\Lambda^* L G_\Lambda \,,$$

and E_Λ^ν are the counter terms:

$$E_\Lambda^\nu(P) := \begin{cases} \sum_{i=1}^M \int_{B_\Lambda} \left| v_{p_i - k}^i(k) \right|^2 (\Theta(k) + \omega(k))^{-1} \, \mathrm{d}k & \nu = 1 \\ \sum_{i=1}^M \int_{B_\Lambda} \left| v_{p_i - k}^i(k) \right|^2 (\Theta(p_i - k) + \omega(k))^{-1} \, \mathrm{d}k & \nu = 2 \,. \end{cases}$$

The constants are bounded and self-adjoint operators on $L^2(\mathbb{R}^{dM})$ by Lemma 3.1. Going through the computation (11) with v_p^i replaced by $\chi_\Lambda v_p^i$, we observe that a similar decomposition of T_Λ into diagonal and off-diagonal terms is possible. Since T_Λ has not yet been modified, it would not converge in the limit $\Lambda \to \infty$, precisely because of the divergence of the integrals that had to be modified in (14). This modification, that seemed to be somewhat ad hoc back then, can be achieved by adding the counter terms to the diagonal part and letting Λ go to infinity. That is, we can decompose into $T_\Lambda + E_\Lambda^\nu = T_{\mathrm{d},\Lambda}^\nu + T_{\mathrm{od},\Lambda}$, where $T_{\mathrm{d},\Lambda}^\nu$ and $T_{\mathrm{od},\Lambda}$ are exactly the operators defined in (16), with v_p^i replaced by $\chi_\Lambda v_p^i$. Recall that the notation $T_{\mathrm{d},\Lambda}^\nu$ differs from the one used in [LS18].

We will state the next Proposition in the case where $\beta = \gamma$. The general case is treated in Proposition A.3.

Proposition 3.10. *Assume Conditions 2.1 and 2.3 and let $\beta = \gamma$. Then $T_\Lambda + E_\Lambda^\nu \to T^\nu$ in norm as operators in $\mathcal{L}(D(T), \mathscr{H})$.*

Proof. This follows by decomposing $T_{\mathrm{od},\Lambda}$ into τ and θ-terms, collecting the results of Lemmas 3.6, 3.7 and 3.8 and estimating $\Omega \leq L$. □

Proof of Proposition 2.4. Let us calculate the difference of resolvents:

$$\begin{aligned}
&(H_\Lambda + E^\nu_\Lambda + \mathrm{i})^{-1} - (H^\nu + \mathrm{i})^{-1} \\
&\quad = (H_\Lambda + E^\nu_\Lambda + \mathrm{i})^{-1}(H^\nu - (H_\Lambda + E^\nu_\Lambda))(H^\nu + \mathrm{i})^{-1} \\
&\quad = (H_\Lambda + E^\nu_\Lambda + \mathrm{i})^{-1}(G_\Lambda - G)^* L(1-G)(H^\nu + \mathrm{i})^{-1} && (25)\\
&\qquad + (H_\Lambda + E^\nu_\Lambda + \mathrm{i})^{-1}(1-G_\Lambda)^* L(G_\Lambda - G)(H^\nu + \mathrm{i})^{-1} && (26)\\
&\qquad + (H_\Lambda + E^\nu_\Lambda + \mathrm{i})^{-1}(T^\nu - (T_\Lambda + E^\nu_\Lambda))(H^\nu + \mathrm{i})^{-1}\,. && (27)
\end{aligned}$$

Because $L(1-G)(H^\nu+\mathrm{i})^{-1}$ is bounded and $G_\Lambda \to G$ in norm according to Proposition 3.2, the expression (25) converges in norm to zero. Clearly, $T_\Lambda + E^\nu_\Lambda$ is relatively bounded by the operator $(1-G_\Lambda)^* L(1-G_\Lambda)$ but more precisely it is bounded with constants inpendent of Λ. This implies that $L(1-G_\Lambda)(H_\Lambda + E^\nu_\Lambda + \mathrm{i})^{-1}$ is bounded uniformly in Λ, so the norm of (26) goes to zero as well. The convergence of (27) follows from Proposition 3.10 or Proposition A.3 and the fact that T^ν is relatively bounded by H^ν on $D(H)$. □

Remark 3.11. Of course the most important result of this article is the Theorem 2.2 – which directly characterises the explicit action and the domain of the Hamiltonian. In earlier works ([Eck70, Wue17]) on these models it was proved that the sequence of cutoff Hamiltonians converges to a self-adjoint and bounded from below operator, and Proposition 2.4 shows that we have identified this very limit. Because the old approach did not succeed in identifiying the limit, it is all the more surprising that the estimates, which are needed in [Wue17], are so similar to the ones that we have proved. Let us explain. In Eckmann's approach, the resolvent of the cutoff Hamiltonian is expanded in a Neumann series

$$(H_\Lambda + E_\Lambda - z)^{-1} = (L-z)^{-1} \sum_{n=0}^{\infty} \Big[-(a(V) + a^*(V_\Lambda) + E_\Lambda)(L-z)^{-1} \Big]^n \,.$$

In [Wue17], where the reordering method due to Eckmann is worked out in detail, it is observed that the terms of the form $a(V_\Lambda)(L-z)^{-1}a^*(V_\Lambda)$ are the ones that do not converge for fixed $z \in \mathbb{C}$. The series is then regrouped in such a way that terms which are of the same order in the form factor v_p are put together. In particular the terms E_Λ and $a(V_\Lambda)(L-z)^{-1}a^*(V_\Lambda)$ both are of order two. The crucial step in the proof is then to show that the sum of these two terms is a Cauchy sequence if the occuring suitable powers of the free resolvent $(L-z)^{-1}$ are taken into account. In our language, for $z = 0$, this is of course nothing but the fact that $a(V_\Lambda)G_\Lambda + E_\Lambda \xrightarrow{\Lambda\to\infty} T$ on the domain of some power of L, which is the statement of Proposition 3.10. In this sense the resolvent approach of Eckmann is more close to the IBC method than, for example, the use of dressing transformations (see also [LS18, Sect. 3.4]).

3.5 Regularity of domain vectors

In this section we will discuss the regularity of vectors in $D(H)$. We already know that $D(H) = (1-G)^{-1}D(L)$. Of course we also have that $D(|H|^{1/2}) = (1-G)^{-1}D(L^{1/2})$, such that the form domain is characterised by the abstract boundary condition $\psi - G\psi \in D(L^{1/2})$.

Corollary 3.12. *Assume the Conditions 2.1. Then for every* $0 \le \eta < \frac{1}{2} - \frac{\mathcal{D}}{2\gamma}$ *we have*

$$D(|H|^{1/2}) \subset D(L^\eta).$$

Proof. Let $\psi \in D(|H|^{1/2})$. Since $\eta \le 1/2$ and therefore $(1-G)\psi \in D(L^{1/2}) \subset D(L^\eta)$, we have to show that $G\psi \in D(L^\eta)$. We may apply Proposition 3.2 with $s = 0$, since $\eta < \frac{1}{2} - \frac{\mathcal{D}}{2\gamma} = \frac{u(0)+1}{2}$. This yields

$$D(|H|^{1/2}) = (1-G)^{-1}D(L^{1/2}) \subset (1-G)^{-1}D(N) \subset D(N^{1/2}) \xrightarrow{G} D(L^\eta)\,.$$

Here we have used Lemma 3.4 in the third step. □

In order to prove the next proposition, we have to add the Condition 2.5 to be able to control $G\psi$ from below.

Proposition 3.13. *Assume Conditions 2.1 and 2.5. Then for any* $\frac{1}{2} - \frac{\mathcal{D}}{2\gamma} \le \eta \le 1$ *we have*

$$D(|H|^{1/2}) \cap D(L^\eta) = \{0\}.$$

Proof. We will show that G maps no $0 \ne \psi \in \mathscr{H}$ into $D(L^{\eta_0})$ where $\eta_0 = \frac{1}{2} - \frac{\mathcal{D}}{2\gamma}$. This will show that $D(|H|^{1/2}) \cap D(L^{\eta_0}) = \{0\}$ because $\eta_0 \le 1/2$ and therefore $(1-G)\psi \in D(L^{\eta_0})$ for $\psi \in D(|H|^{1/2})$. The claim will then follow immediately due to the fact that for any $\eta \le 1$ larger than η_0 it holds that $D(L^\eta) \subset D(L^{\eta_0})$.

Let $n \in \mathbb{N}$ be such that $\psi^{(n)} \ne 0$, and let $R > 0$. Define $U \subset \mathbb{R}^{dM} \times \mathbb{R}^{d(n+1)}$ to be the set

$$U = \{(P,K) | R > |p_j| \text{ and } R > |k_j| \text{ for all } j > 1\} = \mathbb{R}^d \times B_R(0)^{M-1} \times \mathbb{R}^d \times B_R(0)^n\,.$$

We first use that $(a+b)^2 \ge \frac{1}{2}a^2 - b^2$ and obtain the following lower bound:

$$\left|L^\eta G\psi^{(n)}(P,K)\right|^2 \ge \frac{1}{2(n+1)} \frac{\left|v^1_{p_1}(k_1)\right|^2 \left|\psi^{(n)}(P + e_1k_1, \hat{K}_1)\right|^2}{L(P,K)^{2-2\eta}} \tag{28}$$

$$- M \sum_{(i,j)\ne(1,1)} \frac{\left|v^i_{p_i}(k_j)\right|^2 \left|\psi^{(n)}(P + e_ik_j, \hat{K}_j)\right|^2}{L(P,K)^{2-2\eta}}. \tag{29}$$

We will see that the terms (29) have a finite integral over U, while the integral of (28) diverges if $R > 0$ is chosen large enough. In the sum over the tupels $(i, j) \neq (1, 1)$ in (29), have a look at the terms with $i = 1, j > 1$. First of all, we may completely drop L in the denominator, because it is clearly bounded from below by one. Using a change of variables $p_1 \to p_1 + k_j$ we obtain the upper bound

$$\begin{aligned}
&\int\limits_U \frac{\left|v^1_{p_1}(k_j)\right|^2 \left|\psi^{(n)}(P + e_1 k_j, \hat{K}_j)\right|^2}{L(P, K)^{2-2\eta}} \,\mathrm{d}P\mathrm{d}K \\
&\quad \leq \int\limits_U \left|v^1_{p_1 - k_j}(k_j)\right|^2 \left|\psi^{(n)}(P, \hat{K}_j)\right|^2 \,\mathrm{d}P\mathrm{d}K \\
&\quad \leq \int \left|\psi^{(n)}(P, \hat{K}_j)\right|^2 \int_{B_R} \left|v^1_{p_1}(-k_j)\right|^2 \mathrm{d}k_j \,\mathrm{d}P\mathrm{d}\hat{K}_j \,.
\end{aligned}$$

This is finite since $v^1_p(-k)$ is bounded uniformly in p by a function in L^2_{loc}. The terms with $i, j > 1$ can be bounded by enlarging the domain of integration in the variable p_i to $\mathbb{R}^d$. Then we can go on as for $i = 1$. The terms where $j = 1$ but $i > 1$ are estimated in the same way, but the change of variables is performed in k_1 and the remaining integral is then over p_i. This results in

$$\int \left|\psi^{(n)}(\hat{P}_i, K)\right|^2 \int_{B_R} \left|v^i_{p_i}(k_1 - p_i)\right|^2 \mathrm{d}p_i \,\mathrm{d}\hat{P}_i\mathrm{d}K \,.$$

If we employ the fact that $v^i_{p_i}(k_1 - p_i) = v^i_{k_1}(p_i - k_1)$, we can conclude as above.

To bound the integral over the term (28) from below, we first perform the usual change of variables $p_1 \to p_1 + k_1$. Then we restrict the domain of integration to $\{|p_1| < R\} \cap U$ to bound it by

$$\int\limits_{B_R(0)^{M+n}} \left|\psi^{(n)}(P, \hat{K}_1)\right|^2 \int\limits_{\mathbb{R}^d} \frac{\left|v^1_{p_1}(-k_1)\right|^2}{L(P - e_1 k_1, K)^{2-2\eta}} \,\mathrm{d}k_1\mathrm{d}P\mathrm{d}\hat{K}_1 \,. \tag{30}$$

Since we have restricted to $(P, \hat{K}_1) \in B_R(0)^{M+n}$, it holds that $\sum_{i=2}^M \Theta(p_i) + \Omega(\hat{K}_1) \leq C$ for some $C > 0$ that depends on R. Because in particular $|p_1| < R$, we can then estimate by using Condition 2.5

$$L(p_1 - k_1, \hat{P}_1, K) \leq \Theta(k_1 - p_1) + \omega(k_1) + C \leq C'(|k_1|^\gamma + 1) \,,$$

for some $C' > 0$ that depends on R. Condition 2.5 also allows us to bound $\left|v^1_{p_1}(-k_1)\right|^2$ from below by some constant times $(|k_1|^\alpha + 1)^{-2}$. Hence the integral (30) is bounded from below by some constant times

$$\int\limits_{B_R(0)^{M+n}} \left|\psi^{(n)}(P, \hat{K}_1)\right|^2 \mathrm{d}P\mathrm{d}\hat{K}_1 \int_{\mathbb{R}^d} \frac{1}{(|k_1|^\gamma + 1)^{2-2\eta}(|k_1|^\alpha + 1)^2} \,\mathrm{d}k_1 \,.$$

Because $\psi^{(n)} \neq 0$, we can choose an $R > 0$ large enough such that

$$\int\limits_{B_R(0)^{M+n}} \left|\psi^{(n)}(P, \hat{K}_1)\right|^2 \, \mathrm{d}P\mathrm{d}K > 0 \, .$$

But since $(2 - 2\eta)\gamma + 2\alpha \leq d$ by hypothesis, the integral in k_1 is infinite, and we have proved the claim. □

4 Proof of Corollary 2.7

In this section we are going to apply the results obtained in the previous section to the two models we have been discussing in the introduction. That is, we have to check, that the Conditions 2.1 a) – 2.5 are fulfilled. In this way we will prove the Corollary 2.7.

Clearly in both models we have $\gamma = \beta = 1$ and the form factor does not depend on the specific particle, so we will write $v_p^i = v_p$ throughout this section. In Gross' model $v_p = \omega^{-1/2}$ is independent of p, so we may choose $\alpha = 1/2$ for the upper bound. In Eckmann's model, this is less obvious since $v_p(k) = \Theta(p)^{-1/2}\Theta(p + k)^{-1/2}\omega(k)^{-1/2}$. However, for any $0 \leq \delta < 1$ it holds that

$$\Theta(p)^{-1/2}\Theta(p + k)^{-1/2} \leq c(\mu) \, |k|^{-(1-\delta)/2} \tag{31}$$

pointwise on $\mathbb{R}^d \times \mathbb{R}^d$. Here $c(\mu) < \infty$ as long as $\mu > 0$ such that Condition 2.1 a) is still fulfilled if the nucleon mass μ is positive. To see why (31) is true, note that

$$\begin{aligned} 0 &\leq (|k| \, |p| - p^2 + \mu^2)^2 = k^2p^2 - 2 \, |k| \, |p| \, (p^2 + \mu^2) + (p^2 + \mu^2)^2 \\ &\leq k^2p^2 + 2(k \cdot p)(p^2 + \mu^2) + (p^2 + \mu^2)^2 \, . \end{aligned}$$

Adding $\mu^2k^2 + \mu^2$ on both sides, we obtain

$$\begin{aligned} \mu^2(k^2 + 1) &\leq (p^2 + \mu^2)[(k + p)^2 + \mu^2] + \mu^2 \\ &\leq (p^2 + \mu^2)[(k + p)^2 + \mu^2 + 1] \leq (p^2 + \mu^2)(1 + \mu^{-2})[(k + p)^2 + \mu^2] \, . \end{aligned}$$

As a consequence for any $0 \leq \delta < 1$ we have

$$\begin{aligned} \Theta(p)^{-1/2}\Theta(p + k)^{-1/2} &\leq (\mu^{-2} + \mu^{-4})^{1/4}(k^2 + 1)^{-1/4} \\ &\leq (\mu^{-2} + \mu^{-4})^{1/4} \, |k|^{-(1-\delta)/2} \, . \end{aligned}$$

and hence the claimed inequality. That means, since $\omega(k)^{-1/2} \leq |k|^{1/2}$, that the upper bound on v_p is valid with $\alpha = 1 - \delta/2$ and hence $\mathcal{D} = \delta$ in Eckmann's model. In the remainder of this section we will of course choose $\delta = 0$.

The following lemma is inspired by a bound given by Wünsch, see [Wue17, 4.2]. In our case it implies that Condition 2.3 and hence also Condition 2.1 b) is fulfilled in both models.

Lemma 4.1. *Let $\mu, \Lambda \geq 0$, $\omega(k) = \sqrt{k^2+1}$ and $\Theta(p) = \sqrt{p^2+\mu^2}$. For any $\varepsilon > 0$ small enough, there exists a constant $C > 0$ such that pointwise on $\mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} \frac{(1-\chi_\Lambda(q)\,|q|^{-2\alpha}\,|\Theta(q)-\Theta(p-q)|}{(\Theta(p-q)+\omega(q))(\Theta(q)+\omega(q))}\,\mathrm{d}q \leq C\Lambda^{-\varepsilon_\Lambda/2}\left(|p|^{\mathcal{D}+\varepsilon}+1\right). \tag{32}$$

Here $\varepsilon_\Lambda := \varepsilon \cdot (1-\chi_{[0,1]}(\Lambda))$.

Proof. Choose any $R > 0$. We use the reverse triangle inequality to estimate $|\Theta(q)-\Theta(p-q)| \leq \Theta(p)$. Because $\sqrt{q^2+1} \geq c(|q|^{1/2}+1)$ for some constant $c > 0$, we obtain for any $\varepsilon \in (0,1)$ the upper bound

$$C\Theta(p)\int_{\mathbb{R}^d} \frac{\xi_\Lambda(q)\,|q|^{-2\alpha-1+\varepsilon/2}}{|p-q|+|q|^{1/2}+1}\,\mathrm{d}q\,,$$

where we have defined $\xi_\Lambda(q) = 1-\chi_\Lambda(q)$. By applying Lemma 3.1 with $\delta = \varepsilon$, this is clearly bounded by a constant times $\Lambda^{-\varepsilon_\Lambda/2}$ as long as $|p| < R$ and ε is small enough. For larger $|p| \geq R$, we use again the triangle inequality, estimate $\Theta(p-q)+\omega(q) \geq |p-q|+|q| \geq |p|$ and obtain the upper bound

$$C\frac{\Theta(p)}{|p|^{1-\mathcal{D}-\varepsilon}}\int_{\mathbb{R}^d} \frac{\xi_\Lambda(q)\,|q|^{-2\alpha-1+\varepsilon/2}}{(|p-q|+|q|^{1/2}+1)^{\mathcal{D}+\varepsilon}}\,\mathrm{d}q \leq C'|p|^{\mathcal{D}+\varepsilon}\Lambda^{-\varepsilon_\Lambda/2}\,.$$

Since R was arbitrary, this proves the claim. □

Remark 4.2. Note that Condition 2.1 b) can be proved to hold with an improved exponent $\max(\mathcal{D},\varepsilon)$. To do so, decompose the integral for larger $|p| \geq R$ into

$$\int_{B_p} \frac{|q|^{-2\alpha}\,\Theta(p)}{(\Theta(p-q)+\omega(q))(\Theta(q)+\omega(q))}\,\mathrm{d}q + \int_{B_p^C} \frac{|q|^{-2\alpha}\,\Theta(p)}{(\Theta(p-q)+\omega(q))(\Theta(q)+\omega(q))}\,\mathrm{d}q\,,$$

where $B_p \subset \mathbb{R}^d$ is the ball of of radius $|p|$ centered at the origin. For the first term, we obtain for any $\varepsilon \geq 0$ an upper bound of the form

$$\frac{\Theta(p)}{|p|}\int_{B_p} \frac{|q|^{-2\alpha}}{|q|^{1-\varepsilon}}\,\mathrm{d}q \leq C(R)\,|p|^{\mathcal{D}+\varepsilon}\int_{B_1} \frac{1}{|x|^{2\alpha+1-\varepsilon}}\,\mathrm{d}x\,.$$

For $\mathcal{D} > 0$ the integral converges even for $\varepsilon = 0$, such that the term is bounded by a constant times $|p|^{\max(\mathcal{D},\varepsilon)}$. The integral over the complement, we simply estimate by $\Theta(p)\int_{B_p^C} |q|^{-2-2\alpha}\,\mathrm{d}q$. Then by a change of variables $q \to q/\,|p|$ this is seen to be bounded by a constant times $|p|^{\mathcal{D}}$.

Now have a look at Condition 2.5. It is clear that, for $|p| < R$, the inequality $(p-q)^2 \leq c(q^2+1)$ holds for some R-dependent constant, w.l.o.g. $c > 1$. Because the square root is increasing, we have $\Theta(p-q) \leq \sqrt{c(q^2+1)+\mu^2}$. The triangle inequality then yields

$$\sqrt{c(q^2+1)+\mu^2} \leq \sqrt{c}\left(\sqrt{q^2+\mu^2}+1\right) = C(\Theta(q)+1)\,.$$

This already shows Condition 2.5 for Gross' model. In Eckmann's model, this very bound allows us to estimate $v_p(k)$ for $|p| < R$ from below by some constant times $\Theta(p)^{-1/2}(|k|^\gamma+1)^{-1}$, which shows that, since $\gamma = \alpha = 1$, also in this case the Condition 2.5 is fulfilled.

In Gross' model in two dimensions the parameter $\mathcal{D}$ is equal to $d - 2\alpha - \gamma = 0$ and thus smaller than $\gamma/3 = 1/3$. For Eckmann's model in $d = 3$ the same is true. As a consequence, we have checked Codition 2.1 c) and thus proved Corollary 2.7.

5 Massless Bosons

We would like to conclude by briefly discussing a variant of Eckmann's model where the nucleons are massive but the bosons are massless. That is, we would still have $\Theta(p) = \sqrt{p^2+\mu^2}$ and $\mu > 0$ but $\omega(k) = |k|$, such that $v_p(k) = |k|^{-1/2}\,\Theta(p)^{-1/2}\Theta(p+k)^{-1/2}$. While to our knowledge this very model has not yet been considered, the corresponding nonrelativistic massless Nelson model is well known in the literature, see, e.g. [Fro74, BDP12, GW18]. In the following we will sketch the construction of a Hamiltonian for the massless variant. Although L is still invertible in this case, in turns out to be convenient to introduce a positive parameter λ and to define $G_\lambda := (a(V)(L+\lambda)^{-1})^*$. Making use of the resolvent identity, it is easy to show that the domain can be equivalently expressed by $D(H) = (1-G_\lambda)^{-1}D(L)$ and that also

$$H = (1-G_\lambda)^*(L+\lambda)(1-G_\lambda) + T_\lambda - \lambda\,.$$

Here T_λ is the regularised version of $a(V)G_\lambda$. Note that the inequality $N \leq L$, which was used frequently in the massive case, does not hold anymore. That makes it absolutely necessary to obtain n-independent bounds on G_λ and T_λ. To achieve this, in Lemma 3.2 we have to choose $s = 1$, which is not possible for $\mathcal{D} = 0$. In Gross' model, the form factor is just $v(k) = \omega(k)^{-1/2} = |k|^{-1/2}$ and as a consequence it is impossible to choose a different α. In Eckmann's model however, if we are ready to pay the price of a faster diverging renormalisation counter term, the bound (31) allows us to choose $\mathcal{D} = \delta$ for any $\delta \in [0,1)$. Then Lemma 3.2 in particular yields that G_λ maps $D(L^\eta)$ into itself for any $\eta < (1-\delta)/2$. It is easy to see that the norm of G_λ as an operator on $D(L^\eta)$ goes to zero for $\lambda \to \infty$. Therefore $1 - G_\lambda$

is invertible on this domain if λ is chosen large enough. Because we may again set $s = 1$ in Lemma 3.8, the latter together with Lemmas 3.6 and 3.7 yield that T is bounded and symmetric on $D(L^{\varepsilon+\delta})$ for any $\varepsilon > 0$. For $\delta + \varepsilon$ small enough and λ large enough we can use the invertebility of $(1 - G_\lambda)$ on $D(L^{\varepsilon+\delta})$ to obtain the bound

$$\begin{aligned} \|T_\lambda G_\lambda \psi\| \le C \left\| L^{\varepsilon+\delta}\psi \right\| &= C \left\| L^{\varepsilon+\delta}(1 - G_\lambda)^{-1}(1 - G_\lambda)\psi \right\| \\ &\le C' \left(\left\| L^{\varepsilon+\delta}(1 - G_\lambda)\psi \right\| + \|(1 - G_\lambda)\psi\| \right) . \end{aligned}$$

With Young's inequality we conclude that $T_\lambda G_\lambda$ is infinitesimally bounded with respect to $(1 - G_\lambda)^*(L + \lambda)(1 - G_\lambda)$. The same is true of $T_\lambda(1 - G_\lambda)$. Hence we can, in the very same way as in the massive case, prove the self-adjointness of the operator H. In the upcoming work [Sch18], this method will be extended to treat the massless nonrelativistic Nelson model, where the analysis is slightly more involved.

Acknowledgments. I would like to thank Jonas Lampart, Stefan Teufel and Roderich Tumulka for helpful discussions and careful reading of the manuscript. The author was supported by the German Research Foundation (DFG) within the Research Training Group 1838 *Spectral Theory and Dynamics of Quantum Systems*.

A Appendix

We will assume the global Condition 2.1 a) also throughout the Appendix.

Proof of Lemma 3.1.
Set $f_\Lambda := (1 - \chi_\Lambda(k))\,|k|^{-\nu-\delta\beta}$ and $g_p := (|p-k|^\gamma + \Omega)^{-r+\delta}\,|p-k|^{-\sigma}$. By estimating for $\delta < r$ the denominator

$$
\begin{aligned}
(|p-k|^\gamma + |k|^\beta + \Omega)^{-r} &= (|p-k|^\gamma + |k|^\beta + \Omega)^{-r+\delta}(|p-k|^\gamma + |k|^\beta + \Omega)^{-\delta} \\
&\le (|p-k|^\gamma + \Omega)^{-r+\delta}\,|k|^{-\beta\delta}\ ,
\end{aligned}
$$

we observe that the integral under consideration is bounded from above by $\int_{\mathbb{R}^d} f_\Lambda g_p \mathrm{d}k$. The function g_p is vanishing at infinity and is the translated version of a function which is symmetric and decreasing, so clearly its symmetric decreasing rearrangement is just $g_p^* = g_0$. Let us compute the symmetric decreasing rearrangement of f_Λ if $m := \nu + \beta\delta > 0$. It holds that the superlevel sets are

$$
\{f_\Lambda(k) > t\} = \{k \in \mathbb{R}^d | \Lambda < |k| < t^{-1/m}\} = B_{t^{-1/m}} \setminus B_\Lambda\,.
$$

Their volume is $\mathrm{vol}(\{f_\Lambda(k) > t\}) = \mathrm{vol}(B_1 \subset \mathbb{R}^d)(t^{-d/m} - \Lambda^d)$ and therefore their symmetric decreasing rearrangement is equal to $\{f_\Lambda(k) > t\}^* = B_{(t^{-d/m}-\Lambda^d)^{1/d}}$. Recall that

$$
f_\Lambda^*(k) = \int_0^\infty \mathbb{I}_{\{f_\Lambda(k)>t\}^*}\mathrm{d}t\,,
$$

which means that $f_\Lambda^*(k)$ is the solution of $|k|^d = f_\Lambda^*(k)^{-d/m} - \Lambda^d$ which reads $f_\Lambda^*(k) = (|k|^d + \Lambda^d)^{-m/d}$.

If $\Lambda \in [0,1]$ we choose $\delta = 0$. If in addition $\nu = 0$, we can estimate $f_\Lambda \le f_0 = 1$. Then we apply the Hardy-Littlewood inequality to $\int_{\mathbb{R}^d} \sqrt{g_p}\sqrt{g_p}\mathrm{d}k$ and obtain the upper bound

$$
\int_{\mathbb{R}^d} \frac{|k|^{-\sigma}}{(|k|^\gamma + \Omega)^r}\,\mathrm{d}k\,.
$$

If $\Lambda \in [0,1]$ and $0 < \nu$ then f_Λ is vanishing at infinity because $\nu + \beta\delta > 0$. We still choose $\delta = 0$ and recall that $f_\Lambda \le f_0 = f_0^*$. Then apply the inequality to $\int_{\mathbb{R}^d} f_0 g_p \mathrm{d}k$, which yields

$$
\int_{\mathbb{R}^d} f_0 g_p \mathrm{d}k \le \int_{\mathbb{R}^d} f_0 g_0 \mathrm{d}k = \int_{\mathbb{R}^d} \frac{|k|^{-\sigma-\nu}}{(|k|^\gamma + \Omega)^r}\,\mathrm{d}k\,.
$$

If $\Lambda > 1$, let $\delta > 0$. As a consequence $\nu+\beta\delta > 0$. The Hardy-Littlewood inequality yields

$$\int_{\mathbb{R}^d} f_\Lambda g_p \mathrm{d}k \leq \int_{\mathbb{R}^d} f_\Lambda^* g_0 \mathrm{d}k = \int_{\mathbb{R}^d} |k|^{-\sigma} \frac{(|k|^d + \Lambda^d)^{-(\nu+\beta\delta)/d}}{(|k|^\gamma + \Omega)^{r-\delta}} \mathrm{d}k$$
$$\leq \int_{\mathbb{R}^d} |k|^{-\sigma-\nu} \frac{\Lambda^{-\beta\delta}}{(|k|^\gamma + \Omega)^{r-\delta}} \mathrm{d}k \, .$$

Putting these bounds together we obtain

$$\int_{\mathbb{R}^d} |k|^{-\sigma-\nu} \frac{\Lambda^{-\beta\delta_\Lambda}}{(|k|^\gamma + \Omega)^{r-\delta_\Lambda}} \mathrm{d}k$$
$$\leq \Omega^{-r+\delta+(d-\nu-\sigma)/\gamma} \Lambda^{-\beta\delta_\Lambda} \int_{\mathbb{R}^d} \frac{|q|^{-\sigma-\nu}}{(|q|^\gamma + 1)^{r-\delta}} \mathrm{d}q \, .$$

Here we have performed a change of variables $k \to k/\Omega^{1/\gamma} =: q$. The remaining integral is finite, and independent of Λ and Ω, as long as $\nu + \sigma < d$ and $\gamma(r - \delta) + \nu + \sigma > d$. Because $\gamma r + \nu + \sigma > d$, there certainly exists a $\delta_0 \in (0, r)$ such that this holds for all $0 \leq \delta < \delta_0$. □

Proposition A.1. *Assume Conditions 2.1. Set* $u(s) = \frac{\beta}{\gamma} s - \frac{\mathcal{D}}{\gamma}$. *Then for any* $\epsilon > 0$ *small enough, any* $\nu \in \{1, 2\}$ *and all* $s > 0$ *such that the following two conditions are satisfied*

$$u(s) < 1$$
$$0 < u(u(s)) \, ,$$

the operators T^ν *are symmetric on* $D_{s,\epsilon}(T) = D((N + 1)^{\max(0,1-s)} L^{s-u(s)+\epsilon/2})$.

Proof. For $\beta = \gamma$ and $\mathcal{D} < \gamma/2$, the proof has already been given in Proposition 3.9. So let $\beta < \gamma$. We know that (Lemma 3.6) T^ν_{d} defines a symmetric operator on the domain $D(L^{\mathcal{D}/\gamma+\epsilon})$ for any $\epsilon > 0$. We also have

$$s - u(s) = \frac{1}{\gamma}(\gamma - \beta)s + \frac{\mathcal{D}}{\gamma} > \frac{\mathcal{D}}{\gamma} \, ,$$

which means that $s-u(s)+\epsilon/2 > \mathcal{D}/\gamma+\epsilon$ and thus $D((N+1)^{\max(0,1-s)} L^{s-u(s)+\epsilon/2}) \subset D(L^{\mathcal{D}/\gamma+\epsilon})$ if we choose an $\epsilon > 0$ small enough. Here we have estimated $\Omega \leq L$. Therefore Lemmas 3.6, 3.7 and 3.8 together prove the claim. □

Proposition A.2. *Assume the Conditions 2.1. Let* $H_0 := (1 - G)^* L(1 - G)$. *Then the operators* T^ν *are symmetric on* $D(H)$ *and relatively* H_0*-bounded with relative bound smaller than one.*

Proof. Any $\psi \in D(H)$ can be decomposed into $\psi = \psi(1-G) + G\psi$ where the first term belongs to $D(L)$. In Lemma A.4 we will show that, for $(\beta, \mathcal{D})$ satisfying Condition 2.1 c), it is possible to choose an $s > 0$ and a small $\epsilon > 0$ in Proposition A.1 such that $D(L^{\delta_1}) \subset D_{s,\epsilon}(T) = D((N+1)^{\max(0,1-s)} L^{s-u(s)+\epsilon/2})$ for some $\delta_1 < 1$ and additionally that G maps $D(N^{\delta_2})$ into $D_{s,\epsilon}(T)$ for some $\delta_2 < 1$. Because $D(H) \subset D(N)$, this clearly implies that $D(H) \subset D_{s,\epsilon}(T)$ and thus both T^ν are symmetric on $D(H)$.

Because the range of G and the domain of each T^ν match together we conclude that $T^\nu G$ is an operator from $D(N^{\delta_2})$ into $\mathscr{H}$. Making use of Lemma 3.4 we can prove that $T^\nu G$ is relatively H_0-bounded. To prove that $T^\nu(1-G)$ is relatively bounded by H_0 we simply use Young's inequality (see [LS18]). $\square$

Proposition A.3. *Assume Conditions 2.1 and 2.3. Then there exists $s > 0$ and $\epsilon > 0$ admissible in Lemma 3.8 such that $T_\Lambda + E^\nu_\Lambda \to T^\nu$ in norm as operators in $\mathcal{L}(D_{s,\epsilon}(T), \mathscr{H})$.*

Proof. This follows by decomposing $T_{\text{od},\Lambda}$ into τ- and θ-terms and collecting the results of Lemmas 3.6, 3.7 and 3.8 in the same way as in Proposition A.1. $\square$

Lemma A.4. *Assume Conditions 2.1. Let $u(s) = \frac{\beta}{\gamma}s - \frac{\mathcal{D}}{\gamma}$ and $D_{s,\epsilon}(T) = D((N+1)^{\max(0,1-s)} L^{s-u(s)+\epsilon/2})$. Then for any $(\beta, \mathcal{D})$ with $0 \le \mathcal{D} < \frac{\gamma\beta^2}{\beta^2+2\gamma^2}$, there exists an $s > 0$ with $u(s) < 1$ and $u(u(s)) > 0$ and $\delta_1, \delta_2 \in [0,1)$ such that for any $\epsilon > 0$ small enough*

- *$D(L^{\delta_1}) \subset D_{s,\epsilon}(T)$.*
- *G is continuous from $D(N^{\delta_2})$ to $D_{s,\epsilon}(T)$.*

Proof. We can again assume $\beta < \gamma$, since the statement for $\beta = \gamma$ was already proved above. Start by looking at the second part of the statement. Proposition 3.2 states that G maps $D(N^{\max(0,1-\sigma)/2})$ into $D(L^\eta)$ for σ and η that satisfy some conditions. Of course we would like to choose $\eta := s - u(s) + \epsilon/2$ for an $s > 0$ admissible in Proposition A.1 and multiply by $n^{\max(0,1-s)}$ such that

$$D(N^{\max(0,1-\sigma)/2+\max(0,1-s)}) \xrightarrow{G} D(N^{\max(0,1-s)} L^{s-u(s)+\epsilon/2}) = D_{s,\epsilon}(T)\,.$$

First we have to show that the choice of s and σ we want to make is indeed possible. Afterwards the second part of the statement can be proved by showing that

$$\delta_2 := \max(0, 1-s) + \frac{\max(0, 1-\sigma)}{2} < 1\,. \tag{33}$$

The first part of the statement will follow by estimating $N \le L$ and the fact that

$$\delta_1 := \max(0, 1-s) + s - u(s) < 1\,, \tag{34}$$

because we may choose ϵ small enough. We will define a family of pairs of parameters $(s,\sigma) \in (0,\infty) \times [0,\infty)$ that is such that all the following conditions are in fact satisfied:

$$s - u(s) + \frac{\sigma - u(\sigma) - 1}{2} < 0 \tag{35}$$

$$u(\sigma) < 1 \tag{36}$$

$$u(s) < 1 \tag{37}$$

$$u(u(s)) > 0 \tag{38}$$

We set $\eta = s - u(s) + \epsilon/2$ in Proposition 3.2. This leads to the Condition (35) because we may always choose ϵ as small as necessary. As $\sigma - u(\sigma)$ is increasing for $\beta < \gamma$ and $\mathcal{D} \geq 0$, (35) also implies that

$$s - u(s) = \eta < \frac{1 + u(0) - 0}{2} = \frac{1}{2} - \frac{\mathcal{D}}{2\gamma} \leq \frac{1}{2}\,. \tag{39}$$

In Proposition 3.2 we had to choose a parameter $\sigma \geq 0$ which lead to (36). The Conditions (37) and (38) are due to Proposition A.1.

Now we prepare for the definition of our pair (s,σ). To do so we set

$$S_1 := \frac{\gamma + \mathcal{D}}{\beta} \qquad S_2 := \frac{1 - \frac{3}{\gamma}\mathcal{D}}{\gamma - \beta}$$

and note that $u(S_1) = 1$ and $S_1 > 1$ because $\beta < \mathcal{D} + \gamma$. Furthermore, using the upper bound on β and $\mathcal{D}$, we also have that

$$S_2 = \frac{1 - \frac{3}{\gamma}\mathcal{D}}{\gamma - \beta} > \frac{1 - \frac{3\beta^2}{\beta^2 + 2\gamma^2}}{\gamma - \beta} = \frac{\frac{\beta^2 + 2\gamma^2 - 3\beta^2}{\beta^2 + 2\gamma^2}}{\gamma - \beta} = 2\frac{\frac{\gamma^2 - \beta^2}{\beta^2 + 2\gamma^2}}{\gamma - \beta} = 2\frac{\gamma + \beta}{\beta^2 + 2\gamma^2} \tag{40}$$

$$\geq 2\frac{\gamma + \beta}{2\gamma\beta + 2\gamma^2} > \frac{1}{\gamma}\,. \tag{41}$$

We are ready to define a family of pairs $(s_\varepsilon, \sigma_\varepsilon)$ such that they fulfill the conditions (35) - (38) as long as ε is small enough. So for any $\varepsilon > 0$ let

$$(s_\varepsilon, \sigma_\varepsilon) := \begin{cases} (S_1 - \varepsilon, S_1 - \varepsilon) & \gamma S_2 \in [3S_1 - \varepsilon, \infty] \\ (S_1 - \varepsilon, \gamma S_2 - 2S_1) & \gamma S_2 \in (2S_1, 3S_1 - \varepsilon) \\ (\frac{\gamma}{2}S_2 - \varepsilon, 0) & \gamma S_2 \in (1, 2S_1] \end{cases}.$$

We have used the Inequality (41). We can see that in fact $s_\varepsilon > 0$ and $\sigma_\varepsilon \geq 0$ if ε is small enough. To prove that (35) is fulfilled, we start by noting that

$$2s_\varepsilon + \sigma_\varepsilon = \begin{cases} 3S_1 - 3\varepsilon < \gamma S_2 - 2\varepsilon & \gamma S_2 \in [3S_1 - \varepsilon, \infty] \\ \gamma S_2 - 2\varepsilon & \gamma S_2 \in (2S_1, 3S_1 - \varepsilon) \\ \gamma S_2 - 2\varepsilon & \gamma S_2 \in (1, 2S_1] \end{cases},$$

which clearly implies that for ε small enough $2s_\varepsilon + \sigma_\varepsilon < \gamma S_2$. Using this we can prove that (35) is satisfied:

$$\begin{aligned} s_\varepsilon - u(s_\varepsilon) - \frac{1}{2}\left(1 - \sigma_\varepsilon + u(\sigma_\varepsilon)\right) &= \frac{1}{2\gamma}\left(\gamma - \beta\right)\left(2s_\varepsilon + \sigma_\varepsilon\right) + \frac{3\mathcal{D}}{2\gamma} - \frac{1}{2} \\ &< \frac{1}{2\gamma}\left(\gamma - 3\mathcal{D}\right) + \frac{3}{2\gamma}\mathcal{D} - \frac{1}{2} = 0\,. \end{aligned}$$

It is clear that we have $\sigma_\varepsilon < S_1$. Since u is increasing if $\beta > 0$, we conclude that $u(\sigma_\varepsilon) < u(S_1) = 1$. That means that (36) holds. In exactly the same way we can prove that $u(s_\varepsilon) < 1$, so (37) is fulfilled. Now we check that because $\beta > \mathcal{D}$

$$u(u(S_1)) = u(1) = \frac{\beta}{\gamma} - \frac{\mathcal{D}}{\gamma} > 0\,.$$

By using the hypothesis and (40) we also see that

$$\begin{aligned} u\left(u\left(\frac{\gamma}{2}S_2\right)\right) &= \frac{\beta^2}{2\gamma^2}\gamma S_2 - \left(\gamma + \beta\right)\frac{\mathcal{D}}{\gamma^2} \\ &> \frac{\beta^2}{\gamma^2}\gamma\frac{\gamma+\beta}{\beta^2+2\gamma^2} - (\gamma+\beta)\frac{\gamma\beta^2}{\gamma^2(\beta^2+2\gamma^2)} > 0\,. \end{aligned}$$

Both estimates together prove that (38) holds for any ε small enough. In order to finally compute $\delta_2 = \max(0, 1 - s_\varepsilon) + \frac{\max(0,1-\sigma_\varepsilon)}{2}$, note that for ε small enough we have that still $S_1 - \varepsilon > 1$ and therefore

$$\begin{aligned} \delta_2 &= \begin{cases} 0 & \gamma S_2 \in [3S_1 - \varepsilon, \infty] \\ \frac{\max(0,1-\sigma_\varepsilon)}{2} & \gamma S_2 \in (2S_1, 3S_1 - \varepsilon) \\ \max(0, 1 - s_\varepsilon) + \frac{\max(0,1-\sigma_\varepsilon)}{2} & \gamma S_2 \in (1, 2S_1] \end{cases} \\ &= \begin{cases} 0 & \gamma S_2 \in [3S_1 - \varepsilon, \infty] \\ \frac{\max(0,1-\gamma S_2+2S_1)}{2} & \gamma S_2 \in (2S_1, 3S_1 - \varepsilon) \\ \max(0, 1 - \frac{\gamma}{2}S_2 + \varepsilon) + \frac{1}{2} & \gamma S_2 \in (1, 2S_1]\,. \end{cases} \end{aligned}$$

In the second case it holds that $1 - \gamma S_2 \in (1 - 3S_1 + \varepsilon, 1 - 2S_1)$ which implies $\max(0, 1 - \gamma S_2 + 2S_1)/2 < 1/2$. In the third case we can choose ε so small that $(-\gamma S_2 + 2\varepsilon)/2 < -1/2$ and as a consequence $\max(0, 1 - \frac{\gamma}{2}S_2 + \varepsilon) < 1/2$. These estimates imply that

$$\delta_2 = \max(0, 1 - s_\varepsilon) + \frac{\max(0, 1 - \sigma_\varepsilon)}{2} < \begin{cases} 0 & \gamma S_2 \in [3S_1 - \varepsilon, \infty] \\ \frac{1}{2} & \gamma S_2 \in (2S_1, 3S_1 - \varepsilon) \\ 1 & \gamma S_2 \in (1, 2S_1]\,. \end{cases}$$

In order to prove that $\delta_1 < 1$, we have to distinguish only two cases:

$$\delta_1 = s_\varepsilon - u(s_\varepsilon) + \max(0, 1 - s_\varepsilon) = \begin{cases} s_\varepsilon - u(s_\varepsilon) & s_\varepsilon > 1 \\ 1 - u(s_\varepsilon) & 0 < s_\varepsilon \leq 1 \,. \end{cases}$$

If $s_\varepsilon > 1$, using Estimate (39), we conclude that $\delta_1 < 1/2$. If $0 < s_\varepsilon \leq 1$, note that $u(u(s_\varepsilon)) > 0$ implies $u(s_\varepsilon) > 0$ (see also (21)) and therefore we have by (38) that $\delta_1 < 1$ in this case. □

References

[Alb73] S. Albeverio, *Scattering theory in a model of quantum fields. I*, J. Math. Phys. **14** (1973), 1800.

[BDP12] S. Bachmann, D.-A. Deckert, A. Pizzo, *The mass shell of the Nelson model without cut-offs*, J. Funct. Anal. **263** (2012), no. 5, 1224–1282.

[CDF+15] M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli, and A. Teta, *A class of Hamiltonians for a three-particle fermionic system at unitarity*, Math. Phys. Anal. Geom. **18** (2015), no. 1, 32.

[DP14] D.-A. Deckert, A. Pizzo, *Ultraviolet Properties of the Spinless, One-Particle Yukawa Model*, Comm. Math. Phys. **327** (2014), no. 3, 887–920.

[Eck70] J. P. Eckmann, *A model with persistent vacuum*, Comm. Math. Phys. **18** (1970), 247–264.

[Fro74] J. Fröhlich, *Existence of dressed one electron states in a class of persistent models*, Fortschritte der Physik **2** (1974), 159–198.

[GW16] M. Griesemer and A. Wünsch, *Self-adjointness and domain of the Fröhlich Hamiltonian*, J. Math. Phys. **57** (2016), no. 2, 021902.

[GW18] M. Griesemer and A. Wünsch, *On the domain of the Nelson Hamiltonian*, J. Math. Phys. **59** (2018), no. 4, 042111.

[Gro73] L. Gross, *The Relativistic Polaron without Cutoffs*, Comm. Math. Phys. **31** (1973), 25–73.

[Hep69] K. Hepp, *Théorie de la renormalisation*, Lecture Notes in Physics Vol. 2., Springer (1969).

[Lam18] J. Lampart, *A nonrelativistic quantum field theory with point interactions in three dimensions*, arXiv preprint arXiv:1804.08295 (2018).

[LS18] J. Lampart and J. Schmidt, *On Nelson-type Hamiltonians and abstract boundary conditions*, to appear in Comm. Math. Phys. arXiv:1803.00872v3 (2018).

[LSTT17] J. Lampart, J. Schmidt, S. Teufel, and R. Tumulka, *Particle creation at a point source by means of interior-boundary conditions*, Math. Phys. Anal. Geom. **21** (2018) no. 2, 12.

[MS17] T. Moser and R. Seiringer, *Stability of a fermionic N+1 particle system with point interactions*, Comm. Math. Phys. **356** (2017), no. 1, 329–355.

[Mos51] M. Moshinsky, *Boundary Conditions for the Description of Nuclear Reactions*, Physical Review **81** (1951), 347–352.

[Nel64] E. Nelson, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), no. 9, 1190–1197.

[Sch18] J. Schmidt, *The Massless Nelson Hamiltonian and its Domain*, In preparation.

[ST18] J. Schmidt, R. Tumulka, *Complex Charges, Time Reversal Asymmetry, and Interior-Boundary Conditions in Quantum Field Theory*, arXiv preprint arXiv:1810.02173 (2018).

[Slo74] A. D. Sloan, *The polaron without cutoffs in two space dimensions*, J. Math. Phys. **15** (1974), 190.

[TT15] S. Teufel and R. Tumulka, *New type of Hamiltonians without ultraviolet divergence for quantum field theories*, arXiv preprint arXiv:1505.04847 (2015).

[Tho84] L. E. Thomas, *Multiparticle Schrödinger Hamiltonians with point interactions*, Phys. Rev. D **30** (1984), 1233–1237.

[Wue17] A. Wünsch, *Self-adjointness and domain of a class of generalized Nelson models*, PhD-Thesis, Universität Stuttgart (2017). http://dx.doi.org/10.18419/opus-9384.

[Yaf92] D. R. Yafaev, *On a zero-range interaction of a quantum particle with the vacuum*, J. Phys. A: Math. Gen. **25** (1992), no. 4, 963.

4) Article [Sch19]

The Massless Nelson Hamiltonian and its Domain

Julian Schmidt

Abstract In the theory of point interactions, one is given a formal expression for a quantum mechanical Hamiltonian. The interaction terms of the Hamiltonian are singular: they can not be rigorously defined as a perturbation (in the operator or form sense) of an unperturbed free operator. A similar situation occurs in Quantum Field Theory, where it is known as the ultraviolet problem. Recently, it was shown that some of the tools used in the context of point interactions can be adapted to solve the problem of directly defining a Hamiltonian for the Nelson model. This model provides a well studied example of a bosonic quantum field that is linearly coupled to nonrelativistic particles. The novel method employs so called abstract interior-boundary conditions to explicitly characterise the action and the domain of the Hamiltonian without the need for a renormalisation procedure. Here, for the first time, the method of interior-boundary conditions is applied to the massless Nelson model. Neither ultraviolet nor infrared cutoffs are needed.

1 Introduction

In this contribution we will discuss how some of the tools that have been developed in the theory of (many body-)point interactions can be adapted to define Hamiltonians for certain models of Quantum Field Theory. In these models, a nonrelativistic particle interacts linearly with a bosonic quantum field, which means that the interaction term in a formal Hamiltonian is linear in creation and annihilation operators. If one wants to set up a self-adjoint Hamiltonian for such a model, the main obstacle is the fact that this interaction term is in general not small – in the operator or form sense – relative to the free operator L, i.e. the Hamiltonian for the non-interacting system of particles and field. Because the relative bound is given by an integral in

Julian Schmidt
Fachbereich Mathematik, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. e-mail: juls@maphy.uni-tuebingen.de

Fourier space, which does or does not converge for large momenta, this is also called the *ultraviolet problem*. Well studied examples with linear coupling are the so called massive and massless Nelson models. Until recently, the standard approach to overcome the ultraviolet problem was a renormalisation procedure, where the interaction is restricted by hand to momenta $|k| \leq \Lambda$ for some positive Λ in order to render the bound finite. This *UV-cutoff* results in a self-adjoint cutoff Hamiltonian H_Λ. In some models, including the massive and the massless Nelson model, there exists a diverging sequence of so called renormalisation constants E_Λ such that $H_\Lambda + E_\Lambda$ converges for $\Lambda \to \infty$ in norm resolvent sense to a self-adjoint operator H_∞. This is called *removing the UV-cutoff* and the operator H_∞ is called the renormalised Hamiltonian. While the renormalisation method yields that the so obtained operator is bounded from below, neither the action of H_∞ nor its domain $D(H_\infty)$ are obtained in this way. That is why, at the end of his seminal article of 1964, after carrying out the renormalisation procedure sketched above, Edward Nelson posed the following questions:

> It would be interesting to have a direct description of the operator H_∞.
> Is $D(H_\infty) \cap D(L^{1/2}) = \{0\}$? ([Nel64])

In the article [GW18], Griesemer and Wünsch finally gave the answer to the second question: Yes, in fact it even holds form the form domain that $D(|H_\infty|^{1/2}) \cap D(L^{1/2}) = \{0\}$. This was proved with the help of the renormalisation technique. While their result solved the second part of the problem posed by Nelson, it also showed the limitations of this method, for it required considerable technical effort to extract this information.

In the recent article [LS18], Jonas Lampart together with the author gave a complete answer to Nelson's question in the above quote. That is, to provide a direct desription of the operator H_∞ and its domain, from which the answer to the second question can easily be read off. More concretely, a dense domain $D(H)$ on Fock space is constructed, whose elements are the sum of a regular part, which is an element of $D(L)$, and a singular part. Then the action of L is extended to this domain in such a way that it encodes the action of the creation operator. In addition, also the action of the annihilation operator is extended to the domain $D(H)$ and it is shown that their sum defines a self-adjoint operator H, bounded from below. Afterwards it turns out, that this operator is in fact the limit of the sequence of cutoff operators H_Λ, so it becomes clear that H is equal to the renormalised Hamiltonian H_∞.

Characterising elements of $D(H)$ in this way can be viewed as imposing abstract boundary conditions on them. These boundary conditions, which are called *interior-boundary conditions*, are formulated in strong analogy with the theory of point interactions. The main difference being the fact that the boundary space or space of charges of the theory of point interactions is on each sector of Fock space identified with the sector with one boson less. In this way the boundary space can be identified with the Hilbert space $\mathscr{H}$ itself. The singular behaviour of the wave function on one sector is determined by the wave function one sector below. The Skornyakov–Ter-Martyrosyan (STM) operator appears in this construction not as part of a boundary condition and it is therefore not used to label self-adjoint realisations, for the latter alternative see, e.g. [MO17]. Instead, the STM operator T is identified as the correct extension of the annihilation operator to the singular functions and is therefore part of the action of the Hamiltonian. Thus it is not necessary to study T as an operator on the space of charges, but as an operator on $\mathscr{H}$.

In Nelsons original work [Nel64], the so called massive case was treated, where the dispersion relation of the bosonic field is given by $(|k|^2+m^2)^{1/2}$ for some $m>0$. Later, the renormalisation procedure was applied also to the massless case $m=0$ and the properties of the Hamiltonians with and without cutoff were investigated, see e.g. [Fro74, Piz03, BDP12, MM17]. The result of Griesemer and Wünsch equally holds for the massless case.

In [LS18], the case of nonrelativistic particles was considered. In [Sch18], the construction was extended to treat also pseudorelativistic models with dispersion relations $\Theta(p)=\sqrt{p^2+\mu^2}$. If the renormalisation constant E_Λ diverges too fast, the method of [LS18] has to be suitably modified. This was done for the first time in [Lam18a]. In [Lam18b], the enhanced method of the former article is applied to a Polaron-type model.

So far however, these results on interior-boundary conditions were concerned with the massive case: it was always assumed that the dispersion of the bosons is bounded from below by a positive constant. As a consequence, the free operator is bounded from below by the number (of bosons) operator, i.e. $N \leq L$. Now naturally the question arises whether the construction using abstract interior boundary conditions can be extended also to the massless case. After all, within renormalisation schemes, there is no difficulty in treating these cases as well.

In the present note, we will give a more detailed description of the domain $D(H)$ with or without mass. Roughly speaking, we will differentiate Nelson's second question between the full free operator L and the part of it that only acts on the field degrees of freedom, $\mathrm{d}\Gamma(\omega)$. In this way, we will prove self-adjointness of the Hamiltonian H with or without mass. Neither an ultraviolet nor an infrared cutoff is used in the construction, not even in an intermediate step. We will focus on a class of models in three space dimensions where one nonrelativistic particle interacts with the bosonic field.

In [LN18], interior boundary conditions were used in a multi-time formulation for massless Dirac particles in one space dimension. There the number of particles is bounded. As we will explain in more detail below, the main problems with massless fields occur only if an arbitrary number of quanta is allowed.

For physical aspects and more general discussions of the IBC approach, we refer the reader to [KS16, DGS+18, ST18] and [TT16].

2 The Model

In this section we will define the basic objects of our model. Then we will introduce a spectral parameter and justify its use by demonstrating that the domain and the extended annihilation operator are actually parameter-independent.

Our model will be defined on the Hilbert space

$$\mathscr{H} := \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{3n})$$

of the composite system of the particle and the field. We will formulate the model in Fourier representation where elements of the sectors of this Hilbert space are wavefunctions

$$\psi^{(n)}(p,k_1,\dots,k_n)\,,$$

which are symmetric under exchange of either two of the k-variables. The operator that governs the dynamics of the nonrelativistic particle is given by the multiplication operator p^2. The dispersion relation of the field is given by a non-negative function $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$. Its second quantisation will be denoted by $\Omega := \mathrm{d}\Gamma(\omega)$. We can now define the free operator $L = p^2 + \Omega$, which is self-adjoint and non-negative with domain $D(L) \subset \mathscr{H}$. Since $\Omega \geq 0$, the operator $\Omega_\mu := \Omega + \mu$ is invertible for any $\mu > 0$ and so is $L_\mu := p^2 + \Omega_\mu$.

The interaction between the field and the particle is characterised by a coupling function $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, which is called the *form factor*. The formal expression for a Hamiltonian of the model is

$$L + a(V) + a^*(V)\,,$$

where the annihilation operator $a(V)$ acts sector-wise as

$$(a(V)\psi)^{(n)}(p,k_1,\dots,k_n) := \sqrt{n+1}\int_{\mathbb{R}^3} \overline{v(k)}\,\psi^{(n+1)}(p-k,k_1,\dots,k_n,k)\,\mathrm{d}k\,.$$

The creation operator $a^*(V)$ is the formal adjoint of $a(V)$, with action given by

$$(a^*(V)\psi)^{(n)}(p,k_1,\dots,k_n) := n^{-1/2}\sum_{j=1}^{n} v(k_j)\,\psi^{(n-1)}(p+k_j,k_1,\dots,\hat{k}_j,\dots,k_n)\,.$$

As usual, $\hat{k}_j$ means that the j-th variable is omitted. The operator $a^*(V)$ is a densely defined operator on $\mathscr{H}$ if and only if $v \in L^2(\mathbb{R}^d)$. However, in all relevant examples, this is not the case. Often v is in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ but is not decaying fast enough at infinity such that $v \notin L^2$. This is what we will assume in the following.

If we wanted to start with a renormalisation procedure, we would now simply replace v by $\chi_\Lambda v$ where χ_Λ is the characteristic function of a ball of radius Λ in $\mathbb{R}^3$. Instead, we proceed by defining an operator $G^*_\mu := -a(V)L_\mu^{-1}$. Later, we will make assumptions on v which guarantee that this operator is bounded. As a consequence, the symmetric operator $L_{0,\mu} := L_\mu\big|_{\ker a(V)}$ is closed for any $\mu \geq 0$. Because $v \notin L^2$, its domain $\ker a(V)$ is also dense in $\mathscr{H}$, see [LS18, Lem. 2.2]. Therefore the adjoint $L^*_{0,\mu}$ is unique. Observe that the operator G_μ maps elements of $\mathscr{H}$ into $\ker L^*_{0,\mu}$, because for all $\psi \in \ker a(V)$ it holds by definition of G_μ that

$$\langle L_{0,\mu}^* G_\mu \varphi, \psi\rangle = \langle \varphi, G_\mu^* L_{0,\mu} \psi\rangle = -\langle \varphi, a(V)\psi\rangle = 0\,.$$

We will now define a family of subspaces of the adjoint domain $D(L_{0,\mu}^*)$. In order to do so, we decompose elements of $\mathscr{H}$ in the same way as in the theory of point interactions into the sum of two terms: one is regular, i.e. in $D(L)$, and one term is singular, that is, of the form $G_\mu \varphi$. If we would like to define a sum of point interaction domains in $\mathscr{H}$, we would introduce a boundary or charge space where φ lives. But because $\mathscr{H}$ is an infinite sum, there is another possibility, namely to take ψ itself as the charge. This is what we will do. Note that the decomposition $\psi = (1-G_\mu)\psi + G_\mu \psi$ holds for any $\psi \in \mathscr{H}$ and $\mu > 0$. Then the family of domains is given by

$$\mathfrak{D}_\mu := \{\psi \in \mathscr{H} | (1-G_\mu)\psi \in D(L)\}\,.$$

For $\mu, \lambda > 0$, the resolvent identity yields

$$(G_\mu - G_\lambda)^* = -a(V)(\lambda-\mu)L_\mu^{-1}L_\lambda^{-1} = ((\lambda-\mu)L_\mu^{-1}G_\lambda)^*\,.$$

In particular it holds that that $1-G_\mu = (1-G_\lambda) - (\lambda-\mu)L_\mu^{-1}G_\lambda$. Because $L_\mu^{-1}G_\lambda$ maps into $D(L)$, this shows that the domain $\mathfrak{D}_\mu$ is in fact independent of the chosen $\mu > 0$. We will denote it by $\mathfrak{D}$ from now on.

In the next step we have to extend the action of $a(V)$ from $D(L)$ to the enlarged domain $\mathfrak{D}$. The formal action of the annihilation operator on the range of G_μ would read

$$\begin{aligned} a(V)G_\mu \psi^{(n)}(p,k_1,\ldots,k_n) \\ = &-\psi^{(n)}(p,k_1,\ldots,k_n)\int_{\mathbb{R}^3} \frac{|v(k_{n+1})|^2}{L_\mu(p,k_1,\ldots,k_{n+1})}\mathrm{d}k_{n+1} \\ &-\sum_{j=1}^n \int_{\mathbb{R}^3} \overline{v(k_{n+1})}v(k_j)\frac{\psi^{(n)}(p+k_j-k_{n+1},k_1,\ldots,\hat{k}_j,\ldots,k_{n+1})}{L_\mu(p,k_1,\ldots,k_{n+1})}\mathrm{d}k_{n+1}\,. \end{aligned} \tag{1}$$

Here $L_\mu(p,k_1,\ldots,k_{n+1})$ denotes the functions to which the operator L_μ reduce to on one sector of $\mathscr{H}$ in the Fourier representation. The off-diagonal part of this sum, the second line of (1), constitutes an integral operator, which we will denote by T_{od}^μ. The integral in the first line of (1) does in general not converge. In order to regularise this expression, we define the diagonal part of the T-operator

$$T_{\mathrm{d}}^\mu \psi(p,k_1,\ldots,k_n) := -I_\mu(p,k_1,\ldots,k_n)\cdot \psi^{(n)}(p,k_1,\ldots,k_n)\,, \tag{2}$$

$$\text{where}\quad I_\mu(p,k_1,\ldots,k_n) := \int_{\mathbb{R}^3} \frac{|v(k_{n+1})|^2}{L_\mu(p,k_1,\ldots,k_{n+1})} - \frac{|v(k_{n+1})|^2}{k_{n+1}^2+\omega(k_{n+1})}\mathrm{d}k_{n+1}\,. \tag{3}$$

Now define the action of $T^\mu \psi := T_{\mathrm{d}}^\mu \psi + T_{\mathrm{od}}^\mu \psi$ on a (maximal) domain $\mathrm{D}^\mu \subset \mathscr{H}$. At first, this definition seems to depend again on the choice of $\mu > 0$. Note however that, because the second term of the integral I_μ in (3) is independent of the parameter $\mu > 0$, it holds that

$$T^\mu - T^\lambda = a(V)(G_\mu - G_\lambda) = a(V)(\lambda - \mu)L_\mu^{-1}G_\lambda = (\mu - \lambda)G_\mu^* G_\lambda \,. \qquad (4)$$

Because the operators G_μ are continuous, this implies that $\psi \in \mathrm{D}^\lambda$ for any $\lambda > 0$ as soon as $\psi \in \mathrm{D}^\mu$ for some $\mu > 0$. Set $D(T) = \mathrm{D}^\mu$. While the action of T^μ does of course still depend on the chosen parameter, this operator gives rise to the desired extension of $a(V)$. We define the action of the full extension for all $\psi \in D(T) \cap \mathfrak{D}$ as

$$A^\mu \psi := a(V)(1 - G_\mu)\psi + T^\mu \psi \,. \qquad (5)$$

As a consequence of (4), we have

$$A^\mu = a(V)(1 - G_\lambda) + a(V)(G_\lambda - G_\mu) + T^\mu = a(V)(1 - G_\lambda) + T^\lambda = A^\lambda \,.$$

Therefore we can define the operator $(A, \mathfrak{D} \cap D(T))$ by choosing any $\mu > 0$. Finally we may also define the action of our Hamiltonian manifestly independent of the spectral parameter:

$$H := L_{0,0}^* + A \,.$$

Using the definition of G_μ and T^μ, we can rewrite it in a convenient form that contains the positive spectral parameter:

$$H = (1 - G_\mu)^* L_\mu (1 - G_\mu) + T^\mu - \mu \,. \qquad (6)$$

In [LS18], it was assumed that $\omega \geq 1$, and as a consequence of the resulting bound $N \leq L$, it was possible to define $G^* := G_0^* = -a(V)L^{-1}$ without the need for a parameter. We would however like to make clear that the use of a spectral parameter was avoided only for convenience and better readability and is by no means the real benefit of the assumption $\omega \geq 1$.

In order to show self-adjointness of H, we will adopt the strategy of [LS18], where the representation (6) (for $\mu = 0$) was used. At first, we have to show that $H_0^\mu := (1 - G_\mu)^* L_\mu (1 - G_\mu)$ is self-adjoint. In [LS18, Lem. 3.3] the estimate $N \leq L$ was envoked to show directly the continuous invertibility of $(1 - G_0)$, from which the self-adjointness of H_0^0 follows. Since we can not use this estimate, we will show that there exists $\mu_0 >$ such that $\|G_\mu\| < 1$ for all $\mu > \mu_0$. The main problem to overcome is however the inclusion $\mathfrak{D} \subset D(T)$ or, more precisely, the relative boundedness of T^μ with respect to H_0^μ.

The proof of the relative bound for T^0 in [LS18] makes extensive use of the inequality $N \leq L$ and the resulting fact that $(1 - G_0)$ leaves $D(N)$ invariant. For that reason, this strategy is not helpful in the massless case. In fact, because there is no relation between N and L, it will be necessary to use characterisations of the domains $D(T)$ and $\mathfrak{D}$ that are independent of N altogether. We will illustrate the problems that occur with this strategy for the example of the Nelson model. While [LS18, Prop. 3.5] gives – for this specific model – an n-independent inclusion $D(L^{1/2}) \subset$

$D(T)$, the statement of [LS18, Lem. 3.2] yields that G_0 maps $\mathscr{H}$ into $D(L^\eta)$ for any $0 \leq \eta < 1/4$. These exponents do not match together and this is the very problem we have to overcome if we want to define T^μ. Differentiating between the diagonal and the off-diagonal part of T^μ, we easily observe that, what is actually proven in [LS18] is that on the one hand $D(\Omega^{1/2}) \subset D(T_{\mathrm{od}})$, but on the other hand $D(L^\varepsilon) \subset D(T_{\mathrm{d}})$ for all $\varepsilon > 0$. Thus, at least in the Nelson model, the diagonal part of the operator T seems to pose no problems. The off-diagonal part could be dealt with, if the mapping properties of G_μ are such that $\mathfrak{D} \subset D(\Omega^{1/2})$. This is exactly what we will prove in the following for a certain class of models under some assumptions on v and ω in three space dimensions.

3 Assumptions and Theorems

Let the dimension of the physical space be equal to three and assume that there exist $\alpha \in [0,3/2)$ and a constant $c > 0$ such that for $v \in L^2_{\text{loc}}(\mathbb{R}^3)$ it holds that $c(1+|k|^{\alpha})^{-1} \leq |v(k)| \leq |k|^{-\alpha}$. Furthermore, there exists $\beta \in (0,2]$ and a constant $\mathsf{m} \geq 0$ such that for $\omega \in L^\infty_{\text{loc}}(\mathbb{R}^3)$ it holds that $|k|^\beta \leq \omega(k) \leq |k|^\beta + \mathsf{m}$. Defining $D := 1-2\alpha$ we always assume that $0 \leq D < \beta$.

Note that the Nelson model is contained in this class because $v = \omega^{-1/2}$ allows us to choose $\alpha = 1/2$. Clearly β is equal to 1. The upper and lower bounds on ω hold because $\sqrt{k^2 + \mathsf{m}^2} \leq |k| + \mathsf{m}$. It will not be necessary to distinguish between the massive and the massless case, for the only important thing is the pair (β, D), which is equal to $(1,0)$ in the Nelson model. Our first result, Proposition 3.1, is concerned with regularity properties of a family of domains $\mathfrak{D}^\sigma$. Its proof can be found in Section 4.2.

Proposition 3.1. *Let $\beta \in (0,2]$, let $0 \leq D < \beta/2$ if $\beta < 2$ and $0 < D < 1$ if $\beta = 2$. Let $\psi \neq 0$ and $\kappa, \eta \in [0,\sigma]$ for some $\sigma \in (0,1]$.*
If

$$\psi \in \mathfrak{D}^\sigma = \{\psi \in \mathscr{H} | (1-G_\mu)\psi \in D(L^\sigma) \text{ for some } \mu > 0\},$$

then $\psi \in D(L^\kappa)$ if and only if $\kappa < \frac{2-D}{4}$, and $\psi \in D(\Omega^\eta)$ if and only if $\eta < \frac{2-D}{2\beta}$.

Note that also the more general domains $\mathfrak{D}^\sigma$ are independent of the spectral parameter $\mu > 0$. If $\mathfrak{D}$ is written without superscript, it is always understood as $\mathfrak{D}^1$.

To prove self-adjointness of the operator H on $\mathfrak{D}$, we need a more refined condition for the pair of parameters (β, D).

Condition 3.2. Assume that the pair (β, D) satisfies the following inequalities:

$$\begin{aligned}
&0 \leq D < \frac{\beta^2}{2} && \beta \in (0, 2(\sqrt{2}-1)) \\
&0 \leq D < \frac{2\beta}{\beta+4} && \beta \in [2(\sqrt{2}-1), \sqrt{5}-1) \\
&0 \leq D < \frac{\beta^2 - 2\beta + 2}{\beta+1} && \beta \in [\sqrt{5}-1, 2) \\
&0 < D < 2/3 && \beta = 2\,.
\end{aligned}$$

Theorem 3.3 is the main result of this article. It shows, that the only restriction one has to face when extending the construction from massive to massless models is the assumption of the lower bound $D > 0$ for $\beta = 2$. The upper bound on admissible D is weaker than the one of [LS18, Cond 1.1], which is $D < \frac{2\beta^2}{\beta^2+8}$. Therefore the Theorem 3.3 extends the result of the former article to pairs (β, D) fulfilling Condition 3.2.

Theorem 3.3. *If Condition 3.2 holds, then the operator*

$$H := (L|_{\ker a(V)})^* + A\,,$$

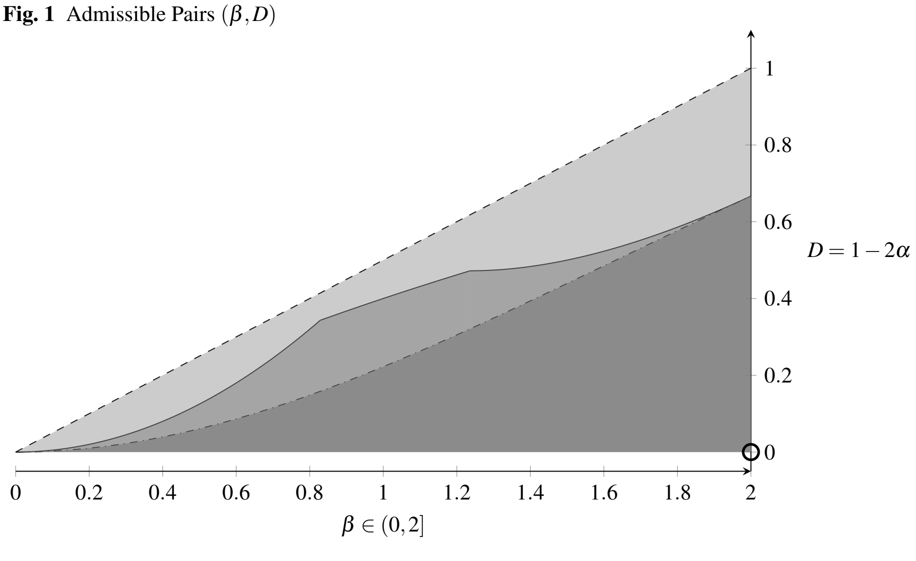
with A defined in (5)*, is self-adjoint and bounded from below on the domain*

$$D(H) := \mathfrak{D} = \{\psi \in \mathscr{H} | (1-G_\mu)\psi \in D(L) \text{ for some } \mu > 0\}\,.$$

The proof of Theorem 3.3 will be given in Section 4.3.

Remark 3.4. The condition $0 \le D < \beta/2$, which was assumed in Proposition 4.2 *does not* ensure that H is self-adjoint. However Condition 3.2 clearly implies that $0 \le D < \beta/2$, so the statement of Proposition 3.1 is in particular valid in cases where $\mathfrak{D} = D(H)$ is the domain of the self-adjoint operator H and $\mathfrak{D}^{1/2}$ is its form domain. The Plot 1 shows the different regions of admissible pairs of parameters. In general, we consider pairs where $0 \le D < \beta$ for $\beta < 2$ and $D \in (0,2)$ if $\beta = 2$. The area below the dotted line, which also excludes the point $(\beta, D) = (2,0)$, is the one for which Proposition 3.1 charaterises the domain $\mathfrak{D}$. It is, in our language, also the area for which [GW18] shows that a renormalisation procedure can be implemented using a Gross transformation. The area below the plain line, again without the point at the right lower corner, is formed by the admissible pairs according to Condition 3.2. The area below the dashdotted line is the one that is allowed in [LS18, Cond 1.1]. Because there only massive models are considered, the point $(2,0)$ is however admissible.

Fig. 1 Admissible Pairs (β, D)



The characterisations of $D(H)$ and $D(|H|^{1/2})$ provide a more detailed answer to Nelson's second question (for the admissible pairs) when compared to the result of Griesemer and Wünsch. First of all, the method in [GW18] only allows for the characterisation of the *form domain* of the limiting Hamiltonian. We can reproduce their earlier result here because setting $\sigma = 1/2$ in Proposition 3.1 yields that

$$D(|H|^{1/2}) = \mathfrak{D}^{1/2} \subset \bigcap_{0 \leq \kappa < \frac{2-D}{4}} D(L^{\kappa}) \cap D(\Omega^{1/2})$$

as long as $2 - D > \beta$, which is in particular fulfilled for the Nelson model. For determining supersets of the *operator domain* $D(H) = \mathfrak{D}$, the IBC method is the only tool avaliable. For the Nelson model, massive or massless, Proposition 3.1 implies that $D(H) \subset D(\Omega^{\eta})$ for all $\eta < 1$ but $D(H) \cap D(\Omega) = \{0\}$.

4 Constructing the Hamiltonian

In the main part of the article we will carry out the program that has been sketched in the introduction. The possibility to set up the operators G and T using positive parameters $\mu > 0$ and the results about the *parameter-independence* of the domains $\mathfrak{D}^{\sigma}$ and the operator A will not be repeated. They can be found in Section 2. We will discuss the mapping properties of G_{μ} and fit them together with those of T^{μ}. In this way, we will prove self-adjointness of the Hamiltonian H (Theorem 3.3) and obtain the characterisation of the domains $\mathfrak{D}^{\sigma}$ in terms of domains of powers of Ω and L (Proposition 3.1).

We will from now on assume that the spectral parameter μ is greater than one, $\mu \geq 1$. When writing $D(L^x)$ without index for some $x \in \mathbb{R} \setminus \{0\}$ we mean the domain $D(L^x_{\mu})$ for any $\mu \geq 1$. Note also that the assumption on μ guarantees monotonicity in the exponent, i.e. $L^x_{\mu} \leq L^y_{\mu}$ if $x \leq y$.

We will denote by K the collection of variables $K := (k_1, \ldots, k_n)$. Consequently $\hat{K}_j := (k_1, \ldots, \hat{k}_j, \ldots, k_n)$ is the collection of variables with the j-th component omitted. We will use the symbols $L_{\mu}(p, K) = p^2 + \Omega_{\mu}(K)$ to denote the functions to which the operators reduce to on one sector of $\mathscr{H}$ in the Fourier representation.

Powers of the self-adjoint operators Ω and L are self-adjoint on their resepctive domains $D(L^{\kappa})$ etc., which are all continuously embedded in $\mathscr{H}$. We will regard the domains as Banach spaces equipped with the norms $\|\psi\|_{D(L^{\kappa})} = \|L^{\kappa}\psi\|_{\mathscr{H}} + \|\psi\|_{\mathscr{H}}$. The intersection of two such subspaces is a Banach space with norm $\|\psi\|_{D(L^{\kappa}) \cap D(\Omega^{\eta})} := \max(\|\psi\|_{D(L^{\kappa})}, \|\psi\|_{D(\Omega^{\eta})})$. We will mostly use the equivalent norm given by the sum, i.e. $\|\psi\|_{D(L^{\kappa})} + \|\psi\|_{D(\Omega^{\eta})}$.

4.1 Mapping Properties of G_μ

Let us begin with a technical lemma that will be useful later on. It is concerned with certain properties of the affine function $u(s) := (\beta s - D)/2$. This function itself plays an important role in the following because many relations between the parameters can be expressed with its help.

Lemma 4.1. *Let* $\beta \in (0,2]$*, let* $0 \leq D < \beta$ *if* $\beta < 2$ *and* $0 < D < 2$ *if* $\beta = 2$*. Let* $\varepsilon_0 > 0$ *be such that* $D + \varepsilon_0 = \beta$*. Define for any* $0 < \varepsilon < \varepsilon_0$ *the function*

$$\theta_\varepsilon(\beta, D) := \begin{cases} \frac{2-D-\varepsilon}{2-\beta} & D > \frac{3\beta-2}{\beta} - \varepsilon \\ \max(1/\beta, 1) & D \leq \frac{3\beta-2}{\beta} - \varepsilon\,. \end{cases} \tag{7}$$

Let the affine transformation u for all $s \in [0,\infty)$ *be defined as* $u(s) := (\beta s - D)/2$*. Then it holds that* $\theta_\varepsilon \geq 1$*. Furthermore* $1 + u(\theta_\varepsilon) - \theta_\varepsilon \geq \varepsilon$ *and* $u(\theta_\varepsilon) < 1$*.*

Proof. If $\theta_\varepsilon = 1$, the hypothesis clearly implies that $u(\theta_\varepsilon) < 1$. When $\theta_\varepsilon = 1/\beta$, then $u(\theta_\varepsilon) = (1-D)/2 \leq 1/2$. If $D > \frac{3\beta-2}{\beta} - \varepsilon$ then, by definition of ε_0, it holds that $\beta^2 > 3\beta - 2$. This implies that $\beta \in (0,1)$, in particular $\beta/(2-\beta) < 1$ and therefore $u(\theta_\varepsilon) < (2 - D - \varepsilon - D)/2 < 1$.

In the upper case of (7), the equality $1 + u(\theta_\varepsilon) - \theta_\varepsilon = \varepsilon$ holds by construction. Because $1 + u(s) - s$ is non-increasing, it remains to prove that $\frac{2-D-\varepsilon}{2-\beta}$ is an upper bound for θ_ε. For $1/\beta$ this is the case if and only if $D \leq \frac{3\beta-2}{\beta} - \varepsilon$. If $\theta_\varepsilon = 1$, this follows easily because by definition $2 - D - \varepsilon > 2 - \beta$.

The last step also proves that $\theta_\varepsilon \geq 1$. □

Now we will consider G_μ as an operator into $D(L^\kappa)$ under some conditions on κ. Later, when the target space will be enlarged to $D(\Omega^\eta)$, we will build on some of the formulas obtained here.

Lemma 4.2. *Let* $\beta \in (0,2]$*, let* $0 \leq D < \beta$ *if* $\beta < 2$ *and* $0 < D < 2$ *if* $\beta = 2$*. Then for any* $0 \leq \kappa < (2-D)/4$ *and any* $\mu \geq 1$ *it holds that* G_μ *is continuous from* $D(\Omega^\kappa)$ *to* $D(L^\kappa)$*. There exists* $\mu_0 \geq 1$ *such that the norm of* G_μ *is smaller than* 1 *for all* $\mu > \mu_0$*.*

Proof. We will show that $\left\|L^\kappa G_\mu \psi\right\| \leq C \left\|\Omega_\mu^{\kappa-(1+u(s)-s)} \psi\right\|$ for some constant $C > 0$ and any $s \geq 1$. In view of Lemma 4.1, this proves the claim because

$$\left\|\Omega_\mu^{\kappa-\varepsilon/2} \psi\right\| \leq \mu^{-\varepsilon/2} \left\|\Omega_\mu^\kappa \psi\right\| \leq \mu^{-\varepsilon/2} \left\|\psi\right\|_{D(\Omega_\mu^\kappa)}\,.$$

For later use, we will write $\Xi_\mu := L_\mu$ at first. To estimate $\left|\Xi_\mu^\eta G_\mu \psi\right|^2$, we multiply by $\omega(k_j)^s/\omega(k_j)^s$ for $s \geq 1$ and use the finite dimensional Cauchy-Schwarz inequality:

$$\left|\Xi_\mu^\kappa G_\mu \psi^{(n)}(p,K)\right|^2 \le \sum_{j=1}^{n+1}\sum_{\nu=1}^{n+1} \frac{\omega(k_\nu)^s}{n+1} \frac{\left|v(k_j)\right|^2 \Xi_\mu(p,K)^{2\kappa} \left|\psi^{(n)}(p+k_j,\hat{K}_j)\right|^2}{L_\mu(p,K)^2 \omega(k_j)^s}$$
$$\le \sum_{j=1}^{n+1} \frac{\omega(k_j)^s + \Omega(\hat{K}_j)^s}{n+1} \frac{\left|v(k_j)\right|^2 \Xi_\mu(p,K)^{2\kappa} \left|\psi^{(n)}(p+k_j,\hat{K}_j)\right|^2}{L_\lambda(p,K)^2 \omega(k_j)^s}.$$

In the second step, the fact that $s \ge 1$ is essential. We now use the assumptions $|v(k)| \le |k|^{-\alpha}$ and $\omega(k) \ge |k|^\beta$. This yields for the translated expression $\left|\Xi_\mu^\kappa G_\mu \psi^{(n)}(p-k_j,K)\right|^2$ the bound

$$\left|\Xi_\mu^\kappa G_\mu \psi^{(n)}(p-k_j,K)\right|^2 \le \sum_{j=1}^{n+1} \frac{\left|\psi^{(n)}(p,\hat{K}_j)\right|^2}{n+1} \frac{\Xi_\mu(p-k_j,K)^{2\kappa} \left|k_j\right|^{-2\alpha-\beta s} \Omega(\hat{K}_j)^s}{L_\mu(p-k_j,K)^2}$$
$$+ \sum_{j=1}^{n+1} \frac{\left|\psi^{(n)}(p,\hat{K}_j)\right|^2}{n+1} \frac{\Xi_\mu(p-k_j,K)^{2\kappa} \left|k_j\right|^{-2\alpha}}{L_\mu(p-k_j,K)^2}.$$

Now we use the symmetry of ψ, L and Ξ to note that we can bound the integral over these sums by the integral over the first term of the sums times $n+1$. That is, we have a bound

$$\left\|\Xi_\mu^\kappa G_\mu \psi^{(n)}\right\|^2 = \int \left|\Xi_\mu^\kappa G_\mu \psi^{(n)}(p-k_j,K)\right|^2 \mathrm{d}K\mathrm{d}p$$
$$\le \int \left|\psi^{(n)}(p,\hat{K}_1)\right|^2 \int_{\mathbb{R}^3} \gamma_{\mathrm{d}}^{\Xi_\mu} + \gamma_{\mathrm{od}}^{\Xi_\mu} \ \mathrm{d}k_1 \ \mathrm{d}\hat{K}_1 \mathrm{d}p$$

where

$$\gamma_{\mathrm{d}}^{\Xi_\mu}(p,K) + \gamma_{\mathrm{od}}^{\Xi_\mu}(p,K) := \frac{\Xi_\mu(p-k_1,K)^{2\kappa}}{L_\mu(p-k_1,K)^2 \left|k_1\right|^{2\alpha}} + \frac{\Xi_\mu(p-k_1,K)^{2\kappa}\Omega(\hat{K}_1)^s}{L_\mu(p-k_1,K)^2 \left|k_1\right|^{2\alpha+\beta s}}. \quad (8)$$

We now specify to $\Xi_\mu = L_\mu$ and estimate it from below by $|p-k_1|^2 + \Omega_\mu(\hat{K}_1)$. Recall that since $D \ge 0$ we have by hypothesis $\kappa < 1/2$. So we can bound the integral over k_1 of the off-diagonal part by

$$\int_{\mathbb{R}^3} \gamma_{\mathrm{od}}^{L_\mu}(p,K)\,\mathrm{d}k_1 \le \int_{\mathbb{R}^3} \frac{\Omega_\mu(\hat{K}_1)^s \left|k_1\right|^{-2\alpha-\beta s}}{(\left|p-k_1\right|^2 + \Omega_\mu(\hat{K}_1))^{2(1-\kappa)}} \mathrm{d}k_1 .$$

If $u(s) < 1$ and $2\kappa < u(s)+1$, this integral is by scaling bounded by a constant times

$$\Omega_\mu(\hat{K}_1)^{s+2(\kappa-1)+\frac{3-2\alpha-\beta s}{2}} = \Omega_\mu(\hat{K}_1)^{2\left(\kappa - \frac{1+u(s)-s}{2}\right)}.$$

If $2\kappa < u(0)+1$, we obtain similarly for some $C > 0$ a bound for the diagonal part:

$$\int_{\mathbb{R}^3} \gamma_{\mathrm{od}}^{L_\mu}(p,K)\,\mathrm{d}k_1 \le C\Omega_\mu(\hat{K}_1)^{2(\kappa-1)+\frac{3-2\alpha}{2}} = C\Omega_\mu(\hat{K}_1)^{2\left(\kappa-\frac{1+u(0)}{2}\right)}.$$

Because $\beta > 0$, the function u is increasing so the hypothesis $2\kappa < u(0)+1 = (2-D)/2$ clearly implies $2\kappa < u(s)+1$. In addition $\beta \le 2$, so we can estimate $1+u(s)-s \le 1+u(0)$. □

The next lemma deals with the most important step of the construction, namely the mapping properties of G_μ into $D(\Omega^\eta)$. It is only here (because more explicit computations are used) where the fact that the dimension is equal to three is relevant.

Lemma 4.3. *Let $\beta \in (0,2]$, let $0 \le D < \beta$ if $\beta < 2$ and $0 < D < 2$ if $\beta = 2$. Assume that there exists $\varepsilon_{\mathrm{od}} > 0$ small enough such that*

$$0 \le \eta \le \begin{cases} \dfrac{\frac{\beta(2-D-\varepsilon_{\mathrm{od}})}{2-\beta} - D + 1}{2\beta} & D > \frac{3\beta-2}{\beta} - \varepsilon_{\mathrm{od}}, \\ \frac{2-D}{2\beta} & D \le \frac{3\beta-2}{\beta} - \varepsilon_{\mathrm{od}}. \end{cases}$$

Define for any $\varepsilon_{\mathrm{d}} \ge 0$ the map $q_{\varepsilon_{\mathrm{d}}}(\eta) := \max(0, \eta + \varepsilon_{\mathrm{d}} - (\beta+2-2D)/(4\beta))$. Then for any $\mu \ge 1$ and any $\varepsilon_{\mathrm{d}} > 0$ it holds that G_μ is continuous from $D(\Omega^\eta) \cap D(L^{q_{\varepsilon_{\mathrm{d}}}(\eta)})$ to $D(\Omega^\eta)$ and there exists $\mu_0 \ge 1$ such that the norm of G_μ as a map between these two spaces is smaller than 1 for all $\mu > \mu_0$.

Proof. To estimate the norm of $\Omega_\mu^\eta G_\mu \psi^{(n)}$, we start directly with the expressions $\gamma_{\mathrm{d}}^{\Omega^\eta}$ and $\gamma_{\mathrm{od}}^{\Omega^\eta}$ as they have been defined in (8). Note that we have replaced the exponent κ by η. By defining the rescaled variables $\tilde{p} := p/\Omega_\lambda^{1/2}$ and $\tilde{k} := k_1/\Omega_\lambda^{1/2}$ we can estimate

$$\begin{aligned}\int_{\mathbb{R}^3} \gamma_{\mathrm{d}}^{\Omega^\eta}(p,K)\,\mathrm{d}k_1 &\le \int_{\mathbb{R}^3} \frac{\left(|k_1|^\beta + \mathsf{m} + \Omega_\mu(\hat{K}_1)\right)^{2\eta} |k_1|^{-2\alpha}}{(|p-k_1|^2 + |k_1|^\beta + \Omega_\mu(\hat{K}_1))^2}\,\mathrm{d}k_1 \\ &= \Omega_\mu(\hat{K}_1)^{2\eta-(u(0)+1)} \int_{\mathbb{R}^3} \frac{\left(|\tilde{k}|^\beta + \mathsf{m} + 1\right)^{2\eta} |\tilde{k}|^{-2\alpha}}{\left(|\tilde{p}-\tilde{k}|^2 + \Omega_\mu(\hat{K}_1)^{\frac{\beta-2}{2}} |\tilde{k}|^\beta + 1\right)^2}\mathrm{d}\tilde{k}.\end{aligned}$$

In the very same way we obtain for the integral over k_1 of the off-diagonal part in (8) the upper bound

$$\Omega_\mu(\hat{K}_1)^{2\eta-(1+u(s)-s)} \int_{\mathbb{R}^3} \frac{\left(|\tilde{k}|^\beta + \mathsf{m} + 1\right)^{2\eta} |\tilde{k}|^{-2\alpha-\beta s}}{\left(|\tilde{p}-\tilde{k}|^2 + \Omega_\mu(\hat{K}_1)^{\frac{\beta-2}{2}} |\tilde{k}|^\beta + 1\right)^2}\mathrm{d}\tilde{k}.$$

Abbreviate $\Omega := \Omega_\mu(\hat{K}_1)$, set $\mathsf{M} := \mathsf{m} + 1 \in (0,\infty)$ and denote the remaining integral by

$$\Upsilon(s,\mu,\tilde{p}) := \int_{\mathbb{R}^3} \frac{\left(\left|\tilde{k}\right|^{\beta} + \mathsf{M}\right)^{2\eta} \left|\tilde{k}\right|^{-2\alpha-\beta s}}{\left(\left|\tilde{p}-\tilde{k}\right|^{2} + \Omega^{\beta/2-1}\left|\tilde{k}\right|^{\beta} + 1\right)^{2}} \mathrm{d}\tilde{k}\,. \tag{9}$$

The integral Υ is clearly bounded for any $\tilde{p} \in \mathbb{R}^3$ as long as $\eta < \frac{1+u(s)}{\beta}$ and $u(s) < 1$. If $|\tilde{p}| \leq 1$, we therefore estimate it simply by a constant. So assume in the following that $|\tilde{p}| > 1$ and compute using spherical coordinates

$$\begin{aligned}
\Upsilon(s,\mu,\tilde{p}) &= 2\pi \int_0^\infty \int_{-1}^{1} \frac{(r^\beta + \mathsf{M})^{2\eta} r^{2-2\alpha-\beta s}}{(r^2 + \tilde{p}^2 - 2r\tilde{p}\sigma + r^\beta \Omega^{\frac{\beta-2}{2}} + 1)^2} \mathrm{d}r\mathrm{d}\sigma \\
&= 2\pi \int_0^\infty \frac{(r^\beta + \mathsf{M})^{2\eta} r^{2-2\alpha-\beta s}}{((r-\tilde{p})^2 + r^\beta \Omega^{\frac{\beta-2}{2}} + 1)((r+\tilde{p})^2 + r^\beta \Omega^{\frac{\beta-2}{2}} + 1)} \mathrm{d}r \\
&\leq 2\pi (\tilde{p}^2)^{\eta\beta-(u(s)+1)} \int_0^\infty \frac{(x^\beta + \mathsf{M})^{2\eta} + x^{2-2\alpha-\beta s}}{((x-1)^2 x^\beta p^{\beta-2} + \tilde{p}^{-2})((x+1)^2 + x^\beta p^{\beta-2} + \tilde{p}^{-2})} \mathrm{d}x\,.
\end{aligned}$$

We have replaced $\mathsf{M}/\tilde{p}^\beta$ simply by M because $|\tilde{p}| > 1$. The integral from $x = 2$ to infinity is bounded by a constant, independent of $\tilde{p}$, for any $\eta < \frac{1+u(s)}{\beta}$. The same is true of the integral from zero to $x = 2^{-1/\beta}$. Consider the integral from $2^{1/\beta} < 1$ to 2. On this interval, the numerator of the integral can be estimated by a constant that depends on M, the factor in the denominator that contains the $(x+1)^2$-term is bounded from below by one. It remains to estimate the factor which has a pole at $x = 1$. This can be done by enlarging the domain and making use of fact that the antiderivative of $(1+x^2)^{-1}$ is the arctan. So we have

$$\begin{aligned}
\int_{2^{-1/\beta}}^{2} \frac{1}{((x-1)^2 + x^\beta p^{\beta-2} + \tilde{p}^{-2})} \mathrm{d}x &\leq \int_{2^{-1/\beta}}^{2} \frac{1}{((x-1)^2 + 1/2 p^{\beta-2} + \tilde{p}^{-2})} \mathrm{d}x \\
&\leq \int_{\mathbb{R}} \frac{1}{((x-1)^2 + 1/2 p^{\beta-2} + \tilde{p}^{-2})} \mathrm{d}x = \pi \left[1/2 p^{\beta-2} + \tilde{p}^{-2}\right]^{-1/2}.
\end{aligned}$$

Recall that the other parts of this integral are bounded by a constant. So, because $\tilde{p} > 1$ implies $p > 1$, we can bound as a whole:

$$\begin{aligned}
\chi_{\{\tilde{p}>1\}} \Upsilon(s,\mu,\tilde{p}) &\leq \chi_{\{\tilde{p}>1\}} \left(C + \left[1/2 p^{\beta-2} + \tilde{p}^{-2}\right]^{-1/2}\right) \\
&\leq C' \chi_{\{\tilde{p}>1\}} (p^{\frac{2-\beta}{2}})^{(1-t)} (\tilde{p})^t\,.
\end{aligned} \tag{10}$$

Here we have introduced a parameter $t \in [0,1]$. Now we have to distinguish between the diagonal term in (8), where we have $s = 0$ and choose $t = 0$ in (10), and the off-diagonal term where we choose $t = 1$ in (10) and observe that $s \geq 1$ is required. The off-diagonal term hence can be bounded by

$$\int_{\mathbb{R}^3} \gamma_{\mathrm{od}}^{\Omega^\eta}(p,K)\,\mathrm{d}k_1 \le C\Omega_\mu(\hat{K}_1)^{2\eta-(u(s)+1-s)}\left(\chi_{\{\tilde{p}\le 1\}} + \chi_{\{\tilde{p}>1\}}\tilde{p}^{2\eta\beta-2(u(s)+1))+1}\right).$$

We would like to have – for the off-diagonal term – a bound independent of p. To achieve this, we apply Lemma 4.1 and choose $s = \theta_{\varepsilon_{\mathrm{od}}}$ for an $\varepsilon_{\mathrm{od}} > 0$ admissible there. Then we can see that our upper bounds on η are such that the exponent of $\tilde{p}$ is non-positive. This is because for $s = \theta_{\varepsilon_{\mathrm{od}}}$ the exponent becomes

$$2\eta\beta - 2(u(\theta_{\varepsilon_{\mathrm{od}}})+1))+1 = 2\beta\eta - 2\beta\begin{cases}\frac{\frac{\beta(2-D)-\beta\varepsilon_{\mathrm{od}}}{2-\beta}-D+1}{2\beta} & D > \frac{3\beta-2}{\beta}-\varepsilon_{\mathrm{od}}\\ \frac{\beta\max(1,1/\beta)-D+1}{2\beta} & D \le \frac{3\beta-2}{\beta}-\varepsilon_{\mathrm{od}},\end{cases}$$

and obviously $1 \le \beta\max(1,1/\beta)$. These considerations imply that the norm of the off-diagonal term is bounded by $\left\|\Omega_\mu^{\eta-\varepsilon_{\mathrm{od}}/2}\psi\right\|^2 \le \mu^{-\varepsilon_{\mathrm{od}}}\left\|\Omega_\mu^\eta\psi\right\|$.

We are not able to obtain a bound independent of p also for the diagonal term in (8). Setting $s = t = 0$ in (10), yields for the integral $\int_{\mathbb{R}^3}\gamma_{\mathrm{d}}^{\Omega^\eta}(p,K)\,\mathrm{d}k_1$ a bound of the form constant times

$$\begin{aligned}&\Omega_\mu(\hat{K}_1)^{2\eta-(u(0)+1)}\left(\chi_{\{\tilde{p}\le 1\}} + \chi_{\{\tilde{p}>1\}}\Omega_\mu(\hat{K}_1)^{-\eta\beta+(u(0)+1)}p^{2\eta\beta-2(u(0)+1)+\frac{2-\beta}{2}}\right)\\ &= \Omega_\mu(\hat{K}_1)^{2\eta-(u(0)+1)}\chi_{\{\tilde{p}\le 1\}} + \chi_{\{\tilde{p}>1\}}\Omega_\mu(\hat{K}_1)^{2\eta-\eta\beta}p^{2\beta(\eta-(\beta+2-2D)/(4\beta))}.\end{aligned}$$

Due to the fact that $D < \beta \le 2$, the first term here is bounded by $\mu^{-u(0)-1}\Omega_\mu^{2\eta}$ for all $\tilde{p} \in [0,\infty)$. To bound the second term, introduce an $\varepsilon_{\mathrm{d}} > 0$, which yields

$$\begin{aligned}&\chi_{\{\tilde{p}>1\}}\Omega_\mu(\hat{K}_1)^{2\eta-\eta\beta}p^{2\beta(\eta-(\beta+2-2D)/(4\beta))}\\ &\qquad\le \chi_{\{\tilde{p}>1\}}\Omega_\mu(\hat{K}_1)^{\eta(2-\beta)}\mu^{-\varepsilon_{\mathrm{d}}\beta}(p^2+\mu)^{\beta(\eta+\varepsilon_{\mathrm{d}}-(\beta+2-2D)/(4\beta))}\\ &\qquad\le \Omega_\mu(\hat{K}_1)^{\eta(2-\beta)}\mu^{-\varepsilon_{\mathrm{d}}\beta}(p^2+\mu)^{\beta q_{\varepsilon_{\mathrm{d}}}(\eta)}\end{aligned}$$

We have used in particular that $\mu \ge 1$ to get rid of the characteristic function. Now we apply Young's inequality with $\nu = 2/(2-\beta)$ and $\xi = 2/\beta$, which leads to the upper bound

$$C\mu^{-\varepsilon\beta}\left(\Omega_\mu(\hat{K}_1)^{2\eta} + (p^2+\mu)^{2q_{\varepsilon_{\mathrm{d}}}(\eta)}\right).$$

Because $\beta > 0$, the norm of this term goes to zero as $\mu \to \infty$. This proves the claim. □

The Neumann series is a candidate for the inverse of the operator $1 - G_\mu$. On domains where the norm of G_μ is decreasing, the series will converge for large enough μ.

Corollary 4.4. *Let* $\beta \in (0,2]$, *let* $0 \le D < \beta$ *if* $\beta < 2$ *and* $0 < D < 2$ *if* $\beta = 2$. *Let* $\eta, \kappa \ge 0$. *Assume that for any* $\varepsilon > 0$ *small enough*

$$0 \le \eta < \begin{cases} \frac{\frac{\beta(2-D-\varepsilon)}{2-\beta}-D+1}{2\beta} & D > \frac{3\beta-2}{\beta}-\varepsilon, \\ \frac{2-D}{2\beta} & D \le \frac{3\beta-2}{\beta}-\varepsilon \end{cases} \tag{11}$$

and $\max(\kappa, q_0(\eta)) < \frac{2-D}{4}$*. Then there exists* $\mu_0 \ge 1$ *such that* $1-G_\mu$ *is continuously invertible on* $D(\Omega^\eta) \cap D(L^{\max(\kappa, q_\varepsilon(\eta))})$ *for any* $\mu > \mu_0$*, possibly for a smaller* $\varepsilon > 0$*.*

Proof. We make $\varepsilon > 0$ possibly smaller, such that also $\max(\kappa, q_\varepsilon(\eta)) < \frac{2-D}{4}$. Then Lemma 4.2 implies that for any $\eta \ge 0$ it holds that

$$\left\|G_\mu \psi\right\|_{D(L^{\max(\kappa,q_\varepsilon(\eta))})} \le c(\mu)\,\|\psi\|_{D(\Omega^{\max(\kappa,q_\varepsilon(\eta))})} \le c(\mu)\,\|\psi\|_{D(\Omega^\eta)\cap D(L^{\max(\kappa,q_\varepsilon(\eta))})}$$

with $c(\mu) < 1$ for μ larger than some $\mu_0 \ge 1$. Due to the assumptions we have made on η, the Lemma 4.3 gives

$$\left\|G_\mu \psi\right\|_{D(\Omega^\eta)} \le C(\mu)\,\|\psi\|_{D(\Omega^\eta)\cap D(L^{q_\varepsilon(\eta)})} \le C(\mu)\,\|\psi\|_{D(\Omega^\eta)\cap D(L^{\max(\kappa,q_\varepsilon(\eta))})}$$

with $C(\mu) < 1$ if $\mu > \mu_0$ for some $\mu_0 \ge 1$. The last inequality simply holds because $\mu \ge 1$ and $q_\varepsilon(\eta) \le \max(\kappa, q_\varepsilon(\eta))$. □

We are now ready to prove that the "free" operator $H_0^\mu := (1-G_\mu)^* L_\mu (1-G_\mu)$ is self-adjoint. To prove self-adjointness of the whole operator H in Section 4.3, the operator T^μ will be regarded as an operator perturbation of H_0^μ.

Corollary 4.5. *Let* $\beta \in (0,2]$*, let* $0 \le D < \beta$ *if* $\beta < 2$ *and* $0 < D < 2$ *if* $\beta = 2$*. Then* H_0^μ *is self-adjoint and positive on* $D(H_0^\mu) = \mathfrak{D} = \{\psi \in \mathscr{H} | (1-G_\mu)\psi \in D(L)$ *for some* $\mu > 0\}$*.*

Proof. Apply Corollary 4.4 with $\eta = \kappa = 0$. This is possible because the upper bounds on η and κ are positive for $D < \beta$ and in addition $q_0(0) \le 0$. That means that $(1-G_\mu)$ is invertible on $\mathscr{H}$ for $\mu \ge 1$ large enough, so $D(H_0^\mu) := \mathfrak{D}$ is dense in $\mathscr{H}$. The operator H_0^μ is clearly symmetric and positive and it is easy to see that $\varphi \in D((H_0^\mu)^*)$ implies $\varphi \in D(H_0^\mu)$. □

4.2 The Domain $\mathfrak{D}$: Proof of Proposition 3.1

In order to determine supersets for $\mathfrak{D}$, we can now build on the results of the previous section. The domain can be characterised as $\mathfrak{D} = (1-G_\mu)^{-1} D(L)$ for any $\mu \ge 1$ admissible in Corollary 4.5. Therefore any subspace of the form $(1-G_\mu)^{-1}\mathscr{S}$ with $D(L) \hookrightarrow \mathscr{S} \subset \mathscr{H}$ is also a superset for $\mathfrak{D}$. If $1-G_\mu$ is invertible on $(\mathscr{S}, \|\cdot\|_{\mathscr{S}})$, we have $(1-G_\mu)^{-1}\mathscr{S} = \mathscr{S}$, which then allows us to explicitly characterise this space. In this section, we will restrict the range of parameters to pairs where $D < \beta/2$ in contrast to β. In this way, the various conditions on η can be significantly simplified.

Proposition 4.6. *Let* $\beta \in (0,2]$*, let* $0 \le D < \beta/2$ *if* $\beta < 2$ *and* $0 < D < 1$ *if* $\beta = 2$*. Define for any* $\sigma \in (0,1]$ *the subspace* $\mathfrak{D}^\sigma = \{\psi \in \mathscr{H} | (1-G_\mu)\psi \in D(L^\sigma)$ *for some* $\mu > 0\}$.

- *For any* $\eta \in [0,\sigma]$ *with* $\eta < \frac{2-D}{2\beta}$ *it holds that* $\mathfrak{D}^\sigma \subset D(\Omega^\eta) \cap D(L^{q_\varepsilon(\eta)})$ *for any* $\varepsilon > 0$ *small enough.*
- *For any* $\kappa \in [0,\sigma]$ *with* $\kappa < \frac{2-D}{4}$ *it holds that* $\mathfrak{D}^\sigma \subset D(L^\kappa)$*.*

Proof. The first task will be to perform the promised simplification of the conditions on η in Corollary 4.4. First, observe that $\eta \le \sigma$ means of course also $\eta \le 1$. We will now prove that $\eta \le 1$ together with $D < \beta/2$ implies that, if $\varepsilon > 0$ can be arbitrarily small, then

$$\eta < \frac{\frac{\beta(2-D-\varepsilon)}{2-\beta} - D + 1}{2\beta} \quad \text{if} \quad D > \frac{3\beta - 2}{\beta} - \varepsilon\,.$$

To show this, observe that $\frac{3\beta-2}{\beta} < D + \varepsilon < \beta/2 + \varepsilon$ means that β has to fulfill the inequality $2\beta\varepsilon > 6\beta - \beta^2 - 4$. This can, for ε small enough, only be satisfied for $\beta < 4/5$. Using again $D < \beta/2$ we bound, possibly making $\varepsilon > 0$ smaller,

$$\begin{aligned}\frac{\frac{\beta(2-D-\varepsilon)}{2-\beta} - D + 1}{2\beta} - 1 &> \frac{\beta(2-\beta/2-\varepsilon) - (\beta/2 - 1 + 2\beta)(2-\beta)}{2\beta(2-\beta)} \\ &= \frac{(1-\beta)^2 - \beta\varepsilon}{\beta(2-\beta)} > \frac{5^{-2} - (4/5)\varepsilon}{2(4/5)} > 0\,.\end{aligned}$$

To sum up, we have shown that if $\eta \le 1$ then the upper case of (11) is fulfilled. The lower case in this very condition is also satisfied by hypothesis.

Our second step is to show that the assumptions $\eta \le 1$ and $\eta < \frac{2-D}{2\beta}$ are such that also $q_0(\eta) < \frac{2-D}{4}$. Note that the latter condition is equivalent to $\eta < \frac{2-D}{4} + \frac{\beta+2(1-D)}{4\beta}$. Using $D < \beta/2$ we now bound from below

$$\begin{aligned}\frac{2-D}{4} + \frac{\beta+2(1-D)}{4\beta} - 1 &> \frac{4-4\beta-\beta^2}{8\beta} \\ \text{and} \quad \frac{2-D}{4} + \frac{\beta+2(1-D)}{4\beta} - \frac{2-D}{2\beta} &> \frac{6\beta-\beta^2-4}{8\beta}\,.\end{aligned}$$

Observe that for any β at least one of these functions is positive. So if either $\eta \le 1$ or $\eta < \frac{2-D}{2\beta}$ then also $q_0(\eta) < \frac{2-D}{4}$. The above considerations allow us to apply the Corollary 4.4 and proceed with the main part of the proof.

For η, κ fulfilling the hypothesis, we define $S_1 := \Omega^\eta$ and $S_2 := L^{\max(\kappa, q_\varepsilon(\eta))}$ and $\mathscr{S} = (D(S_1) \cap D(S_2), \|\cdot\|_{D(S_1)} + \|\cdot\|_{D(S_2)})$. Recall that $\mu \ge 1$ implies $D(L^\sigma) \hookrightarrow D(L^{\min(\eta,\kappa)})$. Therefore we may consider the chain of inclusions $D(L) \hookrightarrow D(L^\sigma) \hookrightarrow \mathscr{S}$. Furthermore $\|S_i\psi\|_{\mathscr{H}} \le \|\psi\|_{\mathscr{S}}$ and denoting $C_\mu := \left\|(1-G_\mu)^{-1}\right\|_{\mathscr{L}(\mathscr{S})}$ we have

$$\begin{aligned}\|S_i\psi\|_{\mathscr{H}} &\le \|\psi\|_{\mathscr{S}} = \left\|(1-G_\mu)^{-1}(1-G_\mu)\psi\right\|_{\mathscr{S}} \le C_\mu \left\|(1-G_\mu)\psi\right\|_{\mathscr{S}} \\ &\le C_\mu C' \left\|(1-G_\mu)\psi\right\|_{D(L^\sigma)} = C_\mu C' \left(\left\|L^\sigma_\mu(1-G_\mu)\psi\right\|_{\mathscr{H}} + \left\|(1-G_\mu)\psi\right\|_{\mathscr{H}}\right).\end{aligned} \tag{12}$$

Inserting $1=(1-G_\mu)^{-*}(1-G_\mu)^*$ yields the desired bound. In order to obtain the first part of the statement, we can set $\kappa=0$. For the second part we choose $\eta=0$ which implies $D(\Omega^\eta)=\mathscr{H}$ and $q_\varepsilon(\eta)\le 0$ for $\varepsilon>0$ small enough. □

Corollary 4.7. *Let $\beta\in(0,2]$, let $0\le D<\beta/2$ if $\beta<2$ and $0<D<1$ if $\beta=2$.*

- *For any $\eta\in[0,1)$ with $\eta<\frac{2-D}{2\beta}$ there exists $\mu_0\ge 1$ such that for any $\mu>\mu_0$ the operator Ω^η_μ is infinitesimally bounded with respect to H^μ_0*
- *For any $\kappa\ge 0$ with $\kappa<\frac{2-D}{4}$ there exists $\lambda_0\ge 1$ such that for any $\lambda>\lambda_0$ the operator L^κ_λ is infinitesimally bounded with respect to H^λ_0.*

Proof. Because $\eta<1$, by Young's inequality, we have

$$\left\|L^\eta_\mu\varphi\right\| \le \tilde{C}(\varepsilon\left\|L_\mu\varphi\right\| + \varepsilon^{-\eta/(1-\eta)}\|\varphi\|). \tag{13}$$

for any $\varepsilon>0$ and any $\varphi\in D(L)$. In (12) we can set $\sigma=\eta$, and because $\varphi=(1-G_\mu)\psi\in(1-G_\mu)\mathfrak{D}^\eta\subset D(L)$, we can use (13) such that

$$\|\Omega^\eta\psi\|_{\mathscr{H}} \le C_\mu C'\tilde{C}\left(\varepsilon\left\|L_\mu(1-G_\mu)\psi\right\|_{\mathscr{H}} + (1+\varepsilon^{-\eta/(1-\eta)})\left\|(1-G_\mu)\psi\right\|_{\mathscr{H}}\right).$$

Using $1=(1-G_\mu)^{-*}(1-G_\mu)^*$, we prove infintesimal boundedness of Ω^η with respect to H^μ_0 if μ is large enough. The case of L^κ can be proved in exactly the same way. □

Now we are well prepared to prove Proposition 3.1.

*Proof (**Proof of Proposition 3.1**).* One of the implications is provided by Proposition 4.6. It remains to prove that $0\ne\psi\in\mathfrak{D}^\sigma$ implies that $\|L^\kappa\psi\|$ or $\|\Omega^\eta\psi\|$ are infinite if $\kappa\ge\frac{2-D}{4}$ or $\eta\ge\frac{2-D}{2\beta}$, respectively. For later use we write Ξ_μ to denote either L_μ or Ω_μ. Decomposing $\Xi^\kappa\psi=\Xi^\kappa_\mu(1-G_\mu)\psi+\Xi^\kappa_\mu G_\mu\psi$ we see that, because in any case $\kappa,\eta\le\sigma$, the norm of the first term is always finite. Recall that we have $\mu\ge 1$. Choose $n\in\mathbb{N}$ such that $\psi^{(n)}\ne 0$. For any $r>0$ we define the set

$$U_r := \{(p,K)\in\mathbb{R}^{3+3(n+1)}\,|\,|p|<r,\ \left|k_j\right|<r \text{ for all } 2\le j\le n+1\}.$$

We will now show that we can choose $r>0$ such that $\left\|\Xi^\kappa_\mu G_\mu\psi^{(n)}\right\|^2_{L^2(U_r)}$ is infinite. To do so we will split the sum that consitutes G_μ and apply the inequality $\frac{t-1}{t}a^2-(t-1)b^2\le|a+b|^2$ for $t=2$. In addition we use that $(\sum_{j=1}^n a_j)^2\le n\sum_{j=1}^n a_j^2$. Taken together, this leads to the lower bound

$$\left|L_\mu G_\mu \psi^{(n)}\right|^2 \geq \frac{|v(k_1)|^2\, \Xi_\mu(p,K)^{2\kappa} \left|\psi(p+k_1,\hat{K}_1)\right|^2}{2(n+1)L_\mu(p,K)^2} - \sum_{j=2}^{n+1} \frac{\left|v(k_j)\right|^2 \Xi_\mu(p,K)^{2\kappa} \left|\psi(p+k_j,\hat{K}_j)\right|^2}{L_\mu(p,K)^2}. \tag{14}$$

We procced by showing that the integral over U_r of the n lower terms in (14), all coming with a minus, is finite, but the integral of the first term is not. We enlarge the domain of integration to all $p \in \mathbb{R}^3$ and perform a change of variables in $p \to p+k_j$ to obtain an upper bound for the integral over one of these terms:

$$\int_{U_r} \frac{\left|v(k_j)\right|^2 \Xi_\mu(p,K)^{2\kappa} \left|\psi(p+k_j,\hat{K}_j)\right|^2}{L_\mu(p,K)^2} \mathrm{d}p\mathrm{d}K \leq \int_{\mathbb{R}\times B_r^n} \left|\psi(p,\hat{K}_j)\right|^2 \int_{|k_j|<r} \frac{\Xi_\mu(p-k_j,K)^{2\kappa}}{L_\mu(p-k_j,K)^2 \left|k_j\right|^{2\alpha}} \mathrm{d}p\mathrm{d}K.$$

Here B_r denotes the ball of radius r in $\mathbb{R}^3$. Specifying to $\Xi_\mu = L_\mu$, we can bound the k_j-integral, using the fact that $\kappa < 1$ and $\mu \geq 1$, by $\int_{|k_j|<r} \left|k_j\right|^{-2\alpha} \mathrm{d}k_j$. This is clearly finite since $\alpha < d/2$ by hypothesis. For $\Xi_\mu = \Omega_\mu$ and $\kappa \to \eta$ we bound $\Omega_\mu(K)^{2\eta} L_\mu(p-k_j,K)^{-2} \leq \Omega_\mu(K)^{2(\eta-1)} \leq 1$ and conclude in the same way.

To bound the integral over the first term in (14) from below, we use the assumption $|v(k)| \geq c(1+|k|^\alpha)^{-1}$ and the fact that $\omega(k) \leq |k|^\beta + \mathsf{m}$ implies $\Omega(\hat{K}_1) \leq C$ for some constant on U_r:

$$\int_{U_r} \frac{|v(k_1)|^2\, \Xi_\mu(p,K)^{2\kappa} \left|\psi(p+k_1,\hat{K}_1)\right|^2}{L_\mu(p,K)^2} \mathrm{d}p\mathrm{d}K \geq c \int_{B_r\times B_r^n} \left|\psi(p,\hat{K}_1)\right|^2 \int_{\mathbb{R}^3} \frac{\Xi_\mu(p-k_1,K)^{2\kappa}}{(1+|k_1|^\alpha)^2((p-k_1)^2+k_1^\beta+\mathsf{m}+C)^2} \mathrm{d}p\mathrm{d}K. \tag{15}$$

When $\Xi_\mu = L_\mu$, we bound the integral over k_1 from below by

$$\int_{\mathbb{R}^3} \frac{(p-k_1)^{4\kappa}}{(1+|k_1|^\alpha)^2((p-k_1)^2+k_1^\beta+\mathsf{m}+C)^2} \mathrm{d}k_1,$$

which does not converge for any fixed $p \in \mathbb{R}^3$ and $\kappa \geq (2-D)/4$. The same is true if $\Xi_\mu = \Omega_\mu \geq |k|^\beta$ and $\eta \geq (2-D)/(2\beta)$. Because $\psi^{(n)} \neq 0$, we can choose $r > 0$ such that the integral (15) is infinite. This proves the claim. □

4.3 Self-Adjointness: Proof of Theorem 3.3

At first we have to make sure that the construction sketched in the introduction is in fact possible in our case. We start by observing that the lower bound $c(1+|k|^{\alpha})^{-1} \leq |v(k)|$ and the restriction $\alpha < 3/2$ implies that $v \notin L^2$. Thus, by [LS18, Lem. 2.2], $\ker a(V)$ is dense in $\mathscr{H}$ and the adjoint of $L_\mu\big|_{\ker a(V)} = L_{\mu,0}$ is well defined. Using the fact that G_μ maps into $\ker L^*_{\mu,0}$, we arrive at the representation (6), which we repeat for the convenience of the reader:

$$H = (1-G_\mu)^* L_\mu (1-G_\mu) + T^\mu - \mu\,.$$

As has been discussed already in the introduction, it is necessary to prove infinitesimal boundedness of T^μ with respect to the self-adjoint operator H_0^μ (see Corollary 4.5) for some $\mu \geq 1$. Then we can conclude with Kato-Rellich. We will not aim at proving new results about $D(T)$ but instead recall that $u(s) := \frac{\beta}{2}s - \frac{D}{2}$ and cite the existing ones.

Lemma 4.8 (Lemma 3.6 of [LS18]). *Assume $D \geq 0$. Then for any $\varepsilon > 0$ the expression T_{d}^μ given by* (2) *defines a symmetric operator on the domain $D(L^{\max(\varepsilon,D/2)})$ for any $\mu \geq 1$.*

Lemma 4.9 (Lemma 3.8 of [Sch18]). *Assume $D \geq 0$. Then, for all $s > 0$ such that $u(s) < 1$ and $0 < u(u(s))$, the operator T_{od}^μ, defined in* (1)*, is bounded from $D(N^{\max(0,1-s)}\Omega^{s-u(s)})$ to $\mathscr{H}$ and is symmetric on this domain for any $\mu \geq 1$.*

In order to apply the result of Lemma 4.9, we clearly have to restrict to $s \geq 1$ as usual.

*Proof (**Proof of Theorem 3.3**).* Decompose into diagonal and off-diagonal terms $T^\mu = T_{\mathrm{d}}^\mu + T_{\mathrm{od}}^\mu$. Due to Lemma 4.8, we have a bound $\left\|T_{\mathrm{d}}^\mu \psi\right\| \leq \left\|L_\mu^{\max(\varepsilon,D/2)}\psi\right\|$. As long as μ is greater than some μ_0 and $D < 2/3$, the second part of Corollary 4.7 implies that the diagonal part of the operator is infinitesimally bounded by H_0^μ. To proceed analogously for the off-diagonal part we need that for $s \geq 1$ Lemma 4.9 is applicable, so necessarily

$$u(s) < 1 \tag{1}$$

$$u(u(s)) > 0\,. \tag{2}$$

In this way we can bound the norm of $T_{\mathrm{od}}^\mu \psi$ by the norm of $\Omega_\mu^{s-u(s)}\psi$. Then we would like to conclude the infinitesimal boundedness by setting $\eta = s - u(s)$ in Corollary 4.7. To do so, we have to make sure that

$$s - u(s) < 1, \tag{3}$$

$$s - u(s) < \frac{2-D}{2\beta} \tag{4}$$

These four condition can be converted into bounds on D that depend on β and s:

$$\begin{aligned} D &> \beta s - 2 =: f_1(s) \\ D &< s\frac{\beta^2}{\beta+2} =: f_2(s) \\ D &< 2 - s(2-\beta) =: f_3(s) \\ D &< \frac{2 - s\beta(2-\beta)}{\beta+1} =: f_4(s) \end{aligned}$$

If $\beta = 2$, we choose $s = 1$ and $D \in (0, 2/3)$ to satisfy all four conditions. For $\beta \in (0,2)$, we assume $D \geq 0$ and set

$$F := \min_{i=3,4,2} f_i : [1, 2/\beta) \to \mathbb{R}. \tag{5}$$

On this interval $[1, 2/\beta)$, the Condition (1) is always satisfied and the Lemma 4.10 below completes the proof of Theorem 3.3 because it confirms the upper bound on D. □

Lemma 4.10. *Let F be as defined in* (5)*. For $\beta \in (0,2)$ it holds that*

$$\max_{s\in[1,2/\beta)} F(s) = \begin{cases} \frac{\beta^2}{2} & \beta \in (0, 2(\sqrt{2}-1)) \\ \frac{2\beta}{\beta+4} & \beta \in [2(\sqrt{2}-1), \sqrt{5}-1) \\ \frac{\beta^2-2\beta+2}{\beta+1} & \beta \in [\sqrt{5}-1, 2). \end{cases}$$

Proof. Closing the interval at the right endpoint we conclude that the supremum is attained, and we denote the point where this happens by s_*. All functions f_i are affine functions on $[1, 2/\beta]$. The fact that $(2-\beta) \geq 0$ implies that f_3 and f_4 are non-increasing whereas f_2 is clearly increasing. Thus we have

$$F(s) = \begin{cases} f_2(s) & f_2(s) < \min(f_3(s), f_4(s)) \\ \min(f_3(s), f_4(s)) & f_2(s) \geq \min(f_3(s), f_4(s)). \end{cases} \tag{6}$$

If $\beta \geq \sqrt{5}-1$ then $f_4(1) \leq \min(f_3(1), f_2(1))$. This however implies that it holds that $F(s) = \min(f_3(s), f_4(s))$ and consequently $s_* = 1$. We can also conclude that

$$F(s_*) = \min(f_3(1), f_4(1)) = f_4(1) = \frac{\beta^2 - 2\beta + 2}{\beta+1}.$$

Now consider the case where $\beta < \sqrt{5}-1$. Because $f_2(1) < f_i(1)$ for $\beta \in (0, \sqrt{5}-1)$, we observe that the first case of (6) is never empty. Consequently

$$s_* := \{s \in [1, 2/\beta] \,|\, f_2(s) = \min(f_3(s), f_4(s))\} = \min_{i=3,4} \{s \in [1, 2/\beta] \,|\, f_2(s) = f_i(s)\}$$

and of course $F(s_*) = f_2(s_*)$. We find that

$$f_2(s) = f_3(s) \iff s = s_3 := \frac{\beta+2}{2}$$
$$f_2(s) = f_4(s) \iff s = s_4 := \frac{2(\beta+2)}{\beta(\beta+4)}$$

and, because both s_3 and s_4 lie in the interval $[0, 2/\beta]$, that means

$$\begin{aligned} s_* = \min(s_3, s_4) &= \begin{cases} s_3 & \beta \in (0, 2(\sqrt{2}-1)) \\ s_4 & \beta \in [2(\sqrt{2}-1), \sqrt{5}-1) \end{cases} \\ &= \frac{\beta+2}{2} \begin{cases} 1 & \beta \in (0, 2(\sqrt{2}-1)) \\ \frac{4}{\beta(\beta+4)} & \beta \in [2(\sqrt{2}-1), \sqrt{5}-1)\,. \end{cases} \end{aligned}$$

Insert this into f_2 and note that $s_* < 2/\beta$. This yields the desired expression for $\max_{s\in[1,2/\beta)} F(s)$. □

5 Concluding Remarks

We would like to adress two points that have not been discussed so far. We have not said anything yet about the connection of the IBC approach to renormalisation procedures in the massless case. In [Sch18, Prop. 3.4], it is shown that for quite general massive models the cutoff operator plus renormalisation constant $H_\Lambda + E_\Lambda$ converges in norm resolvent sense to the Hamiltonian H. In this cutoff operator, the form factor in the formal expression $L + a(V) + a^*(V)$ is replaced by $\chi_\Lambda v$ for the characteristic function χ_Λ of a ball of radius Λ.

As we will argue in the following, such a result does also hold in the case of massive or massless models if Condition 3.2 is satisfied. Denote by $G_{\mu,\Lambda}$ and T^μ_Λ the corresponding operators with v replaced by $\chi_\Lambda v$. The parameter $\mu \geq 1$ is chosen as large as necessary and fixed. For the proof of norm resolvent convergence, convergence of $G_{\mu,\Lambda}$ in the $\mathscr{H}$-norm (to G_μ) is needed. As long as $u(1) \in (0,1)$, this follows in the massless case exactly as in [Sch18, Prop. 4.4] by explicitly computing symmetric decreasing rearrangements. To prove convergence of the STM-operator T^μ, it is convenient to decompose again into diagonal and off-diagonal parts. Using [Sch18, Lem. 3.6 and 3.8], we can prove convergence of $T^\mu_{\mathrm{d},\Lambda} + E_\Lambda$ on $D(L^\kappa)$ and of $T^\mu_{\mathrm{od},\Lambda}$ on $D(\Omega^\eta)$ for some κ, η. It turns out that κ, η are such that $T^\mu_\Lambda + E_\Lambda \to T^\mu$ on $D(H)$. This would complete the proof of norm resolvent convergence.

Although the case of a single particle was considered in this contribution in order to keep the notation simple, the case of $M > 1$ particles could be included in the analysis as well. This is because when bounding norms of $G_\mu \psi$ from above, the relevant estimates are the same for $M = 1$ and $M > 1$. For bounds from below, as in Section 4.2, one has to take care of some more cross-terms because the domain of integration is chosen to be not symmetric under exchange of particles. It is only the T-operator where a significant difference occurs. The off-diagonal part of T consists for $M > 1$ of additional terms, which are called θ-terms in [LS18]. They are however bounded on $D(L^{\max(\varepsilon,D/2)})$ for any $\varepsilon > 0$, exactly as the diagonal part of T, see [Sch18, Lem. 3.7]. In the context of the above analysis, these θ-terms can therefore be put together with T_{d}^{μ} and pose almost no constraints on the allowed pairs (β, D).

References

[BDP12] S. Bachmann, D.-A. Deckert and A. Pizzo, *The mass shell of the Nelson model without cut-offs*, J. Funct. Anal. **263** (2012), no. 5, 1224–1282.

[CDF+15] M. Correggi, G. Dell'Antonio, D. Finco, A. Michelangeli and A. Teta, *A class of Hamiltonians for a three-particle fermionic system at unitarity*, Math. Phys. Anal. Geom. **18** (2015), no. 1, 32.

[DGS+18] D. Dürr, S. Goldstein, S. Teufel, R. Tumulka and N. Zanghì *Bohmian Trajectories for Hamiltonians with Interior-Boundary Conditions* arXiv preprint arXiv:1809.10235

[Fro74] J. Fröhlich, *Existence of dressed one electron states in a class of persistent models*, Fortschritte der Physik **2** (1974), 159–198.

[GW18] M. Griesemer and A. Wünsch, *On the domain of the Nelson Hamiltonian*, J. Math. Phys. **59** (2018), no. 4, 042111.

[KS16] S. Keppeler and M. Sieber, *Particle creation and annihilation at interior boundaries: one-dimensional models*, J. Phys. A **49** (2016), no. 12, 125204.

[Lam18a] J. Lampart, *A nonrelativistic quantum field theory with point interactions in three dimensions*, arXiv preprint arXiv:1804.08295 (2018).

[Lam18b] J. Lampart, *A polaron model with point interactions in three dimensions*, Contribution for the 3rd workshop on *Mathematical Challenges of Zero-Range Physics: rigorous results and open problems*, In preparation.

[LS18] J. Lampart and J. Schmidt, *On Nelson-type Hamiltonians and abstract boundary conditions*, to appear in Comm. Math. Phys., arXiv:1803.00872 (2018).

[LSTT18] J. Lampart, J. Schmidt, S. Teufel and R. Tumulka, *Particle creation at a point source by means of interior-boundary conditions*, Math. Phys. Anal. Geom. **21** (2018), no. 2, 12.

[LN18] M. Lienert and L. Nickel, *Multi-time formulation of particle creation and annihilation via interior-boundary conditions* arXiv preprint arXiv:1808.04192 (2018)

[MM17] O. Matte and J. S. Møller, *Feynman-Kac formulas for the ultra-violet renormalized Nelson model*, arXiv preprint arXiv:1701.02600 (2017).

[MO17] A. Michelangeli and A. Ottolini, *On point interactions realised as Ter-Martyrosyan–Skornyakov operators*, Rep. Math. Phys. **79** (2017), no. 2, 215–260.

[Nel64] E. Nelson, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys. **5** (1964), no. 9, 1190–1197.

[Piz03] A. Pizzo, *One-particle (improper) States in Nelson's Massless Model*, Ann. Henri Poincaré **4** (2003), no. 3, 439.

[Sch18] J. Schmidt, *On a Direct Description of Pseudorelativistic Nelson Hamiltonians*, arxiv preprint arXiv:1810.03313 (2018).

[ST18] J. Schmidt and R. Tumulka, *Complex Charges, Time Reversal Asymmetry, and Interior-Boundary Conditions in Quantum Field Theory*, to appear in J. Phys. A, arXiv:1810.02173 (2018).

[TT16] S. Teufel and R. Tumulka, *Avoiding ultraviolet divergence by means of interior–boundary conditions*, Quantum Mathematical Physics (F. Finster, J. Kleiner, C. Röken, and J. Tolksdorf, eds.), Birkhäuser, 2016, 293–311.