Effective dynamics of interacting bosons: Quasi-low-dimensional gases and higher order corrections to the mean-field description

Dissertation
der Mathematisch-Naturwissenschaftlichen Fakultät
der Eberhard Karls Universität Tübingen
zur Erlangung des Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt von
Lea Boßmann
aus München

Tübingen
2019

Gedruckt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Eberhard Karls Universität Tübingen.

Tag der mündlichen Qualifikation:	11.10.2019
Dekan:	Prof. Dr. Wolfgang Rosenstiel
1. Berichterstatter:	Prof. Dr. Stefan Teufel
2. Berichterstatter:	Prof. Dr. Peter Pickl
3. Berichterstatterin:	Dr. Serena Cenatiempo

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Zusammenfassung

Die vorliegende Dissertation behandelt die Dynamik von Vielteilchen-Quantensystemen wechselwirkender Bosonen für große Teilchenzahlen N. Im ersten Teil wird eine effektive Beschreibung der N-Teilchen-Zeitentwicklung effektiv niedrigdimensionaler Bosegase hergeleitet. Im zweiten Teil konstruieren wir eine beliebig gute Approximation an die Dynamik schwach wechselwirkender Bosonen.

Der erste Teil der Dissertation beschäftigt sich mit der Dynamik N wechselwirkender Bosonen in einer zigarren- oder scheibenförmigen externen Falle, die die Bewegung der Bosonen in zwei Dimensionen bzw. einer Dimension auf ein Gebiet der Größenordnung ε einschränkt. Im gleichzeitigen Limes $(N, \varepsilon) \to (\infty, 0)$ verhält sich das Gas effektiv d-dimensional, wobei $d = 1$ der zigarrenförmigen und $d = 2$ der scheibenförmigen Anordnung entspricht.

Die Wechselwirkung zwischen den Bosonen wird als nichtnegativ und beschränkt angenommen. Da das Gas auf einer Längenskala der Ordnung 1 beschrieben werden soll, betrachten wir ein entsprechend skaliertes Wechselwirkungspotential mit Streulänge der Ordnung $(N/\varepsilon^{3-d})^{-1}$. Die Reichweite der Wechselwirkung wird proportional zu $(N/\varepsilon^{3-d})^{-\beta}$ gewählt, wobei der Skalierungsparameter β die Werte $\beta \in (0, 1]$ annehmen kann. Die Wahl $\beta = 1$ entspricht der physikalisch relevanten Gross–Pitaevskii-Skalierung.

Unter der Annahme, dass das System anfangs als Bose–Einstein-Kondensat vorliegt, zeigen wir, dass die N-Teilchen Dynamik im Limes $(N, \varepsilon) \to (\infty, 0)$ den Zustand der Kondensation erhält. Die zeitentwickelte Wellenfunktion ist Lösung einer d-dimensionalen nichtlinearen Gleichung, wobei die Stärke der Nichtlinearität von β abhängt. Für $\beta \in (0, 1)$ erhalten wir eine kubisch-defokussierende nichtlineare Schrödingergleichung, während $\beta = 1$ einer Gross–Pitaevskii-Gleichung entspricht, die explizit die Streulänge der Wechselwirkung enthält. In beiden Fällen hängt die Kopplungskonstante über einen multiplikativen Faktor von der zigarren- bzw. scheibenförmigen Falle ab.

Der zweiten Teil der Arbeit behandelt die Dynamik N d-dimensionaler Bosonen, die über Paarpotentiale miteinander wechselwirken. Insbesondere betrachten wir Wechselwirkungen der Form $(N-1)^{-1}N^{d\beta}v(N^{\beta}\cdot)$ für Skalierungsparameter $\beta \in [0, \frac{1}{4d})$, was die Situation vieler schwacher Wechselwirkungen modelliert. Das unskalierte Po-

tential v wird als beschränkt angenommen, wobei keine Vereinbarung bezüglich des Vorzeichens von v getroffen wird.

Die betrachteten Systeme sollen anfangs Bose–Einstein-Kondensation aufweisen, wobei die Anzahl der Anregungen aus dem Kondensat im Anfangszustand als ausreichend gering gefordert wird. Unter dieser Annahme konstruieren wir eine Folge von N-Teilchen-Funktionen, die die wahre Vielteilchendynamik bezüglich der $L^2(\mathbb{R}^{dN})$-Norm mit beliebiger Genauigkeit bezüglich Potenzen von N^{-1} annähern. Die approximierenden Funktionen werden als endliche Duhamel-Entwicklungen einer erstquantisierten Bogoliubov-Zeitentwicklung konstruiert. Ein Zwischenresultat bilden Abschätzungen aller endlichen Momente der Anregungsanzahl in der zeitentwickelten Wellenfunktion.

Summary

In this thesis, we study the dynamics of quantum many-body systems of interacting bosons for large particle numbers N. We derive an effective description of the N-body time evolution of quasi-low-dimensional Bose gases and construct an approximation to any order of the dynamics of weakly interacting bosons.

The first part of the thesis is concerned with the dynamics of N interacting bosons in a cigar-shaped or disc-shaped trap, which confines the bosons in two dimensions or one dimension, respectively, to a region of order ε in each direction. In the simultaneous limit $(N,\varepsilon) \to (\infty, 0)$, the gas becomes quasi d-dimensional, where $d = 1$ for the cigar-shaped and $d = 2$ for the disc-shaped confinement.

The interaction between the bosons is assumed non-negative and bounded. To describe the gas on a length scale of order one, the interaction is scaled such that its scattering length is of order $(N/\varepsilon^{3-d})^{-1}$, while its range is proportional to $(N/\varepsilon^{3-d})^{-\beta}$ with scaling parameter $\beta \in (0, 1]$. The choice $\beta = 1$ corresponds to the physically relevant Gross–Pitaevskii scaling regime.

Under the assumption that the system initially exhibits Bose–Einstein condensation, we show that the N-body dynamics preserve condensation in the simultaneous limit $(N,\varepsilon) \to (\infty, 0)$. The time-evolved condensate wave function is the solution of a d-dimensional non-linear equation, where the strength of the non-linearity depends on the scaling parameter β. For $\beta \in (0, 1)$, we obtain a cubic defocusing non-linear Schrödinger equation, while the choice $\beta = 1$ yields a Gross–Pitaevskii equation featuring the scattering length of the interaction. In both cases, the coupling parameter depends on the confining potential.

In the second part of the thesis, we consider the dynamics of N d-dimensional bosons, which interact with each other via a pair potential in the mean-field scaling regime. More precisely, we study interactions of the form $(N-1)^{-1}N^{d\beta}v(N^{\beta}\cdot)$ for $\beta \in [0, \frac{1}{4d})$, which corresponds to the situation of many weak interactions. While we require the unscaled potential v to be bounded, no assumption on the sign of v is made.

We assume that the system initially exhibits Bose–Einstein condensation with sufficiently few excitations from the condensate. We derive a sequence of N-body wave functions which approximate the true many-body dynamics in $L^2(\mathbb{R}^{dN})$-norm to arbitrary precision in powers of N^{-1}. The approximating functions are constructed as

Duhamel expansions of finite order in terms of the first quantised analogue of a Bogoliubov time evolution. As an intermediate result, we prove estimates for finite moments of the number of excitations in the time-evolved wave function.

Acknowledgements

First of all, I thank Stefan Teufel for being a great advisor in every sense. Thank you for all your time and patience, for outstanding support and encouragement, for mathematical help and clarity and physical insight. I will miss our discussions at your blackboard.

I am very grateful to Peter Pickl for being the second advisor of my thesis. Thank you for your extraordinary hospitality in Kunshan and for sharing your brilliant ideas.

Moreover, I thank Detlef Dürr for constant support and advice over all the years and for teaching the importance of getting things clear.

Special thanks go to Serena Cenatiempo for hosting me at GSSI and taking so much of your time explaining your work, and for being a referee for this thesis.

I am grateful to Nataša Pavlović, Peter Pickl and Avy Soffer for our collaboration, to Maximilian Jeblick and Nikolai Leopold for many explanations about Gross–Pitaevskii, and to Sören Petrat for long discussions about Bogoliubov theory.

I thank Roman Bause for always providing the (experimental) physics point of view, and Felix Reihl for graphical support and much more.

Finally, huge thanks are due to my family and friends for your support during all the time, and to the whole Mathematical Physics group in Tübingen, who made it a really great time.

List of publications

a) Accepted publications

1. **Derivation of the 1d nonlinear Schrödinger equation from the 3d quantum many-body dynamics of strongly confined bosons**
   Lea Boßmann
   Published in *Journal of Mathematical Physics* 60, 031902 (2019).
   Cited in the following as [32] and included in Appendix A.1.

2. **Derivation of the 1d Gross–Pitaevskii equation from the 3d quantum many-body dynamics of strongly confined bosons**
   Lea Boßmann and Stefan Teufel
   Published in *Annales Henri Poincaré* 20 (2019), 1003–10049.
   Cited in the following as [35] and included in Appendix A.2.

b) Submitted manuscripts (available as preprints)

1. **Derivation of the 2d Gross–Pitaevskii equation for strongly confined 3d bosons**
   Lea Boßmann
   Preprint, arXiv:1907.04547.
   Cited in the following as [33] and included in Appendix B.1.

2. **Higher order corrections to the mean-field description of the dynamics of interacting bosons**
   Lea Boßmann, Nataša Pavlović, Peter Pickl, and Avy Soffer
   Preprint, arXiv:1905.06164.
   Cited in the following as [34] and included in Appendix B.2.

Personal contribution

In all articles, the authors are ordered alphabetically.

a) Accepted publications

1. I am the single author of the article [32]. Stefan Teufel provided much helpful advice and feedback and was involved in the closely related joint project [35]. Fruitful discussions with Maximilian Jeblick, Nikolai Leopold, Peter Pickl and Christof Sparber are gratefully acknowledged.

   Scientific ideas: 95%. Paper writing: 100%.

2. The project [35] was realised in close collaboration with Stefan Teufel. The proof of Lemma 4.9a-c is mainly due to Stefan Teufel, while I provided most of the proof of Lemma 4.12. We thank Serena Cenatiempo, Maximilian Jeblick, Nikolai Leopold and Peter Pickl for helpful discussions.

   Scientific ideas: 45%. Paper writing: 50%.

b) Submitted manuscripts

1. I am the single author of the article [33]. Many ideas for the proof and the presentation of the material originate from the joint project with Stefan Teufel in [35]. I am thankful for helpful discussions with Serena Cenatiempo and Nikolai Leopold.

   Scientific ideas: 95%. Paper writing: 100%.

2. The original idea for the article [34] is due to Nataša Pavlović, Peter Pickl and Avy Soffer, who proved Theorem 1 for $\beta = 0$ and $a = 3$, including the proof of Lemma 2.6 for $\beta = 0$. During my stay with Peter Pickl at Duke Kunshan University, I generalised this to larger values of β, which in particular requires Proposition 2.4. The proof of this proposition is due to Peter Pickl and me, where Peter Pickl provided the idea to follow the strategy from [146] and I did the technical estimates. I observed that for $\beta > 0$, assumption A3 can be relaxed to parameters $\gamma \leq 1$, and generalised Theorem 1 to arbitrary $a \in \mathbb{N}$. The main part of the technical estimates

and the introduction was written by me, while the review of the literature was mostly contributed by Nataša Pavlović. Helpful discussion with Stefan Teufel, Sören Petrat and Marcello Porta are gratefully acknowledged.

Scientific ideas: 25%. Paper writing: 50%.

Conventions and notation

- We use units where $\hbar = 1$. Besides, except for Section 1.1, the particles are assumed to have mass $m = \frac{1}{2}$.
- The n-fold symmetric product of a one-body Hilbert space $\mathfrak{H} = L^2(\Omega)$ for some $\Omega \subset \mathbb{R}^d$ is denoted as
$$\mathfrak{H}_+^n = \bigotimes_{\mathrm{sym}}^n \mathfrak{H} \,.$$
In particular, $L^2_+(\mathbb{R}^{dN})$ denotes the symmetric subspace of $L^2(\mathbb{R}^{dN})$.
- We use capital letters to denote the interaction W, its scattering length A, and the length scale L of a system without specifying a frame of reference. In the coordinates where $L = 1$ is chosen as length unit, the interaction is denoted as w_N with scattering length a_N.
- An expression C that is independent of the number of particles N and the time t is referred to as a constant. Additionally, in Sections 1.3.3 and 3.1, constants must be independent of the width ε of the confinement.
- We use the notations $A \lesssim B$, $A \gtrsim B$ and $A \sim B$ to indicate that there exists a constant $C > 0$ such that $A \leq CB$, $A \geq CB$ or $A = CB$, respectively.
- The scalar product, norm and operator norm of the N-body Hilbert space are denoted as
$$\langle\!\langle \cdot, \cdot \rangle\!\rangle := \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{dN})} \,, \quad |\!|\!| \cdot |\!|\!| := \|\cdot\|_{L^2(\mathbb{R}^{dN})} \quad \text{and} \quad |\!|\!| \cdot |\!|\!|_{\mathrm{op}} := \|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^{dN}))} \,,$$
where d denotes the spatial dimension. For most of the thesis, we consider $d = 3$.
- The set of all permutations of n elements is denoted as $\mathfrak{S}_n$.
- The symbol $\widehat{\cdot}$ denotes the weighted many-body operators from Definition 1.4.1. The only exceptions are Sections 1.2.5, 1.5.1 and 1.5.2, where $\widehat{\cdot}$ denotes the Fourier transform.
- In Section 3.2, we write x^+ and x^- to denote $(x + \sigma)$ and $(x - \sigma)$ for any fixed $\sigma > 0$, which is to be understood in the following sense: Let the sequence

$(N_n, \varepsilon_n)_{n\in\mathbb{N}} \to (\infty, 0)$. Then

$$\begin{aligned} f(N,\varepsilon) \lesssim N^{-x^-} \quad &:\Leftrightarrow \quad \forall\, \sigma > 0,\ f(N_n,\varepsilon_n) \lesssim N_n^{-x+\sigma} \ \text{ for sufficiently large } n\,,\\ f(N,\varepsilon) \lesssim \varepsilon^{x^-} \quad &:\Leftrightarrow \quad \forall\, \sigma > 0,\ f(N_n,\varepsilon_n) \lesssim \varepsilon_n^{x-\sigma} \ \text{ for sufficiently large } n\,,\\ f(N,\varepsilon) \lesssim \mu^{x^-} \quad &:\Leftrightarrow \quad \forall\, \sigma > 0,\ f(N_n,\varepsilon_n) \lesssim \mu_n^{x-\sigma} \ \text{ for sufficiently large } n\,. \end{aligned}$$

These statements concern fixed σ in the limit $(N, \varepsilon) \to (\infty, 0)$ and do in general not hold uniformly as $\sigma \to 0$.

- The decomposition of a function into its negative and positive part is denoted as $f = f_+ - f_-$ with sign convention $f_+, f_- \geq 0$.
- We denote $\lfloor r \rfloor := \max\{z \in \mathbb{Z} :\ z \leq r\}$ and $\lceil r \rceil := \min\{z \in \mathbb{Z} :\ z \geq r\}$ for $r \in \mathbb{R}$.

In the single papers included in the appendix, the notation may vary and is indicated in each paper separately.

1. Introduction

Macroscopic physical systems are usually extremely complex since they contain a huge number of constituents, whose motion is entangled as the result of interactions. An explicit analytical description of the dynamics of such systems is practically impossible, and in many cases also a numerical solution is far beyond computational reach. Moreover, even if it was feasible to explicitly predict the behaviour of each individual constituent, this vast amount of information would not be very helpful for the understanding of the dynamics of the system as a whole. Much better suited for this purpose is an appropriately coarse-grained approximation, which focuses on relatively few collective degrees of freedom and monitors their time evolution. Such laws of motion are referred to as *effective descriptions*, and it is at the heart of statistical physics to derive them from an underlying fundamental theory.

Since the groundbreaking works of Boltzmann and Maxwell dating back to the 19th century, effective models have many times been successful in the description and prediction of physical phenomena. Most notably, the laws of thermodynamics determine the evolution of macroscopic variables such as the temperature, pressure or volume of an ideal gas consisting of many non-interacting particles, whose individual motion is governed by Newton's laws of classical mechanics. Another famous example is the Boltzmann equation describing a gas of small interacting spheres, or the Vlasov equation, which is applied to analyse the dynamics of stellar matter. Effective theories arising from quantum mechanics are for example Hartree–Fock theory for fermions, and the Hartree and Gross–Pitaevskii equation for the dynamics of interacting bosons.

By its very nature, an effective description is an approximation, and as such needs to be justified from the full many-body theory with mathematical rigour. This means that the solution of the microscopic evolution equation and the solution of the effective equation should coincide in a suitable limit and with respect to an appropriately chosen topology. Moreover, to judge the viability of the approximation, it is desirable to quantify the approximation error in terms of the parameters of the model, such as the number of particles, the size of the system, or the initial conditions.

This thesis contributes to the mathematically rigorous derivation of effective dynamics for systems of indistinguishable, interacting bosons. Macroscopic Bose gases in current experiments contain at least $N \sim 10^6$ particles, whose individual motion is determined by the N-body Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi^N(t) = H_N(t)\psi^N(t)\,, \qquad \psi^N(0) = \psi_0^N \in L^2_+(\mathbb{R}^{3N})\,. \tag{1.1}$$

Here, $L^2_+(\mathbb{R}^{3N})$ denotes the symmetric subspace of the Hilbert space $L^2(\mathbb{R}^{3N})$, whose elements are square integrable and symmetric under the exchange of any two coordinates, i.e.,

$$\psi^N(x_1,...,x_j,...,x_k,...,x_N) \;=\; \psi^N(x_1,...,x_k,...,x_j,...,x_N) \;\text{ for }\; j,k \in \{1,...,N\}\,.$$

We use the normalisation convention $\|\psi^N\|_{L^2(\mathbb{R}^{3N})} = 1$. The N-body Hamiltonian is given as

$$H_N(t) = \sum_{j=1}^{N}\left(-\frac{1}{2m}\Delta_j + V^{\mathrm{ext}}(t,x_j)\right) + \sum_{1\leq i<j\leq N} w^{\mathrm{int}}(x_i - x_j)\,, \tag{1.2}$$

where Δ_j denotes the Laplace operator acting on the jth particle with mass m, V^{ext} is an external trapping potential, and w^{int} describes the interaction between any two particles.

Since we study very dilute gases, we neglect all interactions involving three or more particles. Besides, we analyse the behaviour of the bosons at very low temperatures, where their de Broglie wavelength is sufficiently large that microscopic details of the scattering potential cannot be resolved. Hence, we simply assume w^{int} to be spherically symmetric, i.e., to depend only on the distance between two particles. Due to the presence of the interaction w^{int}, solving (1.1) means to solve a differential equation in N variables, which makes $\psi^N(t)$ practically inaccessible for any further analysis.

At extremely low temperatures, Bose gases display the fascinating phenomenon of Bose–Einstein condensation, experimentally first realised in 1995. In this exceptional state of matter, almost all particles occupy approximately the same quantum state. The N-body wave function is therefore close to an N-fold product of a single wave function $\varphi(t)$ depending on only one spatial variable, i.e.,

$$\psi^N(t) \approx \varphi(t)^{\otimes N}\,. \tag{1.3}$$

Due to the interactions, this is no exact equality but holds asymptotically as $N \to \infty$, with respect to an appropriately chosen measure of distance. Since the great majority of particles condenses into a cloud, where, roughly speaking, all particles behave as one, the dynamics $\varphi(t)$ provide an effective description of the dynamics of the gas as a whole. Due to the inter-particle interactions, the equation of motion for $\varphi(t)$ is non-linear.

The first three projects of this thesis [32, 35, 33] concern the derivation of this effective evolution equation for a very particular setup: we consider the case where V^{ext} in (1.2) confines the particles in one [33] or two [32, 35] spatial dimensions to a region of order ε in each direction. In the simultaneous limit $N \to \infty$ and $\varepsilon \to 0$, we derive a low-dimensional, non-linear equation determining the evolution of $\varphi(t)$. The results of these projects are presented in Section 3.1.

While the overwhelming majority of particles in a Bose–Einstein condensate is approximately in the state $\varphi(t)$, relatively few bosons may be in a different state, forming excitations from the condensate. Hence, approximating the N-body dynamics by $\varphi(t)$ means ignoring these excitations. To obtain a more accurate but still simplifying description of the system, one must additionally account for the dynamics of the excitations. These dynamics can be described by an effective theory, the so-called Bogoliubov approximation. Combining the evolution of the condensate with the dynamics of the excitations, one obtains an effective N-body wave function that approximates the actual dynamics $\psi^N(t)$ with respect to the L^2-norm of the N-body Hilbert space. In the last project of this thesis [34], which is discussed in Section 3.2, we derive higher order corrections to this description.

In the remainder of the introduction, we review the mathematical and physical notions and results that form the foundation for the results obtained in this thesis. Chapter 2 summarises the objectives of the thesis, while the results are presented and discussed in Chapter 3.

1.1. Ideal Bose gas

In this section, we recall the concept of Bose–Einstein condensation for an ideal gas. Herein, we mainly follow [110, §62], [145, Chapter 2] and [153, Chapters 3 and 10].

Let us consider a d-dimensional ideal Bose gas of N indistinguishable, non-relativistic, spinless bosons with mass m in thermodynamic equilibrium. The dynamics $\psi^N(t)$ of the N-body wave function are determined by the Hamiltonian H^{ideal}, which decomposes into a sum of one-body Hamiltonians h,

$$H^{\text{ideal}} = \sum_{j=1}^{N} h_j\,, \qquad h = -\frac{1}{2m}\Delta + V^{\text{ext}}\,,$$

where Δ is the d-dimensional Laplace operator and h_j denotes h acting on the j'th coordinate. Consequently, the N-body eigenfunctions of H^{ideal} are symmetrised products of eigenfunctions of h. At temperature $T = (k_B\beta)^{-1}$, where k_B denotes Boltzmann's constant, the mean occupation number n_j of the single-particle state j with energy ϵ_j

is determined by the Bose distribution function,

$$n_j \;=\; \frac{1}{\mathrm{e}^{\beta(\epsilon_j-\mu)}-1}\,, \qquad j \geq 0\,. \tag{1.4}$$

The chemical potential satisfies $\mu \leq \epsilon_0$ and is implicitly determined by the condition

$$N \;=\; \sum_{j\geq 0} n_j \;=\; \sum_{j\geq 0} \frac{1}{\mathrm{e}^{\beta(\epsilon_j-\mu)}-1} \;=\; N_0 + \sum_{j\geq 1} \frac{1}{\mathrm{e}^{\beta(\epsilon_j-\mu)}-1} \;=:\; N_0 + N_{\mathrm{ex}}\,. \tag{1.5}$$

Here, N_0 counts the particles occupying the ground state, while all particles in excited states contribute to N_{ex}. To calculate the thermodynamic properties of the system for large N, one usually replaces sums over states by integrals over a density of states $g(\epsilon)$,

$$g(\epsilon) = \frac{\mathrm{d}G(\epsilon)}{\mathrm{d}\epsilon} =: C_\alpha \epsilon^{\alpha-1}\,,$$

where $G(\epsilon)$ is the total number of states with energy less than ϵ. Naturally, this quantity depends on the dimension as well as on the external trapping potential, resulting in different values for the real-valued parameters α and C_α. A free particle in d dimensions in a volume V, whose dispersion relation is $|p|(\epsilon) = \sqrt{2m\epsilon}$, yields

$$G(\epsilon) = \frac{V\mathcal{V}_d(\sqrt{2m\epsilon})}{(2\pi)^d}\,,$$

where $\mathcal{V}_d(R) = \frac{\pi^{d/2}R^d}{\Gamma(d/2+1)}$ is the volume of the d-dimensional ball with radius R. Hence,

$$g(\epsilon) = C_{d/2}\epsilon^{\frac{d}{2}-1}\,, \qquad C_{d/2} = \frac{\frac{d}{2}\left(\frac{m}{2\pi}\right)^{\frac{d}{2}} V}{\Gamma(\frac{d}{2}+1)}\,,$$

which corresponds to $\alpha = \frac{d}{2}$. If V^{ext} is a d-dimensional harmonic potential with frequencies ω_i, $i = 1, \ldots, d$, this yields

$$g(\epsilon) = \widetilde{C}_d \epsilon^{d-1}\,, \qquad \widetilde{C}_d = \frac{1}{(d-1)!\,\omega_1 \cdots \omega_d}\,,$$

corresponding to $\alpha = d$.

For sufficiently large N, we may approximate $\epsilon_0 \approx 0$, which implies $\mu \leq 0$. Making use of the density of states, we replace the sum defining N_{ex} in (1.5) by the corresponding integral. Substituting $x = \beta\epsilon$, we obtain

$$N_{\mathrm{ex}} \;=\; \beta^{-\alpha} C_\alpha \int_0^\infty \mathrm{d}x \frac{x^{\alpha-1}}{\mathrm{e}^{x-\beta\mu}-1}\,, \tag{1.6}$$

which is increasing as $\mu \uparrow 0$. For $\alpha > 1$, the integral corresponding to $\mu = 0$ converges and yields

$$N_{\mathrm{ex}} \leq \beta^{-\alpha} C_\alpha \int_0^\infty \mathrm{d}x \frac{x^{\alpha-1}}{\mathrm{e}^x - 1} \; = \; \beta^{-\alpha} C_\alpha \Gamma(\alpha) \zeta(\alpha) \,, \tag{1.7}$$

where $\zeta(\alpha) = \sum_{n=1}^\infty n^{-\alpha}$ is the Riemann zeta function. Since (1.7) is finite and in particular N-independent, the total particle number exceeds this value for sufficiently large N, which implies that all excess particles must occupy the ground state and thus contribute to N_0. This macroscopic occupation of a single one-body state — macroscopic in the sense that the fraction of particles in this state does not vanish in the limit $N \to \infty$ — is called *Bose–Einstein condensation (BEC)*. Its theoretical existence was discovered in 1924 by Einstein [60, 61], building on a work by Bose [31].

Let us remark that it is only justified to replace the sum in (1.5) by an integral because we exclude the first term $j = 0$ in the sum from this replacement and treat it separately. The reason is that n_0 diverges in the limit $\mu \to 0$, whereas all higher terms of the sum converge to a finite value. In the integral (1.7), the density of states $g(\epsilon) \sim \epsilon^{\alpha-1}$ makes the integrand behave as $x^{\alpha-2}$ for $x \to 0$, which diverges for $\alpha \leq 1$. Hence, for $\alpha > 1$, the contribution of infinitesimal x is not appropriately accounted for in the integration.

For $\alpha > 1$, BEC occurs below a critical transition temperature T_c, which is determined by the condition

$$N \; = \; N_{\mathrm{ex}}(T_c, \mu = 0) \; = \; \beta_c^{-\alpha} C_\alpha \Gamma(\alpha) \zeta(\alpha)$$

as

$$\beta_c^{-1} = k_B T_c = \left(\frac{N}{C_\alpha \Gamma(\alpha) \zeta(\alpha)} \right)^{\frac{1}{\alpha}} .$$

For temperatures $T \leq T_c$, the number of particles in the condensate is given by

$$N_0 = N \left[1 - \left(\frac{T}{T_c} \right)^\alpha \right] .$$

In conclusion, BEC at positive temperature occurs in systems whose density of states $g(\epsilon)$ is characterised by a parameter $\alpha > 1$. The most renowned examples are the spatially homogeneous 3d Bose gas ($\alpha = \frac{3}{2}$) and the 3d gas in a harmonic trap ($\alpha = 3$). In low dimensions, the situation changes: whereas the homogeneous 2d Bose gas does not exhibit BEC at $T > 0$, the phenomenon occurs for 2d bosons in harmonic traps ($\alpha = 2$). In case of the ideal 1d Bose gas, not even a harmonic trap suffices for BEC, but one requires a potential that is more confining than parabolic.

A comprehensive analysis of the behaviour of 1d and 2d bosons in power-law traps is given in [11].

1.2. Interacting three-dimensional Bose gas at low temperature

In this section, we summarise results concerning ground states of 3d interacting Bose gases. First, we recall the concept of BEC for interacting particles and introduce the scattering length from both a physical and mathematical point of view. Subsequently, we discuss the case of a homogeneous gas, comment on relevant scaling regimes, and conclude with an overview of ground state properties of a spatially inhomogeneous gas. We mainly follow [109, §45 and §125], [119, Chapters 1,2,5,6,7 and Appendix C], [129, Chapters 2,3,4 and 7], [145, Chapters 5,6], [153, Chapter 4] and [171, Chapter 19]. To keep the notation simple, we will from now on choose the mass of the bosons as $m = \frac{1}{2}$.

1.2.1. Definition of BEC in an interacting Bose gas

The more realistic case of an *interacting* d-dimensional Bose gas is described by the Hamiltonian (1.2). This Hamiltonian does not factorise into a sum of one-body Hamiltonians, and the N-body eigenfunctions can consequently not be expressed as products of single-particle states. To give the concept of a macroscopic occupation of a single one-body state meaning in the interacting context, it is rephrased in terms of reduced densities.

For any $k \in \{1, ..., N\}$, the k-particle reduced density matrix (or marginal) of an N-body function $\psi^N \in L^2(\mathbb{R}^{dN})$ is the positive trace-class operator $\gamma^{(k)}_{\psi^N} \in \mathcal{L}^1(L^2(\mathbb{R}^{dk}))$ with trace one, defined by its kernel

$$\gamma^{(k)}_{\psi^N}(x_1, ..., x_k; y_1, ..., y_k) := \int\limits_{\mathbb{R}^{d(N-k)}} \mathrm{d}x_{k+1} \cdots \mathrm{d}x_N \psi^N(x_1, ..., x_k, x_{k+1}, ..., x_N) \, \overline{\psi^N(y_1, ..., y_k, x_{k+1}, ..., x_N)} \,. \tag{1.8}$$

Equivalently, in Dirac notation,

$$\gamma^{(k)}_{\psi^N} := \mathrm{Tr}_{k+1,...,N} |\psi^N\rangle\langle\psi^N| \,.$$

Any reduced density matrix can be written in diagonal form as

$$\gamma_{\psi^N}^{(k)} = \sum_{j=0}^{J} \lambda_j |\varphi_j\rangle\langle\varphi_j| \,, \qquad \lambda_j \geq \lambda_{j+1} > 0 \,, \qquad \sum_{j=0}^{J} \lambda_j = 1 \,, \tag{1.9}$$

where $\{\varphi_j\}_{j=0}^J$ is an orthonormal system of $L^2(\mathbb{R}^{dk})$ with $0 \leq J \leq \infty$, and where $\{\lambda_j\}_{j=0}^J$ denotes the corresponding set of eigenvalues of $\gamma_{\psi^N}^{(k)}$. Physically, the reduced densities are relevant because the expectation values of k-body operators (sums containing only terms of the form $A^{(k)} = A_k \otimes \mathbb{1}_{N-k}$ that act non-trivially on at most k variables and as identities on all others) are completely determined by the k-particle reduced density matrices since

$$\left\langle \psi^N, A^{(k)} \psi^N \right\rangle = \mathrm{Tr}(\gamma_{\psi^N}^{(k)} A^{(k)}) \,.$$

A particularly relevant example of a (symmetrised) one-body operator is the position of the centre of mass of a system of N identical particles. Besides, note that the particle density

$$n(x) = N \int\limits_{\mathbb{R}^{d(N-1)}} |\psi^N(x, x_2, \dots, x_N)|^2 \, \mathrm{d}x_2 \cdots \mathrm{d}x_N = N \gamma_{\psi^N}^{(1)}(x; x) \tag{1.10}$$

is completely determined by the one-particle reduced density matrix of ψ^N.

In terms of reduced densities, a more general definition of BEC was first proposed by Penrose and Onsager in [144]:

> *An N-body state ψ^N exhibits BEC if and only if the largest eigenvalue λ_0 of its reduced one-particle density matrix $\gamma_{\psi^N}^{(1)}$ is of order one.*

Note that an N-body eigenstate ψ^N of an ideal gas in thermodynamic equilibrium is given as a product of single-body eigenstates φ_j with occupation number n_j, and the corresponding reduced one-particle density matrix can be written as[1]

$$\gamma_{\psi^N}^{(1)} = \sum_{j=0}^{J} \frac{n_j}{N} |\varphi_j\rangle\langle\varphi_j| \,.$$

Hence, for an ideal gas, the eigenvalues of $\gamma_{\psi^N}^{(1)}$ correspond to relative occupation numbers of the single-particle states, and the criterion by Penrose and Onsager is equivalent to a macroscopic occupation of a single-particle state.

In particular in regard of the limit $N \to \infty$, this rather operational criterion requires a more precise asymptotic formulation. In the mathematical literature, there appear

[1] This is shown in [130, Theorem 8.1].

several notions of asymptotic BEC, and we refer to [130, Definition 9.1] for an overview. The standard definition in the context of dynamics of N-body bosonic systems, also referred to as *complete asymptotic BEC*, is as follows:

Definition 1.2.1. *Let* $\{\psi^N\}_N$ *be a sequence of normalised* N*-body wave functions such that* $\psi^N \in L^2_+(\mathbb{R}^{dN})$*. The system is said to exhibit complete asymptotic BEC in the state* $\varphi \in L^2(\mathbb{R}^d)$ *if and only if*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right| = 0\,.$$

Equivalently, we have one of the following:

Lemma 1.2.2. *Let* $\{\psi^N\}_N$ *be a sequence of normalised* N*-body wave functions such that* $\psi^N \in L^2_+(\mathbb{R}^{dN})$*. Further, let* $k \geq 1$ *and let* $\varphi \in L^2(\mathbb{R}^d)$ *be normalised. Then the following are equivalent:*

(a) $\lim\limits_{N\to\infty} \operatorname{Tr}_{L^2(\mathbb{R}^d)}\left|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right| = 0\,,$

(b) $\lim\limits_{N\to\infty} \left\|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right\|_{HS} := \lim\limits_{N\to\infty} \operatorname{Tr}_{L^2(\mathbb{R}^d)}\left(\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right)^2 = 0\,,$

(c) $\lim\limits_{N\to\infty} \left\langle\varphi, \gamma^{(1)}_{\psi^N}\varphi\right\rangle_{L^2(\mathbb{R}^d)} = 1\,,$

(d) $\lim\limits_{N\to\infty} \operatorname{Tr}_{L^2(\mathbb{R}^{dk})}\left|\gamma^{(k)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right| = 0$ *for all* $k \in \mathbb{N}$,

(e) $\lim\limits_{N\to\infty} \left\|\gamma^{(k)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right\|_{\mathcal{L}(L^2(\mathbb{R}^{dk}))} = 0$ *for all* $k \in \mathbb{N}$,

where $\|\cdot\|_{HS}$ *denotes the Hilbert-Schmidt norm.*

These equivalences are well known and proofs are given, i.a., in [130, Theorems 9.4 and 13.2] and [158, Remark 1.4].

The occurrence of BEC in the ground state of an N-body system in a large box of side length L in the thermodynamic limit ($N, L \to \infty$ with fixed density $\varrho := \frac{N}{L^d}$) has so far only been proven for a hard-core gas on a cubic lattice at half-filling, where the particle number is half of the number of sites (see e.g. [4]). For particles in a continuum, the thermodynamic limit has not yet been treated rigorously. However, there are rigorous results proving BEC in the so-called *Gross–Pitaevskii* limit of infinite dilution, and we will comment on this limit and the results in Section 1.2.3.

To define more precisely what is meant by a dilute gas, we first introduce the concept of a scattering length. This parameter is crucial for the analysis of ultra-cold Bose gases since it characterises all interaction-related properties of the gas to leading order. In the following, we will focus on the physically most relevant case of a 3d interacting

Bose gas. Since some crucial low-energy properties depend on the spatial dimension in a non-trivial way, this analysis does not easily generalise to generic dimensions d, and we comment on low dimensional problems in Section 1.3.

1.2.2. Scattering length

The quantum mechanical scattering of two 3d particles with mass $\frac{1}{2}$ and mutual interaction potential W is most conveniently described in the centre-of-mass system. The wave function ϕ_k of the motion relative to the centre of mass with energy $2E = 2k^2$ solves the stationary Schrödinger equation with reduced mass $\frac{1}{4}$,

$$\left(-\Delta + \tfrac{1}{2}W(x)\right)\phi_k(x) = E\phi_k(x)\,, \tag{1.11}$$

where we used the relative coordinates $x = (r, \theta, \phi)$ and multiplied both sides with a factor $\frac{1}{2}$ for later convenience. Let us assume that the interaction potential W is spherically symmetric and decays sufficiently fast to be negligible in the region $r > R$ for some $R > 0$. To solve (1.11) for this region, one makes the ansatz

$$\phi_k(x) = \mathrm{e}^{\mathrm{i}k\cdot x} + f_k(\theta)\frac{\mathrm{e}^{\mathrm{i}|k|r}}{r}\,, \tag{1.12}$$

where the scattering state ϕ_k is modelled as the superposition of an incoming plane wave and an outgoing scattered wave. The latter depends on the scattering angle θ via the scattering amplitude f_k. At very low energies, i.e., as $k \to 0$, the particles cannot resolve the angular dependence of the scattering amplitude. Hence,

$$\lim_{k\to 0} f_k(\theta) =: -A\,, \tag{1.13}$$

and the scattering state has the asymptotic form

$$\phi_0(x) = 1 - \frac{A}{r} \quad \text{for} \quad r \gg R\,. \tag{1.14}$$

To justify this heuristic reasoning, one expands the solution of (1.11) in partial waves with angular momentum l, solves the resulting equation for the radial part of the wave function, and obtains an expansion of f_k in terms of the Legendre polynomials $P_l(\cos\theta)$. Integrating $|f_k(\theta)|^2$ over the whole solid angle yields the total scattering cross-section, which turns out to be dominated by the contribution from $l = 0$, the so-called *s-wave scattering*. This justifies to keep only the term $l = 0$ in $f_k(\theta)$, and a comparison of (1.12) and (1.14) yields (1.13)[2]. As a consequence, the parameter A is referred to as the *s-wave scattering length* of the interaction W.

[2] The full argument can be found in most standard textbooks on quantum mechanics, such as [109, Chapter XVII] and [171, Chapter 19].

Mathematically, the scattering length is defined via a variational principle, and the following rigorous definition is taken from [125, Appendix A] and [168]. Let B_R denote the 3d ball with radius R, $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$, and define

$$\mathcal{E}_R[\phi] := \int_{B_R} \left(|\nabla\phi(x)|^2 + \tfrac{1}{2}W(x)|\phi(x)|^2\right) \mathrm{d}x \,. \tag{1.15}$$

Assume that the interaction potential W is spherically symmetric and compactly supported within the ball B_{R_0} for some $R_0 > 0$. Further, assume that the negative part of W is in $L^{\frac{3}{2}}(\mathbb{R}^3)$ and assume that $\frac{1}{2}W$ has no negative energy bound states in $L^2(\mathbb{R}^3)$, i.e., that $\lim_{R\to\infty} \mathcal{E}_R[\phi] \geq 0$ for all $\phi \in H^1(\mathbb{R}^3)$. Note that we use capital letters to denote the quantities W, L and A without specifying a frame of reference. Later, we will choose L as length unit, and in these coordinates W is expressed as w_N with scattering length a_N.

Definition 1.2.3. *Under the above assumptions on the interaction potential W, the scattering length A of W is defined as*

$$A := \lim_{R\to\infty} A_R \,,$$

where A_R is given by the variational principle

$$4\pi A_R = \inf\left\{\mathcal{E}_R[\phi] \,:\, \phi \in H^1(B_R) \,,\, \phi(x) = 1 \textit{ for } |x| = R\right\} \,.$$

Existence and uniqueness of the minimiser of $\mathcal{E}_R$ were shown by Lieb and Yngvason in [125, Theorem A.1], who also proved some important properties of this minimiser [125, Lemma A.1]. We collect both statements in the following lemma.

Lemma 1.2.4. *Let W satisfy the above assumptions. Then, in the subclass of functions $\phi \in H^1(B_R)$ such that $\phi(x) = 1$ for $|x| = R$, there is a unique function ϕ_0 that minimises $\mathcal{E}_R$. The minimiser has the following properties:*

(a) There exists a function $f_0 : (0, R] \to \mathbb{R}_0^+$ such that $\phi_0(x) = f_0(|x|)$, i.e., ϕ_0 is non-negative and spherically symmetric,

(b) ϕ_0 satisfies

$$-\Delta\phi_0(x) + \tfrac{1}{2}W(x)\phi_0(x) = 0 \tag{1.16}$$

in the sense of distributions on B_R with boundary condition $f_0(R) = 1$,

(c) for $R_0 < r < R$,

$$f_0(r) = f^{\mathrm{asymp}}(r) := \frac{1 - A/r}{1 - A/R} \,,$$

(d) the minimum value of $\mathcal{E}_R[\phi]$ *is*

$$E_R = 4\pi \frac{A}{1 - A/R}\,.$$

If W *is non-negative, it holds additionally for all* $0 < r \leq R$ *that*

(e) $f_0(r) \geq f^{\text{asymp}}(r)$ *and* $f_0(r)$ *is a non-decreasing function of* r*,*

(f) $0 \leq A \leq R_0$*.*

In the following, we will denote the scattering solution on $\mathbb{R}^3$ corresponding to $R \to \infty$ by j. Written in a more compact form, Lemma 1.2.4 states that

$$\begin{cases} j(x) = 1 - \dfrac{A}{|x|} & |x| > R_0\,, \\ j(x) \geq 1 - \dfrac{A}{|x|} & \text{else}\,, \end{cases} \tag{1.17}$$

hence,

$$A = \lim_{|x|\to\infty} |x|\,(1 - j(x)) = \frac{1}{8\pi} \int\limits_{\mathbb{R}^3} W(x) j(x)\,\mathrm{d}x\,. \tag{1.18}$$

To obtain the second equality, one notes that $W(x)j(x) = 2\Delta j(x) = 2(\partial_r^2 + \frac{2}{r}\partial_r)j(r)$ by (1.16) and integrates by parts. The scattering solution j coincides with (1.14), hence Definition 1.2.3 and Lemma 1.2.4 provide a mathematical framework for the analysis of the low-energy scattering of two particles. While this definition of the scattering length is most convenient here, we remark that an alternative definition without variational principle, which includes potentials with bound states but admits less singular local behaviour of the interaction potential, is given in [93, Definition 2].

In conclusion, the scattering of two sufficiently distant and low-energetic particles is, to leading order, entirely characterised by the single parameter A. While the interaction potential determines A uniquely, the converse is false. In fact, very different potentials may have the same scattering length, hence their low-energy scattering is to leading order equivalent. Since the scattering length of a hard sphere potential equals its radius, this statement can be rephrased as follows: outside the range of their mutual scattering potential, two sufficiently low-energetic particles do not resolve the microscopic details of the interaction potential and behave as if scattered at a hard sphere with radius A.

1.2.3. Homogeneous dilute three-dimensional Bose gas

Let us begin with the homogeneous case, i.e., with N bosons in a cubic box Λ with $|\Lambda| = L^3$ and periodic boundary conditions, where the bosons interact via repulsive interactions. More precisely, we assume the two-body interaction potential to be non-negative and compactly supported. The density of particles in the gas is then given by

$$\varrho = \frac{N}{L^3}\,,$$

and the interaction W is characterised by its scattering length $A \geq 0$. *Dilute* means that the mean inter-particle distance $\varrho^{-\frac{1}{3}}$ is much larger than the length scale of the interaction determined by its scattering length A, i.e., that

$$\varrho A^3 \ll 1\,.$$

The ground state energy per particle in the thermodynamic limit is defined as

$$e_0(\varrho) := \lim_{N\to\infty} \frac{E_0(N,(N/\varrho)^{\frac{1}{3}})}{N}\,,$$

where $E_0(N,L)$ denotes the ground state of the N-body Hamiltonian in the cubic box with side length L. It satisfies the low-density asymptotics

$$\lim_{\varrho A^3\to 0} \frac{e_0(\varrho)}{4\pi\varrho A} \;=\; 1\,, \tag{1.19}$$

independently of the boundary conditions on the box. This formula was first proposed by Bogoliubov in [29]. For a rigorous proof, one computes an upper and lower bound and shows that they converge to the same limit. The upper bound was obtained by Dyson in [59], while the lower bound was established more than forty years later by Lieb and Yngvason in [124]. A summary of the proof is given in [119, Chapter 2].

The rigorous proof does not require any assumptions on properties of the ground state. In particular, it does not pre-suppose BEC. However, in order to give a heuristic justification of the formula (1.19), let us for a moment assume that the many-body ground state is a condensate. In this case, Definition 1.2.1 suggests that the many-body wave function ψ^N be close to a factorised state $\varphi^{\otimes N}$, where the condensate wave function φ varies in space on the macroscopic length scale of the system. However, the inter-particle interactions impose on ψ^N a correlation structure on the much shorter length scale determined by the interactions, which is not visible on the level of reduced densities. In [96], Jastrow proposed to model this situation by a trial function consisting of the product state $\varphi^{\otimes N}$ overlaid with a microscopic structure determined by the scattering state j as

$$\psi^N_{\mathrm{cor}}(x_1, \dots, x_N) \;\approx\; \frac{\prod\limits_{j=1}^{N} \varphi(x_j) \prod\limits_{1\le k<l\le N} j(x_k - x_l)}{\left\| \prod\limits_{j=1}^{N} \varphi(x_j) \prod\limits_{1\le k<l\le N} j(x_k - x_l) \right\|_{L^2(\Lambda^N)}} \,. \tag{1.20}$$

In fact, Erdős, Michelangeli and Schlein proved in [63] that this characteristic short-scale structure emerges dynamically within a very short time, even if the system was initially in a pure product state with all particles independent of each other[3].

In this spirit, let us formally estimate the ground state energy $E_0(2, L)$ of two particles in a large cubic box Λ with $|\Lambda| = L^3$. The normalised ground state of the box potential with periodic boundary conditions is given by $\varphi \equiv |\Lambda|^{-\frac{1}{2}}$, hence we make the ansatz $\psi_{\mathrm{cor}}(x_1, x_2) = |\Lambda|^{-1} j(x_1 - x_2)$, where j denotes the scattering solution (1.17). Note that for sufficiently small A/L, we find $\|\psi_{\mathrm{cor}}\|_{L^2(\Lambda^2)} \approx 1$ since $1 - \frac{A}{|x|} \le j(x) \le 1$ implies that

$$\begin{aligned} 1 \;\ge\; \|\psi_{\mathrm{cor}}\|^2_{L^2(\Lambda^2)} \;&\ge\; \frac{1}{|\Lambda|^2} \int\limits_{\Lambda} \mathrm{d}x_1 \int\limits_{\Lambda} \mathrm{d}x_2 \left(1 - \frac{2A}{|x_1 - x_2|}\right) \\ &=\; 1 - \frac{2A}{|\Lambda|} \int\limits_{\Lambda} \mathrm{d}x \frac{1}{|x|} \;\ge\; 1 - \frac{2A}{L^3} 4\pi \int\limits_0^{2L} r \,\mathrm{d}r \;=\; 1 - 16\pi \frac{A}{L} \,. \end{aligned}$$

Hence, we can neglect the normalisation factor in (1.20) for our heuristic argument. This yields

$$\begin{aligned} &E_0(2, L) \\ &\approx \int\limits_{\Lambda\times\Lambda} \left(\left(|\nabla_1 \psi_{\mathrm{cor}}(x_1, x_2)|^2 + |\nabla_2 \psi_{\mathrm{cor}}(x_1, x_2)|^2 \right) + W(x_1 - x_2) |\psi_{\mathrm{cor}}(x_1, x_2)|^2 \right) \mathrm{d}x_1 \,\mathrm{d}x_2 \\ &= \frac{2}{|\Lambda|} \int\limits_{\Lambda} \left(|\nabla j(x)|^2 + \frac{1}{2} |j(x)|^2 W(x) \right) \mathrm{d}x \;=\; \frac{2}{|\Lambda|} \lim_{R\to\infty} \mathcal{E}_R[j] \;=\; \frac{8\pi A}{L^3} \end{aligned}$$

by Lemma 1.2.4d since j is the minimiser of $\mathcal{E}_R$ for $R \to \infty$. For a sufficiently dilute gas, the total energy essentially equals the sum of all $\frac{1}{2}N(N-1)$ such two-particle contributions, and we conclude that

$$E_0(N, L) \approx N \, 4\pi A \varrho$$

[3] This result was obtained for the Gross–Pitaevskii scaling regime, which is explained below.

as in [119, p.13].

The second energy (scale) that is relevant for the analysis of the dilute Bose gas is the energy gap in the box, or, equivalently, the (purely kinetic) ground state energy of a free particle in the box $|\Lambda| = L^3$,

$$\frac{E_{\text{kin}}(N,L)}{N} = \frac{3\pi^2}{L^2} \sim \frac{1}{L^2}\,.$$

Comparing the kinetic energy per particle with the total energy per particle, one observes that for these energies to remain comparable in the limit $N \to \infty$, i.e.,

$$e_0(\varrho) \;\sim\; \varrho A \;=\; \frac{NA}{L^3} \;\overset{!}{\sim}\; \frac{1}{L^2} \;\sim\; \frac{E_{\text{kin}}(N,L)}{N}\,, \tag{1.21}$$

the scaling condition

$$\frac{NA}{L} =: g = \text{const.} \tag{1.22}$$

must be satisfied. The limit $N \to \infty$ such that g is constant is the so-called *Gross–Pitaevskii (GP)* limit. Note that condition (1.22) implies that

$$\varrho A^3 = \frac{N}{L^3}\left(g\frac{L}{N}\right)^3 \sim \frac{1}{N^2}\,,$$

hence the GP limit is a limit of infinite dilution, and the ground state asymptotics (1.19) are valid in this case. Since kinetic and interaction energy remain comparable in this limit $N \to \infty$, it is also called a *dynamical* limit of ultra-high dilution.

The GP scaling condition (1.22) requires that A, L or both quantities scale with N. Among all equivalent realisations of this constraint, we will focus on two cases:

- $A = \text{const.}$, $L \sim N$:
  The N-independence of the scattering length implies that the two-body interactions do not depend on the total number of particles. Hence, to increase the number of particles in the box and remain in the ultra-dilute regime, the box must grow proportionally to N, which is much faster than the rate $L \sim N^{\frac{1}{3}}$ which corresponds to the thermodynamic limit with constant density.

- $A \sim N^{-1}$, $L = \text{const.}$:
  Considering the problem on a fixed length scale implies that the scattering length and thus the pair interaction must be rescaled in an N-dependent way. We will see in Section 1.2.4 that the scattering length a_N of

  $$w_N(x) := N^2 w(Nx)$$

for any N-independent potential w scales as N^{-1}.

Both (and all other) realisations of (1.22) are equivalent in the sense that they correspond to choices of a coordinate system and are related by coordinate transforms. While the first option is more in accordance with physical reality, the second one is more convenient for the mathematical analysis because in these coordinates, all L^p- and H^p-norms of the condensate wave function, which varies on the scale L, are N-independent.

For a homogeneous gas in the GP limit, the question of the occurrence of BEC in the ground state was answered in the affirmative by Lieb and Seiringer in [118]. They proved that if the box Λ is equipped with periodic or Neumann boundary conditions, the one-particle reduced density matrix of the ground state ψ^N satisfies

$$\lim_{N\to\infty} \frac{1}{L^3} \int\limits_{\Lambda\times\Lambda} \gamma^{(1)}_{\psi^N}(x;y)\,\mathrm{d}x\,\mathrm{d}y \;=\; 1\,, \tag{1.23}$$

where the limit is taken such that ϱ and $g = \frac{NA}{L}$ remain fixed. Since the ground state of the free particle in the box Λ is $\varphi_0 \equiv L^{-\frac{3}{2}}$, this statement is equivalent to

$$\lim_{N\to\infty} \left\langle \varphi_0, \gamma^{(1)}_{\psi^N}\varphi_0 \right\rangle = 1\,,$$

which, in turn, means complete asymptotic condensation in the state φ_0 by Lemma 1.2.2. Recently, this statement was extended to positive temperatures by Deuchert and Seiringer in [57]. They showed that BEC occurs below a critical temperature, which, to leading order, coincides with the critical temperature of the ideal gas.

1.2.4. Scaling regimes

As mentioned above, it is mathematically most convenient to keep the length scale L of the system fixed and to rescale the interaction potential such that the scattering length scales as N^{-1}. The standard way of implementing this is to consider the interaction potential

$$w_{N,\beta}(x) := N^{-1+3\beta} w(N^\beta x)\,, \qquad \beta \in (0,1]\,, \tag{1.24}$$

where w is assumed to be non-negative, bounded, spherically symmetric, compactly supported, and, in particular, independent of N. Naturally, this implies that the scattering length a of w is N-independent as well. Let us for simplicity assume that $\operatorname{diam}(\operatorname{supp} w) = 1$ and denote

$$R_{N,\beta} := \operatorname{diam}(\operatorname{supp} w_{N,\beta})\,,$$

which is consequently given by $R_{N,\beta} = N^{-\beta}$. We shall see that the scattering length $a_{N,\beta}$ of $w_{N,\beta}$ shrinks as N^{-1} for the whole range of the scaling parameter β, which can be subdivided into three scaling regimes.

Gross–Pitaevskii regime: $\beta = 1$

This scaling
$$w_N(x) = N^2 w(Nx) \tag{1.25}$$
realises the GP scaling condition (1.22) because the scattering length a_N of w_N is given by
$$a_N = \frac{a}{N}\,,$$
which follows from the scaling behaviour of (1.16): Let w_a be a potential with scattering length a and corresponding scattering solution j_a. Then the potential

$$w_b(|x|) := b^2 w_a(b|x|) \tag{1.26}$$

has scattering length $\frac{a}{b}$, and the corresponding scattering state satisfies

$$j_b(x/b) = j_a(x)\,.$$

Clearly, the GP scaling condition is satisfied by the interaction w_N. Moreover, note that the range of w_N is of order N^{-1} and thus comparable to a_N. Since the scattering length determines the length scale of the inter-particle correlations, which are described by the zero-energy scattering solution (1.17), this means that the correlations vary on the same scale as the interaction. Consequently, they remain visible even in the limit $N \to \infty$, when $Nw_N(x) \to \|w\|_{L^1(\mathbb{R}^3)}\delta(x)$ in the sense of distributions.

Non-linear Schrödinger regime: $0 < \beta < 1$

For $\beta \neq 1$, the interaction $w_{N,\beta}$ is not of the form (1.26), hence its scaling behaviour cannot immediately be deduced from (1.16). Instead, it is shown in [65, Lemma A.1] that for any $0 < \beta < 1$, the scattering length $a_{N,\beta}$ of $w_{N,\beta}$ satisfies

$$\lim_{N\to\infty} N a_{N,\beta} = \frac{b_0}{8\pi}\,, \tag{1.27}$$

where
$$b_0 := N\|w_{N,\beta}\|_{L^1(\mathbb{R}^3)} = \|w\|_{L^1(\mathbb{R}^3)}$$
is the so-called *(first order) Born approximation* to the scattering length. The upper bound for the asymptotics (1.27) follows from the Spruch–Rosenberg inequality [175], which states that the scattering length of a potential not admitting bound states is

always bounded above by its Born approximation. The lower bound is a consequence of the property (1.17) of the scattering solution. Note that for a non-negative potential w, the Spruch–Rosenberg inequality is an immediate consequence of (1.18) because the scattering solution satisfies $0 \leq j(x) \leq 1$.

In view of the integral representation of the scattering length,

$$N 8\pi a_{N,\beta} = N \int w_{N,\beta}(x) j_{N,\beta}(x)\,\mathrm{d}x \;=\; N^{3\beta} \int w(N^\beta x) j_{N,\beta}(x)\,\mathrm{d}x\,,$$

it is clear that the Born approximation can asymptotically coincide with the scattering length only if $j_{N,\beta}$ is approximately constant over the range of $w_{N,\beta}$. In the GP regime, j_N and w_N are both peaked on the scale $|x| \lesssim N^{-1}$, which causes the approximation to break down. To verify this, observe that for all $x \in \operatorname{supp} w_{N,\beta}$,

$$j_{N,\beta}(x) \leq j_{N,\beta}\big|_{|x|=N^{-\beta}} = 1 - N^\beta a_{N,\beta}$$

since $j_{N,\beta}$ is non-decreasing by Lemma 1.2.4e. Hence, one estimates

$$b_0 - 8\pi N a_{N,\beta} \;=\; N^{3\beta} \int\limits_{\operatorname{supp} w_{N,\beta}} w(N^\beta x)(1 - j_{N,\beta}(x))\,\mathrm{d}x \;\geq\; N^{-1+\beta} N a_{N,\beta}\, b_0\,. \quad (1.28)$$

In the GP regime $\beta = 1$, the relation $Na_N = a$ implies that $b_0 - 8\pi N a_N \geq b_0 a = \mathcal{O}(1)$, hence the Born approximation is invalid. For $\beta \in (0,1)$, $Nw_{N,\beta}$ still approximates a δ-distribution as $N \to \infty$, but its range shrinks proportionally to $N^{-\beta}$ and is therefore much larger than the length scale of the correlations. This implies that to leading order, the relevance of the correlations vanishes as $N \to \infty$, or, put differently, that $j_{N,\beta}$ is approximately constant on the length scale $N^{-\beta}$ of the interaction.

In the physics literature, the standard way to justify the Born approximation, which was originally found by Born in [30], is via perturbation theory. Consider the elastic scattering of particles with mass $m = \frac{1}{2}$ at a potential U, which satisfies

$$\|U\|_{L^\infty(\mathbb{R}^3)} R^2 \ll 1\,, \quad (1.29)$$

where R denotes the range of U. Under this condition, U can be seen as a perturbation of the free Schrödinger equation, and perturbation theory leads to the Born approximation (see e.g. [109, §125 and §45]). Equivalently, the Born approximation is given by the first term of the series expansion that is constructed by iterating the Lippmann–Schwinger equation (see e.g. [171, Chapter 19.4] or [145, Chapter 5.2]). To give above condition a physical meaning, note that R^{-2} is the scaling behaviour of the kinetic energy of a free particle in a box with side length R. Clearly, the condition (1.29) is satisfied by $w_{N,\beta}$ as long as $\beta < 1$ because $\|w_{N,\beta}\|_{L^\infty(\mathbb{R}^3)} R^2_{N,\beta} \lesssim N^{-1+\beta}$.

Finally, let us remark that we refer to the scaling $\beta \in (0,1)$ as the *Non-linear Schrödinger (NLS)* regime because the time evolution of the condensate wave function in this scaling regime is determined by an NLS equation. This will be explained in Section 1.4.

Mean-field regime: $\beta < \frac{1}{3}$

Although the Born approximation holds for the whole parameter range $\beta \in (0,1)$, the physical picture changes at the threshold $\beta = \frac{1}{3}$.

- For $\beta > \frac{1}{3}$, it holds that

$$R_{N,\beta} = N^{-\beta} \ll N^{-\frac{1}{3}} = \varrho^{-\frac{1}{3}},$$

  i.e., the range of the interaction is much smaller than the mean inter-particle distance. Besides, the amplitude $N^{-1+3\beta}$ of the interaction diverges as $N \to \infty$. This corresponds to the situation of rare but very strong interactions.

- For $\beta < \frac{1}{3}$, we have on the contrary

$$R_{N,\beta} \gg \varrho^{-\frac{1}{3}}$$

  and $N^{-1+3\beta} \to 0$ as $N \to \infty$. Hence, on average, every particle interacts with many other particles and the interactions are weak, which characterises a mean-field regime.

The limiting case $\beta = 0$, corresponding to the interaction $w_{N,0}(x) = \frac{1}{N}w(x)$, is known as the *Hartree* regime. In contrast to $\beta > 0$, the range of the interaction $w_{N,0}$ does not shrink as N grows but remains of the same order as the system size.

In conclusion, an N-body Hamiltonian with pair interaction w_N ($\beta = 1$) implements the GP scaling condition (1.22), i.e., it describes a system with N-independent ratio of kinetic and potential energy, using coordinates such that the length scale of the system is independent of N. Note that for the dynamical problem, whose solution is determined by the time-dependent Schrödinger equation, this rescaling of space comes with a rescaling of time: the coordinate system (x,t) with $L \sim 1$ arises from the frame (x', t') with $A \sim 1$ by the coordinate transform $(x,t) = (\frac{x'}{N}, \frac{t'}{N^2})$. Hence, times of order one with respect to the frame (x,t) correspond to extremely long times with respect to the frame (x', t'). This relates to the low density of the gas, which causes the average time between two collisions to be very long.

Systems whose interactions are compactly supported and scale with $\beta < 1$ do not emerge via a coordinate transform from systems with N-independent interaction, hence their analysis is rather of mathematical than of physical interest. However, in the next section and in particular in Section 1.4.4, we will explain that such interaction potentials are crucial for the study of the GP regime since they can be used as effective or pseudo-potentials to approximate the GP interaction.

1.2.5. Dilute three-dimensional Bose gas in a trap

While the study of the homogeneous Bose gas is of great theoretical interest, dilute Bose gases in external traps model actual experiments more realistically. Due to the inhomogeneity of the system caused by the trap, we need suitable generalisations of the parameters L, measuring the size of the trap, and ϱ, corresponding to the mean particle density.

The scale L is determined by the characteristic length (or oscillator length) of the trapping potential V^{ext},

$$L_{\mathrm{osc}} := \sqrt{\frac{2}{\omega}}\,,$$

where ω denotes the order of the ground state energy of $-\Delta + V^{\mathrm{ext}}$. If V^{ext} is taken to be a harmonic oscillator, ω is its frequency. Physically, this formula is motivated as follows: consider a cloud of particles with mass $\frac{1}{2}$ in the ground state of a harmonic oscillator with frequency ω. If the extension of the cloud is R, the potential energy of a particle is $E_{\mathrm{pot}} \sim \frac{1}{4}\omega^2 R^2$. Its kinetic energy is given by $E_{\mathrm{kin}} \sim R^{-2}$ since the typical momentum of a particle in the ground state is $p \sim R^{-1}$. The total energy is minimised when $E_{\mathrm{pot}} = E_{\mathrm{kin}}$, which determines the size of the cloud as $R = L_{\mathrm{osc}}$ [145, Chapter 6.2].

As in the homogeneous case, we choose L_{osc} as length unit, i.e., we fix $L_{\mathrm{osc}} = 1$. In these coordinates, the N-body Hamiltonian is given as

$$H_N = \sum_{j=1}^{N} \left(-\Delta_j + V^{\mathrm{ext}}(x_j)\right) + \sum_{i<j} w_N(x_i - x_j)\,, \qquad w_N(x) = N^2 w(Nx)\,, \tag{1.30}$$

where V^{ext} and w are N-independent, w is non-negative, compactly supported and has scattering length a, and w_N has scattering length $a_N = a/N$. The external trap V^{ext} is assumed to be non-negative, measurable and locally bounded, and to tend to infinity as $|x| \to \infty$.

The mean particle density in the ground state is determined by the probability distribution induced by the condensate wave function. To explain what is meant by this, let us introduce the GP energy functional, which depends on the parameter $a = Na_N$ and is defined as

$$\mathcal{E}_a^{\mathrm{GP}}[\varphi] := \int_{\mathbb{R}^3} \left(|\nabla\varphi(x)|^2 + V^{\mathrm{ext}}(x)|\varphi(x)|^2 + 4\pi a|\varphi(x)|^4\right) \mathrm{d}x \tag{1.31}$$

for $\varphi \in \mathcal{D}$ with[4]

$$\mathcal{D} = \left\{ \varphi \in H^1(\mathbb{R}^3) : \; V^{\mathrm{ext}}|\varphi|^2 \in L^1(\mathbb{R}^3), \; \|\varphi\|_{L^2(\mathbb{R}^3)} = 1 \right\} .$$

Mathematically, the functional $\mathcal{E}_a^{\mathrm{GP}}[\varphi]$ is formed by taking the ground state energy per particle for the homogeneous gas as a local energy density for an inhomogeneous system. Since $\varrho(x) = N|\varphi(x)|^2$, this energy density is by (1.19) given as

$$4\pi a_N \varrho(x) = 4\pi N a_N |\varphi(x)|^2 = 4\pi a |\varphi(x)|^2 .$$

To give a heuristic argument for this functional, note first that the system described by the Hamiltonian (1.30) has two well-separated length scales: while the many-body wave function changes slowly in space on the scale $L_{\mathrm{osc}} = 1$, the inter-particle correlations vary on the extremely short length scale $a_N \sim N^{-1}$. In the physics literature, one deals with this separation of scales by absorbing the short-scale spatial variations into an effective interaction U^{eff}, which is then used to describe interactions between the long-wavelength degrees of freedom. As long as U^{eff} and w_N have the same scattering length, they are, to leading order, equivalent when it comes to calculating macroscopic properties of the system, as was argued in Section 1.2.2.

The advantage of using the effective interaction potential U^{eff} is that it can be chosen such that its scattering length is approximated by the first order Born approximation. The standard formal argument[5] is the following: one replaces w_N by an interaction U^{eff} with the same scattering length a_N as w_N, which is sufficiently shallow to satisfy (1.29), hence the Born approximation is valid for U^{eff}. Subsequently, changing to the momentum space representation, one argues that in the low energy regime, it is sufficient to consider the zero momentum component $\widehat{U^{\mathrm{eff}}}(0)$ of the Fourier transform $\widehat{U^{\mathrm{eff}}}$, which is given as

$$\widehat{U^{\mathrm{eff}}}(0) = \int\limits_{\mathbb{R}^3} U^{\mathrm{eff}}(x)\,\mathrm{d}x \;\approx\; 8\pi a_N$$

by the Born approximation. Finally, transforming back to position space, this corresponds to the on-site interaction

$$8\pi a_N \delta(x) \;=:\; \frac{U_0}{N}\delta(x) . \tag{1.32}$$

Instead of the Hamiltonian (1.30) with pair interaction w_N, one then studies the

[4] The definition is sensible as $H^1(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ by the Sobolev embedding theorem, see e.g. [3, Theorem 4.12].

[5] See e.g. [153, Chapter 4.1] and [145, Chapter 5.2.1]. Note that we transferred these arguments to the coordinate frame with fixed length scale to fit in the presentation.

Hamiltonian

$$H^{\text{eff}} = \sum_{j=1}^{N} \left(-\Delta_j + V^{\text{ext}}(x_j)\right) + \frac{U_0}{N} \sum_{i<j} \delta(x_i - x_j) \tag{1.33}$$

containing an effective delta interaction. Note that $U_0 = 8\pi a = \mathcal{O}(1)$, hence all terms in H^{eff} are of the same order with respect to N.

Recall that in view of formula (1.18), the difference between $8\pi a_N$ and $\|w_N\|_{L^1(\mathbb{R}^3)}$ is due to the correlation structure. Hence, by replacing w_N by $8\pi a_N \delta(x)$ and not simply by $\|w_N\|_{L^1(\mathbb{R}^3)}\delta(x)$, although $w_N(x) \approx \|w_N\|_{L^1(\mathbb{R}^3)}\delta(x)$ for sufficiently large N in the sense of distributions, we have "integrated out" the short-wavelength degrees of freedom and incorporated them in the effective interaction potential.

Since the correlation structure as in (1.20) is already taken into account in the effective interaction, one now adopts a mean-field approach and assumes that all particles occupy the same normalised state φ. Evaluated on this product state, the energy corresponding to the Hamiltonian H^{eff},

$$\begin{aligned} &N \int_{\mathbb{R}^3} \left(|\nabla\varphi(x)|^2 + V^{\text{ext}}(x)|\varphi(x)|^2\right) \mathrm{d}x \\ &\qquad + (N-1) \int_{\mathbb{R}^3\times\mathbb{R}^3} \tfrac{1}{2} U_0 |\varphi(x_1)|^2 |\varphi(x_2)|^2 \delta(x_1 - x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ &\approx\; N\mathcal{E}_a^{GP}[\varphi]\,, \end{aligned}$$

is given by the GP energy functional.

The GP energy functional has a unique minimizer φ_a^{GP} (up to a phase), which is positive and continuously differentiable (see e.g. [120, Theorem 2.1]). The corresponding ground state energy is

$$E_a^{\text{GP}} = \inf_{\int |\varphi|^2 = 1} \mathcal{E}_a^{\text{GP}}[\varphi] = \mathcal{E}_a^{\text{GP}}[\varphi_a^{\text{GP}}]\,.$$

The minimiser φ_a^{GP} solves the stationary GP equation,

$$\left(-\Delta + V^{\text{ext}} + 4\pi a|\varphi|^2\right)\varphi = \mu\varphi\,, \tag{1.34}$$

in the sense of distributions. Here, μ is the chemical potential, which is a Lagrange multiplier arising from the normalisation condition for φ_a^{GP}. The stationary GP equation has the form of a stationary Schrödinger equation with non-linear potential term, where the eigenvalue is the chemical potential. Note that for non-interacting particles, the chemical potential and the energy per particle coincide, hence (1.34) reduces to the linear Schrödinger equation.

These considerations lead to the following generalisation of the homogeneous particle density $\varrho = N/L^3$ to the inhomogeneous setting: the mean density is defined as the average

$$\overline{\varrho} := \frac{1}{N} \int_{\mathbb{R}^3} \left(\varrho_a^{\mathrm{GP}}(x)\right)^2 \mathrm{d}x \;=\; N \int_{\mathbb{R}^3} \left|\varphi_a^{\mathrm{GP}}(x)\right|^4 \mathrm{d}x \,,$$

where $\varrho_a^{\mathrm{GP}} = N|\varphi_a^{\mathrm{GP}}|^2$ is the particle density (1.10) of the product state $(\varphi_a^{\mathrm{GP}})^{\otimes N}$. In the inhomogeneous context, *dilute* means that

$$\overline{\varrho}\, a_N^3 \ll 1 \,.$$

The minimiser φ_a^{GP} is N-independent, hence $\overline{\varrho}$ is of order N. Consequently, we obtain $\overline{\varrho}\, a_N^3 \sim N^{-2}$ as in the homogeneous case, which implies that the trapped gas with interactions in the GP scaling regime is ultra-dilute.

In [120], Lieb, Seiringer and Yngvason proved that the minimum of the GP energy functional with parameter a asymptotically describes the ground state energy per particle, $N^{-1}E_0(N, a_N)$, of the Hamiltonian (1.30). More precisely, they showed that

$$\lim_{N\to\infty} \frac{E_0(N, a_N)}{N} \;=\; E_a^{\mathrm{GP}} \tag{1.35}$$

(see also [119, Chapter 6]). Moreover, Lieb and Seiringer showed that the ground state ψ^N of (1.30) exhibits complete asymptotic condensation in the state φ_a^{GP}, i.e.,

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\psi^N}^{(1)} - \left|\varphi_a^{\mathrm{GP}}\right\rangle \left\langle \varphi_a^{\mathrm{GP}}\right| \right| = 0$$

([118] and [119, Chapter 7]). Note that the GP energy functional in a box Λ with periodic or Neumann boundary conditions is minimised by the constant function $\varphi_a^{\mathrm{GP}} = |\Lambda|^{-\frac{1}{2}}$, which yields the corresponding statements (1.19) and (1.23). For positive temperatures, a result similar to the homogeneous case was obtained by Deuchert, Seiringer and Yngvason in [58].

1.3. Interacting Bose gas in one and two dimensions

In this section, we briefly summarise the 2d analogues of the results in Sections 1.2.3 and 1.2.5 and present an exactly solvable model for a 1d gas. In physical reality, low-dimensional gases are realised in highly anisotropic traps which tightly confine the motion of the 3d particles in one or two directions. Such systems are called *quasi-low-dimensional*, and we summarise some of their relevant ground state properties in Sections 1.3.3. The main references for this section are [119, Chapters 3, 6, 8, 9 and Appendix B], [123], [145, Chapter 15.4], [147, Lectures 3 and 4], [153, Chapter 17.3],

[164] and [170].

1.3.1. Two dimensions

As explained in Section 1.1, a homogeneous 2d ideal gas does not exhibit BEC at positive temperature. However, BEC can occur in traps (e.g. in a harmonic trap) since the potential changes the density of states $g(\epsilon)$.

For the analysis of 2d interacting systems, the zero energy scattering solution, i.e., the minimiser of the 2d functional corresponding to $\mathcal{E}_R$ from (1.15), plays an important role. Consider a spherically symmetric, compactly supported potential W with range R_0, whose negative part is contained in $L^{1+\varepsilon}(\mathbb{R}^2)$ for some $\varepsilon > 0$ and where $\frac{1}{2}W$ has no negative energy bound states in $L^2(\mathbb{R}^2)$. If W has scattering length A, the minimiser of $\mathcal{E}_R^{2\mathrm{d}}$ satisfies

$$\begin{cases} \phi_0^{2\mathrm{d}}(x) = \dfrac{\ln(|x|/A)}{\ln(R/A)} & R > |x| > R_0\,, \\ \phi_0^{2\mathrm{d}}(x) \geq \dfrac{\ln(|x|/A)}{\ln(R/A)} & |x| \leq R_0\,, \end{cases}$$

and the minimum of $\mathcal{E}_R^{2\mathrm{d}}$ is

$$E_R^{2\mathrm{d}} = \frac{2\pi}{\ln(R/A)}\,.$$

Both statements are proven in [125, Theorem A.1 and Lemma A.1]. The density of a gas of N particles in a quadratic box Λ with $|\Lambda| = L^2$ is given by

$$\varrho_{2\mathrm{d}} = \frac{N}{L^2}\,.$$

Since the mean inter-particle distance in two dimensions is given by $\varrho_{2\mathrm{d}}^{-\frac{1}{2}}$, the diluteness condition for particles interacting via a potential with scattering length A is

$$\varrho_{2\mathrm{d}} A^2 \ll 1\,.$$

The ground state energy per particle of a dilute homogeneous 2d Bose gas in the thermodynamic limit, $e_0^{2\mathrm{d}}(\varrho_{2\mathrm{d}}) = \lim_{N\to\infty} E_0(N, (N/\varrho_{2\mathrm{d}})^{\frac{1}{2}})/N$ is asymptotically given by

$$\lim_{\varrho_{2\mathrm{d}} A^2 \to 0} \frac{e_0^{2\mathrm{d}}(\varrho_{2\mathrm{d}})}{4\pi\varrho_{2\mathrm{d}} |\ln(\varrho_{2\mathrm{d}} A^2)|^{-1}} = 1\,, \tag{1.36}$$

independently of the boundary conditions on the box. This asymptotic formula was first derived by Schick [161] for a gas of hard discs and rigorously proven by Lieb and Yngvason in [125]. Note that in contrast to the 3d problem, the 2d ground state energy of N particles, $E_0(N, L) \approx N e_0^{2\mathrm{d}}(\varrho_{2\mathrm{d}})$, does not (asymptotically) equal $N(N-1)/2$ times the energy of two particles. The latter can be calculated similarly as in the 3d problem (Section 1.2.3) and yields

$$\frac{N(N-1)}{2}E_0(2,L) = \frac{N(N-1)}{2}\frac{8\pi}{L^2}|\ln(L^{-2}A^2)|^{-1} \approx N4\pi\varrho_{2\mathrm{d}}|\ln(L^{-2}A^2)|^{-1}$$

[119, pp. 27–28]. Here, in contrast to (1.36), the mean inter-particle separation $\varrho_{2\mathrm{d}}^{-\frac{1}{2}}$ in the logarithm is replaced by L, resulting in a much lower energy.

Let us now consider a 2d Bose gas in an external trapping potential. As in Section 1.2.5, we choose the characteristic length of the trap as length unit, hence the scattering length $A = a_N$ depends on N. Analogously to the 3d case, one could take $4\pi N \int_{\mathbb{R}^2} |\ln(|\varphi(x)|^2 a_N^2)|^{-1}|\varphi(x)|^4 \,\mathrm{d}x$ as interaction term in the functional. However, since ln varies only slowly, it turns out that one may, to leading order, replace this complicated expression by $4\pi Ng \int_{\mathbb{R}^2} |\varphi(x)|^4 \,\mathrm{d}x$, i.e.,

$$\mathcal{E}_{Ng}^{\mathrm{GP,2d}}[\varphi] := \int\limits_{\mathbb{R}^2} \left(|\nabla\varphi(x)|^2 + V^{\mathrm{ext}}(x)|\varphi(x)|^2 + 4\pi Ng|\varphi(x)|^4\right) \mathrm{d}x \tag{1.37}$$

with subsidiary condition $\int |\varphi|^2 = 1$. A valid choice for the coupling parameter g is

$$g = \frac{1}{|\ln(\overline{\varrho}_{2\mathrm{d}} a_N^2)|} \,. \tag{1.38}$$

Here, $\overline{\varrho}_{2\mathrm{d}}$ denotes a mean density, which is for simplicity taken as

$$\overline{\varrho}_{2\mathrm{d}} := N \int\limits_{\mathbb{R}^2} |\varphi_N^{\mathrm{GP}}(x)|^4 \,\mathrm{d}x \,, \tag{1.39}$$

where φ_N^{GP} denotes the minimiser of $\mathcal{E}_N^{\mathrm{GP,2d}}$ with coupling parameter $g = 1$. Note that one could also define $\overline{\varrho}_{2\mathrm{d}}$ self-consistently, i.e., in terms of $\varphi_{Ng}^{\mathrm{GP}}$, but the above simpler choice is sufficient for a leading order estimate as $\overline{\varrho}_{2\mathrm{d}}$ exclusively appears in the logarithm.

In [121], Lieb, Seiringer and Yngvason prove that the minimum $E_{Ng}^{\mathrm{GP,2d}}$ of $\mathcal{E}_{Ng}^{\mathrm{GP,2d}}$ asymptotically coincides with the ground state energy per particle of a dilute 2d Bose gas, $N^{-1}E_0(N, a_N)$. More precisely, they show that in the limit $N \to \infty$ such that $a_N^2 \overline{\varrho}_{2\mathrm{d}} \to 0$ and Ng fixed,

$$\lim \frac{E_0^{2\mathrm{d}}(N, a_N)}{N E_{Ng}^{\mathrm{GP,2d}}} = 1 \,. \tag{1.40}$$

To realise the GP scaling regime, where Ng is fixed independently of N, the scattering length a_N must decrease exponentially in N. A slower decrease implies $g \to \infty$, which means that the kinetic term in the GP energy functional becomes negligible and $\mathcal{E}_{Ng}^{\mathrm{GP,2d}}$

simplifies to the so-called *Thomas–Fermi functional* (see [121] for a rigorous proof).

1.3.2. One-dimension: Lieb–Liniger model

In one dimension, BEC does not occur in the homogeneous ideal Bose gas but takes place in external traps which confine the particles stronger than a harmonic potential [11]. A famous model of an interacting 1d Bose gas is the so-called *Lieb–Liniger (LL) model*,

$$H_{N,g}^{1\mathrm{d}} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + g \sum_{1\leq i<j\leq N} \delta(z_i - z_j)\,,$$

which describes a uniform gas of N bosons interacting via repulsive zero-range potentials with strength $g \geq 0$. It was originally proposed and analysed by Lieb and Liniger in [115, 116]. The δ-function potential is equivalent to the boundary conditions

$$(\partial_j - \partial_k)\,\psi\big|_{x_j\downarrow x_k} - (\partial_j - \partial_k)\,\psi\big|_{x_j\uparrow x_k} = g\psi\big|_{x_j=x_k}\,,$$

implying that ψ is continuous at the points where two particles coincide, while the derivative is discontinuous and jumps by the value g. The particles are confined to an interval of length L with periodic boundary conditions, hence the density is given by

$$\varrho_{1\mathrm{d}} = \frac{N}{L}\,.$$

The relevant parameter of the model is

$$\gamma = \frac{g}{\varrho_{1\mathrm{d}}}\,,$$

which corresponds to the ratio of the interaction energy per particle $\sim \varrho_{1\mathrm{d}}\, g$ to the kinetic energy per particle $\sim 1/(\varrho_{1\mathrm{d}}^{-1})^2 = \varrho_{1\mathrm{d}}^2$. For $\gamma \ll 1$, the gas is weakly interacting, with $\gamma = 0$ corresponding to the ideal gas. The case $\gamma \gg 1$ describes a strongly interacting gas, also called *Tonks–Girardeau gas*, where the limit $\gamma = \infty$ describes a gas of impenetrable bosons. In [79], Girardeau proved that the energy spectrum of such a gas coincides with the spectrum of a one-component spinless non-interacting Fermi gas of the same density. Mathematically, this follows since $\gamma = \infty$ implies the boundary condition $\psi\big|_{x_j=x_k} = 0$, which is solved by a (fermionic) Slater determinant, multiplied with a sign function symmetrising the N-body wave function (see e.g. [145, Chapter 15.4.1]).

As the density $\varrho_{1\mathrm{d}}$ decreases, the parameter γ increases, implying that the gas becomes more interacting. This peculiar feature is unique for the 1d case and contrasts with the behaviour of two-and three-dimensional Bose gases.

In [116], Lieb and Liniger computed the eigenfunctions as well as the ground state energy per particle in the thermodynamic limit $N, L \to \infty$ with fixed density $\varrho_{1\mathrm{d}}$. They showed that, irrespective of the boundary conditions on the 1d box, the ground state energy is given by

$$e_0^{1\mathrm{d}}(\varrho_{1\mathrm{d}}) = \varrho_{1\mathrm{d}}^2 e(\gamma) \approx \begin{cases} \frac{1}{2} g \varrho_{1\mathrm{d}} & \gamma = g/\varrho_{1\mathrm{d}} \ll 1\,, \\ \frac{\pi^3}{3} \varrho_{1\mathrm{d}}^2 & \gamma = g/\varrho_{1\mathrm{d}} \gg 1\,, \end{cases} \tag{1.41}$$

where e is the solution of an integral equation (see e.g. [119, Appendix B]) with the specified asymptotic behaviour. For the Tonks–Girardeau gas ($\gamma = \infty$), $e_0^{1\mathrm{d}}$ coincides with the energy of N non-interacting fermions, while one obtains $e_0^{1\mathrm{d}} = 0$ for the ideal gas ($\gamma = 0$).

1.3.3. Quasi-low-dimensional Bose gases

Effectively low-dimensional behaviour of a 3d gas occurs when the motion of the particles in one or two spatial dimensions is frozen out as the result of a sufficiently anisotropic trapping potential. Let L denote the longitudinal length scale and εL the transverse length scale, implying that ε is a measure of the asymmetry of the set-up. Studying the Bose gas for small ε, one observes quasi-low-dimensional behaviour if the energy associated with the motion along the trap is small compared to the energy gap between transverse ground state and excitation spectrum, which scales as $(\varepsilon L)^{-2}$.

For the mathematical analysis of the ground state problem, we use the coordinates

$$z = (x, y) \in \mathbb{R}^3\,, \qquad x \in \mathbb{R}^d\,, \qquad y \in \mathbb{R}^{3-d}\,,$$

where x is the coordinate in the d longitudinal direction(s) and y is the coordinate in the $(3-d)$ transverse direction(s). We consider the 3d Hamiltonian

$$H^{3\mathrm{d}}_{N,L,\varepsilon,A} = \sum_{j=1}^{N} \left(-\Delta_j + V^{\perp}_{\varepsilon L}(y_j) + V^{\parallel}_{L}(x_j) \right) + \sum_{1 \le i < j \le N} w_A(z_i - z_j)\,, \tag{1.42}$$

where

$$V^{\perp}_{\varepsilon L}(y) = \frac{1}{(\varepsilon L)^2} V^{\perp}\left(\frac{y}{\varepsilon L}\right), \qquad V^{\parallel}_{L}(x) = \frac{1}{L^2} V^{\parallel}\left(\frac{x}{L}\right)$$

and

$$w_A(z) = \frac{1}{A^2} w\left(\frac{x}{A}\right).$$

The parameters L, ε and A are scaling parameters, while $V^{\perp}$, $V^{\parallel}$ and w are taken as fixed. We assume that the characteristic lengths of $V^{\perp}$ and $V^{\parallel}$ equal one, hence L and $L\varepsilon$ measure the extensions of the trap, and that w has scattering length one, which implies that the scattering length of w_A equals A since w_A is of the form (1.26). The transverse ground states χ and χ^ε corresponding to the potentials $V^\perp(y)$ and $V^\perp_{\varepsilon L}(y)$ are related by scaling as

$$\left(-\Delta_y + V^\perp\right)\chi = e^\perp \chi\,, \qquad \left(-\Delta_y + V^\perp_{\varepsilon L}\right)\chi^{\varepsilon L} = \frac{e^\perp}{(\varepsilon L)^2}\chi^{\varepsilon L}\,,$$

with

$$\chi^{\varepsilon L}(y) = (\varepsilon L)^{-\frac{3-d}{2}}\chi\left(\frac{y}{\varepsilon L}\right)\,. \tag{1.43}$$

Let $E^{\mathrm{3d}}_{N,L,\varepsilon,A}$ denote the ground state energy of the Hamiltonian $H^{\mathrm{3d}}_{N,L,\varepsilon,A}$. We are interested in the behaviour of this quantity in the limit where $N \to \infty$ and $\varepsilon \to 0$, i.e., the limit of large particle numbers and infinite asymmetry. For $d = 1$, this limit was analysed by Lieb, Seiringer and Yngvason in [122, 123] and [119, Chapter 8] and by Seiringer and Yin in [170], and the analogous problem for $d = 2$ was treated by Schnee and Yngvason in [164] and [119, Chapter 9]. In both cases, the authors assume that the potentials $V^\perp$ and $V^\parallel$ are locally bounded, diverging as $|y|\,, |x| \to \infty$, and homogeneous of degree $s > 0$. The potential w is assumed non-negative and of finite range. In the remainder of this section, we summarise the results obtained in [122, 123, 164], suitably adapted to our notation.

Quasi-one-dimensional Bose gas

Let us begin with $d = 1$, where the confinement is in two directions and consequently $x \in \mathbb{R}$ and $y \in \mathbb{R}^2$. The main result of [122, 123] by Lieb, Seiringer and Yngvason states that in the limit $\varepsilon \to 0$ and $N \to \infty$, the ground state energy $E^{\mathrm{3d}}_{N,L,\varepsilon,A}$ of $H^{\mathrm{3d}}_{N,L,\varepsilon,A}$ equals the minimum of a 1d functional that is obtained from an inhomogeneous LL model with coupling parameter

$$g_{\mathrm{1d}} = \frac{8\pi A}{(\varepsilon L)^2}\int\limits_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y = 8\pi A\int\limits_{\mathbb{R}^2}|\chi^\varepsilon(y)|^4\,\mathrm{d}y\,. \tag{1.44}$$

To construct it, we recall that the effective interaction term $4\pi a|\varphi|^4$ in the 3d GP energy functional $\mathcal{E}^{\mathrm{GP}}_a$ (1.31) is formed by taking the homogeneous ground state energy per particle (1.19) with density $\varrho = N|\varphi|^2$ as local energy density. For the 1d functional, one replaces $4\pi a_N\varrho = 4\pi a|\varphi|^2$ with the ground state energy of the homogeneous LL model, $e^{\mathrm{1d}}_0(x) = \varrho(x)^2 e(g_{\mathrm{1d}}/\varrho(x))$. Here, e is the function which arises from solving the stationary Schrödinger equation for the LL Hamiltonian with asymptotic behaviour as in (1.41). Multiplying the resulting functional by N yields

$$\mathcal{E}^{\mathrm{1d}}_{N,L,g_{\mathrm{1d}}}[\varrho] = \int\limits_{\mathbb{R}}\left(\left|\frac{\partial}{\partial x}\sqrt{\varrho(x)}\right|^2 + V^\parallel_L(x)\varrho(x) + \varrho(x)^3 e\left(\frac{g_{\mathrm{1d}}}{\varrho(x)}\right)\right)\mathrm{d}x \tag{1.45}$$

for ϱ satisfying the normalisation condition $\int_{\mathbb{R}} \varrho(x)\,\mathrm{d}x = N$. The functional (1.45) has a unique minimiser, which is denoted by $\varrho_{N,L,g_{1\mathrm{d}}}$ and defines the mean 1d density

$$\overline{\varrho}_{1\mathrm{d}} := \frac{1}{N} \int_{\mathbb{R}} (\varrho_{N,L,g_{1\mathrm{d}}}(x))^2 \,\mathrm{d}x \,,$$

analogously to the 3d mean density $\overline{\varrho}$ in Section 1.2.5. The corresponding length scale

$$\overline{L}_{1\mathrm{d}} = \frac{N}{\overline{\varrho}_{1\mathrm{d}}}$$

determines the characteristic length of the gas cloud, and the minimum of (1.45) is denoted as

$$E^{1\mathrm{d}}_{N,L,g_{1\mathrm{d}}} := \mathcal{E}^{1\mathrm{d}}[\varrho_{N,L,g_{1\mathrm{d}}}] \,.$$

The physical relevance of the functional (1.45) is established in [123, Theorem 1.1]: With coupling parameter $g_{1\mathrm{d}}$ as in (1.44), it holds in the combined limit

$$N \to \infty \qquad \text{and} \qquad \begin{cases} (a) & \varepsilon \to 0 \,, \\ (b) & \frac{A}{\varepsilon L} \to 0 \,, \\ (c) & (\varepsilon L)^2 \,\overline{\varrho}_{1\mathrm{d}} \,\min\{\overline{\varrho}_{1\mathrm{d}}, g_{1\mathrm{d}}\} \to 0 \end{cases} \tag{1.46}$$

that

$$\lim_{(1.46)} \frac{E^{3\mathrm{d}}_{N,L,\varepsilon,A} - N\frac{e^{\perp}}{(\varepsilon L)^2}}{E^{1\mathrm{d}}_{N,L,g_{1\mathrm{d}}}} = 1 \,. \tag{1.47}$$

Note that the three conditions (a) to (c) of the combined limit (1.46) are not independent of each other. With regard to the asymptotic behaviour (1.41) of $e_0^{1\mathrm{d}}$, condition (c) is equivalent to the requirement that

$$e_0^{1\mathrm{d}}(\overline{\varrho}_{1\mathrm{d}}) \ll \frac{1}{(\varepsilon L)^2} \,,$$

which means that the longitudinal energy per particle must be much smaller than the transverse energy gap.

In conclusion, the 3d ground state energy related to the longitudinal motion (where the ground state energy in the confined directions is subtracted) is asymptotically described by the minimiser of the 1d energy functional containing the LL energy density, provided one chooses the LL-coupling parameter g as in (1.44). In this sense, the ground state of a 3d Bose gas in a highly elongated trap is described by the 1d LL model.

Depending on the ratio $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}}$, the energy functional (1.45) simplifies in different limiting cases, dividing the parameter space into five regions, which can be grouped into two physically very different regimes. To motivate this dichotomy, note that the formula (1.19) for the ground state energy of a homogeneous 3d gas yields for the box $\overline{L}_{1\mathrm{d}} \times (\varepsilon L) \times (\varepsilon L)$ the expression

$$e_0^{3\mathrm{d}} \sim \frac{NA}{(\varepsilon L)^2 \overline{L}_{1\mathrm{d}}}\,.$$

While (1.19) is correct for any fixed ε, it does not hold uniformly as $\varepsilon \to 0$. Let us compare this expression with the 1d energy per particle $e_0^{1\mathrm{d}}(\overline{\varrho}_{1\mathrm{d}})$. Noting that $g_{1\mathrm{d}}$ scales as $A/(\varepsilon L)^2$, we obtain by (1.41) the scaling behaviour

$$e_0^{1\mathrm{d}}(\overline{\varrho}_{1\mathrm{d}}) \sim \begin{cases} g_{1\mathrm{d}}\overline{\varrho}_{1\mathrm{d}} \sim \dfrac{NA}{(\varepsilon L)^2 \overline{L}_{1\mathrm{d}}} & g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \ll 1\,, \\ \overline{\varrho}_{1\mathrm{d}}^2 \sim \dfrac{N^2}{\overline{L}_{1\mathrm{d}}^2} & g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \gg 1\,. \end{cases}$$

While in the first case, the 1d formula for the energy per particle coincides with the 3d formula, this is not true in the second case. Note that the quantity $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}}$ can also be interpreted as the fraction $e_0^{3\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}}^2$. Its behaviour as $N \to \infty$ defines the two regimes mentioned above:

- $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \ll 1$: *The 1d limit of the 3d GP regime*
  In this regime, the predictions of 3d and 1d theory coincide, which was motivated above by the comparison of $e_0^{3\mathrm{d}}$ and $e_0^{1\mathrm{d}}$. Since the LL parameter γ is much smaller than one, the gas is weakly interacting, which is equivalent to a high 1d density. This regime subdivides into three regions, each of which is characterised by a simpler form of the functional (1.45):
  1. *The ideal gas case:* $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \ll N^{-2}$.
     The interactions are so weak that $e_0^{1\mathrm{d}} = (g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}})\overline{\varrho}_{1\mathrm{d}}^2 \ll \overline{L}_{1\mathrm{d}}^{-2}$, i.e., their effect vanishes in comparison with the longitudinal kinetic energy. In (1.47), the minimiser $E^{1\mathrm{d}}_{N,L,g_{1\mathrm{d}}}$ of the LL functional can simply be replaced by $Ne^{\parallel}/L^2$, where $e^{\parallel}$ denotes the ground state energy of $-\partial_x^2 + V^{\parallel}$.
  2. *The 1d GP case:* $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \sim N^{-2}$.
     In this region, the functional (1.45) reduces to the 1d GP functional

     $$\mathcal{E}^{\mathrm{GP,1d}}_{N,L,g_{1\mathrm{d}}}[\varrho] = \int\limits_{\mathbb{R}} \left(|\partial_x \sqrt{\varrho(x)}|^2 + V_L^{\parallel}(x)\varrho(x) + \tfrac{1}{2} g_{1\mathrm{d}}\varrho(x)^2 \right) \mathrm{d}x\,,$$

     where the interaction term is taken as the low-γ-asymptotics (1.41) of the LL ground state energy per particle. To bring this expression into a form which is analogous to the 3d GP functional (1.31), where the dependence on the parameters N, L, $g_{1\mathrm{d}}$ is exclusively encoded in the interaction term, one notes that $\varrho(x) = N|\Phi_L(x)|^2$, where $\Phi_L(x) = L^{-\frac{1}{2}}\Phi(x/L)$ by definition of $V_L^{\parallel}$. Substituting this into $\mathcal{E}^{\mathrm{GP,1d}}_{N,L,g_{1\mathrm{d}}}$ and dividing by N/L^2, one obtains the functional

$$\mathcal{E}^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}L} = \int\limits_{\mathbb{R}} \left(|\partial_x\Phi(x)|^2 + V^{\parallel}(x)|\Phi(x)|^2 + \tfrac{1}{2}Ng_{1\mathrm{d}}L|\Phi(x)|^4\right)\mathrm{d}x \quad (1.48)$$

for Φ with $\int|\Phi(x)|^2\,\mathrm{d}x = 1$. The ground states of both functionals are related by the scaling relation

$$E^{\mathrm{GP,1d}}_{N,L,g_{1\mathrm{d}}} = \frac{N}{L^2}E^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}L}\,.$$

The characteristic length scale $\overline{L}_{1\mathrm{d}}$ of the cloud can be computed via the scaling relation

$$V_L^{\parallel}(\overline{L}_{1\mathrm{d}}) \sim e_0^{1\mathrm{d}}(\overline{\varrho}_{1\mathrm{d}})\,. \quad (1.49)$$

Since $V^{\parallel}$ is homogeneous of degree s and $e_0^{1\mathrm{d}} = (g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}})\overline{\varrho}_{1\mathrm{d}}^2 \sim \overline{L}_{1\mathrm{d}}^{-2}$, this yields $\overline{L}_{1\mathrm{d}} \sim L$, implying that the cloud longitudinally extends over the whole trap, hence $\overline{\varrho}_{1\mathrm{d}} \sim N/L$. Finally, note that $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \sim N^{-2}$ is equivalent to the requirement that $Ng_{1\mathrm{d}}L$ be fixed as $N\to\infty$, and that conditions (b) and (c) in (1.46) are implied by (a). Hence, the statement (1.47) can be simplified as follows:

In the limit $N\to\infty$ and $\varepsilon\to 0$ such that $Ng_{1\mathrm{d}}L$ remains fixed,

$$\lim \frac{E^{\mathrm{3d}}_{N,L,\varepsilon,A} - N\frac{e^{\perp}}{(\varepsilon L)^2}}{\frac{N}{L^2}E^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}L}} = 1\,. \quad (1.50)$$

3. *The 1d Thomas–Fermi case:* $N^{-2} \ll g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \ll 1$.
   Since $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \gg N^{-2}$ is equivalent to $Ng_{1\mathrm{d}}L\to\infty$, it follows that $e_0^{1\mathrm{d}} \gg \overline{L}_{1\mathrm{d}}^{-2}$, i.e., the gradient term in the functional (1.45) becomes negligible. Hence, (1.45) is asymptotically equivalent to the 1d Thomas–Fermi functional, which is given by the GP functional without kinetic term.

The ground state energy of a Bose gas in this first regime can also be obtained as the limit $\varepsilon\to 0$ of the 3d GP energy. More precisely, let $\mathcal{E}^{\mathrm{GP}}_{\varepsilon,\frac{NA}{L}}$ denote the GP energy functional corresponding to the Hamiltonian $H^{3\mathrm{d}}_{N,L,\varepsilon,A}$, i.e.,

$$\mathcal{E}^{\mathrm{GP}}_{\varepsilon,\frac{NA}{L}}[\varphi] = \int\limits_{\mathbb{R}^3} \left(|\nabla\varphi(z)|^2 + \left(V^{\perp}_{\varepsilon}(y) + V^{\parallel}(x)\right) |\varphi(z)|^2 + 4\pi \frac{NA}{L} |\varphi(z)|^4 \right) \mathrm{d}z \,. \tag{1.51}$$

Then it is shown in [123, Theorem 2.6] that in the limit $\varepsilon \to 0$, $A \to 0$,

$$\lim \frac{E^{\mathrm{GP}}_{\varepsilon,\frac{NA}{L}} - \frac{e^{\perp}}{\varepsilon^2}}{E^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}L}} = 1 \tag{1.52}$$

uniformly in $g_{1\mathrm{d}}$, as long as $\varepsilon^2 E^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}L} \to 0$. In view of (1.47), this implies that the ground state energy can be calculated by first taking the limit $N \to \infty$ and subsequently the limit $\varepsilon \to 0$. It implies that the 3d GP result (1.35) holds uniformly as $\varepsilon \to 0$, provided the quantity $NA/(L\varepsilon^2)$ remains bounded. This requirement is in particular satisfied in region 2.

Moreover, it has been shown that the N-body ground state $\psi_0^{N,L,\varepsilon}$ of the Hamiltonian $H^{3\mathrm{d}}_{N,L,\varepsilon,A}$ exhibits BEC in regions 1 and 2, while the problem remains open in region 3. More precisely, [123, Theorem 5.1] states that

$$\mathrm{Tr}_{L^2(\mathbb{R}^3)} \left| \gamma^{(1)}_{\psi_0^{N,L,\varepsilon}} - |\Phi^{\mathrm{GP}}_L \chi^{\varepsilon L}\rangle\langle \Phi^{\mathrm{GP}}_L \chi^{\varepsilon L}| \right| \to 0 \,, \tag{1.53}$$

where $\Phi^{\mathrm{GP}}_L(x) = L^{-\frac{1}{2}} \Phi^{\mathrm{GP}}(x/L)$ and Φ^{GP} is the minimiser of $\mathcal{E}^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}L}$.

- $g/\overline{\varrho}_{1\mathrm{d}} \gtrsim 1$: *The true 1d regime.*
  In this regime, $e_0^{1\mathrm{d}}$ and $e_0^{3\mathrm{d}}$ differ from each other. In contrast to the first regime, the second regime cannot be reached from a 3d energy functional as in (1.52), and is, in this sense, truly 1d. The corresponding LL parameter γ is large, hence the gas is strongly interacting with low 1d density. The ground state is not expected to exhibit BEC; instead, the motion in the longitudinal direction is strongly correlated. The true 1d regime can be split into two regions:

  4. *The LL case:* $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \sim 1$.
     In this region, neither of the asymptotics in (1.41) apply, hence the full LL energy is required in the functional. Since $e_0^{1\mathrm{d}} \sim N^2 \overline{L}^{-2} \gg \overline{L}^{-2}$, one can neglect the gradient term of the functional (1.45).

  5. *The Tonks–Girardeau case:* $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \gg 1$.
     This region is analogous to the LL case, with the only exception that the asymptotics (1.41) for $\gamma \to \infty$ apply. Consequently, the expression $\varrho^3 e(g_{1\mathrm{d}}/\varrho)$ in the functional can be replaced by $\frac{\pi^2}{3}\varrho^3$.

The ground state result (1.47) was extended in [170] by Seiringer and Yin to the lower part of the excitation spectrum. More precisely, let $E^{3\mathrm{d},k}_{N,L,\varepsilon,A}$ for $k = 1, 2, 3, \ldots$, denote the k'th eigenvalue of the Hamiltonian $H^{3\mathrm{d}}_{N,L,\varepsilon,A}$ and denote by $E^{1\mathrm{d},k}_{N,L,g_{1\mathrm{d}}}$ the k'th eigenvalue of the LL Hamiltonian with coupling parameter $g_{1\mathrm{d}}$ and external potential $V^{\|}_L$,

$$H^{1\mathrm{d}}_{N,L,g_{1\mathrm{d}}} = \sum_{j=1}^{N} \left(-\frac{\partial^2}{\partial x_j^2} + V^{\|}_L(x_j) \right) + g_{1\mathrm{d}} \sum_{1\le i<j\le N} \delta(x_i - x_j) \,. \tag{1.54}$$

For fixed N, L and k, the authors prove that in the limit $\varepsilon \to 0$ and $A \to 0$ such that $A/\varepsilon \to 0$,

$$\lim \frac{E^{3\mathrm{d},k}_{N,L,\varepsilon,A} - \frac{Ne^{\perp}}{(\varepsilon L)^2}}{E^{1\mathrm{d},k}_{N,L,g_{1\mathrm{d}}}} = 1 \,, \tag{1.55}$$

as long as $E^{1\mathrm{d},k}_{N,L,g_{1\mathrm{d}}} \le \widetilde{e}^{\perp}/(\varepsilon L)^2$. Here, $\widetilde{e}^{\perp}$ denotes the spectral gap above the ground state energy of $-\Delta^{\perp}_y + V^{\perp}(y)$ ([170, Corollary 1]). This statement follows from the upper and lower bounds

$$E^{3\mathrm{d},k}_{N,L,\varepsilon,A} \ \ge \ \frac{Ne^{\perp}}{(\varepsilon L)^2} + E^{1\mathrm{d},k}_{N,L,g_{1\mathrm{d}}}(1-\eta_L) \left(1 - \frac{(\varepsilon L)^2}{\widetilde{e}^{\perp}} E^{1\mathrm{d},k}_{N,L,g_{1\mathrm{d}}} \right) , \tag{1.56}$$

$$E^{3\mathrm{d},k}_{N,L,\varepsilon,A} \ \le \ \frac{Ne^{\perp}}{(\varepsilon L)^2} + E^{1\mathrm{d},k}_{N,L,g_{1\mathrm{d}}} \left(1-\eta_U\right)^{-1} , \tag{1.57}$$

where

$$\eta_L = D \left(\left(\frac{NA}{\varepsilon L} \right)^{\frac{1}{8}} + N^2 \left(\frac{NA}{\varepsilon L} \right)^{\frac{3}{8}} \right) , \quad \eta_U = C \left(\frac{NA}{\varepsilon L} \right)^{\frac{2}{3}}$$

for some constants $C, D > 0$ ([170, Theorem 1]). Hence, the spectrum of a Bose gas in a cigar-shaped trap in an energy interval of size $\sim (\varepsilon L)^{-2}$ above the ground state asymptotically coincides with the spectrum of the LL model with coupling parameter $g_{1\mathrm{d}}$. Note that this applies to all parameter regions 1 to 5.

Moreover, Seiringer and Yin prove a similar result for the eigenfunctions. For $g_0 \in \mathbb{R}$, let $\psi^{3\mathrm{d},k}$ be an eigenfunction of $H^{3\mathrm{d}}_{N,L,\varepsilon,A}$ with eigenvalue $E^{3\mathrm{d},k}_{N,L,\varepsilon,A}$, and let $P^{1\mathrm{d},k}_{g_0}$ denote the projection onto the eigenspace of $H^{1\mathrm{d}}_{N,L,g_0}$ with eigenvalue $E^{1\mathrm{d},k}_{N,L,g_0}$. Further, let $P^{\perp}_{\varepsilon L}$ denote the projection onto $\prod_{j=1}^{N} \chi^{\varepsilon L}(y_j)$. Then, for fixed N, L, k and g_0, Seiringer and Yin prove that in the limit $\varepsilon \to 0$, $A \to 0$ such that $g_{1\mathrm{d}} \to g_0$,

$$\lim \left\langle \psi^{3\mathrm{d},k}, \left(P^{1\mathrm{d},k}_{g_0} \otimes P^{\perp}_{\varepsilon L} \right) \psi^{3\mathrm{d},k} \right\rangle = 1 \tag{1.58}$$

([170, Corollary 2]), where the tensor product refers to the decomposition $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^{2N})$ of $L^2(\mathbb{R}^{3N})$ into longitudinal and transverse coordinates. This implies that the 3d eigenfunctions $\psi^{3d,k}$ are, in $L^2(\mathbb{R}^{3N})$ sense, asymptotically of the product form

$$\psi^{3d,k}(z_1,\dots,z_N) \approx \psi^{1d,k}(x_1,\dots,x_N) \prod_{j=1}^{N} \chi^{\varepsilon L}(y_j)\,,$$

where $\psi^{1d,k}$ is an eigenfunction of H^{1d}_{N,L,g_0} corresponding to the eigenvalue $E^{1d,k}_{N,L,g_0}$.

Quasi-two-dimensional Bose gas

For $d = 2$, it was shown in [164] by Schnee and Yngvason that in the limit $\varepsilon \to 0$, the ground state energy $E^{3d}_{N,L,\varepsilon,A}$ of the Hamiltonian $H^{3d}_{N,L,\varepsilon,A}$ converges to the ground state energy of a 2d gas with effective 2d scattering length

$$A_{2d} = L\varepsilon \exp\left\{ -\frac{1}{\int_{\mathbb{R}} |\chi(y)|^4 \, dy} \cdot \frac{\varepsilon L}{2A} \right\} . \tag{1.59}$$

In view of (1.40), this 2d ground state energy $E^{2d}_{L,Ng_{2d}}$ can be obtained by minimising the 2d GP functional (1.37) with choice $V^{ext} = V^{\|}_L$,

$$\mathcal{E}^{GP,2d}_{L,Ng_{2d}}[\Phi_L] := \int_{\mathbb{R}^2} \left(|\nabla \Phi_L(x)|^2 + V^{\|}_L(x)|\Phi_L(x)|^2 + 4\pi N g_{2d} |\Phi_L(x)|^4 \right) dx\,, \tag{1.60}$$

where the corresponding coupling parameter g_{2d} is given by

$$g_{2d} = \frac{1}{|\ln(\overline{\varrho}_{2d} A_{2d}^2)|} \tag{1.61}$$

as in (1.38), with mean 2d density $\overline{\varrho}_{2d}$ from (1.39). More precisely, Schnee and Yngvason prove that in the combined limit

$$N \to \infty \qquad \text{and} \qquad \begin{cases} (a) & \varepsilon \to 0\,, \\ (b) & \frac{A}{\varepsilon L} \to 0\,, \\ (c) & (\varepsilon L)^2 \overline{\varrho}_{2d} g_{2d} \to 0\,, \end{cases} \tag{1.62}$$

it holds that

$$\lim_{(1.62)} \frac{\frac{1}{N} E^{3d}_{N,L,\varepsilon,A} - \frac{e^{\perp}}{(\varepsilon L)^2}}{E^{2d}_{L,Ng_{2d}}} = 1 \tag{1.63}$$

by [164, Theorem 1.1]. This statement is the 2d analogue of (1.47), where the role of the LL functional (1.45) is taken by the 2d GP functional (1.60). As in the 1d case, conditions (a) to (c) are not independent of each other. Condition (c) states that the

2d energy per particle corresponding to the density $\overline{\varrho}_{2\mathrm{d}}$, which scales as

$$e_0^{2\mathrm{d}}(\overline{\varrho}_{2\mathrm{d}}) \sim g_{2\mathrm{d}}\overline{\varrho}_{2\mathrm{d}}$$

by (1.36), must be much smaller than the transverse energy gap $\sim (\varepsilon L)^{-2}$.

Similarly to the quasi-1d gas, there is a fundamental division into two parameter regimes, which can be determined by comparing the expressions for the 3d and 2d energy per particle, (1.19) and (1.36). Since the mean 3d density is given by

$$\varrho = \frac{N}{\overline{L}^2(\varepsilon L)} = \frac{\overline{\varrho}_{2\mathrm{d}}}{(\varepsilon L)}\,,$$

this yields

$$\frac{e_0^{3\mathrm{d}}}{e_0^{2\mathrm{d}}} \sim \frac{NA/(\varepsilon L\overline{L}_{2\mathrm{d}}^2)}{g_{2\mathrm{d}}\overline{\varrho}_{2\mathrm{d}}} \sim \frac{A}{\varepsilon L}|\ln(\overline{\varrho}_{2\mathrm{d}}A_{2\mathrm{d}}^2)| = \frac{A}{\varepsilon L}\left|\ln(\overline{\varrho}_{2\mathrm{d}}(L\varepsilon)^2) - \frac{1}{\int|\chi(y)|^4\,\mathrm{d}y}\frac{\varepsilon L}{A}\right|\,,$$

which leads to the following two regimes:

1. $|\ln(\overline{\varrho}_{2\mathrm{d}}(\varepsilon L)^2)| \ll (\varepsilon L)/A$: *The 2d limit of the 3d GP regime.*

   By definition (1.59) of the effective 2d scattering length, this condition yields

   $$g_{2\mathrm{d}} = |\ln(\overline{\varrho}_{2\mathrm{d}}A_{2\mathrm{d}}^2)|^{-1} \sim \frac{A}{\varepsilon L}\,,$$

   hence $e_0^{2\mathrm{d}}(\overline{\varrho}_{2\mathrm{d}}) \sim (A\overline{\varrho}_{2\mathrm{d}})/(\varepsilon L)$. Consequently, the 2d formula (1.36) leads to the same result as the 3d formula (1.19). Moreover, one can replace $g_{2\mathrm{d}}$ by the simplified coupling parameter

   $$g_{2\mathrm{d}}^{(1)} := \int_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y\,\frac{A}{\varepsilon L}\,. \tag{1.64}$$

   Similarly to the 1d problem, the ground state energy in this regime can be understood as the limit $\varepsilon \to 0$ of the minimum of the 3d GP functional $\mathcal{E}^{\mathrm{GP}}_{\varepsilon,\frac{NA}{L}}$ from (1.51) but with 1d confinement. It is shown in [164, Theorem 2.1] that in the limit $\varepsilon \to 0$ with $g_{2\mathrm{d}} = g_{2\mathrm{d}}^{(1)}$,

   $$\lim \frac{E^{\mathrm{GP}}_{\varepsilon,\frac{NA}{L}} - \frac{e^{\perp}}{\varepsilon^2}}{E^{\mathrm{GP,2d}}_{1,Ng_{2\mathrm{d}}^{(1)}/L}} = 1 \tag{1.65}$$

   uniformly in the parameters, as long as condition (c) in (1.62) is satisfied.

2. $|\ln(\overline{\varrho}_{2\mathrm{d}}(\varepsilon L)^2)| \gtrsim (\varepsilon L)/A$: *The true 2d regime.*
   In this regime,
$$g_{2\mathrm{d}} \sim g_{2\mathrm{d}}^{(2)} := |\ln(\overline{\varrho}_{2\mathrm{d}}(\varepsilon L)^2)|^{-1}\,,$$
   and the logarithmic dependence on the density implies that the 2d and 3d predictions of the ground state energy lead to different results.

The relevant parameter for the 2d GP functional is $Ng_{2\mathrm{d}}$. Its size subdivides the parameter space into three regions:

(a) $Ng_{2\mathrm{d}} \ll 1$: *The ideal gas case.*
The interaction term in the GP functional becomes negligible, hence this region describes an ideal gas in an external trapping potential.

(b) $Ng_{2\mathrm{d}} \sim 1$: *The GP case.*
All terms in the GP functional are of the same order. In regime 1, this region corresponds to the scaling $A \sim \frac{\varepsilon L}{N}\int|\chi(y)|^4\,\mathrm{d}y$, while one requires $\overline{\varrho}_{2\mathrm{d}} \sim (\varepsilon L)^{-2}\mathrm{e}^{-2N}$ to reach it from regime 2. In this region, BEC occurs in the ground state $\psi_0^{N,L,\varepsilon}$ of $H^{3\mathrm{d}}_{N,L,\varepsilon,A}$. More precisely, [164, Theorem 1.3] states that in the limit $N \to \infty$ and $\varepsilon \to 0$ with Ng and L fixed,
$$\mathrm{Tr}_{L^2(\mathbb{R}^3)}\left|\gamma^{(1)}_{\psi_0^{N,L,\varepsilon}} - |\Phi_L^{\mathrm{GP}}\chi^{\varepsilon L}\rangle\langle\Phi_L^{\mathrm{GP}}\chi^{\varepsilon L}|\right| \to 0\,, \tag{1.66}$$
where $\Phi_L^{\mathrm{GP}}(x)$ denotes the minimiser of $\mathcal{E}^{\mathrm{GP,2d}}_{Ng_{2\mathrm{d}}}$ from (1.60).

(c) $Ng_{2\mathrm{d}} \gg 1$: *The Thomas–Fermi case.*
The gradient term in the GP functional becomes irrelevant, hence the GP functional simplifies to the 2d Thomas–Fermi functional.

In contrast to the 1d problem, these three regions cannot be understood as subdivisions of the regimes 1 and 2 but all can be reached from both regimes. This situation is different from the 1d case since the splitting into regimes 1 and 2 depends on the parameter $|\ln(\overline{\varrho}_{2\mathrm{d}}(\varepsilon L)^2)|$, whereas the relevant parameter for the 2d GP functional is $Ng_{2\mathrm{d}}$. In contrast, in the 1d case, the regimes 1 and 2 are characterised by the size of $g_{2\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}}$, which is at the same time the relevant parameter for the functional (1.45).

1.4. Effective dynamics of the condensate wave function

Monitoring the dynamical behaviour of a condensed cloud after being released from a trap is an important method in the experimental analysis of BECs. To understand and predict the dynamics of a dilute Bose gas at zero temperature theoretically, one must essentially solve two problems:

- *Persistence of BEC.* Assume that the Bose gas initially exhibits BEC (which is given if the gas is initially prepared in the ground state of a suitable external trap). Show that the gas remains in the condensed phase under time evolution if the trap is varied or completely removed.

- *Evolution equation.* Derive an evolution equation for the condensate wave function, starting from the N-body dynamics. Since a macroscopic fraction of all particles occupies the condensed state, this provides an effective description of the dynamics of the Bose gas.

The second question has been thoroughly discussed in the physical literature, and we begin this section by reviewing the standard formal derivation of the time-dependent GP equation. The main references for this part are [80, Chapter 11], [129, Chapter 5], [145, Chapter 7], and [153, Chapter 5]. Subsequently, we formulate the two problems in precise mathematical terms and give an overview of rigorous results. Finally, we explain in detail the stragety of proof developed by Pickl in [150, 151].

1.4.1. Time-dependent Gross–Pitaevskii equation

Let us consider the Hamiltonian $H_N(t)$ from (1.2) with interactions w_N from (1.25) in the GP scaling regime, i.e.,

$$H_N(t) = \sum_{j=1}^{N} \left(-\Delta_j + V^{\text{ext}}(t, x_j)\right) + \sum_{i<j} w_N(x_i - x_j)\,,$$

where we admit time dependent external potentials $V^{\text{ext}}(t)$ to model the spatial variation of the external trap. To formally derive an evolution equation for the condensate, one absorbs the correlation structure into the effective interaction potential (1.32) as for the static problem and considers the Hamiltonian $H^{\text{eff}}(t)$ (1.33),

$$H^{\text{eff}}(t) = \sum_{j=1}^{N} \left(-\Delta_j + V^{\text{ext}}(t, x_j)\right) + \frac{U_0}{N} \sum_{i<j} \delta(x_i - x_j)\,,$$

acting on a product state $\psi^N = \varphi^{\otimes N}$. Recall that the Schrödinger equation for a Hamiltonian $H = -\Delta_{\mathbb{R}^d} + V$ on $L^2(\mathbb{R}^d)$ can be obtained from the action principle

$$\delta \int_{t_1}^{t_2} L \,\mathrm{d}t = 0$$

for the Lagrange function $L = \int_{\mathbb{R}^d} \mathscr{L}(x)\,\mathrm{d}x$ with Lagrange density

$$\mathscr{L} = \nabla\psi\cdot\nabla\overline{\psi} + V\overline{\psi}\psi - \frac{\mathrm{i}}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right) ,$$

where $\dot{\eta_j} \equiv \partial\eta_j/\partial t$ for $\eta_1 := \psi$, $\eta_2 := \overline{\psi}$. Considering ψ and $\overline{\psi}$ as two independent fields, the action principle leads to the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}}{\partial\dot{\eta_j}} + \sum_{k=1}^{d}\frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial\mathscr{L}}{\partial\left(\frac{\partial\eta_j}{\partial x_k}\right)}\right) - \frac{\partial\mathscr{L}}{\partial\eta_j} = 0\,, \qquad j = 1,2\,, \tag{1.67}$$

which yield the Schrödinger equation and its complex conjugate.

Let us derive the Euler–Lagrange equations corresponding to the effective Hamiltonian $H^{\mathrm{eff}}(t)$ under the assumption that the N-body state factorises as $\psi^N(t) = \varphi(t)^{\otimes N}$.

Denoting $\mathbf{x} = (x_1, \ldots, x_N)$, one computes the resulting Lagrange function as

$$\begin{aligned} L \;=\; & \int\limits_{\mathbb{R}^{3N}} \Bigg(\sum_{j=1}^{N} \left(|\nabla_{x_j}\psi|^2 + V^{\mathrm{ext}}(t,x_j)|\psi|^2 \right) \\ & + \frac{U_0}{N}\sum_{i<j}\delta(x_i - x_j)|\psi|^2 - \frac{\mathrm{i}}{2}\left(\overline{\psi}\dot{\psi} - \dot{\overline{\psi}}\psi\right)\Bigg)\,\mathrm{d}\mathbf{x} \\ \approx\; & N\int\limits_{\mathbb{R}^3}\left(\nabla\overline{\varphi}\cdot\nabla\varphi + V^{\mathrm{ext}}(t,x)\overline{\varphi}\varphi + \frac{U_0}{2}\,\overline{\varphi}^2\varphi^2 - \frac{\mathrm{i}}{2}\left(\overline{\varphi}\dot{\varphi} - \dot{\overline{\varphi}}\varphi\right)\right)\mathrm{d}x\,, \end{aligned}$$

which reduces to a Lagrange function depending on the fields φ, $\overline{\varphi}$ and their respective derivatives. The Euler–Lagrange equation (1.67) for $\overline{\varphi}$ is

$$\mathrm{i}\tfrac{\partial}{\partial t}\varphi(t) = \left(-\Delta + V^{\mathrm{ext}}(t) + U_0|\varphi(t)|^2\right)\varphi(t)\,, \tag{1.68}$$

which is known as *time-dependent Gross–Pitaevskii equation*, named after the two researchers who independently discovered it in 1961 [90, 152]. It is an effective equation which asymptotically describes the dynamics of an interacting Bose gas. The description is valid for sufficiently large particle numbers N and under the assumption of BEC. It requires high dilution as well as sufficiently low temperatures such that the thermal depletion of the condensate is negligible. Under these conditions, the GP equation correctly predicts the behaviour on length scales much larger than the scattering length, while it is not suited to describe phenomena over microscopic distances comparable to the scattering length.

To conclude this section, note that above derivation can only be valid on a formal level. First, we replaced $H_N(t)$ by $H^{\mathrm{eff}}(t)$ without any control of the approximation, i.e., without proving that the time evolutions generated by $H_N(t)$ and $H^{\text{eff}}(t)$ are in any sense close. Besides, to have a well-defined time evolution generated by $H^{\text{eff}}(t)$, one needs to choose a self-adjoint extension of $H^{\text{eff}}(t)$ on an appropriate domain with specific boundary conditions. Note that in our formal derivation of the Euler–Lagrange equations, we simply treated the δ-operator as multiplication operator. A more detailed analysis of the problems arising from using $H^{\text{eff}}(t)$ is given in [129, Chapter 5.2]. For a comprehensive analysis of N-body systems interacting via point interaction, we refer to [55].

Second, by using the product ansatz $\psi^N(t) = \varphi(t)^{\otimes N}$, we tacitly assumed that the first of the two questions mentioned at the beginning of this sections was answered in the affirmative, namely, that condensation is preserved by the the time evolution. Moreover, the splitting of the correlations from the condensate wave function by choosing an appropriate effective interaction requires a more careful justification.

1.4.2. Time-dependent NLS and Hartree equation

Let us formally derive an effective equation for the Hamiltonian $H_{N,\beta}(t)$ with interaction $w_{N,\beta}$ for $\beta \in [0,1)$ as in (1.24). To this end, recall that the inter-particle correlations vary on a length scale that is much shorter than the range of the interaction, hence they become invisible for sufficiently large N (see Section 1.2.4). Ignoring the correlations and assuming a factorised N-body wave function, the interaction energy contributed by two particles in the state $\varphi^{\otimes 2}$ is given by

$$E_{\text{int}}(2) \;=\; \langle \varphi(x_1)\varphi(x_2), w_{N,\beta}(x_1-x_2)\varphi(x_1)\varphi(x_2)\rangle \;=\; \left\langle \varphi, w_{N,\beta} * |\varphi|^2\varphi \right\rangle .$$

As before, one argues that the gas is sufficiently dilute that the total interaction energy equals $\frac{N(N-1)}{2}E_{\text{int}}(2)$. Consequently, for sufficiently large N, the total ground state energy is

$$E_0(N) \;\approx\; N\int \left(|\nabla\varphi(x)|^2 + V^{\text{ext}}(x)|\varphi(x)|^2 + \frac{1}{2}N^{3\beta}\big(w(N^\beta\,\cdot\,) * |\varphi|^2\big)(x)|\varphi(x)|^2 \right) \mathrm{d}x\,.$$

In formal analogy to the GP functional, this leads to the effective dynamical equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left(-\Delta + V^{\text{ext}}(t) + \overline{w}^{\varphi(t)}\right)\varphi(t) =: h^{\varphi(t)}\varphi(t)\,, \tag{1.69}$$

where we introduced the abbreviation

$$\overline{w}^{\varphi(t)} := N^{3\beta}w(N^\beta\cdot) * |\varphi(t)|^2\,. \tag{1.70}$$

For $\beta = 0$, this reduces to the N-independent *Hartree equation*,

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left(-\Delta + V^{\mathrm{ext}}(t) + w * |\varphi(t)|^2\right)\varphi(t)\,. \tag{1.71}$$

For $\beta \in (0,1)$, the effective interaction potential $\overline{w}^{\varphi(t)}$ is N-dependent. However,

$$\overline{w}^{\varphi(t)} \to \|w\|_{L^1(\mathbb{R}^3)}|\varphi(t)|^2 \quad \text{as } N \to \infty$$

for sufficiently regular $\varphi(t)$, hence (1.69) becomes the *(time dependent) NLS equation*

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left(-\Delta + V^{\mathrm{ext}}(t) + \|w\|_{L^1(\mathbb{R}^3)}|\varphi(t)|^2\right)\varphi(t)\,. \tag{1.72}$$

Alternatively, one can derive (1.72) analogously to the GP case. Since the Born approximation is applicable to $w_{N,\beta}$, the parameter of the resulting effective δ-interaction is given by

$$U_0 = 8\pi N a_N \to \|w\|_{L^1(\mathbb{R}^3)} \quad \text{as } \; N \to \infty\,,$$

which follows from (1.27).

1.4.3. Rigorous derivation of the effective dynamics

To rigorously derive an effective description for the N-body dynamics generated by the Hamiltonian

$$H_{N,\beta}(t) = \sum_{j=1}^{N}\left(-\Delta_j + V^{\mathrm{ext}}(t,x_j)\right) + \sum_{i<j} w_{N,\beta}(x_i - x_j)\,, \qquad \beta \in [0,1]\,,$$

it has proved successful to answer the two questions raised at the beginning of this section simultaneously. Besides, it is more convenient to work in terms of reduced densities than to argue on the level of the many-body wave function.

In mathematical terms, the problem is the following: Assume that at time $t = 0$, the N-body state ψ_0^N exhibits complete asymptotic BEC in the state φ_0. Let $\psi^N(t)$ denote the solution of the N-body Schrödinger equation with initial condition ψ_0^N, and let $\varphi(t)$ denote the solution of the effective equation with initial datum φ_0. The goal is to show that $\psi^N(t)$ exhibits BEC in the state $\varphi(t)$, i.e., that

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma^{(1)}_{\psi_0^N} - |\varphi_0\rangle\langle\varphi_0|\right| = 0 \quad \Rightarrow \quad \lim_{N\to\infty} \operatorname{Tr}\left|\gamma^{(1)}_{\psi^N(t)} - |\varphi(t)\rangle\langle\varphi(t)|\right| = 0\,. \tag{1.73}$$

The rigorous derivation of such statements has been a very active field of research in mathematical physics, and a variety of mathematical methods have been applied to this problem. Several lecture notes reviewing different approaches are available, for instance [22, 81, 159, 162]. In the following, we will give a brief overview of the different strategies and comment on the corresponding results, although without any claim to completeness.

BBGKY approach

The *Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY)* approach starts from the Heisenberg equation for the density matrix of an N-body state $\psi^N(t)$. By tracing out $N-k$ particles, one derives an evolution equation for the k-particle reduced density matrices $\gamma^{(k)}_{\psi^N(t)}$, which yields the so-called BBGKY-hierarchy of N coupled equations. Since the BBGKY approach is based on an abstract compactness argument, it does not provide explicit error bounds.

The idea of using the BBGKY hierarchy in this context is due to Spohn, who derived the Hartree equation for bounded pair potentials in [174]. Spohn's approach was used by Bardos, Golse and Maurer in [16] and by Erdős and Yau in [68] to derive the Hartree equation for Coulomb-like potentials, and by Elgart and Schlein in [62] for bosons with relativistic dispersion relation. It was extended by Erdős, Schlein and Yau to interactions $w_{N,\beta}$ with larger scaling parameters in [64, 65, 66]. A different and shorter proof of uniqueness of the hierarchy was provided by Klainerman and Machedon in [105]. Bosons in a quadratic trap were considered by X. Chen in [44].

In [67], Erdős, Schlein and Yau extended their result to the case $\beta = 1$, using again the BBGKY approach with the difference that the solution to the respective infinite hierarchy includes correlations. Part of their proof was simplified by T. Chen, Hainzl, Pavlović and Seiringer in [43].

Concerning low dimensional bosons, the BBGKY approach was used by Adami, Bardos, Golse and Teta in [1, 2] to derive a 1d NLS equation for scalings $\beta < \frac{1}{2}$, and by X. Chen and Holmer in [46] for the 1d focusing case with $\beta \in (0,1)$ and in [48] for the 2d focusing case with $\beta \in (0, \frac{1}{6})$.

Finally, X. Chen and Holmer applied the BBGKY method to derive effective 1d and 2d equations for 3d bosons in highly anisotropic traps [45, 47] (see Section 3.1.3).

Second quantised approach

Based on the works by Hepp [94] and Ginibre and Velo [77, 78] on classical limits of bosonic systems, another approach was developed by Rodnianski and Schlein in [158] and further improved in [42] by L. Chen, Lee and Schlein. The idea is to represent the many-body system on a Fock space and study the time evolution of coherent initial states, which also yields an explicit rate of convergence. In [21], Benedikter, de Oliveira and Schlein extended this method to the GP scaling regime. Recently, Brennecke and Schlein improved it for the GP case and N-body initial data to yield an optimal rate of convergence [39]. For 2d bosons, an NLS equation was derived by Kirkpatrick, Schlein and Staffilani in [104].

Further approaches

We postpone the method by Pickl to the next section, since it is used in this thesis and therefore explained in detail. Apart from this, let us mention the approach suggested by Fröhlich, Graffi and Schwarz in [73] and the related works [74, 75] by Fröhlich, Knowles and Schwarz and Fröhlich, Knowles and Pizzo. Moreover, semiclassical methods were applied by Ammari and Nier in [7], by Ammari and Breteaux in [5], and by Ammari, Falconi and Pawilowski in [6]. For more details, we refer to the lecture notes [22, 81, 159, 162] reviewing the different approaches.

1.4.4. First quantised approach by Pickl

The approach developed by Pickl in [148, 150, 151] is formulated in the first quantised N-body setting, and, as a by-product, yields an explicit (but not optimal) estimate of the rate of convergence. The main idea for Hartree and NLS scaling regime is to define a functional

$$\alpha : \mathbb{R} \times L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}_0^+ \,, \qquad (t, \psi^N(t), \varphi(t)) \mapsto \alpha(t, \psi^N(t), \varphi(t)) =: \alpha(t) \,,$$

such that $\alpha(t)$ counts the (suitably weighted) relative number of particles in $\psi^N(t)$ that are outside the condensed phase. After proving that convergence of $\alpha(t)$ to zero is equivalent to condensation at time t, one derives an estimate of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha(t) \lesssim C(t)\alpha(t) + \mathcal{O}(N) \ ,$$

which yields (1.73) by means of Grönwall's lemma (Lemma 1.4.3). Since $\alpha(t)$ can only be controlled if the argument $\varphi(t)$ of α is the solution of the corresponding NLS/Hartree equation in the limit $N \to \infty$, this implicitly proves the respective effective evolution equation.

For the GP scaling regime, the central idea of the proof is closely connected with the heuristic derivation of the stationary GP equation in Section 1.2.5: using a modified counting functional, one effectively replaces the very singular GP interaction w_N by a softer (but still singular) potential $U_{\widetilde{\beta}}$ in the NLS scaling regime, which is defined such that its scattering length asymptotically coincides with the scattering length of w_N. Roughly speaking, the NLS result covers the auxiliary potential $U_{\widetilde{\beta}}$, and it remains to control the remainders from this substitution.

Before describing this strategy of proof in more detail, let us give an overview of the results obtained using Pickl's approach. In [149], Pickl covered interactions without positivity condition for $\beta < \frac{1}{6}$, which was extended by Jeblick and Pickl in [98] to potentials in the GP regime with sufficiently small negative part. Knowles and Pickl [106] proved convergence of the reduced densities for bosons in the Hartree regime with singular interactions. Mitrouskas, Petrat and Pickl [135, 134] improved this to convergence with respect to the energy trace norm, obtaining an optimal rate of convergence, and Anapolitanos and Hott [8, 95] generalised the analysis to a larger class of kinetic terms. Mixtures of condensates were studied by Anapolitanos, Hott and Hundertmark [9] and by Michelangeli and Olgiati [132, 140], who also considered spinor condensates [131, 133]. The dynamics of a tracer particle interacting with an ideal Bose gas was studied by Deckert, Pickl, Fröhlich and Pizzo in [53], who also considered high density Bose gases in a large box [54]. In [146], their result was improved by Petrat, Pickl and Soffer.

Moreover, the approach was successfully applied to low dimensions: in [97], Jeblick, Leopold and Pickl derived the time-dependent GP equation for 2d bosons, and an effective focusing NLS equation for 2d was proved by Jeblick and Pickl in [99]. Finally, effectively 1d and 2d equations for strongly confined 3d bosons were derived by von Keler and Teufel in [100] and in the three projects [33, 32, 35] of this thesis (see Section 3.1).

Hartree and NLS regime

The functional α can be understood as a measure of the relative number of particles which do not occupy the condensate state φ. To implement this mathematically, one introduces projectors onto the condensate wave function and its orthogonal complement.

Definition 1.4.1. *For any $\varphi \in L^2(\mathbb{R}^3)$, let*

$$p^\varphi := |\varphi\rangle\langle\varphi| \,, \qquad q^\varphi := \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\varphi$$

denote the projectors onto φ and its orthogonal complement. With this, we define the projection operators on $L^2(\mathbb{R}^{3N})$

$$p_j^\varphi := \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{j-1}\otimes p^\varphi \otimes \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{N-j}, \qquad q_j^\varphi := \mathbb{1}_{L^2(\mathbb{R}^{3N})} - p_j^\varphi$$

for $j \in \{1,\dots,N\}$. Further, for $0 \le k \le N$, define the many-body projections

$$P_k^\varphi := \sum_{\substack{J\subseteq\{1,\dots,N\}\\|J|=k}} \prod_{j\in J} q_j^\varphi \prod_{l\notin J} p_l^\varphi = \frac{1}{(N-k)!k!}\sum_{\sigma\in\mathfrak{S}_N} q^\varphi_{\sigma(1)}\cdots q^\varphi_{\sigma(k)} p^\varphi_{\sigma(k+1)}\cdots p^\varphi_{\sigma(N)}$$

and $P_k^\varphi = 0$ for $k<0$ and $k>N$. For any function $f:\mathbb{N}_0\to\mathbb{R}_0^+$ and $d\in\mathbb{Z}$, define *the operators* $\widehat{f^\varphi}, \widehat{f^\varphi_d} \in \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)$ *by*

$$\widehat{f^\varphi} := \sum_{k=0}^{N} f(k) P_k^\varphi \,, \qquad \widehat{f^\varphi_d} := \sum_{j=-d}^{N-d} f(j+d) P_j^\varphi \,.$$

Obviously, $\sum_{k=0}^N P_k^\varphi = \mathbb{1}$, which implies that

$$\frac{1}{N}\sum_{j=1}^{N} q_j^\varphi = \frac{1}{N}\sum_{k=0}^{N}\sum_{j=1}^{N} q_j^\varphi P_k^\varphi = \frac{1}{N}\sum_{k=0}^{N} k P_k^\varphi = (\widehat{n^\varphi})^2 \tag{1.74}$$

for the weight function

$$n(k) := \sqrt{\frac{k}{N}}\,.$$

As a consequence, the expected relative number of particles outside φ in a symmetric N-body state $\psi^N \in L^2_+(\mathbb{R}^{3N})$ is given by

$$\alpha(\psi^N,\varphi) \;:=\; \left\langle\!\left\langle \psi^N, q_1^\varphi \psi^N \right\rangle\!\right\rangle \;=\; \left\langle\!\left\langle \psi^N, (\widehat{n^\varphi})^2 \psi^N \right\rangle\!\right\rangle \,. \tag{1.75}$$

Note that since ψ^N is normalised,

$$\alpha(\psi^N,\varphi) = 1 - \left\langle\!\left\langle \psi^N, p_1^\varphi \psi^N \right\rangle\!\right\rangle = 1 - \left\langle \varphi, \gamma^{(1)}_{\psi^N}\varphi \right\rangle_{L^2(\mathbb{R}^3)} \,.$$

Hence, the convergence $\alpha(\psi^N,\varphi) \to 0$ as $N \to \infty$ is equivalent to complete asymptotic condensation in the state φ by Lemma 1.2.2c. In particular, if the initial N-body state ψ_0^N exhibits BEC in φ_0, this implies

$$\lim_{N\to\infty} \alpha(\psi_0^N,\varphi_0) = 0\,.$$

In fact, one has the freedom to choose any positive power of $n(k)$ as weight in the counting functional (see e.g. [100, Lemma 3.1]):

Lemma 1.4.2. *Let* $\{\psi^N\}_N$ *be a sequence of normalised* N*-body wave functions such that* $\psi^N \in L^2_+(\mathbb{R}^{3N})$ *and let* $\varphi \in L^2(\mathbb{R}^3)$*. Define*

$$\alpha_f(\psi^N,\varphi) := \left\langle\!\!\left\langle \psi^N, \widehat{f^\varphi}\,\psi^N \right\rangle\!\!\right\rangle$$

for any weight function $f : \mathbb{N}_0 \to \mathbb{R}_0^+$*. Then the following statements are equivalent:*

(a) $\lim_{N\to\infty} \alpha_{n^a}(\psi^N,\varphi) = 0$ *for some* $a > 0$*,*

(b) $\lim_{N\to\infty} \alpha_{n^a}(\psi^N,\varphi) = 0$ *for any* $a > 0$*,*

(c) $\lim_{N\to\infty} \mathrm{Tr}_{L^2(\mathbb{R}^3)} \left|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right| = 0.$

In conclusion, the functional α_{n^a} counts the relative number of particles outside the condensate weighted with $n^a(k)$, where the power a may be chosen for convenience. To prove persistence of condensation, one chooses a suitable weight function f and shows that the respective functional $\alpha_f(t, \psi^N(t), \varphi(t))$ satisfies a Grönwall-type inequality at time $t > 0$. Here, the second argument $\psi^N(t)$ of α_f is the solution of the N-body Schrödinger equation, $\frac{\mathrm{d}}{\mathrm{d}t}\psi^N(t) = H_{N,\beta}(t)\psi^N(t)$, and the third argument $\varphi(t)$ is the solution of the respective effective equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = h^{\varphi(t)}\varphi(t) := \left(-\Delta + V^{\mathrm{ext}}(t) + \overline{w}^{\varphi(t)}\right)\varphi(t)$$

with $\overline{w}^{\varphi(t)}$ from (1.70).

While the weight n^2 as in (1.75) is a good choice for the Hartree case, it is not suitable to derive a Grönwall estimate for larger values of β. In this case, the Hartree counting functional (1.75) needs to be modified in two respects, whose necessity will be explained below when we sketch the proof.

- In addition to complete asymptotic condensation of the initial data, one assumes that
$$\lim_{N\to\infty}\left|E^{\psi_0^N}_{w_{N,\beta}}(0) - \mathcal{E}^{\varphi_0}_{\|w_{N,\beta}\|_1}(0)\right| = 0\,, \tag{1.76}$$
where $E^{\psi^N(t)}_{w_{N,\beta}}(t) = \frac{1}{N}\left\langle\!\left\langle\psi^N(t), H_{N,\beta}(t)\psi^N(t)\right\rangle\!\right\rangle$ denotes the energy per particle and $\mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t)$ denotes the GP functional (1.31) where the parameter a is replaced by $g = \frac{N}{8\pi}\|w_{N,\beta}\|_{L^1(\mathbb{R}^3)}$. If ψ_0^N is close to the N-body ground state, this assumption is physically motivated by the heuristic arguments given at the beginning of Section 1.4.2. Note that for a time-independent external field, both $E^{\psi^N(t)}_{w_{N,\beta}}$ and $\mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}$ are constants of motion. The energy difference (1.76) at time t is added to the counting functional, resulting in
$$\left\langle\!\!\left\langle\psi^N(t), \widehat{f^{\varphi(t)}}\,\psi^N(t)\right\rangle\!\!\right\rangle + \left|E^{\psi^N(t)}_{w_{N,\beta}}(t) - \mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t)\right|\,.$$
Since both terms are non-negative, this expression converges to zero as $N \to \infty$ if and only if both complete condensation and the property (1.76) are preserved at time t, given the weight f is chosen appropriately in the sense of Lemma 1.4.2.

- One chooses the weight n instead of n^2 and modifies it by a smooth cut-off for small k. More precisely, one uses the weight m, which is defined as
$$m(k) := \begin{cases} n(k) & \text{for } k \geq N^{1-2\xi}\,, \\ \frac{1}{2}\left(N^{-1+\xi}k + N^{-\xi}\right) & \text{else} \end{cases} \tag{1.77}$$

for some $\xi \in (0, \frac{1}{2})$. Since

$$n(k) \leq m(k) \leq n(k) + \tfrac{1}{2} N^{-\xi} \,,$$

the functional α_m converges to zero as $N \to \infty$ if and only if α_n converges to zero, although the rate of the convergence differs by $\frac{1}{2}N^{-\xi}$.

In conclusion, we use for the NLS scaling the counting functional

$$\alpha^{<}_{\xi, w_{N,\beta}}(t, \psi^N(t), \varphi(t)) := \left\langle\!\!\left\langle \psi^N(t), \widehat{m^{\varphi(t)}} \psi^N(t) \right\rangle\!\!\right\rangle + \left| E^{\psi^N(t)}_{w_{N,\beta}}(t) - \mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t) \right| \,, \tag{1.78}$$

which satisfies

$$\lim_{N\to\infty} \alpha^{<}_{\xi, w_{N,\beta}}(t, \psi^N(t), \varphi(t)) = 0 \quad \Leftrightarrow \quad \begin{cases} \lim\limits_{N\to\infty} \operatorname{Tr} \left| \gamma^{(1)}_{\psi^N(t)} - |\varphi(t)\rangle\langle\varphi| \right| = 0 \\ \lim\limits_{N\to\infty} \left| E^{\psi^N(t)}_{w_{N,\beta}}(t) - \mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t) \right| = 0 \,. \end{cases} \tag{1.79}$$

As a consequence, the asymptotic closeness of the energies at time $t > 0$ comes as a by-product of proving that complete condensation is preserved in time.

Since Grönwall's lemma is at the core of the proof, let us recall its statement (see e.g. [70, Appendix B.2.j]):

Lemma 1.4.3. *Let η be a non-negative, absolutely continuous function on $[0,T]$ such that*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \eta(t) \leq f(t)\eta(t) + g(t) \qquad \textit{for a.e. } t \in [0,T] \,,$$

where f and g are non-negative, summable functions on $[0,T]$. Then

$$\eta(t) \leq \left(\eta(0) + \int_0^t g(s)\,\mathrm{d}s \right) \mathrm{e}^{\int_0^t f(s)\,\mathrm{d}s} \qquad \textit{for all } t \in [0,T] \,.$$

In the NLS case, both terms in the functional $\alpha^{<}_{\xi, w_{N,\beta}}$ contribute to the time derivative of $\alpha^{<}_{\xi, w_{N,\beta}}$. Differentiating the energy term yields for almost every t the estimate

$$\begin{aligned} &\left| \tfrac{\mathrm{d}}{\mathrm{d}t} \left| E^{\psi^N(t)}_{w_{N,\beta}}(t) - \mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t) \right| \right| \\ &\quad = \left| \tfrac{\mathrm{d}}{\mathrm{d}t} \left(E^{\psi^N(t)}_{w_{N,\beta}}(t) - \mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t) \right) \right| \\ &\quad = \left| \left\langle\!\!\left\langle \psi^N(t), \dot{V^{\mathrm{ext}}}(t,x_1) \psi^N(t) \right\rangle\!\!\right\rangle - \left\langle \varphi(t), \dot{V^{\mathrm{ext}}}(t)\varphi(t) \right\rangle_{L^2(\mathbb{R}^3)} \right| \,, \end{aligned} \tag{1.80}$$

where the first equality holds by [117, Theorem 6.17] for almost every t if the map $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(E^{\psi^N(t)}_{w_{N,\beta}}(t) - \mathcal{E}^{\varphi(t)}_{\|w_{N,\beta}\|_1}(t))$ is continuous. This imposes on the external field the condition that the map $t \mapsto V^{\text{ext}}(t)$ be $\mathcal{C}^1$.

The second contribution to the time derivative of $\alpha^<_{\xi,w_{N,\beta}}$ is of the same form as $\frac{\mathrm{d}}{\mathrm{d}t}\alpha$ in the Hartree case. For simplicity dropping the time dependences and indices $\varphi(t)$, one computes

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle \psi^N, \widehat{f}\,\psi^N \right\rangle\!\!\right\rangle \\
&= \mathrm{i}\left\langle\!\!\left\langle \psi^N, \left[H_{N,\beta} - \sum_{j=1}^N h_j, \widehat{f}\right]\psi^N(t) \right\rangle\!\!\right\rangle && (1.81)\\
&= -N(N-1)\Im\left\langle\!\!\left\langle \psi^N, Z_{N,\beta}(x_1,x_2)\widehat{f}\,\psi^N \right\rangle\!\!\right\rangle && (1.82)\\
&= -N(N-1)\Im\left\langle\!\!\left\langle \psi^N, q_1p_2(\widehat{f}-\widehat{f}_{-1})Z_{N,\beta}(x_1,x_2)p_1p_2\psi^N \right\rangle\!\!\right\rangle && (1.83)\\
&\quad -N(N-1)\Im\left\langle\!\!\left\langle \psi^N, q_1q_2(\widehat{f}-\widehat{f}_{-2})w_{N,\beta}(x_1-x_2)p_1p_2\psi^N \right\rangle\!\!\right\rangle && (1.84)\\
&\quad -2N(N-1)\Im\left\langle\!\!\left\langle \psi^N, q_1q_2(\widehat{f}-\widehat{f}_{-1})Z_{N,\beta}(x_1-x_2)p_1q_2\psi^N \right\rangle\!\!\right\rangle, && (1.85)
\end{aligned}$$

where

$$Z_{N,\beta}(x_1,x_2) := w_{N,\beta}(x_1-x_2) - \tfrac{1}{N-1}(\overline{w}^{\varphi(t)}(x_1) + \overline{w}^{\varphi(t)}(x_2))\,. \qquad (1.86)$$

To compute $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f}$, note that $\frac{\mathrm{d}}{\mathrm{d}t}p_j = \mathrm{i}\,[p_j,h_j]$, hence

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{f} = \mathrm{i}\,[\widehat{f}, \sum_{j=1}^N h_j]\,.$$

To obtain (1.83) to (1.85), one inserts identities $\mathbb{1} = (p_i+q_i)(p_j+q_j)$ on both sides of the commutator and uses the symmetry of ψ^N as well as the identity

$$Q_\mu \widehat{f}\, T_{ij} Q_\nu = Q_\mu T_{ij} \widehat{f}_{\mu-\nu} Q_\nu\,,$$

where $Q_0 := p_1p_2$, $Q_1 \in \{p_1q_2, q_1p_2\}$, $Q_0 := q_1q_2$ and T_{ij} denotes an operator acting non-trivially only on coordinates i and j (e.g. [32, Lemma 4.2b])[6]. Note that the differences $\widehat{f}-\widehat{f}_d$ in (1.83) to (1.85) can be understood as operators that are weighted, in the sense of Definition 1.4.1, with the derivative of $f(k)$. For example, we obtain in (1.83)

$$q_1p_2\left(\widehat{f}-\widehat{f}_{-1}\right) = q_1p_2\Big(\sum_{k=1}^N \big(f(k)-f(k-1)\big)P_k + f(0)P_0\Big) = q_1p_2\sum_{k=1}^N f'(k)P_k\,,$$

where $f'(k)$ denotes the discrete derivative of f with respect to k. For the weight

[6] For convenience of the reader, we refer as far as possible to articles of this thesis. This does not imply that these statements were originally proven there.

$f(k) = n^2(k) = k/N$, we find $|f'(k)| = N^{-1}$, hence $\|\widehat{f} - \widehat{f}_{-1}\|_{\text{op}} = N^{-1}$. In contrast, the derivative of the weight $f(k) = n(k) = \sqrt{k/N}$ diverges as $k \to 0$. This is the reason why one introduces the cut-off ξ, which softens this singularity such that one can derive the estimates $\|\widehat{f} - \widehat{f}_{-1}\|_{\text{op}} \lesssim N^{-1+\xi}$ and $\|(\widehat{f} - \widehat{f}_{-1})q_1\psi^N\| \lesssim N^{-1}$ (e.g. [32, Lemma 4.1]). Analogous results hold for $\widehat{f} - \widehat{f}_{-2}$.

Let us now analyse the four contributions (1.80) to (1.85) to the time derivative of the counting functional. For simplicity, we will only discuss the problem of purely repulsive interactions, i.e.,

$$w_{N,\beta}(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^3 .$$

As mentioned above, the method has been extended to include attractive interactions and repulsive interactions with a certain negative part, but this is beyond the scope of this discussion.

- *Energy term* (1.80).
  Recall that this term appears in the NLS regime, while it is not present in the Hartree case. It contains exclusively interactions between the bosons and the external field V^{ext}, which makes it the easiest term to control. The main idea is the observation that for any $f \in L^\infty(\mathbb{R}^3)$,

  $$\left| \langle\!\langle \psi^N, f(x_1)\psi^N \rangle\!\rangle - \langle \varphi, f\,\varphi \rangle_{L^2(\mathbb{R}^3)} \right| \lesssim \|f\|_{L^\infty(\mathbb{R}^3)} \langle\!\langle \psi^N, \widehat{n}\psi^N \rangle\!\rangle \tag{1.87}$$

  (e.g. [32, Lemma 4.7]). Hence, for an external field with bounded time derivative, (1.80) is small if the N-body state is close to a condensate.

- *(qp–pp) term* (1.83).
  Note that $q_1\overline{w}^{\varphi(t)}(x_2)p_1 = 0$, hence (1.83) contains the difference

  $$p_2\Big((N-1)w_{N,\beta}(x_1 - x_2) - \overline{w}^{\varphi(t)}(x_1)\Big)p_2$$

  between the true pair interaction $w_{N,\beta}$ and the effective one-body interaction potential $\overline{w}^{\varphi(t)}$. Since

  $$\begin{aligned} &(N-1)p_2 w_{N,\beta}(x_1 - x_2)p_2 \\ &\quad= \quad (N-1)|\varphi(x_2)\rangle\, \langle \varphi(x_2), w_{N,\beta}(x_1 - x_2)\varphi(x_2)\rangle\, \langle \varphi(x_2)| = \tfrac{N-1}{N}\, \overline{w}^{\varphi(t)}(x_1)p_2\,, \end{aligned}$$

  this difference converges to zero as $N \to \infty$. Since $\|(\widehat{f} - \widehat{f}_{-1})q_1\psi\| \lesssim N^{-1}$ in both NLS and Hartree case, we conclude that (1.83) $\to 0$ as $N \to \infty$. This indirectly proves that the time evolution of $\varphi(t)$ is determined by a non-linear equation, whose non-linearity may differ from the non-linear term $\overline{w}^{\varphi(t)}$ in (1.69) at most by $\mathcal{O}(1)$. In particular, this includes the N-independent NLS equation (1.72) with non-linear term $\|w\|_{L^2(\mathbb{R}^3)}|\varphi|^2$.

- (1.84) *and* (1.85)*, Hartree scaling.*
Both terms can be estimated by straightfoward applications of the Cauchy–Schwarz inequality, using that $\|q_1\psi^N\|^2 = \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle < \alpha(t, \psi^N, \varphi)$. The pre-factor of order $\mathcal{O}(N^2)$ is essentially cancelled since $\|w_{N,\beta=0}\|_{L^2(\mathbb{R}^3)} \lesssim N^{-1}$ and because $\|\widehat{f} - \widehat{f}_{-d}\|_{\text{op}} \lesssim N^{-1}$. The full argument is given in [150].

- *(qq–pp) term* (1.84)*, NLS scaling.*
For $\beta > 0$, an estimate as in the Hartree case does not suffice. Among other obstructions, $\|w_{N,\beta}\|_{L^2(\mathbb{R}^3)} \lesssim N^{-1+\frac{3\beta}{2}}$ is not small enough to compensate for the pre-factor. One solves this by integration by parts, exploiting that the anti-derivative of $w_{N,\beta}$ is less singular than $w_{N,\beta}$. Heuristically speaking, shifting one derivative from the strongly peaked interaction to the N-body wave function yields an improvement because the great majority of particles occupies the condensate wave function, which varies slowly in space. In the course of the integration by parts, derivatives ∇_1 fall upon projectors p_1 and q_1 as well as on the N-body wave function ψ^N. In the first case, note that $\nabla p = |\nabla\varphi\rangle\langle\varphi|$, hence $\|\nabla_1 p_1\|_{\text{op}} = \mathcal{O}(1)$ for sufficiently regular φ. To control $\|\nabla_1\psi^N\|$, one observes that
$$E^{\psi^N}_{w_{N,\beta}}(t) \geq \|\nabla_1\psi^N\|^2 - \left|\langle\!\langle \psi^N, V^{\text{ext}}(t,x_1)\psi^N \rangle\!\rangle\right| \geq \|\nabla_1\psi^N\|^2 - \mathcal{O}(1)$$
because $w_{N,\beta} \geq 0$ and if V^{ext} is assumed bounded. Since $E^{\psi^N_0}_{w_{N,\beta}}(0)$ is of order $\mathcal{O}(1)$ and the time derivative depends only on the N-independent quantity V^{ext}, this yields the *a priori* bound $\|\nabla_1\psi^N\| \lesssim 1$, which is sufficient to control (1.84).

- *(pq–qq) term* (1.85)*, NLS scaling.*
Following the same strategy of integration by parts, one finds that above *a priori* estimate is not sufficient, but a better control of the kinetic energy $\|\nabla_1 q_1\psi^N\|^2$ contributed by a particle outside the condensate is required. More precisely, one needs a bound of the form
$$\|\nabla_1 q_1\psi^N\|^2 \lesssim \alpha^{<}_{\xi,w_{N,\beta}}(t,\psi^N,\varphi) + o(1)\ . \tag{1.88}$$
To this end, one first proves that
$$|E^{\psi^N}_{w_{N,\beta}} - \mathcal{E}^{\varphi}_{\|w_{N,\beta}\|_1}| \gtrsim \|\nabla_1\psi^N\|^2 - \|\nabla\varphi\|^2 - o(1)\ . \tag{1.89}$$
Now one inserts the identity $p_1 + q_1$ after ∇_1, expands the scalar product and

observes that

$$\|\nabla_1 p_1\psi^N\|^2 - \|\nabla\varphi\|^2 = \|\nabla\varphi\|^2\|p_1\psi^N\|^2 - \|\nabla\varphi\|^2 \lesssim \|q_1\psi^N\|^2 = \langle\!\langle \psi^N, \widehat{n}^2\psi^N\rangle\!\rangle$$

and that

$$\begin{aligned}|\langle\!\langle \psi^N, p_1(-\Delta_1)q_1\psi^N\rangle\!\rangle| &= \left|\left\langle\!\!\left\langle \widehat{n}_1^{\frac{1}{2}}\psi^N, p_1(-\Delta_1)q_1\widehat{n}^{-\frac{1}{2}}\psi^N\right\rangle\!\!\right\rangle\right| \leq \|\Delta_1 p_1\|_{\mathrm{op}}\|\widehat{n}^{\frac{1}{2}}\psi^N\|^2 \\ &= \|\nabla\varphi\|^2\|q_1\psi\|^2 = \langle\!\langle \psi^N, \widehat{n}\psi^N\rangle\!\rangle\,\|\nabla\varphi\|^2 \lesssim \langle\!\langle \psi^N, \widehat{n}\psi^N\rangle\!\rangle\end{aligned}$$

since $q_1 = \widehat{n}^2$ in the sense of operators on $L^2_+(\mathbb{R}^{3N})$ as in (1.75). Together, this yields

$$\begin{aligned}|E^{\psi^N}_{w_{N,\beta}} - \mathcal{E}^{\varphi}_{\|w_{N,\beta}\|_1}| &\gtrsim \|\nabla_1 q_1\psi^N\|^2 - \langle\!\langle \psi^N, \widehat{n}\psi^N\rangle\!\rangle - \langle\!\langle \psi^N, \widehat{n}^2\psi^N\rangle\!\rangle - \mathcal{O}(1) \\ &\gtrsim \|\nabla_1 q_1\psi^N\|^2 - \langle\!\langle \psi^N, \widehat{m}\psi^N\rangle\!\rangle - \mathcal{O}(1)\ ,\end{aligned}$$

which, by definition of $\alpha^<_{\xi,w_{N,\beta}}$, is precisely the bound (1.88). In conclusion, this estimate is only possible because the energy term is part of the counting functional and because the weight m is chosen such that $m(k) \geq n(k)$ for any k, which, in particular, excludes the Hartree weight n^2. This finally motivates the form (1.78) of the counting functional.

Altogether, the estimates of the four terms lead to the inequality

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha^<_{\xi,w_{N,\beta}}\big(t,\psi^N(t),\varphi(t)\big) \lesssim \alpha^<_{\xi,w_{N,\beta}}\big(t,\psi^N(t),\varphi(t)\big) + \mathcal{O}(1)\ ,$$

which concludes the proof by Lemma 1.4.3 and the equivalence (1.79).

GP regime

For an interaction w_N in the GP scaling regime, the functional $\alpha^<_{\xi,w_N}$ cannot satisfy a Grönwall inequality. This can be seen from the (*pp–pq*) term (1.83) in the time derivative of $\alpha^<_{\xi,w_N}$, which contains the difference between the full pair interaction w_N and the effective interaction potential, now given by $8\pi a|\varphi|^2$. As explained above, this term is only small if the non-linear term in the effective equation differs from $\overline{w}^{\varphi(t)} = N^3 w(N\cdot) * |\varphi|^2 \approx \|w\|_{L^1(\mathbb{R}^3)}|\varphi|^2$ at most by $\mathcal{O}(1)$. However, $\|w\|_{L^1(\mathbb{R}^3)}$ is precisely the first order Born approximation b_0 to the scattering length a, and we argued in (1.28) that the difference between b_0 and $8\pi a$ is of order one.

The functional $\alpha^<_{\xi,w_N}$ is no suitable counting functional for the GP scaling regime because it counts the (weighted) relative number of particle outside $\varphi^{\otimes N}$. However, due to the inter-particle correlations, the condensate is no product state, and one should instead count the particles outside the correlated state ψ^N_{cor} (1.20). Although this is also true in the NLS scaling regime, it becomes relevant only in the GP regime, where the difference between $\varphi^{\otimes N}$ and ψ^N_{cor} is visible on the length scale of the interaction in the limit $N \to \infty$.

Consequently, one requires a new counting functional that takes the correlations into account, similarly to the heuristic derivation of the GP energy functional (Section 1.2.5), where the correlations were absorbed into the effective δ interaction (1.32). To describe the correlations, we use an auxiliary function $f_{\widetilde{\beta}} \in \mathcal{C}^1(\mathbb{R}^3)$, which asymptotically coincides with the scattering solution j_N on $\operatorname{supp} w_N$ and equals one for sufficiently large $|x|$. It is defined as the solution of the zero energy scattering equation of $w_N - U_{\widetilde{\beta}}$, where $U_{\widetilde{\beta}}$ is an auxiliary potential which is constructed such that the scattering length of $w_N - U_{\widetilde{\beta}}$ equals zero:

Definition 1.4.4. *Let $\widetilde{\beta} \in (0,1)$. Define*

$$U_{\widetilde{\beta}}(x) := \begin{cases} N^{-1+3\widetilde{\beta}} a & \text{for } N^{-\widetilde{\beta}} < |x| < R_{\widetilde{\beta}}\,, \\ 0 & \text{else,} \end{cases}$$

where $R_{\widetilde{\beta}}$ is the minimal value in $(N^{-\widetilde{\beta}}, \infty]$ such that the scattering length of $w_N - U_{\widetilde{\beta}}$ equals zero. Let $f_{\widetilde{\beta}} \in \mathcal{C}^1(\mathbb{R}^3)$ be the solution of

$$\begin{cases} \Big(-\Delta + \tfrac{1}{2}\big(w_N(z) - U_{\widetilde{\beta}}(z)\big)\Big) f_{\widetilde{\beta}}(z) = 0 & \text{for } |x| < R_{\widetilde{\beta}}, \\ f_{\widetilde{\beta}}(x) = 1 & \text{for } |x| \geq R_{\widetilde{\beta}}\,, \end{cases} \tag{1.90}$$

and define

$$g_{\widetilde{\beta}} := 1 - f_{\widetilde{\beta}}\,.$$

Using $f_{\widetilde{\beta}}$ instead of j_N has the technical advantage that $g_{\widetilde{\beta}}$ and $\nabla f_{\widetilde{\beta}}$ are compactly supported. To modify the counting functional such that the role of $\varphi^{\otimes N}$ is taken by the correlated state $\prod_{j=1}^N \varphi(x_j) \prod_{1 \leq k < l \leq N} f_{\widetilde{\beta}}(x_k - x_l)$, one substitutes the first term of $\alpha^<_{\xi, w_N}$ by

$$\begin{aligned} \big\langle\!\big\langle \psi^N, \widehat{m} \psi^N \big\rangle\!\big\rangle \quad &\mapsto \quad \Big\langle\!\!\Big\langle \psi^N, \prod_{k<l} f_{\widetilde{\beta}}(x_k - x_l)\, \widehat{m} \prod_{r<s} f_{\widetilde{\beta}}(x_r - x_s) \psi^N \Big\rangle\!\!\Big\rangle \\ &\approx \quad \big\langle\!\big\langle \psi^N, \widehat{m} \psi^N \big\rangle\!\big\rangle - N(N-1) \Re \big\langle\!\big\langle \psi^N, g_{\widetilde{\beta}}(x_1 - x_2) \widehat{m} \psi^N \big\rangle\!\big\rangle\,. \end{aligned} \tag{1.91}$$

Here, we used the symmetry of $\psi^N(t) \equiv \psi$, expanded both products by writing $f_{\widetilde{\beta}} = 1 - g_{\widetilde{\beta}}$, and kept only the terms which are at most linear in $g_{\widetilde{\beta}}$.

Note that the substitution reproduces the original functional up to a correction term. This additional expression plays a crucial role: it effectively leads to the replacement of

w_N by $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ in the time derivative of the new functional, especially in the problematic term (1.83). To demonstrate this, let us consider the case $N = 2$ with $V^{\text{ext}} = 0$. Abbreviating $Z_2^{(12)} := w_2(x_1 - x_2) - 8\pi a(|\varphi(x_1)|^2 + |\varphi(x_2)|^2)$ analogously to (1.86), and with the notation $F^{(12)} := F(x_1 - x_2)$ for any $F : \mathbb{R}^3 \to \mathbb{R}$, we obtain

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \left\langle\!\left\langle \psi^N, \widehat{m}\psi^N \right\rangle\!\right\rangle &= \mathrm{i} \left\langle\!\left\langle \psi^N, \left[Z_2^{(12)}, \widehat{m} \right] \psi^N \right\rangle\!\right\rangle = -2\Im \left\langle\!\left\langle \psi^N, Z_2^{(12)} \widehat{m}\psi^N \right\rangle\!\right\rangle, \\ -2\tfrac{\mathrm{d}}{\mathrm{d}t}\Re \left\langle\!\left\langle \psi^N, g_{\widetilde{\beta}}^{(12)} \widehat{m}\psi^N \right\rangle\!\right\rangle &= 2\Im \left\langle\!\left\langle \psi^N, \left(g_{\widetilde{\beta}}^{(12)} \left[Z_2^{(12)}, \widehat{m} \right] \psi^N \right\rangle\!\right\rangle \\ &\quad + 2\Im \left\langle\!\left\langle \psi^N, (w_N^{(12)} - U_{\widetilde{\beta}}^{(12)}) f_{\widetilde{\beta}}^{(12)} \widehat{m} + 4\nabla_1 f_{\widetilde{\beta}}^{(12)} \cdot \nabla_1 \widehat{m} \right) \psi^N \right\rangle\!\right\rangle . \end{aligned}$$

Adding these expressions and using that $g_{\widetilde{\beta}} = 1 - f_{\widetilde{\beta}}$, we observe that the term $\left\langle\!\left\langle \psi^N, Z_2^{(12)} \widehat{m}\psi^N \right\rangle\!\right\rangle$ cancels. It remains, among other contributions,

$$-2\Im \left\langle\!\left\langle \psi^N, \left(U_{\widetilde{\beta}}^{(12)} f_{\widetilde{\beta}}^{(12)} - 8\pi a(|\varphi(x_1)|^2 + |\varphi(x_2)|^2) \right) \widehat{m}\psi^N \right\rangle\!\right\rangle . \tag{1.92}$$

This is precisely (1.82) with w_N replaced by $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$, which can be seen as follows: Since $\widetilde{\beta} \in (0, 1)$, $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ is a potential in the NLS scaling regime[7], hence its scattering length is asymptotically given by the Born approximation $\|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)}$. This expression asymptotically coincides with the scattering length of w_N because

$$\int\limits_{\mathbb{R}^3} U_{\widetilde{\beta}}(x) f_{\widetilde{\beta}}(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}^3} w_N(x) f_{\widetilde{\beta}}(x)\,\mathrm{d}x \approx \int\limits_{\mathbb{R}^3} w_N(x) j_N(x)\,\mathrm{d}x = 8\pi a_N$$

by construction of $f_{\widetilde{\beta}}$, which asymptotically equals j_N on $\operatorname{supp} w_N$. Consequently, the coupling parameter for the non-linear evolution generated by $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ equals $8\pi a$, which implies that (1.92) can be controlled by the result from the NLS regime. It only remains to prove that the remainders from the substitution vanish as $N \to \infty$. In this sense, the mathematical understanding of interactions in the NLS regime is a crucial ingredient for obtaining an effective description of the dynamics the GP regime.

Let us remark that the underlying physical idea of this replacement is the same as in the heuristic derivation of the GP energy functional (Section 1.2.5): to leading order, a sufficiently distant and low-energetic particle does not resolve the difference between two scattering potentials whose scattering lengths are (asymptotically) equal. Recall that the heuristic argument consists of a two-stage replacement: first, one replaces the interaction w_N with a softer interaction U^{eff} with the same scattering length for which the Born approximation holds — however, without control of the approximation. Second, U^{eff} is replaced by $U_0/N\delta(x)$, where $U_0 = 8\pi a$. Again, this is

[7] In fact, $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ is not exactly of the form $N^{-1+3\widetilde{\beta}} w(N^{\widetilde{\beta}}\cdot)$. Hence, one must slightly enlarge this class of potentials, as is in Definition 3.1.3.

far from rigorous, and the mathematical problems coming with the replacement by a δ-interaction are sketched at the end of Section 1.4.1. In a sense, Pickl's method can be understood as a rigorous version of these heuristics: the role of U^{eff} is taken by $U_{\widetilde{\beta}}$, and the control of the remainders provides the missing control of the approximation; subsequently, the full proof for the NLS regime takes the place of the second stage of the replacement.

Proving that the new functional (1.91) converges to zero as $N \to \infty$ is only meaningful if the correction term vanishes in the limit, since this ensures the equivalence (1.79) for the new counting functional. Therefore, one replaces $\widehat{m}$ by the weighted many-body operator $\widehat{r}$ defined as

$$\widehat{r} := \widehat{m}^b p_1 p_2 + \widehat{m}^a (p_1 q_2 + q_1 p_2)\,, \tag{1.93}$$

where $\widehat{m}^a$ and $\widehat{m}^b$ denote the operators corresponding to the weight functions

$$m^a(k) \;:=\; m(k) - m(k+1), \qquad m^b(k) \;:=\; m(k) - m(k+2)\,.$$

When replacing $\widehat{m}$ by $\widehat{r}$ in (1.91), one gains an additional projection p_1, which allows the estimate of $g_{\widetilde{\beta}}^{(12)} p_1$ instead of $g_{\widetilde{\beta}}^{(12)}$. Besides, $m^a(k)$ and $m^b(k)$ can be understood as discrete derivatives and thus, as explained above, compensate for powers of N. Note that the change $\widehat{m} \mapsto \widehat{r}$ does not affect the replacement of w_N by $U_{\widetilde{\beta}}$ because one can show that $[Z_N^{(12)}, \widehat{m}] = [Z_N^{(12)}, \widehat{r}]$ (see e.g. [32, Lemma 4.2d]). Hence, the counting functional for the GP scaling of the interaction is defined as

$$\alpha_{\xi,w_N}(t,\psi^N,\varphi) := \alpha^{<}_{\xi,w_N}(t,\psi^N,\varphi) - N(N-1)\Re\left\langle\!\!\left\langle \psi^N, g_{\widetilde{\beta}}(x_1-x_2)\,\widehat{r^{\varphi}}\,\psi^N \right\rangle\!\!\right\rangle \tag{1.94}$$

with $\alpha^{<}_{\xi,w_N}$ as in (1.78).

Following the steps sketched for $N=2$, and for simplicity dropping again all time dependences and indices $\varphi(t)$, the time derivative of α_{ξ,w_N} is bounded by

$$\begin{aligned}
&\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_N}(t,\psi^N,\varphi)\right| && (1.95)\\
\lesssim\; &\left|\left\langle\!\!\left\langle \psi^N, \dot{V^{\text{ext}}}(z_1)\psi^N\right\rangle\!\!\right\rangle - \left\langle \varphi, \dot{V^{\text{ext}}}\varphi\right\rangle_{L^2(\mathbb{R}^3)}\right| + N^2\Im\left\langle\!\!\left\langle \psi^N, \widetilde{Z}^{(12)}_{N,\widetilde{\beta}}\,\widehat{m}\,\psi^N\right\rangle\!\!\right\rangle && (1.96)\\
&+N\Im\left\langle\!\!\left\langle \psi^N, |\varphi(x_1)|^2 g_{\widetilde{\beta}}^{(12)}\,\widehat{r}\,\psi^N\right\rangle\!\!\right\rangle + N^2\Im\left\langle\!\!\left\langle \psi^N, g_{\widetilde{\beta}}^{(12)}\,\widehat{r}\,Z_N^{(12)}\psi^N\right\rangle\!\!\right\rangle && (1.97)\\
&+N^2\Im\left\langle\!\!\left\langle \psi^N, (\nabla_1 g_{\widetilde{\beta}}^{(12)})\cdot\nabla_1\widehat{r}\,\psi^N\right\rangle\!\!\right\rangle && (1.98)\\
&+N^3\Im\left\langle\!\!\left\langle \psi^N, g_{\widetilde{\beta}}^{(12)}\left[|\varphi(x_3)|^2,\widehat{r}\right]\psi^N\right\rangle\!\!\right\rangle + N^3\Im\left\langle\!\!\left\langle \psi^N, g_{\widetilde{\beta}}^{(12)}\left[w_N^{(13)},\widehat{r}\right]\psi^N\right\rangle\!\!\right\rangle && (1.99)\\
&+N^4\Im\left\langle\!\!\left\langle \psi^N, g_{\widetilde{\beta}}^{(12)}\left[w_N^{(34)},\widehat{r}\right]\psi^N\right\rangle\!\!\right\rangle && (1.100)
\end{aligned}$$

$$+N^2\Im\left\langle\!\!\left\langle\psi^N, g^{(12)}_{\widetilde{\beta}}\left[|\varphi(x_1)|^2,\widehat{r}\right]\psi^N\right\rangle\!\!\right\rangle, \tag{1.101}$$

where we used the abbreviation

$$\widetilde{Z}^{(12)}_{N,\widetilde{\beta}} := (U_{\widetilde{\beta}}f_{\widetilde{\beta}})(x_1-x_2) - \tfrac{8\pi a}{N-1}(|\varphi(x_1)|^2+|\varphi(x_2)|^2)\,.$$

Let us analyse the different contributions:

- *NLS term* (1.96).
  The first part of (1.96) is exactly the energy term (1.80), and the second part equals (1.82) with interaction potential $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$. However, one cannot immediately use the result from the NLS case, since it relies on the energy estimate (1.88). Note that although the GP interaction potential is in (1.96) replaced by a potential in the NLS scaling regime, the dynamics of the N-body wave function ψ^N are still generated by the GP Hamiltonian. For these dynamics, $\|\nabla_1 q_1\psi^N\|$ is not asymptotically zero because the microscopic structure described by $f_{\widetilde{\beta}}$ varies on the same length scale as the interaction and thus contributes a kinetic energy of $\mathcal{O}(1)$.

  Since this kinetic energy is localised around the scattering centres, one can show a bound similar to (1.88) for the kinetic energy on a subset $\mathcal{A}_1\subset\mathbb{R}^{3N}$, where appropriate holes around these centres are cut out, namely

  $$\|\mathbb{1}_{\mathcal{A}_1}\nabla_1 q_1\psi^N\|^2 \lesssim \alpha^<_{\xi,w_N}(t,\psi^N,\varphi)+o(1)\,. \tag{1.102}$$

  The main tool of the proof is the inequality

  $$\|\mathbb{1}_{|x_1-x_2|<R_{\widetilde{\beta}}}\nabla_1\psi\|^2 + \tfrac{1}{2}\left\langle\!\!\left\langle\psi,\left(w_N^{(12)}-U_{\widetilde{\beta}}^{(12)}\right)\psi\right\rangle\!\!\right\rangle \geq 0 \quad \text{for } \psi\in\mathcal{D}(\nabla_1) \tag{1.103}$$

  ([151, Lemma 5.1(3)] and [97, Lemma 7.10]). To show (1.103), one first argues that the one-body operator $H^{Z_n} := -\Delta + \frac{1}{2}\sum_{z_k\in Z_n}(w_N - U_{\widetilde{\beta}})(\cdot - z_k)$, where Z_n is an n-elemental subset of $\mathbb{R}^3$ with distance between any two elements larger than $2R_{\widetilde{\beta}}$, is for each $n\in\mathbb{N}$ a positive operator. To see this, one observes that $F^{Z_n}_{\widetilde{\beta}} := \prod_{z_k\in Z_n} f_{\widetilde{\beta}}(\cdot - z_k)$ satisfies $H^{Z_n}F^{Z_n}_{\widetilde{\beta}} = 0$. Besides, recall that H^{Z_n} is positive if and only if all of its eigenvalues are non-negative. If H^{Z_n} had a negative eigenvalue, its ground state ψ_G would be strictly positive, which leads to a contradiction when considering the scalar product $\langle F^{Z_n}_{\widetilde{\beta}}, H^{Z_n}\psi_G\rangle$.

  Using the positivity of H^{Z_n} for any $n\in\mathbb{N}$, one shows that the quadratic form

  $$Q(\psi) := \|\mathbb{1}_{|\cdot|\leq R_{\widetilde{\beta}}}\nabla\psi\|^2 + \tfrac{1}{2}\left\langle\psi,(w_N - U_{\widetilde{\beta}})\psi\right\rangle, \qquad \psi\in H^1(\mathbb{R}^3)$$

is non-negative: Assuming that there exists a $\widetilde{\psi} \in H^1(\mathbb{R}^3)$ such that $Q(\widetilde{\psi}) < 0$, one identifies a set Z_n and a function $\chi_R \in H^1(\mathbb{R}^3)$ such that $\langle \chi_R, H^{Z_n}\chi_R\rangle < 0$ for some n, contradicting the positivity of H^{Z_n}. The function χ_R is constructed in such a way that the part of $\langle \chi_R, H^{Z_n}\chi_R\rangle$ inside a ball with radius R containing a sufficiently large neighbourhood of Z_n equals $nQ(\widetilde{\psi}) < 0$. The decay of χ_R outside the ball is chosen such that its positive kinetic energy is not large enough to cancel this negative contribution, given n is chosen sufficiently large. Finally, one deduces (1.103) from the non-negativity of Q.

Let us now explain why (1.103) is crucial for the derivation of (1.102). Inserting identities $\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}}_1}$, where $\overline{\mathcal{A}}_1 := \mathbb{R}^{3N} \setminus \mathcal{A}_1$, one can show with (1.87) that

$$\begin{aligned} |E^{\psi^N}_{w_N} - \mathcal{E}^{\varphi}_a| \;\gtrsim\; & \|\mathbb{1}_{\mathcal{A}_1}\nabla_1 q_1\psi^N\|^2 + \|\mathbb{1}_{\overline{\mathcal{A}}_1}\nabla_1\psi^N\|^2 - \langle\!\langle \psi^N, \widehat{n}\psi^N\rangle\!\rangle - \mathcal{O}(1) \\ & + \tfrac{N-1}{2}\left\langle\!\!\left\langle \psi^N, \left(w_N^{(12)} - \tfrac{8\pi a}{N-1}|\varphi(x_1)|^2\right)\psi^N\right\rangle\!\!\right\rangle . \end{aligned}$$

The second line contains the difference between the pair potential and the effective one-body potential, hence, it would be small if w_N was replaced by $U_{\widetilde{\beta}}$ or $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$. However, simply adding and subtracting $U_{\widetilde{\beta}}$ does not solve the problem since the remainder $\frac{N-1}{2}\langle\!\langle \psi^N, (w_N^{(12)} - U_{\widetilde{\beta}}^{(12)})\psi^N\rangle\!\rangle$ is neither small nor necessarily non-negative. This is where the inequality (1.103) comes into play: defining $\mathcal{B}_1$ as the set where all particles from $\{2, ..., N\}$ are mutually too distant to iteract with each other and $\overline{\mathcal{B}}_1$ as its complement, one infers from (1.103) that

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_1\psi^N\|^2 + \tfrac{N-1}{2}\left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\psi^N, (w_N^{(12)} - U_{\widetilde{\beta}}^{(12)})\mathbb{1}_{\mathcal{B}_1}\psi^N\right\rangle\!\!\right\rangle \geq 0\,, \tag{1.104}$$

which follows because $\mathbb{1}_{\mathcal{B}_1}\psi^N \in \mathcal{D}(\nabla_1)$ and since $\overline{\mathcal{A}}_1$ is chosen sufficiently large that it contains a ball with radius $R_{\widetilde{\beta}}$ around each scattering centre. Hence, one inserts identities $\mathbb{1}_{\mathcal{B}_1} + \mathbb{1}_{\overline{\mathcal{B}}_1}$, which yields

$$\begin{aligned} \alpha^<_{\xi, w_\mu}(t) + \mathcal{O}(1) \;\gtrsim\; & \|\mathbb{1}_{\mathcal{A}_1}\nabla_1 q_1\psi^N\|^2 \\ & + \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_1\psi^N\|^2 + \tfrac{N-1}{2}\left\langle\!\!\left\langle \psi^N, \mathbb{1}_{\mathcal{B}_1}(w_N^{(12)} - U_{\widetilde{\beta}}^{(12)})\psi^N\right\rangle\!\!\right\rangle \\ & - \tfrac{N-1}{2}\left|\left\langle\!\!\left\langle \psi^N, \mathbb{1}_{\mathcal{B}_1}U_{\widetilde{\beta}}^{(12)}\psi^N\right\rangle\!\!\right\rangle - \langle\!\langle \psi^N, 8\pi a|\varphi(x_1)|^2\psi^N\rangle\!\rangle\right| , \end{aligned}$$

where we dropped some non-negative contributions. By (1.103), the second line is non-negative. The last line can be controlled similarly to the comparable terms in the NLS estimate (1.89) and since the set $\overline{\mathcal{B}}_1$ is sufficiently small.

Finally, recall that the energy estimate (1.88) enters only in the (qq–qp)-term (1.85). Hence, one modifies this term by suitable insertion of identities $\mathbb{1} =$

$\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}}_1}$. The expressions containing $\mathbb{1}_{\mathcal{A}_1}$ can be controlled by (1.102), while one estimates the terms containing $\mathbb{1}_{\overline{\mathcal{A}}_1}$ by exploiting the smallness of $\overline{\mathcal{A}}_1$.

- *Remainder terms* (1.97) *to* (1.101).
  These terms collect the remainders from the substitution $w_N \mapsto U_{\widetilde{\beta}} f_{\widetilde{\beta}}$. To control them, one mainly uses properties of the scattering solution $f_{\widetilde{\beta}}$, for whose proof it is crucial that $g_{\widetilde{\beta}}$ has compact support of diameter $\sim N^{-\widetilde{\beta}}$. To control (1.98), one integrates by parts, using that the condensate wave function varies on a much larger length scale than the microscopic structure $g_{\widetilde{\beta}}$.

In summary, these steps lead to the bound

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_N}\left(t,\psi^N(t),\varphi(t)\right) \lesssim \alpha^<_{\xi,w_N}\left(t,\psi^N(t),\varphi(t)\right) + \mathcal{O}(1)\ .$$

Finally, one shows that the correction term in (1.94) is of order $\mathcal{O}(1)$, which concludes the derivation of the GP equation by Lemma 1.4.3 and (1.79).

1.5. Excitations from the condensate

Recall that complete asymptotic BEC implies that a macroscopic fraction of the bosons occupies the condensate state $\varphi \in L^2(\mathbb{R}^3)$, which is mathematically formulated as the convergence of the reduced density matrices (Definition 1.2.1). Note that this convergence does not imply that the N-body wave function ψ is close to the product state $\varphi^{\otimes N}$ in any stonger sense than in the sense of reduced densities. This can be infered from two reasons:

- Since the bosons interact with each other, the N-body ground state as well as the lower excited states (which are expected to exhibit BEC) feature a microscopic correlation structure around the scattering centres as in (1.20), which minimises the energy. To leading order in N, these correlations are not visible in the reduced density matrix, even in the most singular GP scaling. In contrast, the difference between the N-body state and a pure product is large with respect to the $L^2(\mathbb{R}^{3N})$ norm, and in particular with respect to any stronger norm involving the energy[8].

- A microscopic fraction of the total number of particles can be excited from the condensate without destroying the state of BEC. While this does not affect the leading order behaviour of the reduced densities, it has a huge effect on the

[8] An argument for the mean-field regime is given in [113, Corollary 2]. For larger values of β, where the correlation structure becomes relevant, some more work is required since one needs to prove that the normalisation factor in (1.20) converges to one.

$L^2(\mathbb{R}^{3N})$-norm. To motivate this, consider a non-interacting gas, whose ground state is given by the product state $\varphi^{\otimes N}$. Naturally, a symmetric N-body state

$$\widetilde{\psi}(x_1, ..., x_N) := \frac{1}{N}\sum_{j=1}^{N} \varphi^{\perp}(x_j)\varphi^{\otimes(N-1)}(x_1, ..., x_N \setminus x_j)\,,$$

where one particle occupies a state $\varphi^{\perp} \perp \varphi$ orthogonal to φ in $L^2(\mathbb{R}^d)$-sense and all other particles are in the state φ, fulfils the criterion of complete asymptotic BEC. However, it is not close to $\varphi^{\otimes N}$ in $L^2(\mathbb{R}^{3N})$-sense since

$$\|\widetilde{\psi} - \varphi^{\otimes N}\|^2 \;=\; 2 - 2\Re\left\langle\!\!\left\langle \widetilde{\psi}, \varphi^{\otimes N}\right\rangle\!\!\right\rangle \;=\; 2$$

for normalised functions φ, $\varphi^{\perp}$.

The ground state results such as (1.19) and (1.35) discussed so far, as well as the dynamical statement (1.73), are related to the behaviour of the (correlated) condensate wave function. The validity of these results implies that the effects due to the particles outside the condensate are of higher order with respect to N^{-1}, related to the fact that it is very unlikely to find a relevant number of such particles in the low energy states. Hence, to derive the next-to-leading order corrections to the ground state energy and to obtain a more precise characterisation of the dynamics than in terms of reduce densities, these excitations from the condensate must be included in the description.

Since the excitations are often described in the language of second quantisation, we begin with an overview of this formalism and recall the related notation, which is essentially taken from [22, Chapter 3]. Subsequently, we summarise the results of Bogoliubov theory concerning the energy spectrum as well as the dynamics of the excitations. The main references are [134, Chapter 2], [137], [145, Chapter 8], [146], [163] and [169].

1.5.1. Second quantisation

To describe bosonic states where the number of particles is not fixed, one introduces the bosonic Fock space over the one-body Hilbert space $\mathfrak{H} = L^2(\Omega)$ for some $\Omega \subseteq \mathbb{R}^d$,

$$\mathcal{F} = \bigoplus_{n\geq 0} \bigotimes_{\text{sym}}^{n} \mathfrak{H} = \bigoplus_{n\geq 0} \mathfrak{H}_+^n\,,$$

whose elements are denoted by $\phi = (\phi^{(n)})_{n\geq 0}$. It is equipped with the inner product

$$\langle \psi, \phi\rangle_{\mathcal{F}} = \sum_{n\geq 0} \langle \psi^{(n)}, \phi^{(n)}\rangle_{\mathfrak{H}^n}\,,$$

corresponding to the norm

$$\|\phi\|_{\mathcal{F}} = \sum_{n\geq 0} \|\phi^{(n)}\|^2_{\mathfrak{H}^n} .$$

We consider only normalised vectors $\phi \in \mathcal{F}$, hence $\|\phi^{(n)}\|^2_{\mathfrak{H}^n}$ determines the probability for the state ϕ to have n particles. States with a fixed number of particles, i.e., Fock vectors with exactly one non-zero component, are eigenvectors of the number operator $\mathcal{N}$, which is defined by its action

$$(\mathcal{N}\phi)^{(n)} = n\phi^{(n)} .$$

The vacuum state is denoted by $\Omega = (1, 0, 0, \dots)$.

For $f \in \mathfrak{H}$, the creation and annihilation operators $a^*(f)$ and $a(f)$ are defined as

$$\begin{aligned} (a^*(f)\phi)^{(n)}(x_1, ..., x_n) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(x_j)\phi^{(n-1)}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) , \quad n \geq 1 \\ (a(f)\phi)^{(n)}(x_1, ..., x_n) &= \sqrt{n+1} \int_{\Omega} \mathrm{d}x \overline{f(x)} \phi^{(n+1)}(x, x_1, ..., x_n) , \quad n \geq 0 , \end{aligned}$$

i.e., $a^*(f)$ and $a(f)$ create and annihilate a particle in the state f. To write them in a more compact form, one introduces the operator-valued distributions a^*_x, a_x as

$$a^*(f) = \int_{\Omega} \mathrm{d}x f(x)\, a^*_x , \qquad a(f) = \int_{\Omega} \mathrm{d}x \overline{f(x)}\, a_x .$$

For $f, g \in \mathfrak{H}$, creation and annihilation operator satisfy the canonical commutation relations (CCR)

$$[a(f), a^*(g)] = \langle f, g\rangle_{\mathfrak{H}} , \qquad [a(f), a(g)] = [a^*(f), a^*(g)] = 0 ,$$

which correspond to the relations

$$[a_x, a^*_y] = \delta(x - y) , \qquad [a_x, a_y] = [a^*_x, a^*_y] = 0$$

of the operator-valued distributions.

The second quantisation $\mathrm{d}\Gamma(J^{(1)})$ of a one-body operator $J^{(1)}$ acting on $\mathfrak{H}$ is defined by the requirement that

$$\left(\mathrm{d}\Gamma(J^{(1)})\phi\right)^{(n)} = \sum_{j=1}^{n} J^{(1)}_j \psi^{(n)} ,$$

where $J_j^{(1)} := \mathbb{1}^{\otimes(j-1)} \otimes J^{(1)} \otimes \mathbb{1}^{\otimes(n-j)}$ denotes the operator on $\mathfrak{H}^n$ acting as $J^{(1)}$ on the particle j and as the identity on all other particles. If the operator $J^{(1)}$ has an integral kernel $J^{(1)}(x;y)$, its second quantisation is given by

$$\mathrm{d}\Gamma(J^{(1)}) = \int\limits_{\Omega} \mathrm{d}x \int\limits_{\Omega} \mathrm{d}y\, J^{(1)}(x;y)\, a_x^* a_y \,.$$

For k-particle operators $J^{(k)}$ acting on $\mathfrak{H}_+^k$ with integral kernel $J^{(k)}(x_1,\dots,x_k;y_1,\dots,y_k)$, this generalises to

$$\mathrm{d}\Gamma(J^{(k)}) = \int\limits_{\Omega^k} \mathrm{d}x_1 \cdots \mathrm{d}x_k \int\limits_{\Omega^k} \mathrm{d}y_1 \cdots \mathrm{d}y_k\, J^{(k)}(x_1,\dots,x_k;y_1,\dots,y_k) a_{x_1}^* \cdots a_{x_k}^* a_{y_k} \cdots a_{y_1} \,.$$

Since the number operator is the second quantisation of the identity, it can be expressed as

$$\mathcal{N} = \mathrm{d}\Gamma(\mathbb{1}) = \int\limits_{\Omega} \mathrm{d}x\, a_x^* a_x \,. \tag{1.105}$$

The second quantisation $\mathrm{d}\Gamma(H_N(t)) = \mathcal{H}_N(t)$ of the Hamiltonian $H_N(t)$ from (1.2) is determined by its action $(\mathcal{H}_N(t)\phi)^{(n)} = H_N^{(n)}(t)\phi^{(n)}$ with

$$H_N^{(n)} = \sum_{j=1}^{n} \left(-\Delta_j + V^{\mathrm{ext}}(t,x_j)\right) + \sum_{1\le i<j\le n} w^{\mathrm{int}}(x_i - x_j)\,,$$

hence

$$\mathcal{H}_N(t) = \int\limits_{\mathbb{R}^3} \mathrm{d}x \nabla_x a_x^* \nabla_x a_x + \int\limits_{\mathbb{R}^3} \mathrm{d}x V^{\mathrm{ext}}(t,x) a_x^* a_x + \frac{1}{2} \int\limits_{\mathbb{R}^3} \mathrm{d}x \int\limits_{\mathbb{R}^3} \mathrm{d}y\, w^{\mathrm{int}}(x-y) a_x^* a_y^* a_y a_x \,.$$

Note that the parameter N, which enters, e.g., in the interaction if w^{int} is chosen as w_N or $w_{N,\beta}$, is not related to the number of particles of the system, which can take any value. When restricted to the N particle sector $L^2(\mathbb{R}^{3N}) \subset \mathcal{F}$, $\mathcal{H}_N(t)$ coincides with $H_N(t)$ from (1.2).

To describe a uniform Bose gas in a cubic box Λ with side length L and periodic boundary conditions, it is most convenient to work in momentum space. The plane waves $L^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}p\cdot x}$ for $p \in \Lambda^* := \frac{2\pi}{L}\mathbb{Z}^3$ form a basis of $L^2(\Lambda)$, which leads to the introduction of the operator-valued distributions in momentum space,

$$a_p^* := a^*(\mathrm{e}^{-\mathrm{i}p\cdot x}) = \int\limits_{\Lambda} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}p\cdot x}\, a_x^* \,, \qquad a_p := a(\mathrm{e}^{-\mathrm{i}p\cdot x}) = \int\limits_{\Lambda} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}p\cdot x}\, a_x \,.$$

Hence, a_p^* and a_p can be understood as Fourier transforms of a_x^* and a_x, which create and annihilate a particle with momentum $p \in \Lambda^*$. From the CCR of a_x and a_x^*, one derives the corresponding relations

$$[a_p, a_q^*] = \delta_{p,q}\,, \qquad [a_p, a_q] = [a_p^*, a_q^*] = 0\,.$$

In this representation, the total number operator is

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p\,,$$

and the second quantised Hamiltonian is given as

$$\mathbb{H} = \sum_{p\in\Lambda^*} |p|^2 a_p^* a_p + \frac{1}{2L^3} \sum_{p\in\Lambda^*} \widehat{w^{\mathrm{int}}}(p) \sum_{q,k\in\Lambda^*} a_{q+p}^* a_{k-p}^* a_k a_q\,, \tag{1.106}$$

where

$$\widehat{w^{\mathrm{int}}}(p) = \int_\Lambda w^{\mathrm{int}}(x) \mathrm{e}^{-\mathrm{i}p\cdot x}\,\mathrm{d}x$$

denotes the Fourier transform of w^{int}. For the choice $w^{\mathrm{int}} = w_{N,\beta}$, one finds

$$\widehat{w_{N,\beta}}(p) = \frac{1}{N}\widehat{w}\left(\frac{p}{N^\beta}\right)\,.$$

1.5.2. Bogoliubov theory for ground state energy and lower excitation spectrum

At low energies, one expects the lowest-lying single-particle state to be macroscopically occupied. For the uniform Bose gas in the box Λ, this lowest state is given by the plane wave with momentum zero. Macroscopic occupation means that the expectation value of the number operator counting the particles in the zero-momentum mode is of order N, i.e.,

$$\mathcal{N}_0 = a_0^* a_0 \sim N\,.$$

Motivated by this observation, the following approximation scheme was proposed by Bogoliubov in [29]:

- Since the expectation value of $\mathcal{N}_0$ is much larger than $[a_0, a_0^*] = 1$, one replaces a_0^* and a_0 in (1.106) by $\sqrt{N}$ (c-number substitution). The resulting Hamiltonian contains only creation and annihilation operators corresponding to states with $|p| > 0$, describing excitations from the condensate.

- Subsequently, one neglects all terms that are higher than quadratic in a_p^* and a_p, which issue from interactions among the excitations. This yields a Hamiltonian which is quadratic in a_p^* and a_p for $|p| > 0$ and contains no operators creating or annihilating particles in the condensate.

Applying this so-called *Bogoliubov approximation* to (1.106) leads, with the abbreviation $\Lambda_+^* := \Lambda^* \setminus \{0\}$, to

$$\begin{aligned}
\mathbb{H} \;&=\; \sum_{p\in\Lambda_+^*} |p|^2 a_p^* a_p + \frac{1}{2L^3}\widehat{w^{\mathrm{int}}}(0) \sum_{q,k\in\Lambda^*} \left(a_q^*(a_q a_k^* - \delta_{q,k})a_k\right) \\
&\quad + \frac{1}{2L^3} \sum_{p\in\Lambda_+^*} \widehat{w^{\mathrm{int}}}(p) \left(a_p^* a_{-p}^* a_0 a_0 + a_p^* a_0^* a_p a_0 + a_0^* a_0^* a_p a_{-p} + a_0^* a_{-p}^* a_0 a_{-p}\right) \\
&\quad + \frac{1}{2L^3} \sum_{p,q\in\Lambda_+^*} \widehat{w^{\mathrm{int}}}(p) a_{q+p}^* a_{q-p}^* a_q a_q + \frac{1}{2L^3} \sum_{\substack{p,q\in\Lambda_+^*\\ q\neq -p}} \widehat{w^{\mathrm{int}}}(p) a_{q+p}^* a_{-p}^* a_0 a_q \\
&\quad + \frac{1}{2L^3} \sum_{\substack{p,k\in\Lambda_+^*\\ k\neq p}} \widehat{w^{\mathrm{int}}}(p) a_p^* a_{k-p}^* a_k a_0 + \frac{1}{2L^3} \sum_{\substack{p,q,k\in\Lambda_+^*\\ k\neq p, q\neq -p}} \widehat{w^{\mathrm{int}}}(p) a_{q+p}^* a_{k-p}^* a_k a_q \\
&\approx\; \frac{N(N-1)}{2L^3}\widehat{w^{\mathrm{int}}(0)} + \sum_{p\in\Lambda_+^*} \left(|p|^2 a_p^* a_p + \frac{N}{2L^3}\widehat{w^{\mathrm{int}}}(p) \left(a_p^* a_p + a_{-p}^* a_{-p}\right)\right) \\
&\quad + \sum_{p\in\Lambda_+^*} \frac{N}{2L^3}\widehat{w^{\mathrm{int}}}(p) \left(a_p^* a_{-p}^* + a_p a_{-p}\right) \,,
\end{aligned}$$

where we replaced in the first term the number operator $\mathcal{N} = \sum_{q\in\Lambda^*} a_q^* a_q$ by its value N when evaluated on a state with N particles, and applied the Bogoliubov approximation in the remaining terms. For spherically symmetric interaction potentials w^{int}, this equals the so-called *Bogoliubov Hamiltonian*

$$\begin{aligned}
\mathbb{H}_{\mathrm{Bog}} \;&=\; \tfrac{N-1}{2}\varrho\,\widehat{w^{\mathrm{int}}}(0) \\
&\quad + \sum_{p\in\Lambda_+^*} \left(\left(|p|^2 + \varrho\,\widehat{w^{\mathrm{int}}}(p)\right) a_p^* a_p + \tfrac{1}{2}\varrho\widehat{w^{\mathrm{int}}}(p) \left(a_p^* a_{-p}^* + a_p a_{-p}\right)\right) \,, \qquad (1.107)
\end{aligned}$$

where $\varrho = \frac{N}{L^3}$ denotes the particle density. The first term in (1.107) is the energy of N particles in the zero momentum (condensate) state. The term proportional to $a_p^* a_p$ is the energy of excitations moving in the mean-field created by the interactions with all other particles. It describes the process where simultaneously a particle with momentum p and a particle from the condensate are scattered into the zero-momentum state and the state p, respectively. The last term corresponds to the scattering of two condensate particles into a pair with momenta p and $-p$, and vice versa.

Let us now study interactions $w^{\mathrm{int}}(x)$ of the form

$$w^{\mathrm{int}}(x) = \frac{1}{N-1} v_{N,\beta}(x)\,, \qquad v_{N,\beta}(x) = N^{3\beta} v(N^\beta x)\,, \quad \beta \in [0,1) \tag{1.108}$$

in a cubic box Λ with side length $L = 1$ and periodic boundary conditions, or, alternatively, in an external potential V^{ext} with characteristic length of order one. Consequently, the density is $\varrho \sim N$. We assume that the interaction v is bounded, spherically symmetric and compactly supported. Here, we use a different notation than in (1.24), where the prefactor $\sim N^{-1}$ was included in $w_{N,\beta}$. The reason for this inconsistency is that the notation (1.24) seems to be standard for deriving dynamical results on the level of reduced densities, whereas the notation (1.108) is the usual convention for static and dynamical results in relation with the Bogoliubov approximation. To make the distinction clearer, we use v instead of w to denote the unscaled interaction potential. As explained in Section 1.2.4, the case $\beta = 0$ is the Hartree scaling, parameters $\beta < \frac{1}{3}$ describe a system with mean-field interactions, and the whole regime $\beta < 1$ is referred to as NLS scaling of the interaction.

In the homogeneous case, the Bogoliubov Hamiltonian with interaction (1.108) is

$$\mathbb{H}^\beta_{\mathrm{Bog}} = \tfrac{N}{2}\widehat{v}(0) + \sum_{p \in \Lambda^*_+} \left(|p|^2 + \widehat{v}\left(\tfrac{p}{N^\beta}\right) a^*_p a_p + \tfrac{1}{2}\widehat{v}\left(\tfrac{p}{N^\beta}\right)\left(a^*_p a^*_{-p} + a_p a_{-p}\right)\right)\,, \tag{1.109}$$

where we used that

$$\widehat{v_{N,\beta}}(p) = \widehat{v}\left(\tfrac{p}{N^\beta}\right)$$

and approximated $\frac{N-1}{N} \approx 1$. To describe an inhomogeneous gas in an external potential V^{ext} in $\mathbb{R}^3$ with ground state φ, one uses the position space representation. The corresponding Bogoliubov Hamiltonian can be written as[9]

$$\begin{aligned}\mathcal{H}^\beta_{\mathrm{Bog}} &= \int_{\mathbb{R}^3} \mathrm{d}x\, a^*_x \left(h^\varphi(x) + K^\varphi_1(x)\right) a_x \\ &\quad + \tfrac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}x \int_{\mathbb{R}^3} \mathrm{d}y \left(K^\varphi_2(x,y) a^*_x a^*_y + \overline{K^\varphi_2(x,y)} a_x a_y \right)\,.\end{aligned} \tag{1.110}$$

Here,

$$K^\varphi_1 := q^\varphi \widetilde{K}^\varphi_1 q^\varphi\,, \qquad K^\varphi_2(\cdot,\cdot) := (q^\varphi \otimes q^\varphi)\, \widetilde{K}^\varphi_2(\cdot,\cdot) \tag{1.111}$$

with q^φ from Definition 1.4.1, where $\widetilde{K}_1$ is the Hilbert-Schmidt operator on $L^2(\mathbb{R}^3)$ with kernel

$$\widetilde{K}^\varphi_1(x;y) := \varphi(x) v_{N,\beta}(x-y)\overline{\varphi(y)}\,,$$

[9] See, e.g., [137, Equation (31)].

and where the two-body function $\widetilde{K}_2^\varphi$ is given by

$$\widetilde{K}_2^\varphi(x,y) := \varphi(x) v_{N,\beta}(x-y)\varphi(y)\,.$$

Further,

$$h^\varphi := -\Delta + V^{\mathrm{ext}} + v_{N,\beta} * |\varphi|^2 - \mu^\varphi\,, \tag{1.112}$$

where we abbreviated

$$\overline{v}^\varphi := v_{N,\beta} * |\varphi|^2 \tag{1.113}$$

as in (1.70). The phase parameter $\mu^\varphi \in \mathbb{R}$ is usually chosen as

$$\mu^\varphi = \tfrac{1}{2} \int\limits_{\mathbb{R}^3} \mathrm{d}x \int\limits_{\mathbb{R}^3} \mathrm{d}y |\varphi(x)|^2 |\varphi(y)|^2 v_{N,\beta}(x-y)\,. \tag{1.114}$$

To motivate this particular choice of μ^φ, observe that it implies the compatibility of the energies in the time-dependent setting: under the assumption of condensation $\psi(t) \approx \varphi(t)^{\otimes N}$, the N-body energy per particle can be approximated as

$$\begin{aligned} \frac{1}{N}\left\langle\!\left\langle \psi^N(t), H_{N,\beta}\psi^N(t)\right\rangle\!\right\rangle &= \frac{1}{N}\left\langle\!\left\langle \psi^N(t), \mathrm{i}\partial_t\psi^N(t)\right\rangle\!\right\rangle \approx \langle \varphi(t), \mathrm{i}\partial_t \varphi(t)\rangle_{L^2(\mathbb{R}^3)} \\ &= \left\langle \varphi(t), \left(-\Delta + V^{\mathrm{ext}} + v_{N,\beta} * |\varphi(t)|^2 - \mu^{\varphi(t)}\right)\varphi(t)\right\rangle_{L^2(\mathbb{R}^3)}\,, \end{aligned}$$

which coincides with the effective energy per particle,

$$\left\langle \varphi(t), \left(-\Delta + V^{\mathrm{ext}} + \tfrac{1}{2} v_{N,\beta} * |\varphi(t)|^2\right)\varphi(t)\right\rangle_{L^2(\mathbb{R}^3)}\,,$$

for above choice of $\mu^{\varphi(t)}$ ([113, p. 1615]). Note that the operator h^φ in (1.112) coincides with the expression in (1.69) up to μ^φ. As long as exclusively the dynamics of the condensate wave function were concerned, we could neglect this phase parameter since it cancels in the reduced density matrix.

Since the Bogoliubov Hamiltonian is quadratic, it can be explicitly diagonalised by means of a Bogoliubov transformation. To remove the off-diagonal contributions $a_p^* a_{-p}^*$ and $a_p a_{-p}$ in $\mathbb{H}_{\mathrm{Bog}}$, one introduces a new set b_p^*, b_p of creation and annihilation operators satisfying the CCR in such a way that (1.107) written in terms of b_p^*, b_p is diagonal. This can be achieved by a transformation

$$b_p = u(p) a_p + v(p) a_{-p}^*\,, \qquad b_{-p} = u(p) a_{-p} + v(p) a_p^*\,,$$

where $u(p)$ and $v(p)$ must satisfy the condition

$$u(p)^2 - v(p)^2 = 1$$

for b_p^*, b_p to fulfil the CCR. This is clearly given if

$$u(p) = \cosh(\alpha_p)\,, \qquad v(p) = \sinh(\alpha_p)$$

for any $\alpha_p \in \mathbb{R}$, and one finds that the choice

$$\tanh(\alpha_p) = \frac{|p|^2 + \varrho\widehat{w^{\mathrm{int}}}(p) - \sqrt{|p|^4 + 2|p|^2\varrho\widehat{w^{\mathrm{int}}}(p)}}{\varrho\widehat{w^{\mathrm{int}}}(p)}$$

removes the off-diagonal part in (1.107).

As a consequence, the Hamiltonian $\mathbb{H}^0_{\mathrm{Bog}}$ from (1.109) for $\beta = 0$ can be written as

$$\mathbb{H}^0_{\mathrm{Bog}} = E^0_{\mathrm{Bog}} + \sum_{p\in\Lambda_+^*} e^0(p) b_p^* b_p\,, \tag{1.115}$$

where

$$E^0_{\mathrm{Bog}} = \tfrac{N}{2}\,\widehat{v}(0) - \tfrac{1}{2}\sum_{p\in\Lambda_+^*}\left(|p|^2 + \widehat{v}(p) - e^0(p)\right)\,, \tag{1.116}$$

$$e^0(p) = \sqrt{|p|^4 + 2|p|^2\,\widehat{v}(p)}\,. \tag{1.117}$$

Note that $e^0(p)$ is linear in $|p|$ for small momenta, whereas the dispersion relation is quadratic in the non-interacting case. Moreover, the sum in E^0_{Bog} is absolutely convergent, which can be seen by expanding the square root. In conclusion, the Bogoliubov approximation provides the next-to-leading order correction (order one) to the ground state energy (order N). Moreover, it states that the excitation spectrum, i.e., the spectrum of $\mathbb{H}^0_{\mathrm{Bog}} - E^0_{\mathrm{Bog}}$, is given by

$$\sum_{p\in\Lambda_+^*} e^0(p) n_p\,, \qquad n_p \in \{0,1,2,\dots\}\,, \tag{1.118}$$

which implies that the system behaves like a system of non-interacting bosons with energies $e^0(p)$.

These predictions of the Bogoliubov approximation were rigorously justified by Seiringer in [167] for bosons on the unit torus in the Hartree scaling regime. More precisely, for interactions $v_{N,\beta}$ with $\beta = 0$, where v is assumed to be bounded and of positive type[10], the author shows that (1.116) describes the N-body ground state energy up to errors of order $N^{-\frac{1}{2}}$. Besides, the excitation spectrum below an energy threshold ξ is proven to be of the form (1.118), up to errors of order $\mathcal{O}(\xi^{\frac{3}{2}}N^{-\frac{1}{2}})$.

This result was extended by Grech and Seiringer in [83] to the inhomogeneous

[10]This means that v has only non-negative Fourier coefficients.

setting, and by Dereziński and Napiórkowski in [56] to the case of a large but finite volume in the limit $N, L \to \infty$, provided that the volume does not grow too fast relatively to the number of particles.

The works [83, 167] were generalised by Lewin, Nam, Serfaty and Solovej in [114] to bosons in the Hartree scaling regime which interact via potentials of a generic form and for a range of possible kinetic terms. In this abstract setting, they obtain a list of conditions under which the validity of the Bogoliubov approximation can be rigorously shown. Roughly speaking, it is sufficient to have BEC in the ground state φ_0 with optimal error term, i.e., $\left\langle \varphi_0, \gamma^{(1)}_{\psi^N} \varphi_0 \right\rangle \geq 1 - \mathcal{O}(N^{-1})$.

In [114], the authors introduce a method which can be understood as a rigorous implementation of the c-number substitution. They observe that any symmetric N-body wave function $\psi^N \in \mathfrak{H}^N_+$ can be decomposed as

$$\psi^N = \sum_{k=0}^{N} \varphi^{\otimes(N-k)} \otimes_s \xi^{(k)}_\varphi \tag{1.119}$$

for $\varphi \in \mathfrak{H}$ and $\xi_\varphi = \big(\xi^{(k)}_\varphi\big)_{k=0}^N \in \mathcal{F}^{\leq N}_{\perp\varphi}$, where

$$\mathcal{F}^{\leq N}_{\perp\varphi} := \bigoplus_{k=0}^{N} \bigotimes_{\text{sym}}^{k} \{\varphi\}^\perp \subset \mathcal{F}_{\perp\varphi} := \bigoplus_{k\geq 0} \bigotimes_{\text{sym}}^{k} \{\varphi\}^\perp \tag{1.120}$$

is the truncated bosonic Fock space over the one-body space $\{\varphi\}^\perp$ of excited particles. Here, $\{\varphi\}^\perp$ denotes the orthogonal complement of the one-dimensional subspace spanned by φ in $\mathfrak{H}$. Further, $\otimes_s$ denotes the symmetric tensor product, which is for $\psi_a \in \mathfrak{H}^a$, $\psi_b \in \mathfrak{H}^b$ defined as

$$\begin{aligned}(\psi_a \otimes_s \psi_b)(x_1, \ldots, x_{a+b}) :=&\\ \frac{1}{\sqrt{a!\,b!\,(a+b)!}} \sum_{\sigma \in \mathfrak{S}_{a+b}} &\psi_a(x_{\sigma(1)}, \ldots, x_{\sigma(a)})\, \psi_b(x_{\sigma(a+1)}, \ldots, x_{\sigma(a+b)})\,,\end{aligned} \tag{1.121}$$

where $\mathfrak{S}_{a+b}$ denotes the set of all permutations of $a+b$ elements. The addend $k=0$ in (1.119) describes the condensate, while the terms $k \in \{1, \ldots, N\}$ correspond to the excitations. In the following, we refer to $\xi^{(k)}_\varphi$ as *k-particle excitation*. By construction, every k-particle excitation $\xi^{(k)}_\varphi \in \mathfrak{H}^k_+$ is orthogonal to φ in every coordinate. The relation between the N-body state ψ^N and the corresponding excitation vector ξ_φ is given by the unitary map

$$\mathfrak{U}^\varphi_N : \mathfrak{H}^N_+ \to \mathcal{F}^{\leq N}_{\perp\varphi}\,, \quad \psi^N \mapsto \mathfrak{U}^\varphi_N \psi^N := \xi_\varphi\,. \tag{1.122}$$

For a_0^*, a_0 denoting the creation and annihilation operator corresponding to the condensate φ, its action is explicitly given as

$$\mathfrak{U}_N^{\varphi}\psi^N = \bigoplus_{j=0}^{N}(\mathbb{1} - |\varphi\rangle\langle\varphi|)^{\otimes j}\left(\frac{a_0^{N-j}}{\sqrt{(N-j)!}}\psi^N\right) \tag{1.123}$$

([114, Proposition 4.2]), i.e., it annihilates $N-j$ particles from the condensate and projects the resulting j-particle state onto the orthogonal complement of the condensate wave function. The map $\mathfrak{U}_N^{\varphi}$ can be used to factor out the condensate, as was done by Bogoliubov with the replacement $a_0^*, a_0 \mapsto \sqrt{N}$: Denote by $\mathcal{N}$ the number operator on $\mathcal{F}_{\perp\varphi}^{\leq N}$ and let $p, q \in \Lambda_+^*$. Conjugation with $\mathfrak{U}_N^{\varphi}$ yields

$$\begin{aligned}
\mathfrak{U}_N^{\varphi} a_0^* a_0 (\mathfrak{U}_N^{\varphi})^* &= N - \mathcal{N}\,,\\
\mathfrak{U}_N^{\varphi} a_0^* a_p (\mathfrak{U}_N^{\varphi})^* &= \sqrt{N-\mathcal{N}}\,a_p\,,\\
\mathfrak{U}_N^{\varphi} a_p^* a_0 (\mathfrak{U}_N^{\varphi})^* &= a_p^*\sqrt{N-\mathcal{N}}\,,\\
\mathfrak{U}_N^{\varphi} a_p^* a_q (\mathfrak{U}_N^{\varphi})^* &= a_p^* a_q
\end{aligned}$$

as identities on $\mathcal{F}_{\perp\varphi}^{\leq N}$, where we identified $\psi \in \mathfrak{H}^N$ with the Fock vector $(0, \dots 0, \psi, 0 \dots)$ to make sense of the action of creation and annihilation operator. Hence, all operators a_0^*, a_0 are replaced by a factor $\sqrt{N-\mathcal{N}}$ each, corresponding to the number of particles in the condensed state. Using these relations, one conjugates the Hamiltonian (1.106) with $\mathfrak{U}_N^{\varphi}$, which leads to an excitation Hamiltonian $\mathcal{L}^{\beta}$ acting on the excitation Fock space (see, e.g., [24, Eqn. (3.3)]). The constant and quadratic term (with respect to the number of creation and annihilation operators) of $\mathcal{L}^{\beta}$ correspond to leading order to $\mathbb{H}_{\mathrm{Bog}}^{\beta}$ from (1.109). The sub-leading order contributions are different: in $\mathbb{H}_{\mathrm{Bog}}^{\beta}$, one takes the number operator $\mathcal{N}$ of the excitations to be zero, while $\mathcal{N}$ is explicitly taken into account in $\mathcal{L}^{\beta}$. Besides, $\mathcal{L}^{\beta}$ contains a cubic and a quartic term, which can be shown to be small for $\beta = 0$.

For further related results in the Hartree scaling regime, we refer to the proceedings [112] by Lewin and the references contained therein, as well as to the series of works by Pizzo [154, 155].

This analysis for the mean-field regime was extended to singular interactions $\kappa v_{N,\beta}$ with $\beta \in (0, 1)$ and sufficiently small κ by Boccato, Brennecke, Cenatiempo and Schlein in [24][11]. They consider a homogeneous Bose gas and prove that the N-body ground state energy as well as the lower excitation spectrum can be calculated by the Bogoliubov approximation. For $\beta > 0$, the interaction $v_{N,\beta}(x)$ converges to $\|v\|_{L^1(\mathbb{R}^3)}\delta(x)$ in the sense of distributions as $N \to \infty$, hence $\widehat{v}(p) \approx \widehat{v}(0)$ for sufficiently large N. In conclusion, the formulas (1.116) and (1.118) contain the first order Born approximation to the scattering length of $v_{N,\beta}$. Since this approximation becomes less accurate

[11]Following the strategy developed in [26], this constraint on κ can be removed [41, p. 6].

for larger β, it needs to be replaced by a higher order approximation for larger values of β. Essentially, the authors of [24] show that (1.116) and (1.118) correctly describe the N-body ground state and excitation spectrum if $\widehat{v}(0)$ is replaced by a suitably truncated Born series expansion. More precisely, they prove that

$$\begin{aligned} E^{\beta}_{\mathrm{Bog}} &= 4\pi(N-1)a^{\beta}_{N} - \frac{1}{2}\sum_{p\in\Lambda_+^*}\left(|p|^2+\kappa\widehat{v}(0)-e^{\beta}(p)-\frac{\kappa^2\widehat{v}(0)^2}{2|p|^2}\right)+\mathcal{O}(N^{-\alpha})\,,\\ e^{\beta}(p) &= \sqrt{|p|^4+2|p|^2\kappa\widehat{v}(0)} \end{aligned}$$

for all $0<\alpha<\beta$ such that $\alpha\leq\frac{1-\beta}{2}$. Here, $8\pi a^{\beta}_{N}$ denotes N times the Born expansion for the scattering length of the potential $\kappa v_{N,\beta}$, which is truncated at order $k>\frac{1}{1-\beta}$ ([24, Theorem 1.1]).

The appearance of the higher order terms in the Born series is related to the fact that the map $\mathfrak{U}^{\varphi}_{N}$ factors out the condensate $\varphi^{\otimes N}$. However, the interactions between the particles induce a short-scale correlation structure in the sense of (1.20), which cannot be neglected for larger scaling parameters β. Mathematically, the authors of [24] deal with this by conjugating the excitation Hamiltonian $\mathcal{L}^{\beta}$ by a generalised Bogoliubov transformation

$$T=\exp\left(\tfrac{1}{2}\sum_{p\in\Lambda^*}(\eta_p\,b^*_p b^*_{-p}-\mathrm{h.c.})\right), \tag{1.124}$$

where the coefficients η_p are related to the Fourier transform of the zero-energy scattering solution from Lemma 1.2.4 (see [24, Section 3]). The operators b_p, b_{-p} are modified creation/annihilation operators, which are defined as

$$b_p:=\sqrt{\frac{N-\mathcal{N}}{N}}a_p\,,\qquad b^*_p:=a^*_p\sqrt{\frac{N-\mathcal{N}}{N}}\,,$$

which have the advantage that T leaves the space $\mathcal{F}^{\leq N}_{\perp\varphi}$ invariant.

For interactions in the GP regime, the Born approximation is invalid since all terms in the expansion are of the same order, which implies that the complete Born series must be taken into account. As a consequence, the leading order term of the Bogoliubov ground state energy (1.116) contains the full scattering length of the interaction, which is consistent with the leading order result (1.19). This was made rigorous by Boccato, Brennecke, Cenatiempo and Schlein in [25, 27], who extended their analysis [24] for the NLS regime to the GP scaling of the interaction.

Heuristically, the standard formal argument to derive the Bogoliubov energy for the GP scaling is by considering the effective Hamiltonian H^{eff} as in (1.33), which is constructed by applying the first order Born approximation to a softer potential U^{eff} with the same scattering length. This results in the effective δ-interaction (1.32), which contains the scattering length and thereby takes the correlation structure into account (see Section 1.2.5). Applying the Bogolibubov approximation to H^{eff} yields the Bogoliubov energy with the full scattering length (see e.g. [145, Chapter 8.1]).

1.5.3. Dynamics of the excitations and norm approximation

The rigorous results (1.73) presented in Section 1.4.3 provide an approximation of the N-body dynamics $\psi^N(t)$ in the sense of reduced densities. This approximation corresponds to the control of the majority of all particles, which, up to a relative number that vanishes as $N \to \infty$, occupy the time evolved condensate wave function. A much stronger notion of distance is provided by the $L^2(\mathbb{R}^{3N})$-norm, which requires the control of all N particles. In particular, this implies that the excitations from the condensate can no longer be omitted from the description. In this sense, the norm approximation of $\psi^N(t)$ can be understood as next-to-leading order correction to the description with respect to reduced densities.

In the Fock space setting, i.e., for initial states $\psi^N(0)$ that are no N-body states but belong to an appropriate class of Fock space initial data, a norm approximation was first obtained by Grillakis, Machedon and Margetis in [88, 89], and further results were proven in [49, 77, 78, 86, 87, 108, 158]. To rigorously derive norm approximations for initial N-body states $\psi^N(0) \in L^2_+(\mathbb{R}^{3N})$, two ways are known in the mathematical physics literature:

- One can decompose $\psi^N(t)$ via (1.119) into a (time dependent) condensate $\varphi(t)$ and orthogonal excitations. One then shows that the Fock space time evolution of the excitations is generated by the Bogoliubov Hamiltonian, while the evolution of the condensate is determined by the respective effective equation.

- Alternatively, one can define a first quantised analogue of the Bogoliubov Hamiltonian on the N-body Hilbert space and prove that its time evolution, which describes both condensate and excitations, approximates the full N-body dynamics $\psi^N(t)$.

Let us now consider the dynamics $\psi^N(t)$ generated by the Hamiltonian

$$H_{N,\beta} = \sum_{j=1}^{N} \left(-\Delta_j + V^{\text{ext}}(x_j)\right) + \frac{1}{N-1} \sum_{1 \le i < j \le N} v_{N,\beta}(x_i - x_j) \tag{1.125}$$

with interactions as in (1.108). In the following, we briefly review and compare both approaches.

Second quantised approach

The first norm approximation of the N-body dynamics $\psi^N(t)$ was obtained by Lewin, Nam and Schlein in [113] for $\beta = 0$ and $V^{\text{ext}} = 0$ for interaction potentials v satisfying the operator inequality $v^2 \lesssim (1 - \Delta)$ on $L^2(\mathbb{R}^d)$ in dimension $d \geq 1$. For initial data of the form

$$\psi_0^N = \sum_{k=0}^{N} \varphi_0{}^{\otimes(N-k)} \otimes_s \chi_0^{(k)} , \tag{1.126}$$

where the initial excitation vector $\chi_0 \in \mathcal{F}_{\perp\varphi_0}$ is assumed to be normalised and such that

$$\langle \chi_0, \mathrm{d}\Gamma(\mathbb{1} - \Delta)\chi_0 \rangle_{\mathcal{F}_{\perp\varphi_0}} < \infty ,$$

they prove that

$$\lim_{N\to\infty} \left\| \psi^N(t) - \sum_{k=0}^{N} \varphi(t)^{\otimes(N-k)} \otimes_s \chi^{(k)}(t) \right\|_{L^2(\mathbb{R}^{dN})} = 0$$

for all times $t \geq 0$. The time-evolved excitation vector $\chi(t)$ solves the Bogoliubov equation, which will be explained below. The optimal rate of convergence is expected to be of order $N^{-\frac{1}{2}}$ for every fixed $t \in \mathbb{R}$ (see [113, Remark 3]).

For $V^{\text{ext}} = 0$, this result was extended in a series of works [137, 138, 139] by Nam and Napiórkowski. They consider initial data of the form (1.126) for appropriate initial excitation vectors $\chi_0 \in \mathcal{F}_{\perp\varphi_0}$ and show that that there exists a parameter $\delta > 0$ and a function $f : \mathbb{R} \to \mathbb{R}^+$ such that

$$\left\| \psi(t) - \sum_{k=0}^{N} \varphi(t)^{\otimes(N-k)} \otimes_s \chi^{(k)}(t) \right\|^2_{L^2(\mathbb{R}^{dN})} \leq f(t) N^{-\delta} , \tag{1.127}$$

where δ and f depend on the particular situation:

- The work [137] concerns 3d bosons with non-negative interaction potentials scaling with $\beta \in [0, \frac{1}{3})$. The initial excitation vector is assumed to be quasi-free (see below). In this case, (1.127) holds with the parameter $\delta = 1 - 3\beta$ and with $f(t) = \mathrm{e}^{Ct}(1 + \langle \chi_0, \mathcal{N}_{\varphi_0} \chi_0 \rangle)^4$ for some constant $C > 0$ depending only on $\|\varphi_0\|_{H^2(\mathbb{R}^3)}$.
- In [139], this analysis for the 3d defocusing case is extended to the scaling regime $\beta \in [0, \frac{1}{2})$. The initial excitation vector is required to be quasi-free and to satisfy

$$\langle \chi_0, \mathcal{N}_{\varphi_0} \chi_0 \rangle \leq \kappa_\varepsilon N^\varepsilon , \qquad \langle \chi_0, \mathrm{d}\Gamma(\mathbb{1} - \Delta)\chi_0 \rangle \leq \kappa_\varepsilon N^{\beta+\varepsilon} \tag{1.128}$$

  for all $\varepsilon > 0$, where $\kappa_\varepsilon > 0$ is independent of N. The authors derive (1.127) for all $\varepsilon > 0$ with $\delta = (1-2\beta-\varepsilon)/2$ and $f(t) = C_\varepsilon(1+t)^{1+\varepsilon}$, where the constant $C_\varepsilon > 0$ depends only on κ_0 and ε.

- Finally, in [138], the authors consider the focusing case with non-positive interactions in dimensions $d = 1$ for $\beta > 0$ and $d = 2$ for $\beta \in (0,1)$. The initial data are assumed such that
$$\langle \chi_0, \mathrm{d}\Gamma(\mathbb{1} - \Delta)\chi_0\rangle \leq C$$
for some constant $C > 0$. The resulting parameter δ is given as $\delta = \frac{1}{2}$ for the 1d and $0 < \delta < \frac{1}{3}(1-\beta)$ for the 2d case, while the explicit form of $f(t)$ is not specified.

The excitations $\chi(t) = (\chi^{(k)}(t))_{k=0}^{\infty} \in \mathcal{F}_{\perp\varphi(t)}$ contained in the approximating wave function are determined by the Bogoliubov evolution,

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\chi(t) = \mathcal{H}^{\beta}_{\mathrm{Bog}}(t)\chi(t)\,, \tag{1.129}$$

with $\mathcal{H}^{\beta}_{\mathrm{Bog}}(t)$ as in (1.110). Written explicitly, (1.129) equals the coupled equations for the components $\chi^{(k)}(t)$

$$\begin{aligned}
&\mathrm{i}\partial_t\chi^{(k)}(t, x_1, ..., x_k)\\
&= \sum_{j=1}^{k}\Big(h^{\varphi(t)}(x_j) + K_1^{\varphi(t)}(x_j)\Big)\,\chi^{(k)}(t, x_1, ..., x_k)\\
&\quad + \frac{1}{2}\frac{1}{\sqrt{k(k-1)}}\sum_{1\leq i<j\leq k} K_2^{\varphi(t)}(x_i, x_j)\chi^{(k-2)}(t, x_1, ..., x_k\setminus x_i\setminus x_j)\\
&\quad + \frac{1}{2}\sqrt{(k+1)(k+2)}\int \mathrm{d}x\,\mathrm{d}y\,\overline{K_2^{\varphi(t)}(x,y)}\chi^{(k+2)}(t, x_1, ..., x_k, x, y)
\end{aligned} \tag{1.130}$$

for $k \geq 0$, with $K_1^{\varphi(t)}$, $K_2^{\varphi(t)}$ and $h^{\varphi(t)}$ as defined in (1.111) and (1.112). Note that the time dependence of $\mathcal{H}^{\beta}_{\mathrm{Bog}}(t)$ is due to the time dependence of the condensate wave function $\varphi(t)$. As a consequence, the vacuum of the excitation Fock space varies in time, and, moreover, the operators $K_1^{\varphi(t)}$ and $K_2^{\varphi(t)}$ from (1.111) are time dependent via $\varphi(t)$.

Let us now recall the notion of quasi-free states and comment on their relevance in the context of the Bogoliubov time evolution. This part is taken from [137, Lemma 8] and [173, Theorem 10.4]. A more thorough discussion is given, e.g., in the lecture notes [173] by Solovej.

Definition 1.5.1. *Let $\phi \in \mathcal{F}$ be a normalised vector in a bosonic Fock space $\mathcal{F}$ over a Hilbert space $\mathfrak{H}$ such that*
$$\langle \phi, \mathcal{N}\phi\rangle < \infty\,.$$

Define the generalised one-particle density matrix of ϕ as

$$\Gamma_\phi := \begin{pmatrix} \gamma_\phi & \alpha_\phi \\ \alpha_\phi^* & \mathbb{1} + \gamma_\phi^T \end{pmatrix}$$

on $\mathfrak{H} \oplus \overline{\mathfrak{H}}$, where $\overline{\mathfrak{H}}$ denotes the dual of $\mathfrak{H}$. The one-body density matrices $\gamma_\phi : \mathfrak{H} \to \mathfrak{H}$ and $\alpha_\phi : \overline{\mathfrak{H}} \to \mathfrak{H}$ are defined as

$$\langle f, \gamma_\Phi g\rangle_{\mathfrak{H}} = \langle \phi, a^*(g)a(f)\phi\rangle_{\mathcal{F}} , \qquad \langle f, \alpha_\phi\, \overline{g}\rangle_{\mathfrak{H}} = \langle \phi, a(g)a(f)\phi\rangle$$

for $f, g \in \mathfrak{H}$. Then it holds that

$$\gamma_\phi \geq 0 , \qquad \mathrm{Tr}(\gamma_\phi) = \langle \phi, \mathcal{N}\phi\rangle , \qquad \alpha_\phi = \alpha_\phi^T , \qquad \Gamma_\phi \geq 0 .$$

The state ϕ is called quasi-free if and only if

$$\gamma_\phi \alpha_\phi = \alpha_\phi \gamma_\phi^T , \qquad \alpha_\phi \alpha_\phi^* = \gamma_\phi(\mathbb{1} + \gamma_\phi) ,$$

or, equivalently, if and only if

$$\Gamma_\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma_\phi = -\Gamma_\phi .$$

Note that for $\phi \in \mathfrak{H}^N$, i.e., for ϕ with fixed particle number N, γ_ϕ is the usual reduced one-particle density matrix (1.8). Quasi-free states satisfy the so-called Wick property: for $a^\sharp \in \{a^*, a\}$, $n \geq 1$ and $f_1, ..., f_{2n} \in \mathfrak{H}$,

$$\begin{cases} \left\langle \phi, a^\sharp(f_1)a^\sharp(f_2)\cdots a^\sharp(f_{2n-1})\phi \right\rangle_{\mathcal{F}} = 0 , \\ \left\langle \phi, a^\sharp(f_1)a^\sharp(f_2)\cdots a^\sharp(f_{2n})\phi \right\rangle_{\mathcal{F}} = \sum\limits_{\sigma \in P_{2n}} \prod\limits_{j=1}^{n} \left\langle \phi, a^\sharp(f_{\sigma(2j-1)})a^\sharp(f_{\sigma(2j)})\phi \right\rangle_{\mathcal{F}} , \end{cases} \tag{1.131}$$

where P_{2n} denotes the set of pairings

$$P_{2n} = \{\sigma \in \mathfrak{S}_{2n} \,:\, \sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\} \text{ for all } j\} .$$

Hence, all expectation values with respect to a quasi-free state ϕ can be computed from the mere knowledge of its one-body densities $(\gamma_\phi, \alpha_\phi)$. Moreover, finite moments of the number operator are determined by its expectation value: for all $\ell \geq 1$, there exists a constant $C_\ell > 0$ such that

$$\langle \phi, \mathcal{N}^\ell \phi\rangle_{\mathcal{F}} \leq C_\ell \left(1 + \langle \phi, \mathcal{N}\phi\rangle\right)^\ell \tag{1.132}$$

for all quasi-free states ϕ in $\mathcal{F}$ (see e.g. [137, Lemma 5]). Finally, it is well known that the unique ground state of a Bogoliubov Hamiltonian is quasi-free (see e.g. [114, Theorem A.1]).

To characterise the dynamics of the excitations, it is crucial to note that the time evolution generated by $\mathcal{H}^{\beta}_{\mathrm{Bog}}$ preserves the quasi-free property: if $\chi_0 \in \mathcal{F}_{\perp\varphi_0}$ is a quasi-free state, then the solution $\chi(t) \in \mathcal{F}_{\perp\varphi(t)}$ of (1.129) is quasi-free for all $t \in \mathbb{R}$ and

$$\left\langle \chi(t), \mathcal{N}_{\varphi(t)}\chi(t)\right\rangle \leq \mathrm{e}^{Ct}\left(1 + \langle \chi_0, \mathcal{N}_{\varphi_0}\chi_0\rangle\right)^2 \tag{1.133}$$

for a constant C depending only on $\|\varphi_0\|_{H^2(\mathbb{R}^3)}$ ([137, Proposition 4]). Note that in combination with (1.132), this implies a bound on the growth of finite moments of the number of excitations in the wave function evolving under the Bogoliubov time evolution.

Since the quasi-free property is preserved by the Bogoliubov time evolution, the excitation vector $\chi(t)$ at any time $t \in \mathbb{R}$ is characterised by its one-body densities $(\gamma_{\chi(t)}, \alpha_{\chi(t)})$. As a consequence, it was shown in [137] that the Bogoliubov equation (1.129) is for initial quasi-free states equivalent to the coupled system of equations

$$\begin{cases} \mathrm{i}\partial_t\gamma_{\chi(t)} &= (h^{\varphi(t)} + K_1^{\varphi(t)})\gamma_{\chi(t)}(t) - \gamma_{\chi(t)}(h^{\varphi(t)} + K_1^{\varphi(t)}) \\ &\quad + K_2^{\varphi(t)}\alpha_{\chi(t)} - \alpha^*_{\chi(t)}(K_2^{\varphi(t)})^* \\ \mathrm{i}\partial_t\alpha_{\chi(t)} &= (h^{\varphi(t)} + K_1^{\varphi(t)})\alpha_{\chi(t)} + \alpha_{\chi(t)}(h^{\varphi(t)} + K_1^{\varphi(t)})^T \\ &\quad + K_2^{\varphi(t)} + K_2^{\varphi(t)}\gamma^T_{\chi(t)} + \gamma_{\chi(t)}K_2^{\varphi(t)} \end{cases} \tag{1.134}$$

with initial datum $(\gamma_{\chi_0}, \alpha_{\chi_0})$. A comparable system of equations was derived by Grillakis and Machedon in [86, Eqns. (17a-b)] for the Fock space setting. Note that also for an initial state χ_0 which is not quasi-free, the solution $\chi(t)$ of (1.129) solves (1.134). However, (1.134) is not equivalent to (1.129) since $\chi(t)$ is not quasi-free and consequently not uniquely determined by its one-body densities.

For larger values of the scaling parameter β, the evolutions of $\varphi(t)$ and $\xi_{\varphi(t)}$ do not (approximately) decouple any more as a consequence of the short-scale structure related to the two-body scattering process. In [108], it is argued that in the 3d defocusing problem, this is the case for $\beta \geq \frac{1}{2}$ ([108, Section 2, following (35)]). For the range $\beta \in (0,1)$, an accordingly adjusted variant of (1.127) for appropriately modified initial data was obtained by Brennecke, Nam, Napiórkowski and Schlein in [38] for the 3d defocusing case. Here, the dynamics of the condensate wave function are described by a modified N-dependent Hartree equation with nonlinearity $v_{N,\beta}f_N * |\varphi(t)|^2$, where f_N is related to the zero energy scattering solution, similarly to $f_{\widetilde{\beta}}$ in Definition 1.4.4

(see [38, Eqn. (23)]). For $\beta \in (0,1)$ and $N \to \infty$, this converges to the N-independent NLS equation (1.72). A similar estimate for the many-body evolution of appropriate classes of Fock space initial data for $\beta \in (0,1)$ was obtained by Boccato, Cenatiempo and Schlein in [28].

First quantised approach

An alternative way of decomposing the N-body wave function $\psi^N(t)$ into a condensate $\varphi(t)^{\otimes N}$ and orthogonal excitations is by means of the projections $p^{\varphi(t)}$ and $q^{\varphi(t)}$ onto $\varphi(t)$ and its orthogonal complement (Definition 1.4.1). In terms of the many-body projection operators $P_k^{\varphi(t)}$ on $L^2(\mathbb{R}^{3N})$,

$$P_k^{\varphi} = \frac{1}{(N-k)!k!} \sum_{\sigma \in \mathfrak{S}_N} q^{\varphi}_{\sigma(1)} \cdots q^{\varphi}_{\sigma(k)} p^{\varphi}_{\sigma(k+1)} \cdots p^{\varphi}_{\sigma(N)} \,,$$

the part of $\psi^N(t)$ in the condensate is given by $P_0^{\varphi(t)}\psi^N(t)$. The part of $\psi^N(t)$ corresponding to k-particle excitations equals $P_k^{\varphi(t)}\psi^N(t)$ for $k \geq 1$. By construction, $P_k^{\varphi(t)} P_{k'}^{\varphi(t)} = \delta_{k,k'} P_k^{\varphi(t)}$, and the identity $\sum_{k=0}^N P_k^{\varphi(t)} = \mathbb{1}$ implies the decomposition

$$\psi^N(t) \;=\; \sum_{k=0}^{N} P_k^{\varphi(t)} \psi^N(t) \,. \tag{1.135}$$

Note that as opposed to the decomposition (1.119), $P_k^{\varphi(t)}\psi^N$ is an N-body wave function, i.e., it contains both the condensate and the excitation part.

In [135, 134], Mitrouskas, Petrat and Pickl introduced an effective Hamiltonian $\widetilde{H}^{\varphi(t)}$, which is constructed as follows:

- First, one adds and subtracts from $H_{N,\beta}$ in each coordinate the mean-field Hamiltonian $h^{\varphi(t)}$ from (1.112), resulting in

$$\begin{aligned} H_{N,\beta} \;&=\; \sum_{j=1}^{N} h_j^{\varphi(t)} + \frac{1}{N-1} \sum_{i<j} v_{N,\beta}^{(ij)} - \sum_{j=1}^{N} \overline{v}^{\varphi(t)}(x_j) + N\mu^{\varphi(t)} \\ &=\; \sum_{j=1}^{N} h_j^{\varphi(t)} + \frac{1}{N-1} \sum_{i<j} \Big(v_{N,\beta}^{(ij)} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)} \Big) \,. \end{aligned}$$

  Here, we used the notation $\overline{v}^{\varphi}$ as in (1.113) and abbreviated $v_{N,\beta}^{(ij)} := v_{N,\beta}(x_i - x_j)$.

- Second, inserting identities

$$\mathbb{1} = (p_i^{\varphi(t)} + q_i^{\varphi(t)})(p_j^{\varphi(t)} + q_j^{\varphi(t)})$$

before and after the expression in the brackets yields with the relations

$$p_i^{\varphi(t)} v_\beta^{(ij)} p_i^{\varphi(t)} = \overline{v}^{\varphi(t)}(x_j) p_i^{\varphi(t)}, \qquad p_i^{\varphi(t)} \overline{v}^{\varphi(t)}(x_i) p_i^{\varphi(t)} = 2\mu^{\varphi(t)} p_i^{\varphi(t)} \tag{1.136}$$

the decomposition

$$H_{N,\beta} = \widetilde{H}^{\varphi(t)} + \mathcal{C}^{\varphi(t)} + \mathcal{Q}^{\varphi(t)} ,$$

where

$$\begin{aligned} \widetilde{H}^{\varphi(t)} &:= \sum_{j=1}^{N} h_j^{\varphi(t)} + \frac{1}{N-1} \sum_{i<j} \Big(p_i^{\varphi(t)} q_j^{\varphi(t)} v_\beta^{(ij)} q_i^{\varphi(t)} p_j^{\varphi(t)} \\ &\qquad + p_i^{\varphi(t)} p_j^{\varphi(t)} v_{N,\beta}^{(ij)} q_i^{\varphi(t)} q_j^{\varphi(t)} + \text{h.c.} \Big) , \qquad (1.137) \\ \mathcal{C}^{\varphi(t)} &:= \frac{1}{N-1} \sum_{i<j} \Big(q_i^{\varphi(t)} q_j^{\varphi(t)} \Big(v_{N,\beta}^{(ij)} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) \Big) \times \\ &\qquad \times (q_i^{\varphi(t)} p_j^{\varphi(t)} + p_i^{\varphi(t)} q_j^{\varphi(t)}) + \text{h.c.} \Big) , \qquad (1.138) \\ \mathcal{Q}^{\varphi(t)} &:= \frac{1}{N-1} \sum_{i<j} q_i^{\varphi(t)} q_j^{\varphi(t)} \times \\ &\qquad \times \Big(v_{N,\beta}^{(ij)} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)} \Big) q_i^{\varphi(t)} q_j^{\varphi(t)} . \qquad (1.139) \end{aligned}$$

- Finally, discarding all terms from $H_{N,\beta}$ which are cubic ($\mathcal{C}^{\varphi(t)}$) or quartic ($\mathcal{Q}^{\varphi(t)}$) in the number of projections $q^{\varphi(t)}$ yields the effective Hamiltonian $\widetilde{H}^{\varphi(t)}$.

The resulting Hamiltonian $\widetilde{H}^{\varphi(t)}$ has a quadratic structure comparable to $\mathcal{H}_{\text{Bog}}^{\beta}(t)$: all terms in $H_{N,\beta} - \sum_j h_j^{\varphi(t)}$, which form an effective two-body potential, contain exactly two projectors $q^{\varphi(t)}$ onto the complement of the condensate wave function, while $\mathcal{H}_{\text{Bog}}^{\beta}(t)$ is quadratic in the creation and annihilation operators of the excitations.

The Hamiltonian $\widetilde{H}^{\varphi(t)}$ is particle number conserving and acts on the N-body Hilbert space $L^2(\mathbb{R}^{3N})$, i.e., it determines the evolution of both condensate wave function and excitations. In contrast, $\mathcal{H}_{\text{Bog}}^{\beta}(t)$ operates on the excitation Fock space $\mathcal{F}_{\perp\varphi(t)}$ and does not conserve the particle number. It exclusively concerns the dynamics of the excitations with respect to the condensate wave function, which, in turn, evolves according to the non-linear dynamics generated by (1.112).

Making use of $\widetilde{H}^{\varphi(t)}$, Mitrouskas, Petrat and Pickl derive in [135] a norm approximation for the N-body dynamics $\psi^N(t)$. They consider 3d bosons in the Hartree regime $\beta = 0$ for interactions v satisfying the operator inequality $v^2 \lesssim (1 - \Delta)$. The initial N-body state ψ_0^N is assumed such that

(a) $|E^{\psi_0^N} - \mathcal{E}^{\varphi_0}| \lesssim N^{-1}$, where $E^{\psi} := \frac{1}{N} \langle \psi, H_{N,\beta}\psi \rangle_{L^2(\mathbb{R}^{3N})}$ is the energy per particle and $\mathcal{E}^\varphi := \langle \varphi, h^\varphi \varphi \rangle_{L^2(\mathbb{R}^3)}$ denotes the Hartree energy,

(b) $\left\langle\!\!\left\langle \psi_0^N, \Big(\prod\limits_{j=1}^n q_j^{\varphi_0} \Big) \psi_0^N \right\rangle\!\!\right\rangle \lesssim N^{-n}$ for $n = 1, 2, 3$,

(c) $\| P_{\mathrm{odd}}^{\varphi_0} \psi_0^N \| \lesssim N^{-\frac{1}{2}}$, where $P_{\mathrm{odd}}^{\varphi} := \sum\limits_{k \text{ odd}} P_k^\varphi$ projects onto the subspace of $L^2(\mathbb{R}^{3N})$ with an odd number of particles outside the condensate.

For initial states satisfying (a) to (c), the authors prove that the time evolution $\widetilde{U}_\varphi(t, s)$ generated by $\widetilde{H}^{\varphi(t)}$ approximates the N-body dynamics. More precisely, they show that there exists a constant $C > 0$ such that

$$\left\| \psi^N(t) - \widetilde{U}_\varphi(t, 0) \psi_0^N \right\|^2 \leq \mathrm{e}^{C(1+t)^2} N^{-1} \tag{1.140}$$

[135, Theorem 2.1]. A related result for Bose gases with large volume and large density was proved by Petrat, Pickl and Soffer in [146, Theorem 1.2].

Comparison of both approaches

Let us first compare the decompositions (1.119) and (1.135) of an N-body wave function ψ^N. Note that by definition of the projectors P_k^φ (Definition 1.4.1),

$$\begin{aligned} &P_k^\varphi \psi^N(x_1, \ldots, x_N) \\ &\quad= \frac{1}{(N-k)!k!} \sum_{\sigma \in \mathfrak{S}_N} \varphi(x_{\sigma(k+1)}) \cdots \varphi(x_{\sigma(N)})\, q_{\sigma(1)}^\varphi \cdots q_{\sigma(k)}^\varphi \times \\ &\qquad \times \int\limits_{\mathbb{R}^3} \mathrm{d}y_1 \cdots \int\limits_{\mathbb{R}^3} \mathrm{d}y_{N-k}\, \overline{\varphi(y_1)} \cdots \overline{\varphi(y_{N-k})}\, \psi^N(x_{\sigma(1)}, \ldots, x_{\sigma(k)}, y_1, \ldots, y_{N-k}) \\ &\quad=: \left(\varphi^{\otimes(N-k)} \otimes_s \xi_\varphi^{(k)} \right)(x_1, \ldots, x_N)\,, \end{aligned}$$

where, by definition (1.121) of the symmetric tensor product,

$$\begin{aligned} \xi_\varphi^{(k)}(x_1, \ldots, x_k) := \sqrt{\tbinom{N}{k}}\, q_1^\varphi \cdots q_k^\varphi \int\limits_{\mathbb{R}^3} \mathrm{d}y_{k+1} \cdots \int\limits_{\mathbb{R}^3} \mathrm{d}y_N\, \overline{\varphi(y_{k+1})} \cdots \times \\ \times \overline{\varphi(y_N)}\, \psi^N(x_1, \ldots, x_k, y_{k+1}, \ldots, y_N)\,. \end{aligned} \tag{1.141}$$

Obviously, $\xi_\varphi^{(k)}$ is symmetric under permutations of its arguments, and $\xi_\varphi^{(k)}$ is orthogonal to φ in every coordinate, i.e.,

$$\int_{\mathbb{R}^3} \overline{\varphi(x_j)}\, \xi_\varphi^{(k)}(x_1, \ldots, x_j, \ldots, x_k)\, \mathrm{d}x_j = 0\,, \qquad p_j^\varphi \xi_\varphi^{(k)} = 0\,, \qquad q_j^\varphi \xi_\varphi^{(k)} = \xi_\varphi^{(k)}$$

for every $j \in \{1,\dots,k\}$. Hence, $\xi_\varphi := (\xi_\varphi^{(k)})_{k=0}^N \in \mathcal{F}_{\perp\varphi}^{\leq N}$, and (1.141) determines precisely the elements of the excitation Fock vector $\mathfrak{U}_N^\varphi \psi^N$ for $\mathfrak{U}_N^\varphi$ from (1.122). In fact, (1.141) can be understood as the translation of (1.123) into the first quantised language.

Consequently, the probability of finding k particles in ψ^N outside the condensate $\varphi^{\otimes N}$ is given equivalently by

$$\|\xi_\varphi^{(k)}\|^2_{L^2(\mathbb{R}^{3k})} = \tbinom{N}{k}\|q_1^\varphi \cdots q_k^\varphi p_{k+1}^\varphi \cdots p_N^\varphi \psi^N\|^2 = \|P_k^\varphi \psi^N\|^2 .$$

The expected number of excitations from $\varphi^{\otimes N}$ in the state ψ^N is

$$\begin{aligned}\langle \xi_\varphi, \mathcal{N}_\varphi\, \xi_\varphi\rangle_{\mathcal{F}_{\perp\varphi}^{\leq N}} &= \sum_{k=0}^N k\|\xi_\varphi^{(k)}\|^2_{L^2(\mathbb{R}^{3k})} = \sum_{k=0}^N k\|P_k^\varphi \psi^N\|^2 \\ &= N\left\langle\!\!\left\langle \psi^N, \sum_{k=0}^N \tfrac{k}{N} P_k^\varphi \psi^N\right\rangle\!\!\right\rangle = N\|\widehat{n^\varphi}\psi^N\|^2 ,\end{aligned}$$

where $\widehat{n^\varphi}$ denotes the weighted operator from Definition 1.4.1 with weight function

$$n(k) = \sqrt{\frac{k}{N}} .$$

For $a \in \mathbb{N}$, the a'th moment of the number of excitations is given as

$$\left\langle \xi_\varphi, \mathcal{N}_\varphi^a\, \xi_\varphi\right\rangle_{\mathcal{F}_{\perp\varphi}^{\leq N}} = N^a \left\langle\!\!\left\langle \psi^N, \sum_{k=0}^N (\tfrac{k}{N})^a P_k^\varphi \psi^N\right\rangle\!\!\right\rangle = N^a\|(\widehat{n^\varphi})^a\psi^N\|^2 . \tag{1.142}$$

As a consequence, assumption (b) by Mitrouskas, Petrat and Pickl can equivalently be expressed as the requirement that the first three moments of the initial number of excitations be bounded uniformly in N.

In [135], the authors prove that the excitations in $\widetilde{U}_\varphi(t,0)\psi_0^N$ asymptotically coincide with the solutions of the Bogoliubov evolution equation (1.129) as $N \to \infty$. More precisely,

- let $\xi_{\varphi_0} = \big(\xi_{\varphi_0}^{(k)}\big)_{k=0}^N = \mathfrak{U}_N^{\varphi_0}\psi_0^N$ denote the excitations from ${\varphi_0}^{\otimes N}$ in the initial state ψ_0^N,
- let $\widetilde{\xi}_{\varphi(t)} = \big(\widetilde{\xi}_{\varphi(t)}^{(k)}\big)_{k=0}^N = \mathfrak{U}_N^{\varphi(t)}\widetilde{U}_\varphi(t,0)\psi_0^N$ denote the excitations from $\varphi(t)^{\otimes N}$ in the time evolved state $\widetilde{U}_\varphi(t,0)\psi_0^N$,
- let $\chi(t) = \big(\chi^{(k)}(t)\big)_{k\geq 0}$ denote the solutions of the Bogoliubov equation (1.129) with initial datum $(\chi(0))_{k=0}^N = \xi_{\varphi_0}$.

Then there exists a constant $C > 0$ such that

$$\sum_{k=0}^{N} \left\| \widetilde{\xi}^{(k)}_{\varphi(t)} - \chi^{(k)}(t) \right\|^2_{L^2(\mathbb{R}^{3k})} \leq \mathrm{e}^{C(1+t)^2} N^{-1} \tag{1.143}$$

[135, Lemma 2.3]. Following the lines of the proof of [146, Theorem 2.2] for Bose gases with large volume, this bound can presumably be improved to an error of order N^{-2}.

To prove (1.143), the authors of [135, 146] use (1.141) to extract the excitations

$$\widetilde{\xi}_{\varphi(t)} = \mathfrak{U}_N^{\varphi(t)} \widetilde{U}_{\varphi}(t,0)\psi_0^N$$

from the solution $\widetilde{U}_{\varphi}(t,0)\psi_0^N =: \widetilde{\psi}^N(t)$ of the N-body Schrödinger equation $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\psi}^N(t) = \widetilde{H}^{\varphi(t)}\widetilde{\psi}^N(t)$. This leads to the system of coupled equations

$$\begin{aligned}
&\mathrm{i}\partial_t \widetilde{\xi}^{(k)}_{\varphi(t)}(x_1,\ldots,x_k) \\
&= \sum_{j=1}^{k} \left(h_j^{\varphi(t)} + C_k^{(1)} K_1^{\varphi(t)}(x_j) \right) \widetilde{\xi}^{(k)}_{\varphi(t)}(x_1,\ldots,x_k) \\
&\quad + \frac{1}{2} C_{k-2}^{(2)} \frac{1}{\sqrt{k(k-1)}} \sum_{1 \leq i < j \leq k} K_2^{\varphi(t)}(x_i,x_j) \widetilde{\xi}^{(k-2)}_{\varphi(t)}(x_1,\ldots,x_k \setminus x_i \setminus x_j) \\
&\quad + \frac{1}{2} C_k^{(2)} \sqrt{(k+1)(k+2)} \int \mathrm{d}x\,\mathrm{d}y \overline{K_2^{\varphi(t)}(x,y)} \widetilde{\xi}^{(k+2)}_{\varphi(t)}(x_1,\ldots,x_k,x,y)
\end{aligned} \tag{1.144}$$

for $0 \leq k \leq N$, where

$$C_k^{(1)} = \frac{N-k}{N}, \qquad C_k^{(2)} = \frac{\sqrt{(N-k)(N-k-1)}}{N}.$$

For $k = 0$, the first two lines are defined as zero, and for $k = 1$ the second line equals zero by definition. Since $C_N^{(2)} = C_{N-1}^{(2)} = 0$, the third line does not contribute for $k \in \{N-1, N\}$.

A comparison of this hierarchy (1.144) with the Bogoliubov hierarchy (1.130) reveals two differences:

- In (1.144), additional combinatorial factors $C_k^{(1)}$ and $C_k^{(2)}$ appear. Note that these factors are approximately given by $1 + \frac{k}{N}$. Hence, for $k = \mathcal{O}(1)$, these factors are asymptotically one.

- The equations (1.144) and (1.130) do not coincide for $k \geq N-1$. By construction, $\widetilde{\xi}^{(k)}_{\varphi(t)} = 0$ for $k > N$, whereas the components $\chi^{(k)}(t)$ of the solution of the Bogoliubov solution do not necessarily vanish for $k > N$.

Since $\|\chi^{(k)}(t)\|_{L^2(\mathbb{R}^{3k})}$ is very small for k of order N, the error terms from both sources can be controlled sufficiently well to prove the statement (1.143).

1.6. Experiments

In this section, we briefly account for the experimental perspective on BEC, mainly based on [23], [101], [111, Chapter II], [145, Chapter 1] and [153, Chapters 1 and 9]. Subsequently, we collect some experimental results concerning quasi-low-dimensional Bose gases.

BEC in dilute atomic gases was first realised in 1995 with rubidium [10] at Boulder and with sodium [52] at MIT, for which Cornell, Ketterle and Wieman were awarded the 2001 Nobel Prize in Physics. Moreover, also in 1995, first evidence of BEC was found in lithium [37, 36]. Since then, many dilute atomic gases have been confirmed to exhibit BEC, such as ^{1}H, ^{7}Li, ^{23}Na, ^{39}K, ^{41}K, ^{52}Cr, ^{85}Rb, ^{87}Rb, ^{133}Cs, ^{170}Yb and ^{174}Yb and superfluid ^{4}He.

To create a BEC, one needs to cool the gas until the de Broglie wavelength of the atoms is comparable to their average separation. However, the thermodynamic equilibrium at the given conditions of temperature and pressure usually corresponds to a crystal. To observe BEC, one must prevent the gas from solidifying during this cooling process, which is possible for extremely dilute gases: At low temperatures, the decay of the gas phase is mainly due to three-body recombinations, which lead to the formation of molecules. If the density of the gas is sufficiently low, three-body collisions occur only very rarely, and one can observe a metastable gaseous phase that lasts several seconds to minutes. Typically, the particle density required for BEC is 10^{13}–10^{15}cm^{-3}, which is by several orders of magnitude smaller than the density of molecules in air at room temperature and atmospheric pressure, $\sim 10^{19}$cm^{-3}.

Due to this extreme dilution and the corresponding large inter-particle distances, one requires temperatures of order 10^{-5}K or less[12] to observe BEC. To reach such low temperatures in alkali atoms with sufficiently many atoms remaining in the cloud to be observed, one combines different cooling and trapping methods [50, 101]:

- First, the gas is pre-cooled by so-called laser cooling in a magneto-optical trap, where three pairs of counter-propagating laser beams along the three axes are tuned below the atomic resonance frequency (i.e., the wave length is red-shifted with respect to the resonance wave length). Due to the Doppler effect, an atom moving in the opposite direction as a laser beam blue-shifts the incoming photons closer to the resonance, while a co-propagating atom red-shifts the light away from the resonance. Hence, on average, each atom absorbs more photons

[12]These values are taken from [145, Chapter 1].

opposing its motion. Since the emitted photons have no preferred direction, this leads to a net decelerating force.

- Subsequently, one applies so-called evaporative cooling. By reducing the depth of the trap, one removes the more energetic atoms, which carry more than the average energy. As a consequence, the remaining atoms thermalise at a lower temperature.

The duration of this cooling and trapping cycle varies between some seconds and some minutes, and the resulting condensates usually contain between 10^2 and 10^9 atoms[13]. A detailed explanation of different trapping and cooling techniques is, for example, given in [145, Chapter 4] and [72, Chapters 9–10].

By creating and observing BECs, a wide range of physical phenomena has been explored over the last two decades. From the experimental point of view, BECs are very attractive since they can be manipulated by lasers and magnetic fields. Due to the low density, the microscopic length scales are sufficiently large that the condensate wave function is directly observable by optical means and interference phenomena can be studied. Moreover, if the atom species has a Feshbach resonance, it is possible to precisely tune the interaction by changing an external electric or magnetic field, which in particular allows the study of strongly correlated many-body systems.

Most closely connected to the projects [32, 33, 35] of this thesis are experiments with quasi-low-dimensional Bose gases, which are realised in highly anisotropic traps satisfying the condition

$$\hbar\omega^{\perp} \gg k_B T \,,$$

where $\omega^{\perp}$ denotes the frequency of a confining harmonic potential. The cross-over from a 3d gas to quasi-1d and quasi-2d condensates was experimentally first realised in 2001, i.a., by Görlitz *et al.* in [82]. In this work, the authors studied sodium atoms in anisotropic magnetic (1d) and optical (2d) traps and increased the aspect ratios up to values of 50–100 while reducing the number of atoms. The condensates were quasi-low-dimensional, while the thermal component of the gas remained 3d.

Subsequently, a series of works focusing on various features of these systems followed. To avoid problems arising from the detection of very low particle numbers due to the very low densities, the strong confinement was in many experiments realised by optical lattices, which allows the simultaneous study of many copies of the 1d/2d system. The optical lattices are created by the superposition of counter-propagating laser beams, which form standing waves. A 1d lattice is created by a single interference pattern from a pair of laser beams, which yields a periodic array of disc-shaped potentials. To build a 2d optical lattice, one uses two orthogonal standing waves, which results in

[13] Values taken from [101, Section I].

a 2d periodic array of cigar-shaped trapping potentials (see, e.g., the review [23] by Bloch).

Quasi-1d gases are particularly well suited for absorption imaging. For 3d systems, this method entails an integration over one spatial direction, which can be avoided with a quasi-1d gas. Physically, it is very interesting to observe the crossover to the Tonks–Girardeau (TG) regime (see Section 1.3.3). In this regime, which corresponds to a very large LL parameter γ, the repulsive interactions between the atoms are so strong that the wave function vanishes whenever the positions of two particles coincide, implying that the bosons acquire fermionic properties. However, due to the symmetry of the wave function, some bosonic behaviour remains, such as the characteristic bosonic momentum distribution. Besides, quasi-1d gases can also be used to realise physical models such as the Heisenberg spin chain [76, 156]. Exemplary and without any claim to completeness, we collect in the following some interesting experiments that produced and studied quasi-1d Bose gases:

- In [165], Schreck *et al.* prepared a BEC of ^{7}Li atoms immersed in a Fermi sea of ^{6}Li atoms, with the effect that the ^{7}Li condensate behaved as quasi-1d BEC.

- Greiner *et al.* [84] stored rubidium atoms in a 2d optical lattice of $\sim 10^3$ tightly confining potential tubes. When suddenly released from the trap, the single condensate wave functions expand and interfere. Note that the tunnelling of atoms in the thermal cloud is irrelevant at low temperatures due to the small energies, whereas tunnelling of ground state atoms is enhanced due to the macroscopic occupation. Hence, the BECs at the optical lattice sites form a phase-coherent ensemble and interference patterns can be studied. As one result, the authors observed that the quasi-1d nature of the individual BECs was preserved over much longer times than the lifetime of the phase coherence between neighbouring lattice sites.

- Moritz *et al.* [136] also produced cigar-shaped BECs of rubidium in a 2d optical lattice and experimentally confirmed that the gas could locally be well described by a local LL model, even though the whole sample was 3d. They realised thermal quasi-1d gases, where not only the atoms in the condensate but also the thermal cloud behave one-dimensionally. To study the crossover to a 1d thermal gas, they heated an initially pure BEC for some time by means of off-resonant photon scattering. While the radial size was unaffected, the axial width of the cloud increased with the trapping time.

- Esteve *et al.* [69] realised a quasi-1d Bose gas of rubidium atoms within a highly anisotropic magnetic trap created by an atom chip. By increasing the density at fixed temperature, they let the gas pass through the first regime of the quasi-1d gas described in Section 1.3.3, starting from the ideal gas case and ending at the GP case. They observed the density fluctuations by taking absorption images in the transverse direction. In the GP regime, the density fluctuations are given by the Bogoliubov approximation, which was experimentally confirmed for thermal energies approximately equal to the confinement energy.

- Kinoshita, Wenger and Weiss [103] created an array of "quantum Newton's cradles" out of 3000 parallel tubes of quasi-1d Bose gases with an average of 100 rubidium atoms per tube. While maintaining the transverse confinement, they put each atom in a superposition of two states with opposite (longitudinal) momentum and let the system evolve in the longitudinal direction. Although the two momentum groups collided with each other thousands of times, the systems did not approach equilibrium.

- Meinert *et al.* [128] prepared a system of 3500 cigar-shaped gases of cesium atoms in tubes created by a pair of interfering laser beams. They immersed into each tube a single strongly interacting impurity, which was accelerated by gravity. By adiabatically tuning the scattering length using a Feshbach resonance, they obtained a quasi-1d Bose gas with large LL parameter γ (see Sections 1.3.2 and 1.3.3). Although the systems were translation invariant, the authors observed Bragg reflections, which are expected to arise from strong correlations of the bosons that lead to a lattice-like behaviour. Moreover, this resulted in periodic dynamics of the impurity, comparable to Bloch oscillations.

- The first TG gas of rubidium atoms in a 2d optical lattice was realised by Paredes *et al.* in [141], who used a 1d periodic potential along the third axis to reach the required large values of γ. Due to the spatial modulation, the atoms can be interpreted as quasi-particles with an increased effective mass, which yields effective values of γ up to $\gamma = 100$. This procedure resulted in an array of quasi-1d tubes of TG gases consisting of about 20 atoms each.

  In a different experimental setup, Kinoshita, Wenger and Weiss [102] created an array of 20 quasi-1d rubidium gases in the TG regime, using two independent laser traps. At extremely low temperatures and fixed longitudinal confinement, the authors studied the atoms at increasingly strong transverse confinement, reaching values of γ up to $\gamma = 5.5$. Eventually, the axial trapping potential was removed and the free 1d motion of the atoms within the quasi-1d tubes was analysed.

Quasi-2d BECs in disc-shaped geometries were, for instance, created in the following experiments:

- Burger *et al.* [40] considered rubidium atoms in a 3d cigar-shaped static magnetic trap whose axis was superimposed by a 1d optical lattice, resulting in an array of 2d discs. As a consequence of the magnetic potential, the central lattice sites contained a higher number of atoms, leading to a higher critical temperature for the central clouds. Hence, when lowering the temperature, BEC occured first in the central lattice sites and successively spread in the radial direction.
- Rychtarik *et al.* [160] realised quasi-2d BECs of cesium atoms in a so-called gravito-optical surface trap, consisting of an evanescent laser wave on the surface of a horizontally aligned prism in combination with gravity pushing the atoms onto the prism. They reached the BEC phase transition in a 3d situation via evaporative cooling and subsequently increased the trap anisotropy to bring the condensate into the 2d regime.
- In [166], Schweikhard *et al.* produced quasi-2d gases of rubidium atoms in a rapidly rotating trap, where the centrifugal force was so large that it nearly cancelled the radial confining force.
- Smith *et al.* [172] created and studied quasi-2d condensates of rubidium atoms in extremely anisotropic combinations of magnetic and optical traps, where the trap anisotropy was gradually increased up to an aspect ratio of 700. Since no atoms were discarded in this process, this resulted in relatively large quasi-2d BECs of up to 10^5 atoms.

An intriguing 2d-specific phenomenon is the Berezinskii–Kosterlitz–Thouless (BKT) phase transition to a superfluid state at low temperatures. Recall that at positive temperature, the phase transition to BEC is impossible for a uniform 2d Bose gas in the thermodynamic limit. As BEC is associated with long-range order, its absence in 2d means that the (two-point) correlation functions decay with increasing distance. Whereas the decay is exponential in space at high temperatures, it becomes algebraic below a finite critical temperature if the atoms interact repulsively. As a consequence, the system exhibits quasi-long-range order and forms a so-called superfluid "quasi-condensate". This behaviour only occurs for repulsively interacting systems, hence interactions in 2d cannot be regarded as corrections to the ideal gas case as in 3d but fundamentally change the physical situation.

Microscopically, the BKT phase transition is related to the emergence of a topological instead of a long-range order. Below the critical temperature, vortices (phase defects around which the phase varies by a multiple of 2π) can only exist as bound vortex-antivortex pairs, which create no net circulation along larger contours. Above the critical temperature, the pairs break up into free vortices, which destroys the quasi-long-range order. For a detailed explanation, we refer to the lecture notes [91] by Hadzibabic and Dalibard.

For trapped 2d condensates, BEC is possible at finite temperature, and the relation of BEC and BKT phase transitions depends on the size of the system and the strength of the interactions. In presence of repulsive interactions and in sufficiently large systems, BEC is suppressed and replaced by the BKT transition. The BKT crossover has experimentally been studied in several works, for instance in the following experiments:

- Hadzibabic *et al.* [92] prepared disc-shaped gases of rubidium atoms. They observed long-range coherence at low temperatures, whose loss at higher temperatures coincided with the onset of the formation of free vortices. In a follow-up experiment, Krüger, Hadzibabic and Dalibard [107] measured the critical atom number of quasi-2d Bose gases in harmonic trapping potentials at different temperatures. While being about five times higher than the critical numbers corresponding to BEC in a 2d ideal gas, they were in agreement with the predictions of the BKT phase transition.

- Studying the behaviour of sodium atoms in a quasi-2d optical trap, Cladé *et al.* observed in [51] a theoretically predicted intermediate non-superfluid quasi-condensate regime between the thermal and the superfluid phase.

- Fletcher *et al.* [71] created a quasi-2d potassium gas with tunable interactions and experimentally confirmed that BKT and BEC phase transition unify in the limit of vanishing interactions.

2. Objectives

Low-dimensional Gross–Pitaevskii equation for strongly confined bosons

The starting point for the first part of this thesis was the work [100] by von Keler and Teufel. Here, the authors consider N interacting bosons in three dimensions that are in two spatial dimensions strongly confined to a region of order ε, which is modelled by a quantum waveguide with non-trivial geometry. They prove that in the limit $(N, \varepsilon) \to (\infty, 0)$, the dynamics of the system are effectively described by a 1d NLS equation. Since the analysis in [100] is restricted to the parameter range $\beta \in (0, \frac{1}{3})$, the first objective was to extend this result beyond the mean-field regime. Naturally, since the physically relevant case is the GP scaling of the interaction, the long-term goal was the derivation of a 1d GP equation for $\beta = 1$.

It turned out that Pickl's strategy [151] could be adapted to the situation with strong confinement, leading to a proof for the full NLS regime $\beta \in (0, 1)$ in [32], and eventually to a proof of the 1d GP equation in [35]. We decided to focus on straight and untwisted waveguides, and could therefore replace the Dirichlet boundary conditions from [100] by a more realistic confining potential.

The natural next question was to extend this result to a disc-shaped confinement, which would lead to a 2d effective equation. This was finally established in [33] for the full range $\beta \in (0, 1]$.

Higher order corrections to the mean-field dynamics

The objective of the second part of the thesis was the derivation of higher order corrections to the norm approximation of the dynamics of weakly interacting bosons. Approximations with respect to the L^2-norm of the N-body Hilbert space have recently been proved in different settings, and the corresponding results are summarised in Section 1.5.3.

To obtain higher order corrections to the norm approximation with respect to N^{-1}, Pavlović, Pickl and Soffer developed the idea to extend the first quantised approach introduced by Mitrouskas, Petrat and Pickl in [135] by using Duhamel expansions that are truncated after finitely many terms.

The original draft by Pavlović, Pickl and Soffer covered the next order correction to the norm approximation for the Hartree scaling $\beta = 0$ in $d = 3$ dimensions. Hence, the aim was to extend this analysis to arbitrary order with respect to N^{-1} and to a range of scaling parameters β as large as possible.

3. Results and Discussion

In this chapter, we present and discuss the results obtained in this thesis. Section 3.1 collects the results obtained in [32, 33, 35], which are partially joint work with Stefan Teufel. In Section 3.2, we report on the project [34], which is joint work with Nataša Pavlović, Peter Pickl and Avy Soffer. For convenience of the reader, we partially adapted the notation to present the results in a consistent way.

3.1. Low-dimensional Gross–Pitaevskii equation for strongly confined bosons

3.1.1. Results

We consider a gas of N 3d bosons in an extremely asymmetric set-up, where the particles are strongly confined in one or two spatial directions. To describe such systems mathematically, we use the coordinates

$$z = (x, y) \in \mathbb{R}^3, \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^{3-d}, \quad d = 1, 2,$$

where $x \in \mathbb{R}^d$ denotes the longitudinal direction(s) and $y \in \mathbb{R}^{3-d}$ is the coordinate in the confined direction(s). The relevant length scales of the problem are

- L: the length scale in the longitudinal direction,
- εL: the length scale in the transverse direction,
- A: the length scale of the scattering length.

The parameter $0 < \varepsilon \ll 1$ measures the spatial asymmetry. For convenience, we choose L as length unit, which implies that the transverse length scale is ε. The confinement is modelled by the rescaled potential

$$\frac{1}{\varepsilon^2} V^{\perp}\left(\frac{y}{\varepsilon}\right),$$

where $V^{\perp} : \mathbb{R}^{3-d} \to \mathbb{R}$ acts only on the y-coordinates. We impose on $V^{\perp}$ suitable assumptions to ensure that the ground state χ of the operator $-\Delta_y + V^{\perp}$ with eigenvalue E_0 is localised on a length scale of order one. The normalised ground state

$\chi^\varepsilon \in L^2(\mathbb{R}^{3-d})$ of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$ is then given by

$$\chi^\varepsilon(y) = \varepsilon^{-\frac{3-d}{2}}\chi\left(\frac{y}{\varepsilon}\right), \qquad \left(-\Delta_y + \frac{1}{\varepsilon^2}V^\perp\left(\frac{y}{\varepsilon}\right)\right)\chi^\varepsilon(y) = \frac{E_0}{\varepsilon^2}\chi^\varepsilon(y), \tag{3.1}$$

which in particular implies the localisation of χ^ε on the scale ε. Since the energy gap between ground state and first excited state scales as ε^{-2}, transverse excitations are, for sufficiently small ε, strongly suppressed. As as consequence, the great majority of particles remains in the transverse ground state under time evolution, merely undergoing phase oscillations.

As explained in Section 1.2.3, the choice $L = 1$ coerces a rescaling of the interaction to remain in the physically relevant GP scaling regime. As in (1.21), this rescaling is determined by the requirement that the ground state and kinetic energy per particle be comparable, where the relevant kinetic energy is in this case the longitudinal kinetic energy. Since the density of the gas scales as

$$\varrho = \frac{N}{\varepsilon^{3-d}},$$

the total ground state energy is by (1.19) of order $AN\varepsilon^{-(3-d)}$, while the longitudinal kinetic energy is of order one. Hence, the GP scaling condition (1.22) reads

$$A \sim \frac{\varepsilon^{3-d}}{N}. \tag{3.2}$$

As shown in Section 1.2.4, this condition is implemented by the interaction potential

$$w_\mu(z) = \frac{1}{\mu^2}w\left(\frac{z}{\mu}\right)$$

for a compactly supported, bounded and non-negative interaction potential w with scattering length a, where we introduced the parameter

$$\mu := \frac{\varepsilon^{3-d}}{N}$$

measuring the range of the interaction. It coincides with the scale of the scattering length of w_μ, which is given as

$$A = a_\mu = \mu a,$$

where a denotes the scattering length of w.

Finally, we admit an additional, possibly time-dependent external field varying on the length scale $L^\parallel = 1$, which may act on both x an y coordinates. To emphasize the distinction from $V^\perp$, this potential is called $V^\parallel : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. The full N-body

Hamiltonian is given as

$$H_\mu(t) = \sum_{j=1}^{N} \left(-\Delta_j + \frac{1}{\varepsilon^2} V^\perp \left(\frac{y_j}{\varepsilon}\right) + V^\parallel(t, z_j) \right) + \sum_{i<j} w_\mu(z_i - z_j) \,. \qquad (3.3)$$

We study the dynamics of the system in the simultaneous limit $(N, \varepsilon) \to (\infty, 0)$, i.e., in the joint limit of infinite particle number and infinite spatial asymmetry. The state $\psi^{N,\varepsilon}(t)$ at time t is determined by the solution of the N-body Schrödinger equation with Hamiltonian $H_\mu(t)$ and initial datum $\psi_0^{N,\varepsilon} \in L^2_+(\mathbb{R}^{3N})$, which is assumed to exhibit complete asymptotic BEC in the state $\varphi_0^\varepsilon \in L^2(\mathbb{R}^3)$. Given the strong confinement, the condensate wave function φ_0^ε is assumed to factorise into the transverse ground state χ^ε and a longitudinal part $\Phi_0 \in L^2(\mathbb{R}^d)$,

$$\varphi_0^\varepsilon(z) = \Phi_0(x)\chi^\varepsilon(y) \,.$$

The goal of this project is to prove that complete asymptotic condensation in a factorised one-body state is preserved by the time evolution, i.e., that

$$\lim_{(N,\varepsilon)\to(\infty,0)} \operatorname{Tr}\left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}} - |\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|\right| = 0 \quad \Rightarrow \quad \lim_{(N,\varepsilon)\to(\infty,0)} \operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| = 0$$

for $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$, where the longitudinal part $\Phi(t)$ is the solution of the effective d-dimensional GP equation

$$\mathrm{i}\tfrac{\partial}{\partial t}\Phi(t,x) = \left(-\Delta_x + V^\parallel(t,(x,0)) + b|\Phi(t,x)|^2\right)\Phi(t,x)\,, \qquad \Phi(0) = \Phi_0, \qquad (3.4)$$

with coupling parameter

$$b \;:=\; 8\pi a \int\limits_{\mathbb{R}^{3-d}} |\chi(y)|^4 \,\mathrm{d}y \;=\; 8\pi a\, \varepsilon^{3-d} \int\limits_{\mathbb{R}^{3-d}} |\chi^\varepsilon(y)|^4 \,\mathrm{d}y \,. \qquad (3.5)$$

Note that this parameter b is precisely $Ng_{1\mathrm{d}}$ from (1.44) for $d = 1$ and $8\pi N g^{(1)}_{2\mathrm{d}}$ from (1.64) for $d = 2$, respectively, with choices $L = 1$ and $A = a_\mu$. Consequently, (3.4) is the time-dependent GP equation corresponding to the 1d-/2d- GP functionals (1.48) and (1.60) with potential $V^\parallel = V^\parallel(t,(x,0))$.

To heuristically motivate the evolution equation (3.4) with coupling parameter (3.5), note that $\mu \ll \varepsilon$ as long as N is sufficiently large, implying that the interaction appears δ-like even on the scale ε. Formally replacing w_μ by $8\pi a_\mu \delta(x_1 - x_2)\delta(y_1 - y_2)$ as in the heuristic argument in Section 1.2.5, we absorb the short-scale correlation structure into the effective interaction. Further, note that $|\chi^\varepsilon|^2$ acts δ-like on the scale length where $V^{\|}$ varies, in the sense that

$$\int_{\mathbb{R}^{3-d}} \mathrm{d}y\, |\chi^\varepsilon(y)|^2 V^{\|}(t,(x,y)) = \int_{\mathbb{R}^{3-d}} \mathrm{d}y\, |\chi^\varepsilon(y)|^2 V^{\|}(t,(x,0)) + \mathcal{O}(\varepsilon) \approx V^{\|}(t,(x,0))$$

for sufficiently regular $V^{\|}$. By (3.1), with $a_\mu = \mu a$ and since Φ, χ and χ^ε are normalised, this yields the total energy

$$\begin{aligned} E_0(N,\varepsilon) &= N \left\langle \Phi(x)\chi^\varepsilon(y), \left(-\Delta_x - \Delta_y + \tfrac{1}{\varepsilon^2} V^\perp(\tfrac{y}{\varepsilon}) + V^{\|}(t,z)\right) \Phi(x)\chi^\varepsilon(y) \right\rangle_{L^2(\mathbb{R}^3)} \\ &\quad + \tfrac{N(N-1)}{2} 8\pi a_\mu \left\langle \chi^\varepsilon(y_1)\chi^\varepsilon(y_2), \delta(y_1-y_2)\chi^\varepsilon(y_1)\chi^\varepsilon(y_2)\right\rangle_{L^2(\mathbb{R}^{3-d}\times\mathbb{R}^{3-d})} \times \\ &\qquad \times \left\langle \Phi(x_1)\Phi(x_2), \delta(x_1-x_2)\Phi(x_1)\Phi(x_2)\right\rangle_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \\ &\approx N\tfrac{E_0}{\varepsilon^2} + \langle \Phi, (-\Delta_x)\Phi\rangle_{L^2(\mathbb{R}^d)} + N\left\langle \Phi(x), V^{\|}(t,(x,0))\Phi(x)\right\rangle \\ &\quad + N\varepsilon^{3-d} 4\pi a \left\langle \Phi, |\Phi|^2\Phi\right\rangle_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}^{3-d}} |\chi^\varepsilon(y)|^4\,\mathrm{d}y \\ &= N\left(\tfrac{E_0}{\varepsilon^2} + \int_{\mathbb{R}^d} \left(|\nabla_x\Phi(x)|^2 + V^{\|}(t,(x,0))|\Phi(x)|^2 + \tfrac{b}{2}|\Phi(x)|^4\right)\mathrm{d}x \right), \end{aligned}$$

which equals the d-dimensional GP energy functional with coupling parameter $\frac{b}{8\pi}$, plus the transverse ground state energy. The time-dependent d-dimensional GP equation (3.4) can then be formally justified as argued in Section 1.4.1.

As explained in Section 1.4.4, Pickl's strategy of proof requires not only the assumption that the system initially exhibits complete asymptotic BEC in the state $\varphi^\varepsilon = \Phi\chi^\varepsilon$ but also an estimate of the initial energy of the N-body wave function. The corresponding quantities in the situation with strong confinement are

- the *"renormalised" energy per particle*: for $\psi \in \mathcal{D}(H_\mu(t)^{\frac{1}{2}})$,

$$E^\psi_{w_\mu}(t) := \tfrac{1}{N} \langle\!\langle \psi, H_\mu(t)\psi \rangle\!\rangle - \tfrac{E_0}{\varepsilon^2}\,, \tag{3.6}$$

- the *effective longitudinal energy per particle*: for $\Phi \in H^1(\mathbb{R}^d)$,

$$\mathcal{E}^\Phi_b(t) := \left\langle \Phi, \left(-\Delta_x + V^{\|}(t,(x,0)) + \tfrac{b}{2}|\Phi|^2\right)\Phi \right\rangle_{L^2(\mathbb{R}^d)}. \tag{3.7}$$

To model the situation in real experiments, we consider the two limits $N \to \infty$ and $\varepsilon \to 0$ simultaneously. Our analysis does not cover all possible sequences $\{(N_n, \varepsilon_n)\}_{n\in\mathbb{N}}$ in $\mathbb{N}\times(0,1)$ with limiting behaviour $(N_n,\varepsilon_n) \to (\infty, 0)$ as $n\to\infty$ but requires certain restrictions on the relation of the two parameters N and ε. In particular, ε must shrink sufficiently fast compared to N^{-1} to ensure that the spectral gap in the transverse direction grows fast enough to sufficiently suppress transitions into transverse excited states. This is regulated by the so-called *admissibility condition*. Moreover, if the confinement is in one spatial dimension ($d = 2$), we require a second, although very weak condition, referred to as *moderate confinement condition*, which states that ε cannot shrink too fast either. Both conditions seem to be rather of technical than of physical nature, and we comment on their necessity in detail in Section 3.1.3. More precisely, we consider the following sequences:

Definition 3.1.1. *Let* $\{(N_n, \varepsilon_n)\}_{n\in\mathbb{N}} \subset \mathbb{N}\times(0,1)$ *such that* $\lim_{n\to\infty}(N_n, \varepsilon_n) = (\infty, 0)$*, and let* $\mu_n := \varepsilon_n^{3-d}/N_n$*. Then the sequence is called*

- *(Θ -) admissible, if*

$$\lim_{n\to\infty} \frac{\varepsilon_n^{\Theta}}{\mu_n} = \lim_{n\to\infty} N_n \varepsilon_n^{\Theta+d-3} = 0\,, \tag{3.8}$$

- *(Γ-) moderately confining, if*

$$\lim_{n\to\infty} \frac{\varepsilon_n^{\Gamma}}{\mu_n} = \lim_{n\to\infty} N_n \varepsilon_n^{\Gamma+d-3} = \infty\,. \tag{3.9}$$

The admissibility condition (3.8) can only be satisfied for $\Theta > 3 - d$. Clearly, the larger Θ, the weaker the condition. Moreover, it is less restrictive for $d = 2$ than for $d = 1$. The moderate confinement condition (3.9) is automatically fulfilled for $\Gamma \leq 3-d$, and we require $\Gamma < \Theta$ to ensure the compatibility with (3.8). The condition is weaker for smaller Γ and smaller d. In conclusion, Θ and Γ can take the values

$$\Theta \in (3-d, \infty]\,, \qquad \Gamma \in [3-d, \Theta)\,,$$

where $\Theta = \infty$ and $\Gamma = 3 - d$ mean imposing no condition at all.

To prove that (3.4) effectively describes the dynamics of the condensate, we require restrictions on the parameters Θ and Γ, which depend on the dimension d. To express these choices in a more compact way, we use the notation x^+ and x^- to denote $(x+\sigma)$ and $(x-\sigma)$ for any fixed $\sigma > 0$, which is to be understood in the following sense: Let the sequence $(N_n, \varepsilon_n)_{n\in\mathbb{N}} \to (\infty, 0)$ and $\sigma > 0$. Then

$$\begin{aligned}
f(N,\varepsilon) \lesssim N^{-x^-} \quad &:\Leftrightarrow \quad f(N_n,\varepsilon_n) \lesssim N_n^{-x+\sigma} \text{ for sufficiently large } n\,,\\
f(N,\varepsilon) \lesssim \varepsilon^{x^-} \quad &:\Leftrightarrow \quad f(N_n,\varepsilon_n) \lesssim \varepsilon_n^{x-\sigma} \text{ for sufficiently large } n\,,\\
f(N,\varepsilon) \lesssim \mu^{x^-} \quad &:\Leftrightarrow \quad f(N_n,\varepsilon_n) \lesssim \mu_n^{x-\sigma} \text{ for sufficiently large } n\,.
\end{aligned}$$

Using this notation, the weakest possible restrictions covered by our analysis are given by

$$(\Theta, \Gamma)_d = \begin{cases} \left(\frac{12}{5}^-, 2\right) & \text{for } d = 1\,, \\ (3, 1^+) & \text{for } d = 2\,. \end{cases} \tag{3.10}$$

Note that for $d = 1$, we may choose $\Gamma = 2 = 3 - d$, which means that our proof does not require any moderate confinement condition. These constraints are discussed in detail in Section 3.1.3.

Finally, our analysis is only sensible for times where the condensate wave function $\Phi(t)$ exists, and, moreover, we require $H^{2d}(\mathbb{R}^d)$-regularity of $\Phi(t)$ for our proof. Since the evolution equation (3.4) is non-linear, the regularity of the initial datum Φ_0 is not necessarily preserved globally in time. Hence, let us define the maximal time of $H^{2d}(\mathbb{R}^d)$-existence,

$$T^{\text{ex}}_{d,V^\parallel} := \sup\left\{ t \in \mathbb{R}^+_0 \,:\, \|\Phi(t)\|_{H^{2d}(\mathbb{R}^d)} < \infty \right\}\,,$$

which depends on the dimension d of the non-linear equation and on the external potential $V^\parallel(\cdot, (\cdot, 0))$. Conditions on $V^\parallel$ under which the existence is global in time are specified in [32, Assumption A3 and Appendix A] for $d = 1$ and in [33, Remark 1] for $d = 2$.

In conclusion, we make the following assumptions on the model (3.3) and on the initial data:

A1 *Interaction potential.*
Let the unscaled potential $w : \mathbb{R}^3 \to \mathbb{R}$ be bounded uniformly in N and ε, spherically symmetric and non-negative and let $\operatorname{diam}(\operatorname{supp} w) = 1$.

A2 *Confining potential.*
Let $V^\perp : \mathbb{R}^{3-d} \to \mathbb{R}$ such that $-\Delta_y + V^\perp$ is self-adjoint on its domain $\mathcal{D} \subset L^2(\mathbb{R}^{3-d})$ and has a non-degenerate ground state χ with ground state energy $E_0 < \sigma_{\text{ess}}(-\Delta_y + V^\perp)$.
Assume further that the negative part of $V^\perp$ is bounded and that $\chi \in \mathcal{C}^2_b(\mathbb{R}^{3-d})$, i.e., that χ is bounded and twice continuously differentiable with bounded derivatives. We choose χ normalised and real.

A3 *External field.*
Let $V^\parallel : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that for fixed $z \in \mathbb{R}^3$, $V^\parallel(\cdot, z) \in \mathcal{C}^1(\mathbb{R})$. Further, assume that for each fixed $t \in \mathbb{R}$, $V^\parallel(t, \cdot), \dot{V}^\parallel(t, \cdot) \in L^\infty(\mathbb{R}^3) \cap \mathcal{C}^1(\mathbb{R}^3)$ and $\nabla_y V^\parallel(t, \cdot)$, $\nabla_y \dot{V}^\parallel(t, \cdot) \in L^\infty(\mathbb{R}^3)$.

A4 *Initial data.*
Let $(N, \varepsilon) \to (\infty, 0)$ be an admissible and moderately confining sequence with

parameters $(\Theta,\Gamma)_d$ given by (3.10). Assume that the family of initial data, $\psi_0^{N,\varepsilon}\in\mathcal{D}(H_\mu(0))\cap L^2_+(\mathbb{R}^{3N})$ with $\|\psi_0^{N,\varepsilon}\|^2=1$, is close to a condensate with condensate wave function $\varphi_0^\varepsilon=\Phi_0\chi^\varepsilon$ for some normalised $\Phi_0\in H^{2d}(\mathbb{R}^d)$, i.e.,

$$\lim_{(N,\varepsilon)\to(\infty,0)}\operatorname{Tr}_{L^2(\mathbb{R}^3)}\left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}}-|\Phi_0\chi^\varepsilon\rangle\langle\Phi_0\chi^\varepsilon|\right|=0\,. \tag{3.11}$$

Further, let

$$\lim_{(N,\varepsilon)\to(\infty,0)}\left|E^{\psi_0^{N,\varepsilon}}_{w_\mu}(0)-\mathcal{E}^{\Phi_0}_b(0)\right|=0\,. \tag{3.12}$$

Under these assumptions, we prove that condensation in a factorised one-body state is preserved by the N-body time evolution, and that the longitudinal part of the condensate wave function evolves according to a d-dimensional GP equation.

Theorem 3.1.2. *Let $d\in\{1,2\}$ and assume that the potentials w, $V^\perp$ and $V^\parallel$ satisfy* A1 – A3. *Let $\psi_0^{N,\varepsilon}$ be a family of initial data satisfying* A4, *and let $\psi^{N,\varepsilon}(t)$ denote the solution of the N-body Schrödinger equation with Hamiltonian* (3.3) *and initial datum $\psi_0^{N,\varepsilon}$. Then, for any $0\le T<T^{\mathrm{ex}}_{d,V^\parallel}$,*

$$\begin{aligned}\lim_{(N,\varepsilon)\to(\infty,0)}\sup_{t\in[0,T]}\operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)}-|\Phi(t)\chi^\varepsilon\rangle\langle\Phi(t)\chi^\varepsilon|\right| &= 0, &(3.13)\\ \lim_{(N,\varepsilon)\to(\infty,0)}\sup_{t\in[0,T]}\left|E^{\psi^{N,\varepsilon}(t)}_{w_\mu}(t)-\mathcal{E}^{\Phi(t)}_b(t)\right| &= 0\,, &(3.14)\end{aligned}$$

where the limits are taken along the sequence (N,ε) from A4. *Here, $\Phi(t)$ is the solution of* (3.4) *with initial datum $\Phi(0)=\Phi_0$ from* A4.

Theorem 3.1.2 combines the statements of [35, Theorem 1] for $d=1$ and [33, Theorem 1] for $d=2$. In fact, we prove (3.13) and (3.14) for a larger class of interaction potentials, including not only interactions in the GP scaling regime but also interactions in the NLS regime such as

$$w_{\mu,\beta}(z)=\mu^{1-3\beta}w(\mu^{-\beta}z)\,,\qquad \beta\in(0,1) \tag{3.15}$$

(see [32, Theorem 1] and [33, Theorem 1]). The main motivation to consider such interactions is that they are crucial for the proof in the GP regime, where the central idea is the replacement of w_μ by an appropriate, softer scaling interaction in the NLS regime, as explained in Section 1.4.4. Therefore, we postpone the discussion of these interactions to the next section.

Finally, our proof provides an estimate of the rate of the convergence of the reduced densities. Since this rate is not optimal, we do not state it here, but it can easily be recovered from the estimates of the single contributions to the time derivatives of the respective counting functional, together with the quantitative version of Lemma 1.4.2 ([32, Lemma 3.6]). For $d = 1$, the rates are collected in [32, Corollary 3.9] and [35, Corollary 3.5].

3.1.2. Strategy of proof

Pickl's method (Section 1.4.4) was first adapted to the situation with strong confinement by von Keler and Teufel in [100], who cover the mean-field scaling regime $\beta \in (0, \frac{1}{3})$. Our proof can be understood as an extension of these ideas to the whole range $\beta \in (0, 1]$.

In addition to the projectors

$$p := p^{\varphi^\varepsilon} := |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\,, \qquad q := q^{\varphi^\varepsilon} := \mathbb{1}_{L^2(\mathbb{R}^3)} - p$$

onto the condensate and its complement, one introduces projectors onto its longitudinal and transverse part. Define the orthogonal projections on $L^2(\mathbb{R}^3)$

$$\begin{aligned} p^\Phi &:= |\Phi(t)\rangle\langle\Phi(t)| \otimes \mathbb{1}_{L^2(\mathbb{R}^{3-d})}\,, & q^\Phi &:= \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\Phi, \\ p^{\chi^\varepsilon} &:= \mathbb{1}_{L^2(\mathbb{R}^d)} \otimes |\chi^\varepsilon\rangle\langle\chi^\varepsilon|\,, & q^\chi &:= \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\chi\,, \end{aligned}$$

which are lifted to many-body projections on $L^2(\mathbb{R}^{3N})$ as in Definition 1.4.1. They satisfy the relations

$$p = p^\Phi p^{\chi^\varepsilon}\,, \quad q^\Phi q = q^\Phi\,, \quad q^{\chi^\varepsilon} q = q^{\chi^\varepsilon}\,, \quad q = q^{\chi^\varepsilon} + q^\Phi p^{\chi^\varepsilon} = q^\Phi + p^\Phi q^{\chi^\varepsilon}. \tag{3.16}$$

As explained in Section 1.4.4, one of the key ideas of Pickl's strategy of proof is the substitution of the GP interaction by a softer interaction in the NLS scaling regime. More precisely, this interaction should be contained in the following set:

Definition 3.1.3. *Let $\beta \in (0,1)$ and $\eta > 0$. Define $\mathcal{W}_{\beta,\eta}$ as the set containing all families*

$$w_{\mu,\beta} : (0,1) \to L^\infty(\mathbb{R}^3, \mathbb{R}), \quad \mu \mapsto w_{\mu,\beta},$$

such that for any $\mu \in (0,1)$

(a) $w_{\mu,\beta} \geq 0$ is spherically symmetric with $\|w_{\mu,\beta}\|_{L^\infty(\mathbb{R}^3)} \lesssim \mu^{1-3\beta}$ and with $R_{\mu,\beta} := \mathrm{diam}(\mathrm{supp}\, w_{\mu,\beta}) \sim \mu^\beta$,

(b) $\lim\limits_{(N,\varepsilon)\to(\infty,0)} \mu^{-\eta}\, |b_{\beta,N,\varepsilon} - b_\beta| = 0\,,$

where

$$b_{\beta,N,\varepsilon} := N \int\limits_{\mathbb{R}^3} w_{\mu,\beta}(z)\,\mathrm{d}z \int\limits_{\mathbb{R}^{3-d}} |\chi^\varepsilon(y)|^4\,\mathrm{d}y = \mu^{-1} \int\limits_{\mathbb{R}^3} w_{\mu,\beta}(z)\,\mathrm{d}z \int\limits_{\mathbb{R}^{3-d}} |\chi(y)|^4\,\mathrm{d}y\,,$$

and

$$b_\beta := \lim_{(N,\varepsilon)\to(\infty,0)} b_{\beta,N,\varepsilon} \,.$$

The parameter $b_{\beta,N,\varepsilon}$ determines the coupling parameter of the effective (N,ε)-dependent non-linear evolution equation for $\Phi(t)$ in the NLS scaling regime, analogously to $\overline{w}^{\varphi(t)}$ in (1.69). In the limit $(N,\varepsilon)\to(\infty,0)$, it converges to an (N,ε)-independent evolution equation with coupling parameter $b_\beta := \lim_{(N,\varepsilon)\to(\infty,0)} b_{\beta,N,\varepsilon}$, analogously to $\|w\|_1$ in (1.72). The parameter η measures how fast $b_{\beta,N,\varepsilon}$ converges to this limit. Note that the interaction $w_{\mu,\beta}(z)=\mu^{1-3\beta}w(\mu^{-\beta}z)$ is contained in $\mathcal{W}_{\beta,\eta}$ for every $\eta>0$ and corresponds to the coupling parameter

$$b_\beta = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^{3-d}} |\chi(y)|^4 \,\mathrm{d}y \,. \tag{3.17}$$

As for the GP scaling, we must restrict our analysis to sequences (N,ε) satisfying an admissibility as well as a moderate confinement condition (Definition 3.1.1). For $\beta\in(0,1)$, the corresponding parameters Θ and Γ are given by

$$(\Theta,\Gamma)_{d,\beta} = \begin{cases} \left(\frac{2}{\beta},\frac{1}{\beta}\right) & \text{for } d=1\,,\\ \left(\frac{3}{\beta}^-,\frac{1}{\beta}\right) & \text{for } d=2\,. \end{cases} \tag{3.18}$$

In both cases, we may choose $\Gamma=\frac{1}{\beta}$, which implies that the moderate confinement condition can be written as

$$\lim_{(N,\varepsilon)\to(\infty,0)} \frac{\mu^\beta}{\varepsilon} = 0\,.$$

Since the range of $w_{\mu,\beta}\in\mathcal{W}_{\beta,\eta}$ is of order μ^β, this condition $\mu^\beta \ll \varepsilon$ implies that the interaction is supported well within the confining potential. Hence, in the NLS scaling regime, the moderate confinement condition is physically motivated, whereas the admissibility condition is a technical restriction also in this regime (see also Section 3.1.3).

Although our goal is to derive an evolution equation in $d<3$ dimensions, the problem is still three-dimensional, in the sense that the condensate wave function $\varphi^\varepsilon=\Phi\chi^\varepsilon$ is a 3d object. Hence, the counting functionals for NLS and GP scaling regime are defined analogously to (1.78) and (1.94) from the 3d case without strong confinement, namely

$$\alpha^<_{\xi,w_{\mu,\beta}}(t) := \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \widehat{m^{\varphi^\varepsilon}}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle + \left| E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t) - \mathcal{E}^{\Phi(t)}_{b_\beta}(t)\right|, \tag{3.19}$$

$$\alpha_{\xi,w_\mu}(t) := \alpha^<_{\xi,w_\mu}(t) - N(N-1)\Re\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g^{(12)}_{\widetilde{\beta}}\,\widehat{r^{\varphi^\varepsilon}}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \tag{3.20}$$

with $\widehat{m^{\varphi^\varepsilon}}$ from (1.77) and $\widehat{r^{\varphi^\varepsilon}}$ from (1.93). Here, $\mathcal{E}^{\Phi(t)}_{b_\beta}(t)$ is defined as (3.7) with b_β as in Definition 3.1.3. The function $g^{(12)}_{\widetilde{\beta}} := g_{\widetilde{\beta}}(z_1 - z_2)$ denotes the complement of the zero energy scattering solution $f_{\widetilde{\beta}}$ of the potential $w_\mu - U_{\widetilde{\beta}}$ as in(1.90). Here, $U_{\widetilde{\beta}}$ is defined analogously to Definition 1.4.4, i.e.,

$$U_{\widetilde{\beta}}(z) := \mu^{1-3\widetilde{\beta}} a \mathbb{1}_{\mu^{\widetilde{\beta}} < |z| < R_{\widetilde{\beta}}}(z) \,,$$

where $R_{\widetilde{\beta}}$ is defined as the minimal value such that the scattering length of $w_\mu - U_{\widetilde{\beta}}$ equals zero.

To prove the convergence of these functionals, we must eventually compare the 3d pair interaction $w_{\mu,\beta}$ and the d-dimensional effective one-body potential $b_\beta|\Phi|^2$. To cope with this dimensional difference, we construct an effectively d-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$ by integrating out the transverse degrees of freedom in $w_{\mu,\beta}$, i.e.,

$$\overline{\overline{w_{\mu,\beta}}}(x_1 - x_2) := \int\limits_{\mathbb{R}^{3-d}} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \int\limits_{\mathbb{R}^{3-d}} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 w_\mu(z_1 - z_2) \,.$$

As an immediate consequence of this definition, we find

$$p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_{\mu,\beta}(z_1 - z_2) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} = \overline{\overline{w_{\mu,\beta}}}(x_1 - x_2) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} \,. \tag{3.21}$$

We now suitably insert identities $(p_1 + q_1)(p_2 + q_2)$ in the time derivatives of the functionals (3.19) and (3.20) on both sides of the scalar products and decompose them, using the relations (3.16). By (3.21), the contribution with $p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}$ on both sides produces the effectively d-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$, while the other contributions can be understood as remainders from this substitution.

As in Section 1.4.4, we first derive an estimate of the time derivative of $\alpha^<_{\xi,\, w_{\mu,\beta}}(t)$ for interactions $w_{\mu,\beta} \in \mathcal{W}_{\beta,\eta}$ in the NLS scaling regime. Subsequently, we use this result for the GP case.

NLS regime

For interactions in the NLS scaling regime as in Definition 3.1.3, we obtain

$$\left| \tfrac{\mathrm{d}}{\mathrm{d}t} \alpha^<_{\xi,\, w_{\mu,\beta}}(t) \right| \;\lesssim\; \gamma_{a,<}(t) + \gamma^{(1)}_{b,<}(t) + \gamma^{(2)}_{b,<}(t) + \gamma_{r,d}(t)$$

for almost every $t \in [0, T^{\mathrm{ex}}_{d,V^\parallel})$. The terms $\gamma_{a,<}$, $\gamma^{(1)}_{b,<}$ and $\gamma^{(2)}_{b,<}$ contain the quasi-d-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$, hence they are comparable to (1.82) from the fully 3d case. Moreover, they are of the same form for both $d = 1, 2$. For the last term, $\gamma_{r,d}$, which collects the remainders from the substitution $w_\mu \mapsto \overline{\overline{w_{\mu,\beta}}}$, we distinguish between $d = 1$ and $d = 2$. To write the expressions in a more compact form, we abbreviate

$$Z^{(12)}_{\mu,\beta} := w^{(12)}_{\mu,\beta} - \tfrac{b_\beta}{N-1}\left(|\Phi(t,x_1)|^2 + |\Phi(t,x_2)|^2\right)$$

and drop all superscripts $\varphi^\varepsilon(t)$. This yields

$$\begin{aligned}
\gamma_{a,<}(t) &:= \left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \dot{V}^{\|}(t,z_1)\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle - \left\langle \Phi(t), \dot{V}^{\|}(t,(x,0))\Phi(t)\right\rangle_{L^2(\mathbb{R}^d)}\right| \\
&\quad -2\Im\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \widehat{l} q_1\big(V^{\|}(t,z_1) - V^{\|}(t,(x_1,0))\big)p_1\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \\
\gamma^{(1)}_{b,<}(t) &:= N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \widehat{l}\, q_1^\Phi p_1^{\chi^\varepsilon} p_2 Z^{(12)}_{\mu,\beta} p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \\
\gamma^{(2)}_{b,<}(t) &:= N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^\Phi q_2^\Phi\, \widehat{l}\, \overline{\overline{w_{\mu,\beta}}}(x_1-x_2)p_1^\Phi p_2^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^\Phi q_2^\Phi \widehat{l}\, \overline{\overline{w_{\mu,\beta}}}(x_1-x_2)p_1^\Phi q_2^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +b_\beta\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1q_2\,\widehat{l}\,|\Phi(t,x_1)|^2 p_1q_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|.
\end{aligned}$$

where

$$\widehat{l} \in \left\{N\widehat{m}^a_{-1},\, N\widehat{m}^b_{-2}\right\}$$

is in each term chosen such that the term becomes maximal. For $d=1$, we obtain the remainder term

$$\begin{aligned}
\gamma_{r,1}(t) &:= N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^{\chi^\varepsilon} t_2\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (q_1^{\chi^\varepsilon} q_2 + q_1^\Phi p_1^{\chi^\varepsilon} q_2^{\chi^\varepsilon})\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1q_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^\Phi q_2^\Phi\,\widehat{l}\, p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w^{(12)}_{\mu,\beta} p_1 q_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|,
\end{aligned}$$

where $t_2 \in \{p_2, q_2, q_2^\Phi p_2^{\chi^\varepsilon}\}$ such that the first line becomes maximal. For reasons explained below, the case $d=2$ requires one more splitting of the projections q_2 in the second line into $q_2 = q_2^{\chi^\varepsilon} + q_2^\Phi p_2^{\chi^\varepsilon}$. This yields

$$\begin{aligned}
\gamma^{(1)}_{r,2}(t) &:= N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^{\chi^\varepsilon} t_2\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (q_1^{\chi^\varepsilon} q_2 + q_1^\Phi p_1^{\chi^\varepsilon} q_2^{\chi^\varepsilon})\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1 q_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^{\chi^\varepsilon} q_2^{\chi^\varepsilon}\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1 p_2^{\chi^\varepsilon} q_2^\Phi\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \\
\gamma^{(2)}_{r,2}(t) &:= N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (q_1^{\chi^\varepsilon} q_2^\Phi p_2^{\chi^\varepsilon} + q_1^\Phi p_1^{\chi^\varepsilon} q_2^{\chi^\varepsilon})\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1 p_2^{\chi^\varepsilon} q_2^\Phi\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad +N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^\Phi q_2^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}\,\widehat{l}\, w^{(12)}_{\mu,\beta} p_1 q_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|.
\end{aligned}$$

Let us analyse the single terms and, in particular, comment on the differences to the fully 3d problem from Section 1.4.4.

- *Energy term* $\gamma_{a,<}(t)$.
  Similarly to the 3d expression (1.80), this term contains the interactions between the bosons and the external field $V^{\|}$. The second line is specific for the problem with strong confinement and is due to the fact that the N-body Hamiltonian contains the 3d external field $V^{\|}(t)$, while only the value of $V^{\|}(t)$ at $y=0$ enters in the effective equation. In addition to the argument for (1.80), one expands $V^{\|}(t,(x,\cdot))$ around zero and estimates the remainders, which entails the regularity assumptions in assumption *A3*.

- *($q^\Phi p^\Phi - p^\Phi p^\Phi$) term* $\gamma^{(1)}_{b,<}(t)$.
  The 3d counterpart of $\gamma^{(1)}_{b,<}(t)$ is (1.83). Since $q_1^\Phi |\Phi(x_2)|^2 p_1^\Phi = 0$, it follows that $\gamma^{(1)}_{b,<}(t)$ contains the difference

$$p_2^\Phi \left(\overline{\overline{w_{\mu,\beta}}}(x_1 - x_2) - \tfrac{b_\beta}{N-1} |\Phi(x_1)|^2 \right) p_2^\Phi$$

  between the quasi-d-dimensional pair interaction and the effective d-dimensional one-body interaction potential. The estimate works analogously to the 3d case and crucially requires the moderate confinement condition. This constraint ensures that $w_{\mu,\beta}$ is localised well within the region accessible to the confined bosons, implying that the full interaction potential contributes to $\overline{\overline{w_{\mu,\beta}}}$. If it were instead that $\mu^\beta \gg \varepsilon$, the predominant part of $w_{\mu,\beta}$ would be localised in a practically inaccessible area, hence one expects $p_2^\Phi \overline{\overline{w_{\mu,\beta}}}(x_1 - x_2)\, p_2^\Phi \to 0$ as $(N,\varepsilon) \to (\infty, 0)$ (see also the discussion of the moderate confinement condition in Section 3.1.3).

- *($q^\Phi q^\Phi - p^\Phi p^\Phi$) and ($q^\Phi q^\Phi - q^\Phi p^\Phi$) terms* $\gamma^{(2)}_{b,<}(t)$.
  The first two lines of $\gamma^{(2)}_{b,<}(t)$ correspond to the 3d expressions (1.84) and (1.85), while the third line is a remainder, which is easily controlled since it does not contain $w_{\mu,\beta}$. As in the 3d problem, the first two lines are estimated by integration by parts, which, however, is now only in x as $\overline{\overline{w_{\mu,\beta}}}$ is a function on $\mathbb{R}^d$. Since the explicit form of Green's function depends on the dimension, the estimates for $d = 1, 2$ are mutually different and differ from the 3d problem.

  For $d = 1$, we implement the integration by parts by defining the function $\overline{\overline{h}}_{\beta_1}$ as the solution of the equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2} \overline{\overline{h}}_{\beta_1} = \overline{\overline{w_{\mu,\beta}}}$ on the interval $[-N^{-\beta_1}, N^{\beta_1}]$ with Dirichlet boundary conditions for some $\beta_1 \in [0, 1]$. To prevent contributions from the boundary upon integrating by parts on this interval, we insert a smoothed step function, whose derivatives can be controlled (see [32, Definition 4.18]).

For $d = 2$, we define $\overline{\overline{h}}_\rho$ as the solution of the equation $\Delta\overline{\overline{h}}_\rho = \overline{\overline{w_{\mu,\beta}}} - \overline{\overline{v}}_\rho$ on $\mathbb{R}^2$. Here, $\overline{\overline{v}}_\rho$ is a potential with $\|\overline{\overline{v}}_\rho\|_{L^1(\mathbb{R}^2)} = \|\overline{\overline{w_{\mu,\beta}}}\|_{L^1(\mathbb{R}^2)}$ which is supported within a 2d ball B_ρ with radius ρ for some $\rho \in (\mu^\beta, 1]$ ([33, Definition 5.4]). As a consequence of Newton's theorem, $\overline{\overline{h}}_\rho$ is supported in B_ρ, and we can integrate by parts in x without the appearance of boundary terms. To cope with the logarithmic divergence of the 2d Green's function, we integrate by parts twice. This is the reason why we define $\overline{\overline{h}}_\rho$ not on a ball with Dirichlet boundary conditions as was done for $d = 1$: while the results are the same when integrating by parts once, the additional factors ρ^{-1} from a second derivative falling on the smoothed step function are too large.

As in the 3d problem, an *a priori* estimate $\|\nabla_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|^2 \lesssim 1$ suffices for the first but not for the second line of $\gamma_{b,<}^{(2)}(t)$, both for $d = 1$ and $d = 2$. Here, we derive the improved estimate

$$\|\nabla_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|^2 \lesssim \alpha^<_{\xi,\,w_{\mu,\beta}}(t) + \mathcal{O}(1)$$

([32, Lemma 4.21]) by adapting the proof of the corresponding 3d lemma, which again involves a splitting of the interaction by means of the projectors p^{χ^ε} and q^{χ^ε} and an integration by parts. Naturally, remainder terms similar to $\gamma_{r,<}(t)$ appear, and they are controlled as explained below.

- *Remainder term $\gamma_{r,1}(t)$ for $d = 1$.*
  This term does not have any 3d counterpart since it collects all terms without $p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}$ on both sides of the scalar product, i.e., the remainders from the substitution $w_{\mu,\beta} \mapsto \overline{\overline{w_{\mu,\beta}}}$. The integration by parts, which is now in three dimensions, is realised via the function h_ε solving $\Delta h_\varepsilon = w_{\mu,\beta}$ on a 3d ball with radius ε and Dirichlet boundary conditions, in combination with a suitable smoothed step function. In contrast to the integration by parts in x, we must now handle derivatives ∇_{y_1} hitting ψ^N or φ^ε, which contribute a factor ε^{-1} each.

  To compensate for these factors, one observes that transverse excitations are extremely suppressed due to the strong confinement. Since the interaction $w_{\mu,\beta}$ is non-negative and the external potential $V^\parallel$ is bounded, one finds, for simplicity dropping all time-dependencies, that

$$\begin{aligned}\mathcal{O}(1) = E^{\psi^{N,\varepsilon}}_{w_{\mu,\beta}}(t) &\gtrsim \left\langle\!\!\left\langle \psi^{N,\varepsilon}, \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right)\psi^{N,\varepsilon}\right\rangle\!\!\right\rangle - \mathcal{O}(1)\\ &= \left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}, \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right) q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}\right\rangle\!\!\right\rangle - \mathcal{O}(1)\end{aligned}$$

  because $\left(-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2}\right)p_1^{\chi^\varepsilon} = 0$ by (3.1). On the one hand, the spectral gap between the ground state and the first excited state scales as ε^{-2}, hence

$$\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}, \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right) q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}\right\rangle\!\!\right\rangle \gtrsim \tfrac{1}{\varepsilon^2}\left\langle\!\!\left\langle \psi^{N,\varepsilon}, q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}\right\rangle\!\!\right\rangle .$$

On the other hand, assumption *A2* states that $\|(V^\perp - E_0)_-\|_{L^\infty(\mathbb{R}^{3-d})} \lesssim 1$, implying that

$$\begin{aligned}&\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}, \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right) q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}\right\rangle\!\!\right\rangle \\ &\qquad \geq \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}\|^2 - \tfrac{1}{\varepsilon^2}\|(V^\perp - E_0)_-\|_{L^\infty(\mathbb{R}^{3-d})}\|q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}\|^2 .\end{aligned}$$

In conclusion, we obtain the *a priori* estimates

$$\|q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon , \qquad \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \lesssim 1 .$$

Note that each term in $\gamma_{r,1}(t)$ contains a projection $q_1^{\chi^\varepsilon}$, hence one gains a factor ε in each expression. Moreover, by means of the admissibility condition, small positive powers of N can be compensated for by powers of ε, which is crucial for the estimate.

- *Remainder terms $\gamma_{r,2}^{(1)}(t)$ and $\gamma_{r,2}^{(2)}(t)$ for $d = 2$.*
  The term $\gamma_{r,2}^{(1)}(t)$ can be estimated by a 3d integration by parts, similarly to $\gamma_{r,1}(t)$. Note that the second and third line each contain two projections q^{χ^ε}, each of which contributes a factor ε. While one ε cancels the factor ε^{-1} from the derivative, the second ε compensates for all surplus positive powers of N.

  Since the two terms in $\gamma_{r,2}^{(2)}(t)$ contain only one projection q^{χ^ε} each, this strategy of a 3d integration by parts does not work here. Note that this was different for $d = 1$ due a different ratio of ε and N. Instead, one controls $\gamma_{r,2}^{(2)}(t)$ by observing that both lines contain the expression

$$p_1^{\chi^\varepsilon} w_{\mu,\beta}(z_1 - z_2) p_1^{\chi^\varepsilon} =: \overline{w_{\mu,\beta}}(x_1 - x_2, y_2) p_1^{\chi^\varepsilon} ,$$

  which defines a function where one of the y variables of the pair interaction is integrated out, while it still depends on the second one. Now we integrate by parts only in the x-variable as explained for $d = 2$ in $\gamma_{b,<}^{(2)}(t)$. First, this has the advantage that ∇_x does not generate factors ε^{-1}. Second, the x-anti-derivative of $\overline{\overline{w_{\mu,\beta}}}(\cdot, y)$ diverges only logarithmically in μ^{-1}, which can be compensated for by any positive power of ε or N^{-1}, due to admissibility and moderate confinement condition.

In conclusion, the estimates described above lead to the following theorem, which combines the statements of [32, Theorem 1] for $d = 1$ and [33, Theorem 1] for $d = 2$:

Theorem 3.1.4. *Let $\beta \in (0,1)$, $d \in \{1,2\}$ and $w_{\mu,\beta} \in \mathcal{W}_{\beta,\eta}$ for some $\eta > 0$. Assume that the potentials $V^\perp$ and $V^\parallel$ satisfy assumptions* A2 *and* A3*. Let $\psi_0^{N,\varepsilon}$ be a family of initial data satisfying* A4 *with parameters $(\Theta,\Gamma)_{d,\beta}$ as in* (3.18) *and with $\mathcal{E}_b^{\Phi_0}(0)$ replaced by $\mathcal{E}_{b_\beta}^{\Phi_0}(0)$. Let $\psi^{N,\varepsilon}(t)$ denote the solution of the N-body Schrödinger equation with Hamiltonian*

$$H_{\mu,\beta}(t) = \sum_{j=1}^{N} \left(-\Delta_j + \tfrac{1}{\varepsilon^2} V^\perp\left(\tfrac{y_j}{\varepsilon}\right) + V^\parallel(t,z_j) \right) + \sum_{i<j} w_{\mu,\beta}(z_i - z_j) \tag{3.22}$$

and initial datum $\psi_0^{N,\varepsilon}$. Then, for any $0 \leq T \leq T^{\mathrm{ex}}_{d,V^\parallel}$,

$$\begin{aligned} \lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[0,T]} \operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\Phi(t)\chi^\varepsilon\rangle\langle\Phi(t)\chi^\varepsilon|\right| &= 0, \\ \lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[0,T]} \left|E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t) - \mathcal{E}^{\Phi(t)}_{b_\beta}(t)\right| &= 0\,, \end{aligned}$$

where the limits are taken along the sequence from A4*. Here, $\Phi(t)$ is the solution of the NLS equation* (3.4) *but with coupling parameter b_β from Definition 3.1.3.*

GP regime

Let us now turn to the proof of Theorem 3.1.2 for the interaction w_μ in the GP scaling regime. Abbreviating

$$\begin{aligned} Z^{(12)}_\mu &:= w_\mu(z_1 - z_2) - \tfrac{b}{N-1}(|\Phi(t,x_1)|^2 + |\Phi(t,x_2)|^2), \\ \widetilde{Z}^{(12)}_{\mu,\widetilde{\beta}} &:= \left(U_{\widetilde{\beta}} f_{\widetilde{\beta}}\right)(z_1 - z_2) - \tfrac{b}{N-1}(|\Phi(t,x_1)|^2 + |\Phi(t,x_2)|^2) \end{aligned}$$

and dropping all superscripts $\varphi^\varepsilon(t)$, the time derivative of the GP counting functional (3.20) can be estimated as

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}(t)\right| \lesssim \gamma^<(t) + \gamma_a(t) + \gamma_b(t) + \gamma_c(t) + \gamma_d(t) + \gamma_e(t) + \gamma_f(t)$$

for almost every $t \in [0, T^{\mathrm{ex}}_{d,V^\parallel})$, where

$$\begin{aligned} \gamma^<(t) &:= \left| \left\langle\!\left\langle \psi^{N,\varepsilon}(t), \dot{V}^\parallel(t,z_1)\psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle - \left\langle \Phi(t), \dot{V}^\parallel(t,(x,0))\Phi(t) \right\rangle_{L^2(\mathbb{R}^d)} \right| \\ &\quad + N \left| \left\langle\!\left\langle \psi^{N,\varepsilon}(t), q_1 \widehat{m}^a_{-1}\big(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\big) p_1 \psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle \right| \\ &\quad + N^2 \left| \left\langle\!\left\langle \psi^{N,\varepsilon}(t), \widetilde{Z}^{(12)}_{\mu,\widetilde{\beta}} \widehat{m} \psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle \right| , \\ \gamma_a(t) &:= N^3 \left| \left\langle\!\left\langle \psi^{N,\varepsilon}(t), g^{(12)}_{\widetilde{\beta}} \left[V^\parallel(t,z_1) - V^\parallel(t,(x_1,0)), \widehat{r} \right] \psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle \right| , \end{aligned}$$

$$\begin{aligned}
\gamma_b(t) &:= N\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), |\Phi(t,x_1)|^2 g_{\widetilde{\beta}}^{(12)}\,\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad + N^2\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\,\widehat{r}\,Z_\mu^{(12)}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \\
\gamma_c(t) &:= N^2\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (\nabla_1 g_{\widetilde{\beta}}^{(12)})\cdot\nabla_1\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \\
\gamma_d(t) &:= N^3\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\left[|\Phi(t,x_3)|^2,\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \\
&\quad + N^3\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\left[w_\mu^{(13)},\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \\
\gamma_e(t) &:= N^4\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\left[w_\mu^{(34)},\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \\
\gamma_f(t) &:= N^2\left|\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\left[\Phi(t,x_1)|^2,\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|.
\end{aligned}$$

As in the 3d case, these expressions fall into two categories:

- *NLS term* $\gamma^<(t)$.
  This term is the counterpart of (1.96), i.e., it corresponds to the time derivative of the NLS counting functional with interaction $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$. To show that the result from the NLS case can be transferred, one first observes that $U_{\widetilde{\beta}}f_{\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$ for $\eta \in (0, 1-\widetilde{\beta})$. Second, one needs to choose $\widetilde{\beta}$ such that the admissibility/moderate confinement condition with parameters $(\Theta,\Gamma)_d$ from (3.10) implies that the same sequence (N,ε) is also admissible/moderately confining with parameters $(\Theta,\Gamma)_{d,\widetilde{\beta}}$ from (3.18).

  For $d=1$, we have $(\Theta,\Gamma)_1 = (\frac{12}{5}^-, 2)$ and $(\Theta,\Gamma)_{1,\widetilde{\beta}} = (2/\widetilde{\beta}, 1/\widetilde{\beta})$, which implies the sufficient condition $\widetilde{\beta} \in [\frac{1}{2}, \frac{5}{6}^+]$ since

  $$\begin{aligned}
  \widetilde{\beta} \le \tfrac{5}{6}^+ &\quad\Rightarrow\quad N\varepsilon^{2/\widetilde{\beta}-2} = N\varepsilon^{12/5^- -2}\varepsilon^{2/\widetilde{\beta}-12/5^-} \to 0\,, \\
  \widetilde{\beta} \ge \tfrac{1}{2} &\quad\Rightarrow\quad N^{-1}\varepsilon^{2-1/\widetilde{\beta}} \to 0\,.
  \end{aligned}$$

  For $d=2$, the respective parameters are $(\Theta,\Gamma)_2 = (3,1^+)$ and $(\Theta,\Gamma)_{2,\widetilde{\beta}} = (3/\widetilde{\beta}, 1/\widetilde{\beta})$, hence the compatible range of $\widetilde{\beta}$ is $[1^-,1]$ because

  $$\begin{aligned}
  \widetilde{\beta} \le 1 &\quad\Rightarrow\quad N\varepsilon^{3/\widetilde{\beta}-1} = N\varepsilon^2\varepsilon^{3/\widetilde{\beta}-3} \to 0\,, \\
  \widetilde{\beta} \ge 1^- &\quad\Rightarrow\quad N^{-1}\varepsilon^{1-1/\widetilde{\beta}} = N^{-1}\varepsilon^{1-1^+}\varepsilon^{1^+-1/\widetilde{\beta}} \to 0\,.
  \end{aligned}$$

  Note, however, that these conditions are further restricted by requirements in the proof.

As explained in Section 1.4.4, the estimate of the second line of $\gamma^{(2)}_{b,<}$ needs to be adapted to the GP dynamics. First, one proves the estimate

$$\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|^2 \lesssim \alpha^<_{\xi,\,w_\mu}(t) + \mathcal{O}(1)\ ,$$

where $\mathcal{A}_1$ is the subset of $\mathbb{R}^{3N}$ where appropriate 3d holes around the scattering centres are cut out. The basic idea of the proof is the same as for the fully 3d estimate (1.102), which crucially relies on the inequality (1.103). However, it becomes more involved due to the confinement: To be able to apply (1.103), we must show that

$$\begin{aligned}&\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{x_1}\psi^{N,\varepsilon}(t)\|^2 + \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right)\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle \\ &\gtrsim \ \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_1\psi^{N,\varepsilon}(t)\|^2 - \mathcal{O}(1)\ ,\end{aligned}$$

i.e., that the positive term $\|\nabla_{y_1}\psi^{N,\varepsilon}(t)\|^2$ compensates not only for a sufficient share of the negative part of $\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \mathbb{1}_{\mathcal{B}_1}(w_\mu - U_{\widetilde{\beta}})^{(12)}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle$ but also for the large negative part of $\frac{1}{\varepsilon^2}\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (V^\perp(\frac{y_1}{\varepsilon}) - E_0)\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle$. To this end, we introduce a new set $\mathcal{A}_1^x$ as the projection of $\mathcal{A}_1$ onto the hypersurface $y = 0$. Since the corresponding characteristic functions $\mathbb{1}_{\overline{\mathcal{A}}_1^x}$ and $\mathbb{1}_{\mathcal{A}_1^x}$ act non-trivially only on the x variables and $\mathbb{1}_{\mathcal{B}_1}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$ act non-trivially only on the variables $z_2, \ldots, z_N$, the corresponding multiplication operators commute with Δ_{y_1}. In particular, $\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\overline{\mathcal{B}}_1}\psi^{N,\varepsilon}(t)$ and $\mathbb{1}_{\mathcal{A}_1^x}\psi^{N,\varepsilon}(t)$ are contained in the domain of Δ_{y_1}. Hence, by suitable insertion of $\mathbb{1}_{\overline{\mathcal{A}}_1^x} + \mathbb{1}_{\mathcal{A}_1^x}$, by positivity of the operator $-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2}$ and since $\mathbb{1}_{\overline{\mathcal{A}}_1^x} \geq \mathbb{1}_{\overline{\mathcal{A}}_1}$ in the sense of operators, one extracts the required contribution from the scalar product. To control the remaining terms, we exploit the smallness of $\overline{\mathcal{A}}^x$ by means of the Gagliardo–Nirenberg–Sobolev inequality in the x variables. Consequently, the estimates depend non-trivially on the dimension d. For $d = 1$, the resulting expressions can be controlled by the admissibility condition alone, while for $d = 2$ the moderate confinement condition is additionally required.

Finally, we estimate $\gamma^{(2)}_{b,<}$ by integration by parts and insertion of $\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}}_1}$. The contributions with $\mathbb{1}_{\mathcal{A}_1}$ are controlled by the new energy estimate, while one uses the 3d Sobolev inequality to exploit the smallness of $\overline{\mathcal{A}}_1$ to bound the terms with $\mathbb{1}_{\overline{\mathcal{A}}_1}$. For $d = 2$, this integration by parts is done in two stages to cope with the logarithmic divergences from the 2d Green's function, which is similar to the 3d problem. More precisely, one introduces two auxiliary potentials $\overline{\overline{v}}_{\mu^{\beta_1}}$ and $\overline{\overline{\nu}}_1$, which are supported on balls with radius μ^{β_1} and 1, respectively, and defined such that their $L^1(\mathbb{R}^2)$-norms coincide with $\|\overline{\overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}}\|_{L^1(\mathbb{R}^2)}$. Subsequently, the differences $\overline{\overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}} - \overline{\overline{v}}_{\mu^{\beta_1}}$ and $\overline{\overline{v}}_{\mu^{\beta_1}} - \overline{\overline{\nu}}_1$ are integrated by parts, and finally we exploit different properties of the solution of the respective Poisson's equation.

- *Remainder terms* $\gamma_a(t)$ *to* $\gamma_f(t)$. The terms $\gamma_b(t)$ to $\gamma_f(t)$ are equivalent to (1.97) to (1.101), while $\gamma_a(t)$ is particular for the situation with confinement, similarly to the second line of $\gamma_{a,<}(t)$ in the NLS case.

  The main difference ub comparison with the 3d problem is the term $\gamma_c(t)$. While (1.98) is estimated via integration by parts, this does not work in the situation with confinement, where each derivative ∇_1 carries a factor ε^{-1}. For $d = 1$, this is circumvented by proving suitable estimates for $\nabla g_{\widetilde{\beta}}$. In the case $d = 2$, one splits the scalar product into its x and y components, where the moderate confinement condition is crucial to control for the control of the y component. Moreover, the admissibility condition is required for both $d = 1$ and $d = 2$.

3.1.3. Discussion

To the best of our knowledge, the problem of deriving a low-dimensional NLS equation directly from the 3d N-body dynamics has been studied in three cases, while a low-dimensional GP equation has not been derived before:

- in [45], Chen and Holmer consider the case $d = 2$ with repulsive interactions for $\beta \in (0, \frac{2}{5})$,
- in [47], the same authors study $d = 1$ with attractive interactions for scaling parameters $\beta \in (0, \frac{3}{7})$,
- in [100], von Keler and Teufel cover repulsive interactions for $d = 1$ and $\beta \in (0, \frac{1}{3})$, where the confinement is realised by a waveguide with non-trivial geometry.

Let us briefly present these results, suitably adapted to our notation.

Chen and Holmer consider the Hamiltonian $H_{\mu,\beta}$ as in (3.22) without external trap and with a harmonic confining potential in $3 - d$ dimensions,

$$V^{\perp}(y) = y^2 \,, \qquad \tfrac{1}{\varepsilon^2} V^{\perp}(\tfrac{y}{\varepsilon}) = \varepsilon^{-4} y^2,$$

where the frequency of the rescaled potential is $\omega = \varepsilon^{-2}$. They consider interactions $w_{\mu,\beta}$ as in (3.15) for w a Schwartz function. In [45], it must be non-negative, while the authors assume in [47] that $\int w(z)\,\mathrm{d}z \leq 0$ but w may not be negative everywhere. They admit initial data satisfying (3.11) and assume that the initial renormalised energy per particle is bounded uniformly in N and ε, i.e., that

$$\sup_{N,\varepsilon} \left(\frac{1}{N} \left\langle\!\!\left\langle \psi_0^{N,\varepsilon}, H_{\mu,\beta} \psi_0^{N,\varepsilon} \right\rangle\!\!\right\rangle - \frac{E_0}{\varepsilon^2} \right) \lesssim 1 \,. \tag{3.23}$$

Under these conditions, the authors prove that that (3.13) is satisfied, i.e., that the condensation in a factorised state is preserved by the time evolution, where the longitudinal part $\Phi(t)$ solves an NLS equation. Their proof uses the method of BBGKY hierarchies sketched in Section 1.4.3 and, as a consequence, does not provide any estimate of the rate of the convergence. In [45], the coupling parameter is given by $b_\beta = \int w(z)\,\mathrm{d}z \int_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y$, and in [47], they obtain the coupling parameter $b_\beta = -\left|\int w(z)\,\mathrm{d}z\right| \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y$. As in our case, Chen and Holmer do not consider all possible sequences $(N,\varepsilon) \to (\infty, 0)$ but impose the following constraints:

- For the focusing problem $d = 1$ in [47], they assume that

  $$N\varepsilon^{\frac{2}{\beta}-2} \lesssim 1 \,, \qquad N^{-1}\varepsilon^{-\frac{2}{\nu_2(\beta)}} \lesssim 1 \,, \tag{3.24}$$

  where

  $$\nu_2(\beta) = \min\left\{ \frac{1-\beta}{\beta}\,,\, \frac{\frac{3}{5}-\beta}{\beta-\frac{1}{5}} \mathbb{1}_{\beta \geq \frac{1}{5}} + \infty \cdot \mathbb{1}_{\beta < \frac{1}{5}}\,,\, \frac{2\beta^-}{1-2\beta}\,,\, \frac{\frac{7}{8}-\beta}{\beta} \right\} .$$

  Note that the first condition in (3.24) plays the role of an admissibility condition, while the second one is a moderate confinement condition.

- For the de-focusing problem $d = 2$ in [45], the condition is

  $$N^{-1}\varepsilon^{-2\nu(\beta)} \leq \varepsilon^{2\sigma} \text{ for all } \sigma > 0 \,, \tag{3.25}$$

  where

  $$\nu(\beta) := \max\left\{ \frac{1-\beta}{2\beta}\,,\, \frac{\frac{5}{4}\beta - \frac{1}{12}}{1-\frac{5}{2}\beta}\,,\, \frac{\frac{1}{2}\beta + \frac{5}{6}}{1-\beta}\,,\, \frac{\beta+\frac{1}{3}}{1-2\beta} \right\} .$$

  The inequality (3.25) is a moderate confinement condition, while no admissibility condition is imposed.

Below, we comment on the relation with our conditions (3.18).

The work [100] by von Keler and Teufel concerns a Bose gas which is confined to a quantum waveguide with non-trivial geometry, i.e., to a region of space contained in an ε-neighbourhood of a curve in $\mathbb{R}^3$. The confinement is modelled via Dirichlet boundary conditions. The authors consider the interaction (3.15) for $\beta \in (0, \frac{1}{3})$, where w is assumed bounded, spherically symmetric, compactly supported and non-negative.

Under the assumptions (3.11) and (3.12), they prove (3.13) and (3.14), where the longitudinal part of the wave function evolves according to the 1d NLS equation (3.4) with coupling b_β and with additional potential terms from the twisting and bending of the waveguide. Their proof uses Pickl's first quantised method, and our proof can be understood as an extension of the ideas in [100]. Von Keler and Teufel impose the admissibility condition $\varepsilon^{\frac{4}{3}}/\mu^\beta \to 0$, as well as the moderate confinement condition $\mu^\beta/\varepsilon \to 0$. Moreover, they also consider sequences $(N, \varepsilon) \to (\infty, 0)$ with $\mu^\beta/\varepsilon \to \infty$. This is possible for $\beta \in (0, \frac{1}{2})$ and leads to $b_\beta = 0$ in the effective equation (see the discussion of the moderate confinement condition below).

In the remainder of this section, we discuss the assumptions of our model as well as the obtained results and compare them to [45, 47, 100].

Assumptions on the potentials

We consider interaction potentials that are bounded, spherically symmetric, compactly supported and non-negative. With regard to actual inter-atomic interaction potentials, it would be more realistic to describe the interactions by potentials with positive scattering length but with a certain negative part. Since Pickl's approach was recently adapted to such potentials in [98], it is likely that our result can be extended in a similar way.

In comparison to all previous works [45, 47, 100], which are restricted to values of β strictly smaller than $\frac{1}{2}$, our result covers more singular scalings of the interaction, and in particular includes the physically relevant GP scaling.

Assumption *A2* is fulfilled by a harmonic potential as considered by Chen and Holmer but includes also, for example, any smooth and bounded potential with at least one bound state below the essential spectrum. In particular, it is not necessary that the potential diverges as $|y| \to \infty$ since the confining effect of the potential is due to the rescaling by ε: by [85, Theorem 1], the transverse ground state χ^ε is exponentially localised on the scale ε.

Moreover, our result can easily be modified to a confinement via Dirichlet boundary conditions as in [100]. The main difference in the proof is the estimate of expressions such as $\gamma_b^{(1)}$, which contain the difference between the quasi-d-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$ and the effective one-body potential. To take the boundary of the waveguide into account, one divides the dy-integral into an integral over those y sufficiently distant from the boundary that $\operatorname{supp} w_{\mu,\beta}((x,y) - \cdot)$ is completely contained in the waveguide, and into an integral over the rest, which is easily estimated. The extension of our result to quantum waveguides with non-trivial geometry is not straightforward, since a Taylor expansion was used in [100] and the kinetic term contains an additional vector potential.

Finally, assumption *A3* includes any bounded and sufficiently regular external potential $V^{\parallel}$. In contrast to the works by Chen and Holmer, we admit time-dependent potentials $V^{\parallel}$ as in [100], which is important to observe non-trivial dynamics.

Assumption on the initial data

For the GP scaling of the interaction, an external potential $V^{\parallel}(t,z) = V^{\parallel}(x)$ for some homogeneous $V^{\parallel}$ of degree $s \in [2,\infty]$ and a confining potential diverging at infinity, the Hamiltonian (3.3) coincides with the Hamiltonian (1.42) with parameters $L = 1$ and $A = a_\mu$. In this case, the two parts (3.11) and (3.12) of assumption *A4* were proven for the N-body ground state in [122, 123] for $d = 1$ and in [164] for $d = 2$ (see Section 1.3.3):

- For $d = 1$ and $\beta = 1$, the parameter $g_{1\mathrm{d}}$ from (1.44) is in our model given by

  $$g_{1\mathrm{d}} = N^{-1}b = \frac{8\pi a}{N} \int\limits_{\mathbb{R}^2} |\chi(y)|^4 \,\mathrm{d}y \sim \frac{1}{N}\,.$$

  Besides, $\overline{\varrho}_{1\mathrm{d}} \sim N$, hence $g_{1\mathrm{d}}/\overline{\varrho}_{1\mathrm{d}} \sim N^{-2}$, which implies that our model is in parameter region 2 ("The 1d GP case") of the quasi-1d gas. The 1d GP functional $\mathcal{E}^{\mathrm{GP,1d}}_{1,1,Ng_{1\mathrm{d}}}$ coincides with (3.7), hence (1.50) yields the second part (3.12) of *A4* if $\psi_0^{N,\varepsilon}$ is the N-body ground state of the Hamiltonian (3.3) and Φ_0 is the minimiser of the GP energy functional (3.7). Moreover, the first part (3.11) of *A4* follows from (1.53).

- For $d = 2$ and $\beta = 1$, our model is contained in parameter regime 1 ("The 2d limit of the 2d GP regime") of the quasi-2d gas, which is characterised by the simplified coupling parameter $g^{(1)}_{2\mathrm{d}}$ from (1.64). In our model,

  $$g^{(1)}_{2\mathrm{d}} = \frac{b}{8\pi N} = \frac{a}{N} \int\limits_{\mathbb{R}} |\chi(y)|^4 \,\mathrm{d}y \sim \frac{1}{N}\,,$$

  implying that the gas is part of region (b) ("The GP case"). Consequently, the two parts (3.11) and (3.12) of assumption *A4* follow from (1.63) and (1.66) if $\psi_0^{N,\varepsilon}$ is chosen as the N-body ground state and Φ_0 as the minimiser of the 2d GP functional (3.7).

In conclusion, the two parts of *A4* concerning condensation and the energy per particle are fulfilled at least for the N-body ground state in the GP scaling. These assumptions coincide with the assumptions made in [100]. Besides, the first part (3.11) concerning condensation is also required in [45, 47]. While (3.12) is stronger than the corresponding assumption (3.23) by Chen and Holmer, let us remark that assumptions like (3.12) are rather standard in the literature as soon as larger values of β are concerned.

In addition, *A4* restricts the choice of the limiting sequence $\{(N_n, \varepsilon_n)\}_{n\in\mathbb{N}}$ to sequences satisfying moderate confinement and admissibility condition. In the remainder of this section, we discuss these constraints, quantify the coverage of the parameter range, and compare this to the related works [45, 47, 100].

Restrictions on the limiting sequence for $\beta \in (0,1)$

Let us begin the discussion with Theorem 3.1.4 for scalings $\beta \in (0,1)$. Here, the admissibility condition (3.18) can be expressed as

$$\begin{cases} \dfrac{\varepsilon^2}{\mu^\beta} \ll 1 & d = 1 \\ \dfrac{\varepsilon^{3-}}{\mu^\beta} \ll 1 & d = 2 \end{cases}$$

for sufficiently large N and small ε. The moderate confinement condition (3.18) is given by the requirement

$$\frac{\mu^\beta}{\varepsilon} \ll 1\,, \qquad d = 1,2\,,$$

for sufficiently large N and small ε. Figure 3.1 shows the parameter space $\mathbb{N} \times [0,1]$, where we plot for clarity the parameters N^{-1} and ε. A sequence $(N, \varepsilon) \to (\infty, 0)$ can pass through this space in an arbitrary way from the top right to the bottom left corner. The two boundaries correspond to the two-stage limits where first $\lim_{N\to\infty}$ at constant ε and subsequently $\varepsilon \to 0$ (dark solid line) and vice versa (light solid line). In actual experiments, the confinement is often by a harmonic potential, whose frequency $\omega = \varepsilon^{-2}$ is roughly proportional to the number of particles N.[1] This relation is drawn as black dashed line in Figure 3.1.

Our analysis covers a subset of $\mathbb{N} \times [0,1]$. The admissibility condition bounds the possible sequences away from the edge case $\lim_{\varepsilon\to 0} \lim_{N\to\infty}$, while the moderate confinement condition obstructs them from approaching the edge case $\lim_{N\to\infty} \lim_{\varepsilon\to 0}$. The dark region in Figures 3.2 and 3.3 shows the parameter range covered by our analysis for $d = 1,2$ and some exemplary values of $\beta \in (0,1)$. The white area is prohibited by the admissibility condition, while the light grey area is ruled out as a consequence of the moderate confinement condition. Naturally, these restrictions are meaningful only for sufficiently large N and small ε. This implies that only the section of the plot around the bottom left corner is of importance, whereas the elements of the sequence around the top right corner are not constrained by any admissibility or moderate confinement condition.

For $d = 1$, the moderate confinement condition imposes a restriction only for $\beta < \frac{1}{2}$.

[1]This statement is taken from [45, p. 915] and [47, p. 592].

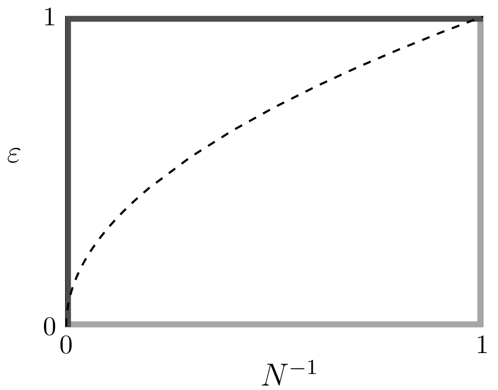
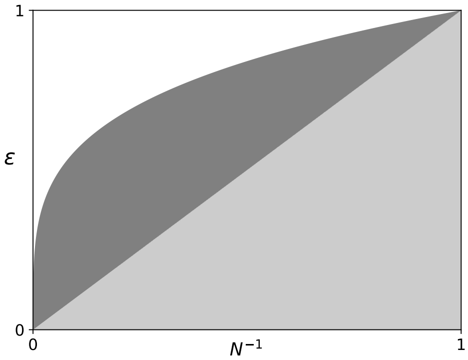


Figure 3.1.: Parameter space $\mathbb{N} \times [0,1]$ containing all sequences $(N,\varepsilon) \to (\infty, 0)$. The dark solid line corresponds to the limit $\lim_{\varepsilon\to 0}\lim_{N\to\infty}$, while the light solid line describes the case $\lim_{N\to\infty}\lim_{\varepsilon\to 0}$. The black dashed line depicts the relation $N \sim \varepsilon^{-2}$, which corresponds to a harmonic confining potential with frequency proportional to N.

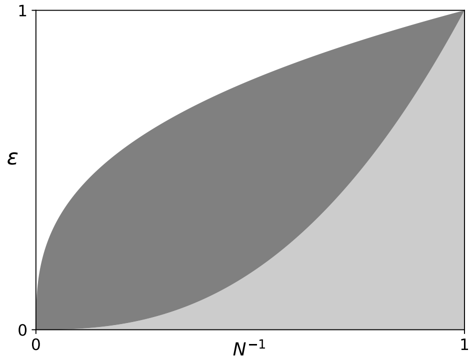
This follows immediately from the definition of μ, which implies that $\mu^\beta/\varepsilon \ll 1$ is trivially true for $\beta \geq \frac{1}{2}$. For $d = 2$, the condition is meaningful for the full range $\beta \in (0,1)$, becoming less restrictive with increasing β.

We expect the moderate confinement condition to be optimal, in the sense that we expect the correct effective equation (3.4) to be a linear evolution with coupling parameter $b_\beta = 0$ if the limiting sequence is such that $\mu^\beta/\varepsilon \to \infty$. This was shown in [100] for $d = 1$, $\beta \in (0, \frac{1}{3})$ and a confinement by Dirichlet boundary conditions. As remarked earlier, we expect this to extend to $\beta < \frac{1}{2}$ and to hold also for $d = 2$ and $\beta < 1$. To motivate this expectation, recall that the moderate confinement condition enters the proof exclusively in the estimate of $\gamma^{(1)}_{b,<}$ and in the energy estimate via a term of the same form as $\gamma^{(1)}_{b,<}$. Let us consider this expression for a confinement by Dirichlet boundary conditions on some sufficiently nice subset $\Omega_{r\varepsilon} \subset \mathbb{R}^{3-d}$ with diameter $r\varepsilon$ for some fixed $r \geq 0$. For $b_\beta = 0$, this leads to the estimate

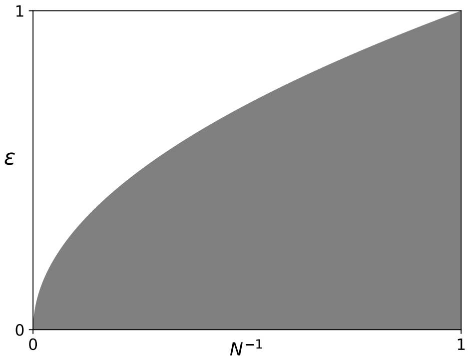
$$\begin{aligned}
\gamma^{(1)}_{b,<} \lesssim N\|p_2 w^{(12)}_{\mu,\beta} p_2\|_{\text{op}} &= N \sup_{\substack{x_1\in\mathbb{R}^d \\ y_1\in\Omega_{r\varepsilon}}} \int\limits_{\mathbb{R}^d} \mathrm{d}x_2 |\Phi(x_2)|^2 \int\limits_{\Omega_{r\varepsilon}} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 w_{\mu,\beta}(z_1 - z_2) \\
&\leq N\mu \, \|\Phi\|^2_{L^\infty(\mathbb{R}^d)} \|\chi^\varepsilon\|^2_{L^\infty(\Omega_{r\varepsilon})} \int\limits_{\mathbb{R}^d} \mathrm{d}x \int\limits_{\Omega_{r\varepsilon/\mu^\beta}} \mathrm{d}y \, w(z) \\
&\lesssim \left(\tfrac{\varepsilon}{\mu^\beta}\right)^{3-d},
\end{aligned}$$



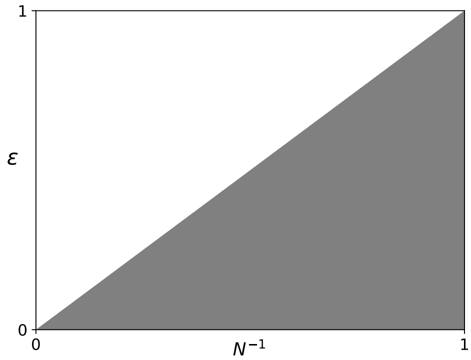
(a) $d = 1$, $\beta = \frac{1}{3}$



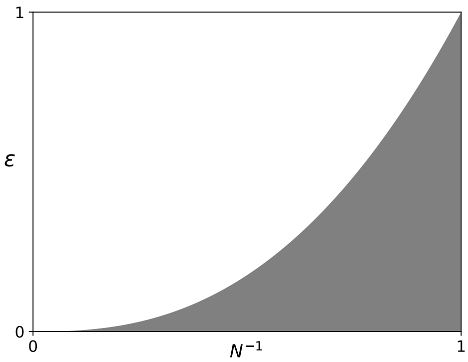
(b) $d = 1$, $\beta = \frac{5}{12}$



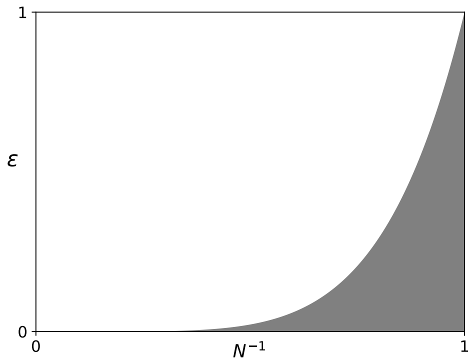
(c) $d = 1$, $\beta = \frac{1}{2}$



(d) $d = 1$, $\beta = \frac{2}{3}$

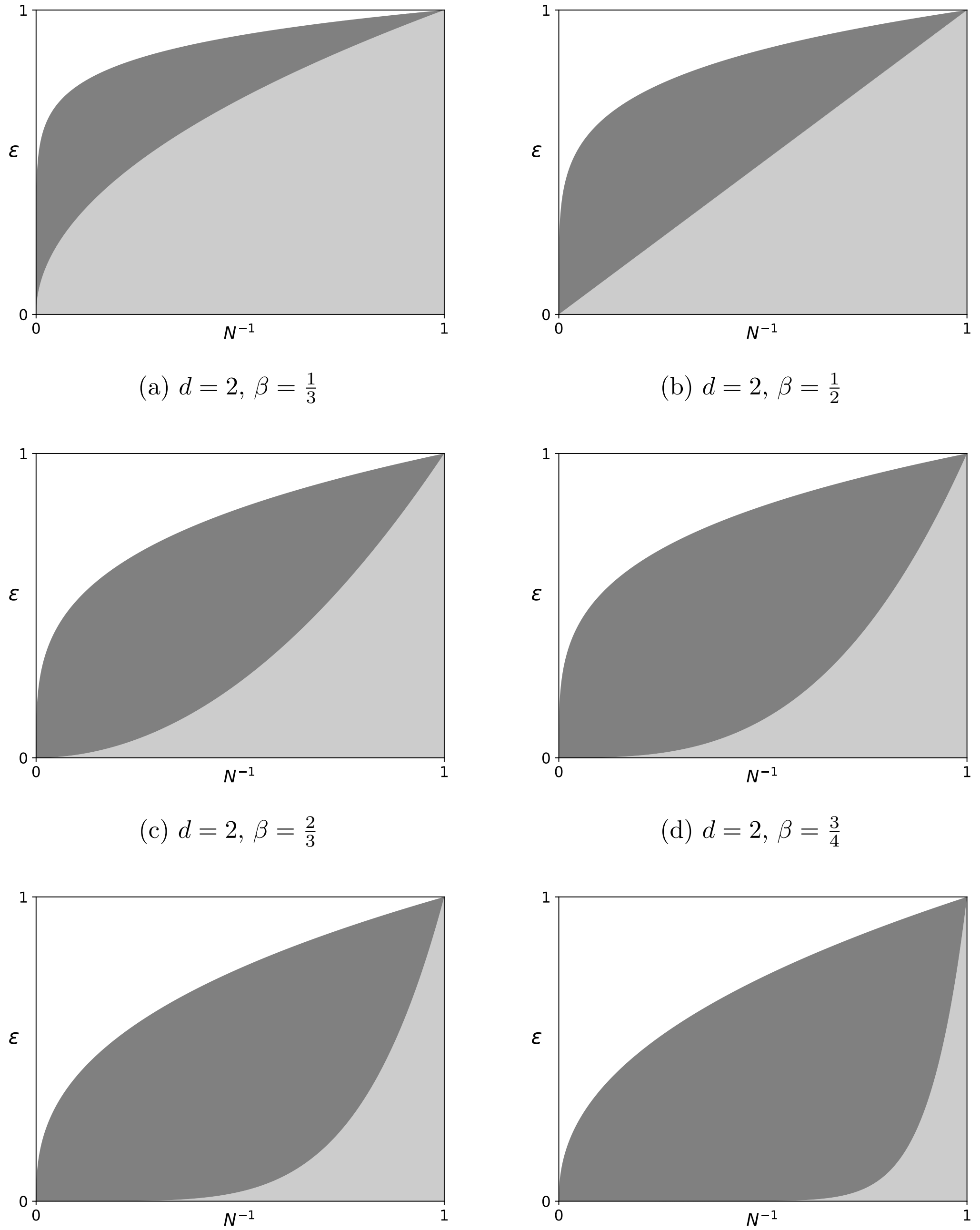


(e) $d = 1$, $\beta = \frac{5}{6}$



(f) $d = 1$, $\beta = \frac{11}{12}$

Figure 3.2.: Subset of the parameter space $\mathbb{N} \times [0, 1]$ covered by our result for some exempary values of $\beta \in (0, 1)$ for $d = 1$. Our analysis covers the dark region. The white region is prohibited as a consequence of the admissibility condition, and the light grey region cannot be reached due to the moderate confinement condition.



(a) $d = 2$, $\beta = \frac{1}{3}$

(b) $d = 2$, $\beta = \frac{1}{2}$

(c) $d = 2$, $\beta = \frac{2}{3}$

(d) $d = 2$, $\beta = \frac{3}{4}$

(e) $d = 2$, $\beta = \frac{5}{6}$

(f) $d = 2$, $\beta = \frac{11}{12}$

Figure 3.3.: Coverage of the parameter space $\mathbb{N} \times [0, 1]$ for $d = 2$ and some exemplary choices of $\beta \in (0, 1)$. While our result applies in the dark grey area, the white and light grey region are prohibited by admissibility and moderate confinement condition, respectively.

which is small by assumption. Moreover, note that the condition $\mu^\beta/\varepsilon \to \infty$ implies the admissibility condition, hence the other estimates remain valid. To extend this argument to a confinement by potentials, recall that χ^ε is by assumption *A2* localised on the scale ε.

The admissibility condition is more restrictive for larger β and means a much stronger constraint for $d = 1$ than for $d = 2$ (see in particular Figures 3.2 and 3.3). Note that the curve corresponding to a harmonic confinement with frequency proportional to N (Figure 3.1) is contained in the region included by the admissibility condition for $d = 1$ with $\beta < \frac{1}{2}$ and for $d = 2$ and all $\beta < 1$.

The stronger admissibility condition for $d = 1$ is, at least to some extent, due to the fact that the estimates of the earlier work [32] ($d = 1$) are not optimal and can presumably be improved with the ideas from [33] for $d = 2$. To see this, recall that the admissibility condition is required for $d = 1$ because of the remainder term $\gamma_{r,1}(t)$ and because of the energy estimate, where it enters via a term that is essentially $\gamma_{r,1}(t)$. A comparison with the corresponding expression $\gamma^{(1)}_{r,2}(t)$ for $d = 2$ leads to the following result:

- The respective first lines of $\gamma_{r,1}(t)$ and $\gamma^{(1)}_{r,2}(t)$ coincide. For $d = 1$, one obtains

$$\begin{aligned}\gamma_{r,1}(t) &\lesssim \left(\frac{\varepsilon^2}{\mu^\beta}\right)^{\frac{1}{2}} (\varepsilon + N^{-1})^{\frac{1}{2}} N^\xi \lesssim \left(\varepsilon^{\frac{3-2\beta}{2}} N^{\frac{\beta}{2}} + \varepsilon^{1-\beta} N^{-\frac{1-\beta}{2}}\right) N^\xi \\ &= \left(\left(N\varepsilon^{\frac{3}{\beta}-2}\right)^{\frac{\beta}{2}} + \mu^{\frac{1-\beta}{2}}\right) N^\xi\end{aligned}$$

  (see [32, Section 4.4.3, estimate of (20)]), which can be controlled by the weaker admissibility condition $N\varepsilon^{3^-/\beta-2} \to 0$, corresponding to the choice $\Theta = \frac{3}{\beta}^-$ as in the case $d = 2$ (see [33, Section 5.2.2, estimate of (23)]).

- The second and third line of $\gamma_{r,1}(t)$ lead to the worse estimate

$$\sim \left(\frac{\varepsilon^2}{\mu^\beta}\right)^{\frac{1}{2}}$$

  (see [32, Section 4.4.3, estimates of (21) and (22)]), resulting in the admissibility condition $\Theta = 2/\beta$. To estimate the corresponding terms for $d = 2$, we split the projector q_2 into a term with $q_2^{\chi^\varepsilon}$, which effectively gains a factor ε, and remainder terms, which we control by defining the interaction $\overline{w_{\mu,\beta}}$ and integrating by parts only in the x coordinate. We expect this strategy to be applicable also for $d = 1$, which should lead to a weaker admissibility condition.

We require the admissibility condition to estimate the remainders $\gamma_{r,<}$ from the substitution of $w_{\mu,\beta}$ by the quasi-d-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$. It is needed to control terms like the expression $\varepsilon^2/\mu^\beta = N^\beta\varepsilon^{2-2\beta}$, where surplus factors of N must be compensated for by powers of ε. Since ε^{-2} is the scale of the energy gap between transverse ground state and excitation spectrum, this condition can be understood as the requirement that this gap must grow sufficiently fast compared to N.

Let us compare moderate confinement and admissibility condition to the constraints imposed in the related works [45, 47, 100].

In [100], von Keler and Teufel impose the same moderate confinement condition and moreover prove that sequences with $\mu^\beta/\varepsilon \to \infty$ yield a free evolution equation (see above discussion). In [45, 47], Chen and Holmer require the moderate confinement conditions (3.24) ($d = 1$) and (3.25) ($d = 2$).

- For $d = 1$ and $\beta \leq \frac{7}{22}$, their exponent $\nu_2(\beta)$ equals $\frac{2\beta^-}{1-2\beta}$, which corresponds to our arguably optimal condition, while they impose a much stronger condition for all larger values of β (see also [47, Figure 1]).

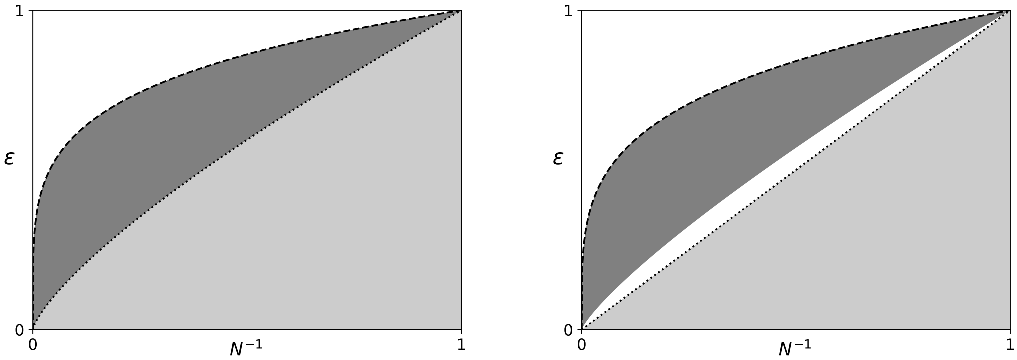
- For $d = 2$, their parameter $\nu(\beta)$ equals $\frac{1-\beta}{2\beta}$ for $\beta < \frac{3}{11}$, which coincides with our condition, while they constrain the parameter range much stronger for larger β (see also [45, Figure 1]).

An admissibility condition is required in the two papers [47, 100] concerning $d = 1$. While the constraint $\varepsilon^{\frac{4}{3}}/\mu^\beta \ll 1$ in [100] is stronger than our condition, the requirement (3.24) in [47] can be expressed as $\varepsilon^{2^+}/\mu^\beta \ll 1$, which is slightly weaker than our condition. For $d = 2$ in [45], Chen and Holmer do not require any admissibility condition.

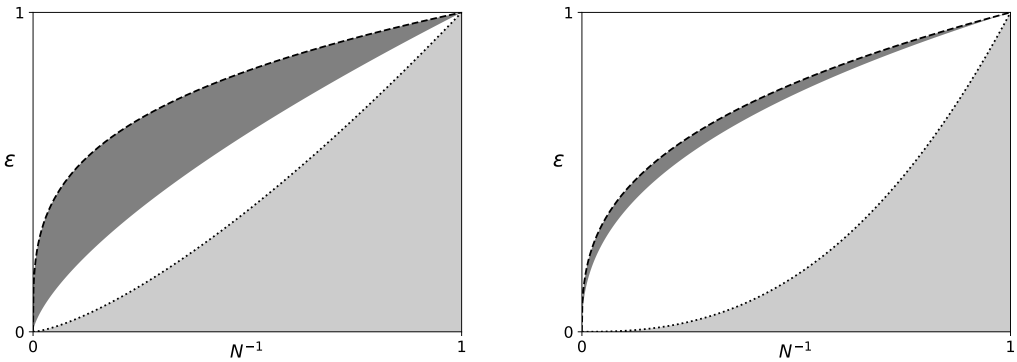
The parameter regions covered by Chen and Holmer in [45, 47] are plotted in Figures 3.4 and 3.5. Sequences (N, ε) within the dark grey regions are admitted by their results, while the white and light grey regions are excluded. As explained above, we expect a free evolution equation for limiting sequences within the light grey regions.

In comparison, our analysis (Theorem 3.1.4) covers the region between the dashed and dotted black lines. As remarked before, especially the region of the parameter space around the bottom left corner is of relevance, which implies that for larger values of β, our restrictions are considerably weaker than the conditions imposed by Chen and Holmer.

Note that for both $d = 1, 2$, our moderate confinement condition becomes weaker with increasing β. In contrast, the moderate confinement condition by Chen and Holmer becomes more restrictive as β increases and thereby limitates the range of β: for $d = 1$, their analysis can only cover β for which the moderate confinement curve lies below the admissibility curve, which is the case for $\beta < \frac{3}{7}$. For $d = 2$, there is no admissibility condition but the moderate confinement condition becomes infinitely restrictive (i.e., $\nu(\beta) = \infty$ for $\nu(\beta)$ from (3.25)) for $\beta = \frac{2}{5}$.

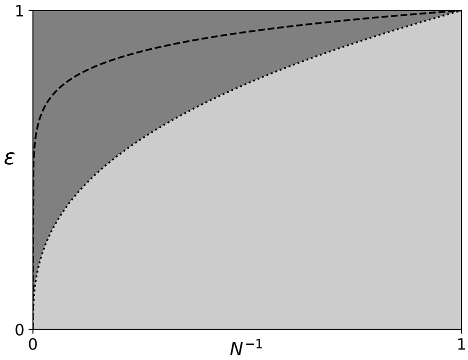


(a) $d = 1$, $\beta = \frac{3}{10}$ (b) $d = 1$, $\beta = \frac{1}{3}$

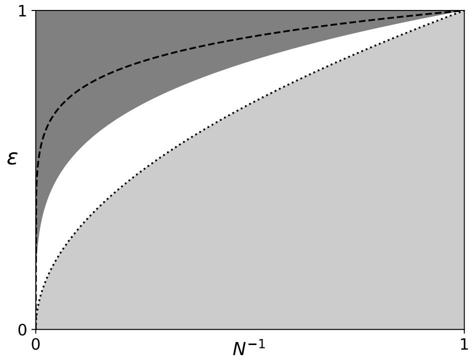


(c) $d = 1$, $\beta = \frac{7}{19}$ (d) $d = 1$, $\beta = \frac{5}{12}$

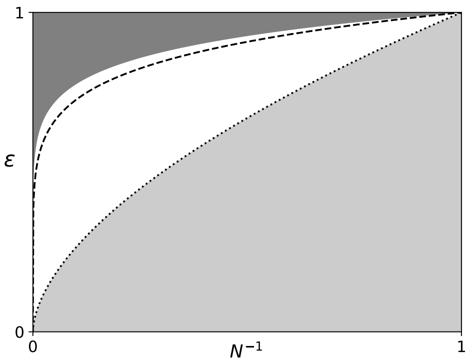
Figure 3.4.: Coverage of the parameter space $\mathbb{N} \times [0, 1]$ for $d = 1$ for some exemplary choices of $\beta \in (0, \frac{3}{7})$. The result by Chen and Holmer in [47] covers the dark grey region, while the white and light grey region are excluded from their analysis. In comparison, our admissibility and moderate confinement conditions (3.18) are drawn as black dashed line and black dotted line, respectively, hence our Theorem 3.1.4 applies in the region enclosed by these curves. For limiting sequences within the light grey region, we expect a free evolution as effective equation.



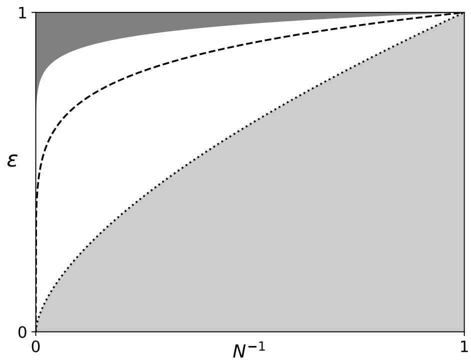
(a) $d = 2$, $\beta = \frac{3}{11}$



(b) $d = 2$, $\beta = \frac{1}{3}$



(c) $d = 2$, $\beta = \frac{11}{30}$



(d) $d = 2$, $\beta = \frac{23}{60}$

Figure 3.5.: Coverage of the parameter space $\mathbb{N} \times [0, 1]$ for $d = 2$ and some exemplary choices of $\beta \in (0, \frac{2}{5})$. In [45], Chen and Holmer cover sequences within the dark grey region, while the white and light grey area are excluded. In comparison, Theorem 3.1.4 applies to all sequences between the black dashed line and the black dotted line, where the dashed line corresponds to the admissibility and the dotted line to the moderate confinement condition (3.18). Limiting sequences within the light grey region are expected to yield a free effective evolution equation.

Physically, we expect that no admissibility condition should occur at all. To motivate this, one first observes that for the GP scaling $\beta = 1$, the gas is for both $d = 1, 2$ in a scaling regime where the ground state energy is described by the 3d GP functional uniformly in ε (see Section 1.3.3, (1.52) and (1.65))[2]. More precisely, the ground state energy can be calculated by minimising the 3d GP functional (1.51) corresponding the Hamiltonian H_μ at fixed ε, which corresponds to taking the limit $N \to \infty$ first and subsequently letting $\varepsilon \to 0$.

Second, on the dynamical side, it is known for $\beta \in (0, 1]$ that the limit $\lim_{\varepsilon\to 0} \lim_{N\to\infty}$, corresponding to a dimensional reduction on the level of the effective equation, yields precisely the effective evolution equation (3.4) with the correct coupling parameter (3.17). The Hamiltonian $H_{\mu,\beta}(t)$ can be written as

$$H_{\mu,\beta}(t) = \sum_{j=1}^{N} \left(-\Delta_j + V^{(\varepsilon)}(t, z_j) \right) + N^{-1+3\beta} \sum_{i<j} w_\beta^{(\varepsilon)} \left(N^\beta (z_i - z_j) \right) , \tag{3.26}$$

where

$$V^{(\varepsilon)}(t, z) := \tfrac{1}{\varepsilon^2} V^\perp(\tfrac{y}{\varepsilon}) + V^\parallel(t, z) , \qquad w_\beta^{(\varepsilon)}(z) := \varepsilon^{(3-d)(1-3\beta)} w(\varepsilon^{-(3-d)\beta} z)$$

for a potential w with (N, ε)-independent scattering length a. Let us now fix ε as a parameter and study the (ε-dependent) dynamics in the limit $N \to \infty$. If the system originally exhibits complete asymptotic BEC in some one-body state $\varphi_0^{(\varepsilon)}$, the result [151] implies[3] that this property is preserved in time, provided ε remains fixed. The condensate wave function at time t is then given as the solution of the 3d NLS (1.72) or GP equation (1.68) with ε-dependent coupling parameter. Note that

$$\|w_\beta^{(\varepsilon)}\|_{L^1(\mathbb{R}^3)} = \varepsilon^{3-d} \|w\|_{L^1(\mathbb{R}^3)} ,$$

and for $\beta = 1$, the scaling relation (1.26) implies that $w_1^{(\varepsilon)}(z) = \varepsilon^{-2(3-d)} w(\varepsilon^{-(3-d)} z)$ has scattering length $\varepsilon^{3-d} a$. Hence, for each fixed ε, the effective evolution equation for $\varphi^{(\varepsilon)}$ is given by

$$\mathrm{i}\tfrac{\partial}{\partial t} \varphi^{(\varepsilon)}(t, z) = \left(-\Delta + V^{(\varepsilon)}(t, z) + \varepsilon^{3-d} b_\beta |\varphi^{(\varepsilon)}(t, z)|^2 \right) \varphi^{(\varepsilon)}(t, z) , \tag{3.27}$$

[2]To be precise, this was shown under the assumption that the external field $V^\parallel$ acts only on the x coordinate and is a homogeneous function, and that the confining potential $V^\perp(y)$ tends to ∞ as $|y| \to \infty$

[3]Note that Pickl's method as described in Section 1.4.4 requires a bounded external field. Therefore, to be precise, this holds only for bounded confining potentials $V^\perp$.

where

$$b_\beta = \begin{cases} \|w\|_{L^1(\mathbb{R}^3)} & \beta \in (0,1)\,, \\ 8\pi a & \beta = 1\,. \end{cases}$$

Note that for $\beta = 1$, (3.27) is precisely the time-dependent equation corresponding to the 3d ε-dependent GP functionals (1.51) and (1.60). Naturally, this result holds for every fixed $\varepsilon > 0$ but, since the rate of convergence is not uniform in ε, it does not extend to the simultaneous limit $(N, \varepsilon) \to (\infty, 0)$. In fact, deriving an estimate that is uniform both in N and in ε was precisely the purpose of the projects [32, 33, 35] of this thesis.

To take the limit $\varepsilon \to 0$ of (3.27), one writes (3.27) in the rescaled coordinates $y \mapsto \widetilde{y} := \varepsilon^{-1} y$. With $\widetilde{\varphi}^{(\varepsilon)}(t,(x,\widetilde{y})) := \varepsilon^{\frac{3-d}{2}} \varphi^{(\varepsilon)}(t,(x,\varepsilon\widetilde{y}))$, this equation yields

$$\mathrm{i}\tfrac{\partial}{\partial t}\widetilde{\varphi}^{(\varepsilon)}(t) = \left(-\Delta_x + \tfrac{1}{\varepsilon^2}\left(-\Delta_{\widetilde{y}} + V^\perp(\widetilde{y})\right) + V^{\parallel}(t,(x,\varepsilon\widetilde{y})) + b|\widetilde{\varphi}^{(\varepsilon)}(t)|^2\right)\widetilde{\varphi}^{(\varepsilon)}(t)\,. \tag{3.28}$$

For $V^\perp(y) = |y|^2$ and $V^{\parallel}(t,z) = |x|^2$, the limit $\varepsilon \to 0$ of (3.28) was studied by Ben Abdallah, Méhats, Schmeiser and Weishäupl in [19]. [4] They assume that the initial 3d wave function,

$$\widetilde{\varphi}_0^{(\varepsilon)}(x,\widetilde{y}) = \Phi_I(x)\chi(y)\,,$$

factorises exactly into some normalised function $\Phi_I \in \mathcal{D}\big((-\Delta_x + |x|^2)^{\frac{1}{2}}\big)$ and the ground state χ of $-\Delta_y + V^\perp$ with eigenvalue E_0. Under this condition, the authors prove that for every $T < \infty$, there exists a constant c_T depending on T such that

$$\sup_{t\in(0,T)} \|\widetilde{\varphi}^{(\varepsilon)}(t) - \mathrm{e}^{-\mathrm{i}E_0 t/\varepsilon^2}\Phi(t)\chi\|_{L^2(\mathbb{R}^3)} \le c_T\varepsilon\,,$$

where $\Phi(t)$ is the solution of

$$\mathrm{i}\tfrac{\partial}{\partial t}\Phi(t) = \left(-\Delta + V^{\parallel}(x) + \overline{b}|\Phi(t)|^2\right)\Phi(t)\,, \quad \overline{b} = b\int\limits_{\mathbb{R}^{3-d}} |\chi(y)|^4\,\mathrm{d}y\,.$$

Hence, in the confinement limit $\varepsilon \to 0$, the 3d ε-dependent one-body dynamics (3.27) converge to the d-dimensional NLS/GP equation (3.4) with coupling parameter (3.5) or (3.17), respectively. In the de-focusing case, the analysis in [19] is valid for both $d = 1, 2$, while only $d = 1$ is included in the focusing case.

Let us remark that the above two-stage process does not yet prove that (3.4) effectively describes the N-body dynamics in the limit $\lim_{\varepsilon\to 0}\lim_{N\to\infty}$ since the confine-

[4] In fact, they consider a $(d+n)$-dimensional non-linear Schrödinger equation with a rather generic non-linearity, which includes the cubic focusing and de-focusing case. The confinement is realised by an anisotropic harmonic potential, where the quotient of the trap frequency in the d unconfined directions and the trap frequency in the n confined directions tends to zero.

ment limit $\varepsilon \to 0$ is proven under the assumption that the condensate wave function $\varphi^{(\varepsilon)}$ factorises exactly for $\varepsilon > 0$. In contrast, we assume with *A4* merely factorisation in the limit $\varepsilon \to 0$. To rigorously prove Theorem 3.1.2 for the limit $\lim_{\varepsilon\to 0}\lim_{N\to\infty}$, the proof of the dimensional reduction needs to be adapted to admit more generic initial conditions. However, also this incomplete argument suggests that our analysis should hold without imposing any admissibility condition.

A comparable result without the strict factorisation assumption was obtained by Méhats and Raymond in [126]. In this work, the authors study the cubic NLS equation in a 2d waveguide, i.e., an ε-tube with Dirichlet boundary conditions around some curve in $\mathbb{R}^2$. They consider a 2d initial datum $\varphi^{(\varepsilon)}$ which is close to its projection onto the transverse ground state χ^ε, up to an error of order ε with respect to the L^2-norm. The authors show that in the limit $\varepsilon \to 0$, the non-linear evolution is in L^2-sense well approximated by the 1d cubic NLS equation (3.4) with coupling parameter (3.17), with an additional potential term from the curvature of the waveguide.

For further analytical results concerning the dimensional reduction of different types of non-linear Schrödinger equations, we refer to [13, 14, 17, 18, 127]. Moreover, numerical treatments are given in [12, 15].

Restrictions on the limiting sequence for the GP scaling

Our main result, Theorem 3.1.2 for the GP scaling of the interaction, holds for sequences $(N,\varepsilon) \to (\infty, 0)$ satisfying assumption *A4* with parameters $(\Theta,\Gamma)_d$ given by (3.10). The admissibility condition states that for any fixed $\sigma > 0$ and sufficiently large N and small ε,

$$\begin{cases} N\varepsilon^{\frac{2}{5}-\sigma} \ll 1 & d = 1\,, \\ N\varepsilon^2 \ll 1 & d = 2\,. \end{cases}$$

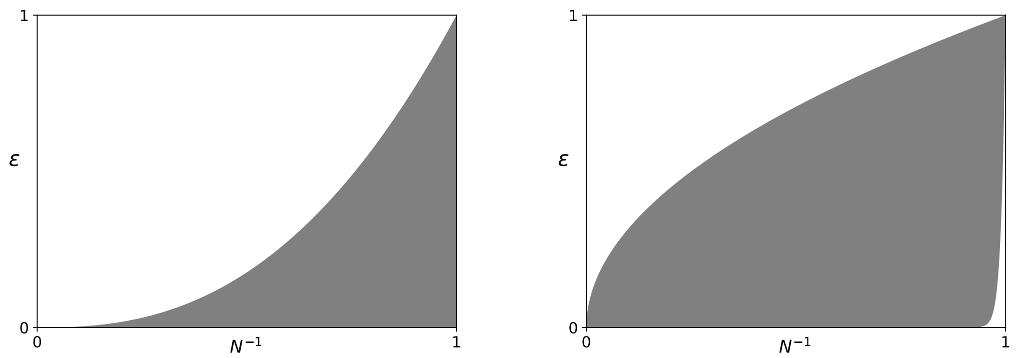
The moderate confinement condition appears only for $d = 2$ and implies that for any fixed σ,

$$N^{-1}\varepsilon^{-\sigma} \ll 1, \qquad d = 2,$$

for sufficiently large N and small ε. Figure 3.6 visualises the corresponding subsets of $\mathbb{N}\times[0,1]$. As in the case $\beta \in (0,1)$, the admissibility condition is much more restrictive for $d = 1$ than for $d = 2$. The harmonic confinement with frequency $\sim N$ (Figure 3.1) is excluded in $d = 1$, while it coincides with the boundary of the admissible region for $d = 2$. Moreover, note that although we impose two constraints on the limiting sequence for $d = 2$, the area covered by our analysis is larger than for $d = 1$.

The moderate confinement condition occurs only for $d = 2$. We require this condition for the following reasons:

- Recall that $\gamma^<(t)$ is controlled by applying the result for the NLS scaling for



(a) $d = 1$, $\beta = 1$ (b) $d = 2$, $\beta = 1$

Figure 3.6.: Coverage of the parameter space $\mathbb{N} \times [0, 1]$ for the GP scaling and dimensions $d = 1, 2$. Theorem 3.1.2 holds for sequences (N, ε) within the dark grey area. The moderate confinement condition for $d = 2$ is realised for the choice $\sigma = 0.01$.

some parameter $\widetilde{\beta}$. Hence, the moderate confinement condition with parameter Γ_d must ensure that the sequence (N, ε) is at the same time moderately confining with a parameter $\Gamma_{d,\widetilde{\beta}}$ satisfying (3.18). While this is automatically given for $d = 1$ as long as $\widetilde{\beta} > \frac{1}{2}$ since $N\varepsilon^{1/\widetilde{\beta}-2} \to \infty$ for all $\widetilde{\beta} \geq \frac{1}{2}$, the case $d = 2$ requires the a moderate confinement condition with parameter

$$\Gamma \geq \widetilde{\beta}^{-1}$$

which ensures that

$$N\varepsilon^{1/\widetilde{\beta}-1} = N^{\Gamma-1}\varepsilon^{1/\widetilde{\beta}-\Gamma} \to \infty\,.$$

- Besides, the moderate confinement condition is required for the GP energy estimate (see [33, Section 6.3]) and enters in the estimate of the remainder term $\gamma_c(t)$ (see [33, Section 6.6.2]).

As a consequence of the first point, the moderate confinement condition (3.18) restricts the possible choices of $\widetilde{\beta}$ for the GP case. Moreover, $\widetilde{\beta}$ must be chosen compatible with the admissibility condition, which enters the proof at several places:

- The admissibility condition is required for the estimate of the remainder term $\gamma_a(t)$ and, in the case $d = 1$, also for the control of $\gamma_d(t)$ (see [35, Sections 4.5.2 and 4.5.5] and [33, Section 6.6.2]).
- The admissibility condition with parameter Θ_d must imply that the sequence (N, ε) is admissible with a parameter $\Theta_{d,\widetilde{\beta}}$ that satisfies (3.18). This can also be

seen graphically when comparing Figure 3.6 with Figures 3.2 and 3.3: $\widetilde{\beta}$ must be chosen such that the dark grey region in Figure 3.6a is completely contained in the respective dark grey region in Figure 3.2 for $d = 1$, and analogously for Figure 3.6b and Figure 3.3 for $d = 2$.

- The admissibility condition is required for the GP energy estimate (see [35, Section 4.3] and [33, Section 6.3]). Besides, one needs the $\Theta_{d,\widetilde{\beta}}$-admissibility of the sequence (N, ε) to control the term $\gamma^{<}(t)$ by means of the GP energy lemma ([35, Section 4.5.1] and [33, Section 6.6.1]).

Moreover, the GP energy estimate restricts the possible choices of $\widetilde{\beta}$ by the requirement that it must be larger than the diameter of the holes around the scattering centres that constitute the set $\overline{\mathcal{A}}$. For presumably merely technical reasons, the diameter of this hole must scale as μ^{δ} with $\delta > \frac{5}{6}$, independently of the dimension. Besides, to contain the full microscopic structure, it must be larger than the support of the scattering solution $g_{\widetilde{\beta}}$, which scales as $\mu^{\widetilde{\beta}}$. Hence, we require

$$\widetilde{\beta} > \tfrac{5}{6}\,.$$

For $d = 1$, this condition determines the weakest possible admissibility condition for which Theorem 3.1.2 holds: For $\widetilde{\beta} = \frac{5}{6}$, we find $\Theta_{1,\widetilde{\beta}} = 2/\widetilde{\beta} = \frac{12}{5}$, hence the sequence (N, ε) is $\Theta_{1,\widetilde{\beta}}$-admissible in the sense of (3.18) if $\Theta = \frac{12}{5}$. Hence, the choice $\widetilde{\beta} > \frac{5}{6}$ leads to our condition with $\Theta = \frac{12}{5}^{-}$. Finally, it turns out that this condition is sufficient to also control the remainder terms.

As mentioned above, we require for $d = 2$ also the moderate confinement condition to obtain the GP energy estimate, which leads to the additional constraint

$$\widetilde{\beta} > \frac{\Gamma + 1}{2\Gamma}\,.$$

The weakest possible moderate confinement condition is given by $\Gamma = 1^{+}$, implying $\widetilde{\beta} > 1^{-}$. As a consequence, the weakest possible admissibility condition is the one that makes any Θ_2-admissible sequence also $\Theta_{2,\widetilde{\beta}}$-admissible with $\widetilde{\beta} = 1^{-}$ in the sense of (3.18), which leads to our choice $\Theta = 3$.

As for scalings $\beta \in (0, 1)$, we understand the admissibility condition in the GP case as a purely technical restriction. In the previous section, we motivated the expectation that our result should hold without such a constraint, although a rigorous proof certainly requires some new ideas.

Whereas the moderate confinement condition for $\beta \in (0, 1)$ is physically motivated and presumably ideal, the corresponding condition for the GP scaling and $d = 2$ is, to our understanding, a technical constraint, although a less restrictive one than the admissibility condition. For $d = 1$, our result is not obstructed by this condition and extends to the full region up to the edge case corresponding to the limit $\lim_{N\to\infty}\lim_{\varepsilon\to 0}$.

The limiting case itself is not included in our model. However, the result [170] by Seiringer and Yin for $d = 1$, which is summarised in Section 1.3.3, suggests that the statement of Theorem 3.1.2 should extend to this edge case. Recall that by (1.55) and (1.58), the LL Hamiltonian $H^{1\mathrm{d}}_{N,1,b/N}$ (1.54) with coupling parameter $g_{1\mathrm{d}} = b/N$, for b as in (3.17), can be understood as the limit $\varepsilon \to 0$ of the Hamiltonian H_μ. In their paper, the authors remark that, as a consequence, H_μ converges to $H^{1\mathrm{d}}_{N,1,b/N} \otimes P^\perp_\varepsilon$, where $P^\perp_\varepsilon$ denotes the projector onto $(\chi^\varepsilon)^{\otimes N}$. This convergence is meant in the following norm resolvent sense: Denote by $E^{1\mathrm{d},1}_{N,1,b/N}$ the lowest eigenvalue of $H^{1\mathrm{d}}_{N,1,b/N}$ and let $\lambda \in \mathbb{C} \setminus [E^{1\mathrm{d},1}_{N,1,b/N}, \infty)$ be fixed. Then

$$\lim_{\varepsilon\to 0}\left\| \frac{1}{\lambda - \left(H_\mu - \frac{NE_0}{\varepsilon^2}\right)} - \frac{1}{\lambda - H^{1\mathrm{d}}_{N,1,b/N}} \otimes P^\perp_\varepsilon \right\| = 0\,. \tag{3.29}$$

By Trotter's theorem (e.g. [157, Theorem VIII.21]), norm resolvent convergence of two operators implies that the unitary time evolutions generated by these operators converge strongly. However, since (3.29) is not quite a norm resolvent convergence in the standard sense, it merely indicates that the dynamics generated by H_μ and by $H^{1\mathrm{d}}_{N,1,b/N}$, respectively, should be asymptotically equal. Moreover, to complete the argument, one needs to take the limit $N \to \infty$ of the resulting 1d N-body wave function and show that initial condensation is preserved by the dynamics. The time-dependent GP equation (3.4) should then emerge as Euler–Lagrange equation (see Section 1.4.1).

Finally, let us remark that the statement (1.55) concerning the excitation spectrum does not hold uniformly in N. However, regarding the lower and upper bound (1.56) and (1.57), one realises that it remains true in the simultaneous limit $(N,\varepsilon) \to (\infty, 0)$ if the parameters η_U and η_L are bounded uniformly in N. For our scaling of the interaction, $NA/(\varepsilon L) \sim \varepsilon$, this would be the case if $\varepsilon^{\frac{3}{8}}N^2 \to 0$ as $(N,\varepsilon) \to \infty$, which corresponds to the admissibility condition $N\varepsilon^{\frac{3}{16}} \ll 1$.

3.2. Higher order corrections to the mean-field dynamics of interacting bosons

3.2.1. Results

We consider a system of N d-dimensional bosons with weak interactions in the mean-field scaling regime, described by the Hamiltonian

$$H_{N,\beta}(t) := \sum_{j=1}^{N} \left(-\Delta_j + V^{\text{ext}}(t, x_j)\right) + \frac{1}{N-1} \sum_{i<j} v_{N,\beta}(x_i - x_j) \,. \tag{3.30}$$

Here, V^{ext} denotes some possibly time-dependent external potential, which is chosen such that $H_{N,\beta}(t)$ is self-adjoint on the time-independent domain $H^2(\mathbb{R}^{dN})$. The interaction is given by

$$v_{N,\beta}(x) := N^{d\beta} v(N^\beta x) \,, \qquad \beta \in [0, \tfrac{1}{d}) \,, \tag{3.31}$$

where $v : \mathbb{R}^d \to \mathbb{R}$ is assumed bounded, spherically symmetric and compactly supported. As explained in Section 1.2.4, the scaling (3.31) is a mean-field scaling: the range of $v_{N,\beta}$ is much larger than the mean inter-particle distance $N^{-1/d}$ and the total prefactor $(N-1)^{-1}N^{d\beta}$ tends to zero as $N \to \infty$. The scaling $\beta = 0$ corresponds to the Hartree regime.

The dynamics of the N-body system are described by the unitary time evolution $\{U(t,s)\}_{t,s\in\mathbb{R}}$, which satisfies the Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t} U(t,s) = H_{N,\beta}(t) U(t,s) \,, \qquad U(s,s) = \mathbb{1} \,. \tag{3.32}$$

The N-body wave function at time $t \in \mathbb{R}$ is denoted as

$$\psi(t) = U(t,0)\psi_0 \,, \qquad \psi_0 \in L^2_+(\mathbb{R}^{dN}) \,. \tag{3.33}$$

We consider systems which initially exhibit BEC. As explained in Section 1.4.2, the dynamics of the condensate wave function are determined by the Hartree equation (1.112),

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left(-\Delta + V^{\text{ext}}(t) + \overline{v}^{\varphi(t)} - \mu^{\varphi(t)}\right) \varphi(t) =: h^{\varphi(t)}(t)\varphi(t) \,, \tag{3.34}$$

where $\overline{v}^{\varphi(t)}$ and $\mu^{\varphi(t)}$ are defined as in (1.113) and (1.114),

$$\overline{v}^{\varphi(t)} = v_{N,\beta} * |\varphi(t)|^2 \,, \qquad \mu^{\varphi(t)} = \tfrac{1}{2} \int\limits_{\mathbb{R}^3} \mathrm{d}x \int\limits_{\mathbb{R}^3} \mathrm{d}y |\varphi(t,x)|^2 |\varphi(t,y)|^2 v_{N,\beta}(x-y) \,.$$

The Hartree evolution of the condensate characterises the N-body dynamics $\psi(t)$ on the level of reduced densities in the sense of (1.73). A more precise characterisation of the dynamics is an approximation with respect to the $L^2(\mathbb{R}^{dN})$-norm, for which also the excitations from the condensate need to be regarded. The corresponding results (1.127) and (1.140) are presented in Section 1.5.3.

The goal of this project is to derive higher order corrections to this norm approximation, i.e., to approximate the N-body wave function in norm to arbitrary order in powers of N^{-1}. More precisely, we construct a sequence of N-body wave functions $\{\psi_\varphi^{(a)}(t)\}_{a\in\mathbb{N}} \subset L^2(\mathbb{R}^{dN})$ such that, for sufficiently large N,

$$\|\psi(t) - \psi_\varphi^{(a)}(t)\|^2 \leq C(t) N^{-a\delta(\beta,\gamma)}\,, \qquad \beta \in [0, \tfrac{1}{4d})\,, \tag{3.35}$$

for some time-dependent constant $C(t)$. The exponent $\delta(\beta,\gamma)$ is positive and depends on β and on a parameter γ which is introduced below. For our analysis, we apply and extend the first quantised framework introduced in Section 1.5.3, which is based on the works [135, 134, 146].

To derive an approximation with higher precision, we require stronger bounds on the initial excitations. More precisely, we assume that the first A moments of the number of excitations from the condensate in the initial state are sub-leading, where the choice of A depends on the index a of the sequence $\psi_\varphi^{(a)}(t)$ in (3.35).

Recall that the excitations from the condensate $\varphi(t)^{\otimes N}$ are given by the truncated Fock vector

$$\xi_{\varphi(t)} = \mathfrak{U}_N^{\varphi(t)} \psi^N(t) \in \mathcal{F}_{\perp\varphi(t)}^{\leq N}$$

with components (1.141). The excitation Fock space $\mathcal{F}_{\perp\varphi(t)}^{\leq N}$ was defined in (1.120), and the map $\mathfrak{U}_N^{\varphi(t)}$ was introduced in (1.122). The a'th moment of the number of excitations from ${\varphi_0}^{\otimes N}$ contained in the initial state ψ_0 is

$$\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \xi_{\varphi_0} \right\rangle_{\mathcal{F}_{\perp\varphi_0}^{\leq N}} = \sum_{k=0}^{N} k^a \|\xi_{\varphi_0}^{(k)}\|^2_{L^2(\mathbb{R}^{dk})}\,,$$

where $\mathcal{N}_{\varphi(t)}$ denotes the number operator on $\mathcal{F}_{\perp\varphi(t)}^{\leq N}$. Our assumption on the initial data can be formulated as follows: Let $\gamma \in (0,1]$. We assume that for all $a \in \{0, \ldots, A\}$, there exists some constant $C(a)$ depending only on a such that

$$\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \xi_{\varphi_0} \right\rangle_{\mathcal{F}_{\perp\varphi_0}^{\leq N}} \leq C(a) N^{(1-\gamma)a}\,. \tag{3.36}$$

Note that $\gamma = 0$ corresponds to the trivial bound $\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \xi_{\varphi_0} \right\rangle \leq N^a$, while $\gamma = 1$ implies that the bound is uniform in N. Hence, (3.36) states that the expected number of excitations contained in ψ_0 must be sub-leading, in the sense that the moments of the relative number of excitations, i.e., the expectation values of $(\mathcal{N}_{\varphi_0}/N)^a$, must vanish as $N \to \infty$.

Note that (3.36) provides a bound on the high components of the excitation vector since, for example,

$$\sum_{k=0}^{N} k^A \|\xi_{\varphi_0}^{(k)}\|^2_{L^2(\mathbb{R}^{dk})} \lesssim N^{(1-\gamma)A} \quad \Rightarrow \quad \|\xi_{\varphi_0}^{(N)}\|^2_{L^2(\mathbb{R}^{dk})} \lesssim N^{-\gamma A}.$$

In other words, it must be very unlikely to find significantly many particles outside the condensate, whereas no such restriction is imposed on excitations involving only very few particles (with respect to N).

For our analysis, it is more convenient to write (3.36) in a different way. By (1.142), the inequality (3.36) is equivalent to

$$\|(\widehat{n^{\varphi_0}})^a \psi_0\|^2 = N^{-a} \left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \xi_{\varphi_0} \right\rangle_{\mathcal{F}_{\perp\varphi_0}^{\leq N}} \leq C(a) N^{-\gamma a} ,$$

where $\widehat{n^{\varphi}}$ denotes the weighted operator from Definition 1.4.1 with weight function $n(k) = \sqrt{\frac{k}{N}}$. We now introduce a second weight function,

$$m(k) := \sqrt{\frac{k+1}{N}} ,$$

such that the corresponding operator $\widehat{m^{\varphi}}$ is related to $\widehat{n}^{\varphi}$ via

$$(\widehat{n^{\varphi}})^{2a} \leq (\widehat{m^{\varphi}})^{2a} \leq 2^a (\widehat{n^{\varphi}})^{2a} + N^{-a} \tag{3.37}$$

in the sense of operators. In terms of $\widehat{m^{\varphi}}$, (3.36) can equivalently be expressed as

$$\|(\widehat{m^{\varphi_0}})^a \psi_0\|^2 \leq C'(a) N^{-\gamma a} \tag{3.38}$$

for some constant $C'(a)$ depending on a. In the following, we prefer to work with the version (3.38), since this simplifies many statements, in particular Proposition 3.2.3b below.

Our analysis is valid for times where the solution $\varphi(t)$ of the Hartree equation exists in $H^k(\mathbb{R}^d)$-sense for $k = \lceil \frac{d}{2} \rceil$. The maximal time of $H^k(\mathbb{R}^d)$-existence is defined as

$$T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}} := \sup \left\{ t \in \mathbb{R}_0^+ : \|\varphi(t)\|_{H^k(\mathbb{R}^d)} < \infty \text{ for } k = \lceil \tfrac{d}{2} \rceil \right\} .$$

It depends on the dimension d, the sign of $\overline{v}^{\varphi(t)}$, and the regularity of the external trap $V^{\mathrm{ext}}(t)$.

Our assumptions on the model (3.30) and on the initial data can be summarised as follows:

A1 *Interaction potential.* Let $v : \mathbb{R}^d \to \mathbb{R}$ be spherically symmetric and bounded uniformly in N. Further, assume that $\operatorname{supp} v \subseteq \{x \in \mathbb{R}^d : |x| \lesssim 1\}$.

A2 *External potential.* Let $V^{\mathrm{ext}} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $V^{\mathrm{ext}}(\cdot, x) \in \mathcal{C}(\mathbb{R})$ for each $x \in \mathbb{R}^d$ and $V^{\mathrm{ext}}(t, \cdot) \in L^\infty(\mathbb{R}^d)$ for each $t \in \mathbb{R}$.

A3 *Initial data.* Let $\psi_0 \in H^2(\mathbb{R}^{dN}) \cap L^2_+(\mathbb{R}^{dN})$ and $\varphi_0 \in H^k(\mathbb{R}^d)$, $k = \lceil \frac{d}{2} \rceil$, both be normalised. Let $\gamma \in (0, 1]$ and $A \in \mathbb{N}$. Assume that for any $a \in \{0, \dots, A\}$, there exists a set of non-negative, a-dependent constants $\{\mathfrak{C}_a\}_{0 \le a \le A}$ with $\mathfrak{C}_0 = 1$ such that, for sufficiently large N,

$$\left\| (\widehat{m^{\varphi_0}})^a \psi_0 \right\|^2 \le \mathfrak{C}_a \, N^{-\gamma a} \, .$$

To construct the approximating sequence $\{\psi^{(a)}_\varphi(t)\}_{a \in \mathbb{N}}$, we recall the effective Hamiltonian $\widetilde{H}^{\varphi(t)}(t)$ from (1.137),

$$\begin{aligned} \widetilde{H}^{\varphi(t)} \quad := \quad & \sum_{j=1}^N h_j^{\varphi(t)} + \frac{1}{N-1} \sum_{i<j} \Big(p_i^{\varphi(t)} q_j^{\varphi(t)} v_{N,\beta}^{(ij)} q_i^{\varphi(t)} p_j^{\varphi(t)} \\ & \qquad + p_i^{\varphi(t)} p_j^{\varphi(t)} v_{N,\beta}^{(ij)} q_i^{\varphi(t)} q_j^{\varphi(t)} + \text{h.c.} \Big) \, , \end{aligned}$$

which generates the time evolution $\widetilde{U}_\varphi(t, s)$. Since $\widetilde{U}_\varphi(t, 0)\psi_0$ is close to $\psi(t)$ in norm by (1.140), we define the first element $\psi^{(1)}_\varphi(t)$ as

$$\psi^{(1)}_\varphi(t) := \widetilde{U}_\varphi(t, 0)\psi_0 \, . \tag{3.39}$$

With $\psi^{(1)}_\varphi(t)$ as starting point, the higher elements are constructed as Duhamel expansions in terms of the cubic and quartic terms $\mathcal{C}^{\varphi(t)}$ and $\mathcal{Q}^{\varphi(t)}$ given in (1.138) and (1.139),

$$\begin{aligned} \mathcal{C}^{\varphi(t)} \quad := \quad & \frac{1}{N-1} \sum_{i<j} \Big(q_i^{\varphi(t)} q_j^{\varphi(t)} \Big(v_{N,\beta}^{(ij)} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) \Big) \times \\ & \qquad \times (q_i^{\varphi(t)} p_j^{\varphi(t)} + p_i^{\varphi(t)} q_j^{\varphi(t)}) + \text{h.c.} \Big) \, , \end{aligned}$$

$$
\begin{aligned}
\mathcal{Q}^{\varphi(t)} \quad := \quad & \frac{1}{N-1}\sum_{i<j} q_i^{\varphi(t)} q_j^{\varphi(t)} \times \\
& \times \left(v_{N,\beta}^{(ij)} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)} \right) q_i^{\varphi(t)} q_j^{\varphi(t)} \,.
\end{aligned}
$$

The next two elements of the sequence are defined as

$$
\psi_\varphi^{(2)}(t) \quad := \quad \widetilde{U}_\varphi(t,0)\psi_0 - \mathrm{i}\int_0^t \mathrm{d}s\, \widetilde{U}_\varphi(t,s)\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0 \,, \tag{3.40}
$$

$$
\begin{aligned}
\psi_\varphi^{(3)}(t) \quad := \quad & \widetilde{U}_\varphi(t,0)\psi - \mathrm{i}\int_0^t \mathrm{d}s\, \widetilde{U}_\varphi(t,s)\left(\mathcal{C}^{\varphi(s)} + \mathcal{Q}^{\varphi(s)}\right)\widetilde{U}_\varphi(s,0)\psi_0 \\
& - \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2\, \widetilde{U}_\varphi(t,s_2)\,\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\,\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0 \,.
\end{aligned} \tag{3.41}
$$

The a'th approximating function is constructed as follows:

Definition 3.2.1. *Let $I_1^{\varphi(t)} := \mathcal{C}^{\varphi(t)}$ and $I_2^{\varphi(t)} := \mathcal{Q}^{\varphi(t)}$. Define the set*

$$
\mathcal{S}_n^{(k)} := \left\{ (j_1, \ldots, j_n) : \; j_\ell \in \{1,2\} \;\textit{ for }\; \ell = 1, \ldots, n \;\textit{ and }\; \sum_{\ell=1}^n j_\ell = k \right\},
$$

i.e., the set of n-tuples with elements in $\{1,2\}$ such that the elements of each tuple add up to k. For $n \in \mathbb{N}$ and $n \leq k \leq 2n$, define

$$
\begin{aligned}
T_n^{(k)} \quad := \quad & \sum_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}} (-\mathrm{i})^n \prod_{\nu=1}^n \left(\int_{s_{\nu-1}}^t \mathrm{d}s_\nu \right) \widetilde{U}_\varphi(t,s_n) \times \\
& \times \prod_{\ell=0}^{n-1} \left(I_{j_{n-\ell}}^{\varphi(s_{n-\ell})} \widetilde{U}_\varphi(s_{n-\ell}, s_{n-\ell-1}) \right) \psi_0 \\
= \quad & (-\mathrm{i})^n \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n\, \widetilde{U}_\varphi(t,s_n) \times \\
& \times \sum_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}} \left(I_{j_n}^{\varphi(s_n)} \widetilde{U}_\varphi(s_n,s_{n-1}) I_{j_{n-1}}^{\varphi(s_{n-1})} \cdots \widetilde{U}_\varphi(s_2,s_1) I_{j_1}^{\varphi(s_1)} \right) \widetilde{U}_\varphi(s_1,0)\psi_0 \,,
\end{aligned}
$$

where $s_0 := 0$. The products are understood as ordered, i.e., $\prod_{\ell=0}^L P_\ell := P_0 P_1 \cdots P_L$ for $L \in \mathbb{N}$ and any expressions P_ℓ. Besides, let $T_0^{(0)} := \widetilde{U}_\varphi(t,0)\psi_0$ for $n = k = 0$, and $T_n^{(k)} := 0$ for $k < n$ and $k > 2n$.

The elements of the sequence $\{\psi_\varphi^{(a)}\}_{a\in\mathbb{N}}$ *are defined as*

$$\psi_\varphi^{(a)}(t) \quad := \quad \sum_{k=0}^{a-1} \sum_{n=\lceil \frac{k}{2} \rceil}^{k} T_n^{(k)} \;=\; \sum_{n=0}^{a-1} \sum_{k=n}^{\min\{2n,a-1\}} T_n^{(k)}.$$

In the main result of this project, we prove that, given any desired precision of the approximation with respect to N^{-1}, there exists an $a \in \mathbb{N}$ such that the corresponding function $\psi_\varphi^{(a)}(t)$ approximates the actual N-body dynamics $\psi(t)$ to that order. To compute $\psi_\varphi^{(a)}(t)$, an a-dependent but N-independent number of steps is required, as well as the knowledge of the (first order) norm approximation $\widetilde{U}_\varphi(t,0)\psi_0$.

Theorem 3.2.2. *Let* $\beta \in [0, \frac{1}{4d})$ *and assume* A1 – A3 *with* $A \in \{1, \ldots, N\}$ *and with* $\gamma \in (\frac{2+d\beta}{3}, 1]$. *Let* $\psi(t)$ *and* $\varphi(t)$ *denote the solutions of* (3.33) *and* (3.34) *with initial data* ψ_0 *and* φ_0 *from* A3, *respectively, and let* $\psi_\varphi^{(a)}(t)$ *be defined as in Definition 3.2.1. Then for sufficiently large* N, $t \in \left[0, T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\right)$ *and* $a \in \{1, \ldots, \lfloor \frac{A}{6} \rfloor\}$, *there exists a constant* $c(a)$ *such that*

$$\|\psi(t) - \psi_\varphi^{(a)}(t)\|^2 \lesssim \mathrm{e}^{c(a)\int_0^t \|\varphi(s)\|^2_{H^k(\mathbb{R}^d)}\,\mathrm{d}s}\, N^{-a\delta(\beta,\gamma)}, \tag{3.42}$$

where

$$\delta(\beta,\gamma) = \begin{cases} 1 - 4d\beta & \textit{for } \; 1 - d\beta \leq \gamma \leq 1\,, \\ 3\gamma - 2 - d\beta & \textit{for } \; \frac{2+d\beta}{3} < \gamma \leq 1 - d\beta\,. \end{cases} \tag{3.43}$$

3.2.2. Strategy of proof

The first part of the proof consists of estimating the growth of the number of excitations under the time evolutions $U(t,s)$ and $\widetilde{U}_\varphi(t,s)$ (Proposition 3.2.3). While the statement for $U(t,s)$ characterising the N-body dynamics is an interesting result on its own, the corresponding assertion for $\widetilde{U}_\varphi(t,s)$ is crucial for the proof of our main result. As a second step, we use these bounds to prove Theorem 3.2.2.

Growth of higher moments of the number of excitations

Let us begin with a general statement concerning the growth of the number of excitations under the dynamics $U(t,s)$ and $\widetilde{U}_\varphi(t,s)$, irrespective of the initial number of excitations.

Proposition 3.2.3. *Let* $j \in \mathbb{N}$, $\beta \in [0, \frac{1}{d})$ *and assume* A1 *and* A2. *Let* $\psi \in L^2_+(\mathbb{R}^{dN})$, $s \in \mathbb{R}$, $\varphi(s) \in H^k(\mathbb{R}^d)$ *for* $k = \lceil \frac{d}{2} \rceil$, *and let* $\varphi(t)$ *be the solution of* (3.34) *with initial datum* $\varphi(s)$. *Then it holds for* $t \in \left[s, s + T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\right)$ *and sufficiently large* N *that* *(a) for any $b \in \mathbb{N}_0$,*

$$\left\| \left(\widehat{m^{\varphi(t)}}\right)^j U(t,s)\psi \right\|^2 \lesssim c(t,s) \sum_{n=0}^{j} N^{n(-1+d\beta)} \left\| \left(\widehat{m^{\varphi(s)}}\right)^{j-n} \psi \right\|^2 + c(t,s) \sum_{n=0}^{b} N^{n(-1+d\beta)+d\beta b} \left\| \left(\widehat{m^{\varphi(s)}}\right)^{b-n} \psi \right\|^2 ,$$

(b)

$$\left\| \left(\widehat{m^{\varphi(t)}}\right)^j \widetilde{U}_\varphi(t,s)\psi \right\|^2 \lesssim c(t,s) \sum_{n=0}^{j} N^{n(-1+d\beta)} \left\| \left(\widehat{m^{\varphi(s)}}\right)^{j-n} \psi \right\|^2 ,$$

where $c(t,s) \lesssim \exp\left\{ C \int_s^t \|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)} \, \mathrm{d}s_1 \right\}$ for some $C > 0$.

This proposition is proven in [34, Proposition 2.4]. Note that part (a) concerning the full time evolution $U(t,s)$ contains two sums. The first sum runs from zero to j, whereas the summation in the second sum may be chosen for convenience (see below).

As a consequence of (3.37), we can equivalently express Proposition 3.2.3 in terms of $\widehat{n^\varphi}$ instead of $\widehat{m^\varphi}$. For instance, part (b) can be formulated as

$$\|(\widehat{n^\varphi})^j \widetilde{U}_\varphi(t,s)\psi\|^2 \lesssim c(t,s) \sum_{n=0}^{j} N^{n(-1+d\beta)} \left(2^{j-n} \|(\widehat{n^\varphi})^{j-n}\psi\|^2 + N^{-j+n} \right) ,$$

which contains an additional term N^{-j+n}. Since the proof of Theorem 3.2.2 requires an iteration of this proposition, the version with $\widehat{m^\varphi}$ is more convenient.

Under the additional assumption A3 on the initial data, Proposition 3.2.3 implies that the first A moments of the number of excitations remain sub-leading under the dynamics $U(t,0)$ and $\widetilde{U}_\varphi(t,0)$ ([34, Corollary 2.5]): Denote $c(t,0) \equiv c(t)$ and

$$\xi_{\varphi_0} = \mathfrak{U}_N^{\varphi_0} \psi_0 \,, \qquad \xi_{\varphi(t)} = \mathfrak{U}_N^{\varphi(t)} \psi(t) \,, \qquad \widetilde{\xi}_{\varphi(t)} = \mathfrak{U}_N^{\varphi(t)} \widetilde{U}_\varphi(t,0)\psi_0 \,.$$

Assume that

$$\|(\widehat{m^{\varphi_0}})^a \psi_0\|^2 \lesssim N^{-\gamma a} \,, \quad \text{or, equivalently, that} \quad \left\langle \xi_{\varphi_0}, \mathcal{N}^a_{\varphi_0} \, \xi_{\varphi_0} \right\rangle_{\mathcal{F}^{\leq N}_{\perp \varphi_0}} \lesssim N^{(1-\gamma)a} \,.$$

Then it follows

- for the time evolution $U(t,s)$ that

$$\|(\widehat{m^{\varphi(t)}})^a\psi(t)\|^2 \lesssim c(t)\begin{cases} N^{-a(1-d\beta)} & \text{for } \beta\in[0,\tfrac{1}{2d})\,,\ 1-d\beta\leq\gamma\leq 1\,,\\ N^{-\gamma a} & \text{for } \beta\in[0,\tfrac{1}{d})\,,\ d\beta<\gamma\leq 1-d\beta\,,\end{cases}$$

or, equivalently, that

$$\left\langle \xi_{\varphi(t)},\mathcal{N}^a_{\varphi(t)}\xi_{\varphi(t)}\right\rangle_{\mathcal{F}^{\leq N}_{\perp\varphi(t)}} \lesssim c(t)\begin{cases} N^{d\beta a} & \text{for } \beta\in[0,\tfrac{1}{2d})\,,\ 1-d\beta\leq\gamma\leq 1\,,\\ N^{(1-\gamma)a} & \text{for } \beta\in[0,\tfrac{1}{d})\,,\ d\beta<\gamma\leq 1-d\beta\,,\end{cases}$$

- for the time evolution $\widetilde{U}_\varphi(t,0)$ and $\beta\in[0,\frac{1}{d})$ that

$$\|(\widehat{m^{\varphi(t)}})^a\widetilde{U}_\varphi(t,0)\psi_0\|^2 \lesssim c(t)\begin{cases} N^{-a(1-d\beta)} & \text{for } 1-d\beta\leq\gamma\leq 1\,,\\ N^{-\gamma a} & \text{for } 0<\gamma\leq 1-d\beta\,,\end{cases}$$

or, equivalently, that

$$\left\langle \widetilde{\xi}_{\varphi(t)},\mathcal{N}^a_{\varphi(t)}\widetilde{\xi}_{\varphi(t)}\right\rangle_{\mathcal{F}^{\leq N}_{\perp\varphi(t)}} \lesssim c(t)\begin{cases} N^{d\beta a} & 1-d\beta\leq\gamma\leq 1\,,\\ N^{(1-\gamma)a} & 0<\gamma\leq 1-d\beta\,.\end{cases}$$

The leading order terms in the sums in Proposition 3.2.3 change at $\gamma=1-d\beta$: for initial data satisfying A3, we obtain

$$N^{n(-1+d\beta)}\|(\widehat{m^{\varphi(s)}})^{j-n}\psi_0\|^2 \lesssim N^{n(\gamma-1+d\beta)-\gamma j}$$

and

$$N^{n(-1+d\beta)+d\beta b}\|(\widehat{m^{\varphi(s)}})^{b-n}\psi_0\|^2 \lesssim N^{n(\gamma-1+d\beta)-b(\gamma-d\beta)}\,,$$

hence the term corresponding to $n=0$ is leading for $\gamma<1-d\beta$, while the addend with maximal n is the dominant contribution for $\gamma>1-d\beta$. Consequently, we obtain different estimates for values of γ below and above this threshold. The additional restrictions on β and γ for the time evolution $U(t,0)$ are due to the second sum in Proposition 3.2.3a: if $\beta<\frac{1}{2d}$ or $\gamma>d\beta$, it is possible to choose b sufficiently large that the first sum dominates for large N.

For $\beta=0$, both time evolutions preserve the property A3 exactly with respect to N, up to a time dependent constant. For $\beta>0$, the conservation is exact only for small γ, whereas one looses some power of N for larger γ. Further, note that for the range $\gamma\in(0,d\beta)$, we do not obtain a non-trivial estimate for the excitations $\xi_{\varphi(t)}$ contained in $\psi(t) = U(t,0)\psi_0$.

To prove Proposition 3.2.3, we essentially adapt the proofs of [135, Lemma 2.1] and [146, Corollary 4.2] to our situation. The basic idea is to derive a hierarchy of j Grönwall estimates for a variant of the counting functional used by Pickl for Hartree and NLS regime (Section 1.4.4).

Let us consider the functional

$$\left\langle\!\!\left\langle \Psi(t), \widehat{f^{\varphi(t)}}\Psi(t) \right\rangle\!\!\right\rangle , \qquad \Psi(t) \in \{\psi(t), \widetilde{U}_\varphi(t,0)\psi_0\} ,$$

for some appropriate weight function $f(k)$, where $\Psi(t)$ stands for the wave function evolving under either of the two dynamics covered by Proposition 3.2.3. If we choose the weight $f(k)$ as $m(k)^{2j}$, the analysis presented in Section 1.4.4 leads to the estimate

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\|(\widehat{m^{\varphi(t)}})^j\Psi(t)\|^2 &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle \Psi(t), (\widehat{m^{\varphi(t)}})^{2j}\Psi(t) \right\rangle\!\!\right\rangle \\ &\lesssim \left\langle\!\!\left\langle \Psi(t), (\widehat{m^{\varphi(t)}})^{2j}\Psi(t) \right\rangle\!\!\right\rangle + \mathcal{O}(1) \; . \end{aligned} \tag{3.44}$$

The remainder term $\mathcal{O}(1)$ is at best of order N^{-1}, hence (3.44) can at most lead to the j-independent estimate

$$\|(\widehat{m^{\varphi(t)}})^j\Psi(t)\|^2 \lesssim \mathrm{e}^{Ct}N^{-1}$$

by Grönwall's Lemma 1.4.3, even if $\|(\widehat{m^{\varphi_0}})^j\Psi_0\|^2 = 0$. To improve this, we will modify the estimates leading to (3.44) to yield a bound of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|(\widehat{m^{\varphi(t)}})^j\Psi(t)\|^2 \lesssim \|(\widehat{m^{\varphi(t)}})^j\Psi(t)\|^2 + \mathcal{O}(1)\,\|(\widehat{m^{\varphi(t)}})^{j-1}\Psi(t)\|^2 . \tag{3.45}$$

By Grönwall's lemma, this leads to a statement of the kind

$$\|(\widehat{m^{\varphi(t)}})^j\Psi(t)\|^2 \lesssim \mathrm{e}^{Ct}\left(\|(\widehat{m^{\varphi_0}})^j\Psi_0\|^2 + \mathcal{O}(1)\int_0^t \|(\widehat{m^{\varphi(s)}})^{j-1}\Psi(s)\|\,\mathrm{d}s \right) , \tag{3.46}$$

to which we can again apply Grönwall's lemma, using again (3.45) but now with the choice $j-1$. Iterating this procedure j times results in a bound of the form

$$\|(\widehat{m^{\varphi(t)}})^j\Psi(t)\|^2 \lesssim \mathrm{e}^{Ct}\sum_{n=0}^{j}\mathcal{O}(1)^n\,\|(\widehat{m^{\varphi_0}})^{j-n}\Psi_0\|^2 , \tag{3.47}$$

which suffices to prove Proposition 3.2.3.

It remains to derive a suitable bound of the form (3.45). Recall that the time derivative of the functional $\left\langle\!\left\langle \Psi(t), \widehat{f^{\varphi(t)}}\Psi(t)\right\rangle\!\right\rangle$ was computed in Section 1.4.4 as the sum of the three expressions (1.83) to (1.85). Note that the Hamiltonian entering (1.81) is given by $H_{N,\beta}(t)$ for the choice $\Psi(t)=\psi(t)$ and by $\widetilde{H}^{\varphi(t)}(t)$ for $\Psi(t)=\widetilde{U}_\varphi(t,0)\psi_0$, while h in (1.81) is taken as $h^{\varphi(t)}(t)$. Due to the relations (1.136), the *(qp–pp)*-term (1.83) equals zero. Moreover, in the case $\Psi(t)=\widetilde{U}_\varphi(t,0)\psi_0$, also the *(pq–qq)*-term (1.85) vanishes. In the sequel, we treat the two cases $\Psi(t)=\psi(t)$ and $\Psi(t)=\widetilde{U}_\varphi(t,0)\psi_0$ separately.

Case 1: $\Psi(t)=\widetilde{U}_\varphi(t,0)\psi_0$.
This case is simpler since only the *(qq–pp)*-term (1.84) contributes. For simplicity dropping all indices $\varphi(t)$, this term can be estimated as

$$\begin{aligned}&\left|N\Im\left\langle\!\left\langle \Psi(t), q_1q_2(\widehat{f}-\widehat{f}_{-2})^{\frac12}v_{N,\beta}^{(12)}p_1p_2(\widehat{f}_2-\widehat{f})^{\frac12}\Psi(t)\right\rangle\!\right\rangle\right|\\ &\qquad\lesssim N\left\langle\!\left\langle \Psi(t),(\widehat{f}-\widehat{f}_{\pm2})\widehat{n}^2\Psi(t)\right\rangle\!\right\rangle + N^{d\beta}\left\langle\!\left\langle \Psi(t),(\widehat{f}-\widehat{f}_{\pm2})\Psi(t)\right\rangle\!\right\rangle\,, \end{aligned}\tag{3.48}$$

which is to be understood as taking the maximum over $(\widehat{f}-\widehat{f}_{-2})$ and $(\widehat{f}-\widehat{f}_2)$ (see [34, Eqns. (45)-(46)]). Choosing $f(k)$ as $m(k)^{2j}$, one observes that

$$|m(k)^{2j}-m(k\pm2)^{2j}|\lesssim\frac{m(k)^{2(j-1)}}{N}\,,\quad m(k)^{2(j-1)}n(k)\lesssim m(k)^{2j}\,.\tag{3.49}$$

Consequently, instead of estimating $\|\widehat{f}-\widehat{f}_{\pm2}\|_{\mathrm{op}}$ by means of the derivative $|f'(k)|$ as in Section 1.4.4, (3.48) is bounded in terms of $\|\widehat{m}^j\Psi(t)\|$ and $\|\widehat{m}^{j-1}\Psi(t)\|$, namely

$$|(3.48)|\lesssim\|\widehat{m}^j\Psi(t)\|^2+N^{-1+d\beta}\|\widehat{m}^{j-1}\Psi(t)\|^2\,.\tag{3.50}$$

This is precisely the required bound (3.45), and part (b) of Proposition 3.2.3 follows from (3.47) with $\mathcal{O}(1)=N^{-1+d\beta}$.

Case 2: $\Psi(t)=\psi(t)$.
If $\Psi(t)$ denotes the wave function evolving under the full dynamics $U(t,s)$, the situation becomes more involved since the *(pq–qq)*-term (1.84) does not vanish. Hence, additionally to (3.48), we must control the expression

$$\begin{aligned}&N\Im\left\langle\!\left\langle \psi(t), q_1q_2(\widehat{f}-\widehat{f}_{-1})^{\frac12}Z_{N,\beta}^{(12)}p_1q_2(\widehat{f}_1-\widehat{f})^{\frac12}\psi(t)\right\rangle\!\right\rangle\\ &\lesssim\ N^{1+\frac{d\beta}{2}}\left\langle\!\left\langle \psi(t),(\widehat{f}-\widehat{f}_{-1})\widehat{n}^4\psi(t)\right\rangle\!\right\rangle^{\frac12}\left\langle\!\left\langle \psi(t),(\widehat{f}-\widehat{f}_{-1})\widehat{n}^2\psi(t)\right\rangle\!\right\rangle^{\frac12}\end{aligned}\tag{3.51}$$

[34, Eqn. (47)], where $Z_{N,\beta}^{(12)}:=v_{N,\beta}^{(12)}-\overline{v}^{\varphi(t)}(x_i)-\overline{v}^{\varphi(t)}(x_j)+2\mu^{\varphi(t)}$. Choosing simply $f(k)=m(k)^{2j}$ as before only leads to the insufficient estimate $|(3.51)|\lesssim N^{\frac{d\beta}{2}}\|\widehat{m}^j\Psi(t)\|^2$.

To amend this, we introduce an auxiliary weight function

$$w_\lambda(k) := \begin{cases} \dfrac{k+1}{N^\lambda} & 0 \le k \le N^\lambda - 1 \\ 1 & \text{else} \end{cases} \qquad \text{for } \lambda \in (0,1)\,.$$

For $\lambda = 1$, this essentially reduces to the weight $m^2(k)$. More precisely, the relation between $w_\lambda(k)$ and $m(k)$ is given by

$$w_\lambda^j(k) \;\le\; N^{j(1-\lambda)} m^{2j}(k) \tag{3.52}$$

and

$$m^{2j}(k) \;\le\; \begin{cases} N^{-j(1-\lambda)} w_\lambda^j(k) & 0 \le k \le N^\lambda - 1 \\ 2^j \;=\; 2^j w_\lambda^b(k) \;\lesssim\; w_\lambda^b(k) & N^\lambda - 1 \le k \le N\,, \end{cases} \tag{3.53}$$

hence

$$m^{2j}(k) \lesssim N^{-j(1-\lambda)} w_\lambda^j(k) + w_\lambda^b(k) \tag{3.54}$$

for any $b \in \mathbb{N}$, where we exploited that $w_\lambda(k) = 1$ for $k \ge N^\lambda - 1$.

We now choose $f(k)$ as $w_\lambda^j(k)$. Similarly to (3.49), one finds for $n = 1, 2$ that

$$|w_\lambda(k) - w_\lambda(k \pm n)| \lesssim \left\{ \begin{matrix} \dfrac{w_\lambda(k)^{j-1}}{N^\lambda} & 0 \le k \le N^\lambda \\ 0 & \text{else} \end{matrix} \right\} =: \ell_\lambda^{(j)}(k)\,.$$

Since $\ell_\lambda^{(j)}(k)$ satisfies

$$\ell_\lambda^{(j)}(k) n^2(k) \lesssim N^{-1} w_\lambda^j(k)\,, \qquad \ell_\lambda^{(j)}(k) n^4(k) \lesssim N^{-2+\lambda} w_\lambda^j(k)\,,$$

we obtain

$$\begin{aligned} |(3.48)| \;&\lesssim\; \left\langle\!\left\langle \psi(t), \widehat{w_\lambda}^j \psi(t) \right\rangle\!\right\rangle + N^{d\beta-\lambda} \left\langle\!\left\langle \psi(t), \widehat{w_\lambda}^{j-1} \psi(t) \right\rangle\!\right\rangle\,, \\ |(3.51)| \;&\lesssim\; N^{\frac{-1+d\beta+\lambda}{2}} \left\langle\!\left\langle \psi(t), \widehat{w_\lambda}^j \psi(t) \right\rangle\!\right\rangle\,. \end{aligned}$$

With the choice $\lambda = 1 - d\beta$, the expression (3.51) is controllable, and we obtain a bound of the form (3.45) with $\mathcal{O}(1) = N^{-1+2d\beta}$ for the weight $w_\lambda^j(k)$. Hence, (3.47) implies

$$\left\langle\!\left\langle \psi(t), \widehat{w_\lambda}^j \psi(t) \right\rangle\!\right\rangle \lesssim \mathrm{e}^{Ct} \sum_{n=0}^{j} N^{n(-1+2d\beta)} \left\langle\!\left\langle \psi_0, \widehat{w_\lambda}^{j-n} \psi_0 \right\rangle\!\right\rangle\,.$$

Finally, using the relations (3.52) and (3.53), one obtains an estimate in terms of $\widehat{m}^j$, which leads to part (a) of Proposition 3.2.3b. In particular, note that this construction via w_λ and (3.54) explain why we obtain two sums in part (a), while part (b) is of an easier form.

Higher order corrections to the norm approximation

Let us now turn to the proof of the main result. The first element of the approximating sequence $\{\psi_\varphi^{(a)}\}_{a\in\mathbb{N}}$ is given by (3.39) as

$$\psi_\varphi^{(1)}(t) = \widetilde{U}_\varphi(t,0)\psi_0 \,.$$

To prove Theorem 3.2.2 for $a = 1$ and to construct the next higher element of the sequence corresponding to $a = 2$, we require three steps:

1. Expand the difference $\big(U(t,0) - \widetilde{U}_\varphi(t,0)\big)\psi_0$ using Duhamel's formula.
2. Estimate all contributions to this difference and identify the leading order term. Its size yields (3.42) for $a = 1$ and fixes the exponent $\delta(\beta,\gamma)$.
3. To construct $\psi_\varphi^{(2)}(t)$, substitute $U(t,s)$ by $\widetilde{U}_\varphi(t,s)$ in the leading order contribution(s) and add the resulting expression as a correction term to $\psi_\varphi^{(1)}(t)$.

Step 1.
Recall that by construction of $\widetilde{H}^{\varphi(t)}(t)$,

$$H_{N,\beta}(t) = \widetilde{H}^{\varphi(t)}(t) + \mathcal{C}^{\varphi(t)} + \mathcal{Q}^{\varphi(t)} \,,$$

hence, Duhamel's formula yields

$$U(t,s)\psi = \widetilde{U}_\varphi(t,s)\psi - \mathrm{i}\int_s^t U(t,r)\left(\mathcal{C}^{\varphi(r)} + \mathcal{Q}^{\varphi(r)}\right)\widetilde{U}_\varphi(r,s)\psi \,\mathrm{d}r \tag{3.55}$$

for any $\psi \in L^2(\mathbb{R}^{dN})$. Consequently,

$$\begin{aligned} \|\psi(t) - \psi_\varphi^{(1)}(t)\| &= \left\| -\mathrm{i}\int_0^t U(t,s)\left(\mathcal{C}^{\varphi(s)} + \mathcal{Q}^{\varphi(s)}\right)\widetilde{U}_\varphi(s,0)\psi_0 \,\mathrm{d}s \right\| \\ &\le \int_0^t \|\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|\,\mathrm{d}s + \int_0^t \|\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|\,\mathrm{d}s \end{aligned} \tag{3.56}$$

by unitarity of $U(t,s)$.

Step 2.
To identify the leading order contributions in (3.56), we combine Proposition 3.2.3b with the following lemma ([34, Lemma 2.6]):

Lemma 3.2.4. *Let $\psi \in L^2_+(\mathbb{R}^{dN})$ and denote by $\varphi(t)$ the solution of* (3.34) *with initial datum $\varphi_0 \in H^k(\mathbb{R}^d)$ for $k = \lceil \frac{d}{2} \rceil$. Then it holds for any $j \in \mathbb{N}_0$ and $t \in \big[0, T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\big)$ that*

(a) $\|\big(\widehat{m^{\varphi(t)}}\big)^j \mathcal{Q}^{\varphi(t)}\psi\|^2 \lesssim N^{2+2d\beta}\|\big(\widehat{m^{\varphi(t)}}\big)^{4+j}\psi\|^2$,

(b) $\|\big(\widehat{m^{\varphi(t)}}\big)^j \mathcal{C}^{\varphi(t)}\psi\|^2 \lesssim \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)} N^{2+d\beta}\|\big(\widehat{m^{\varphi(t)}}\big)^{3+j}\psi\|^2$.

At the core of the proof is the observation that $\mathcal{Q}^{\varphi(t)}/\mathcal{C}^{\varphi(t)}$ contain four/three projections $q^{\varphi(t)}$, each of which contributes an operator $\widehat{n^{\varphi(t)}}$ by (1.74). By (3.37), this is equivalent to gaining four/three factors $\widehat{m^{\varphi(t)}}$. The prefactors stem from combinatorial considerations as well as from the $L^\infty(\mathbb{R}^d)/L^2(\mathbb{R}^d)$-norm of $v_{N,\beta}$.

Let us make this more precise at the example of $\mathcal{Q}^{\varphi(t)}$ and $j = 0$: making use of the abbreviation $Z^\beta_{ij} := v^{(ij)}_{N,\beta} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)}$ with $\|Z^\beta_{ij}\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{d\beta}$, we expand

$$\begin{aligned}
\|\mathcal{Q}^{\varphi(t)}\psi\|^2 &= \tfrac{1}{(N-1)^2}\sum_{i<j}\sum_{k<l}\Big\langle\!\Big\langle \psi, q_i q_j Z^\beta_{ij} q_i q_j q_k q_l Z^\beta_{kl} q_k q_l \psi \Big\rangle\!\Big\rangle \\
&\lesssim \Big\langle\!\Big\langle \psi, q_1 q_2 Z^\beta_{12} q_1 q_2 Z^\beta_{12} q_1 q_2 \psi \Big\rangle\!\Big\rangle + N\Big\langle\!\Big\langle \psi, q_1 q_2 Z^\beta_{12} q_1 q_2 q_3 Z^\beta_{13} q_1 q_3 \psi \Big\rangle\!\Big\rangle \\
&\quad + N^2\Big\langle\!\Big\langle \psi, q_1 q_2 Z^\beta_{12} q_1 q_2 q_3 q_4 Z^\beta_{34} q_3 q_4 \psi \Big\rangle\!\Big\rangle \\
&\lesssim N^{2d\beta}\left(\|q_1 q_2 \psi\|^2 + N\|q_1 q_2 q_3 \widehat{m}^a \psi\|^2 + N^2\|q_1 q_2 q_3 q_4 \psi\|^2\right).
\end{aligned}$$

Since

$$\begin{aligned}
\tbinom{N}{2}\|q_1 q_2 \psi\|^2 &= \sum_{i<j}\langle\!\langle \psi, q_i q_j \psi \rangle\!\rangle < \sum_{i,j,k,l}\langle\!\langle \psi, q_i q_j q_k q_l \psi \rangle\!\rangle \\
&= N^4\Big\langle\!\Big\langle \psi, \Big(\tfrac{1}{N}\sum_{j=1}^N q_j\Big)^4 \psi \Big\rangle\!\Big\rangle < N^4\|\widehat{m}^4\psi\|^2
\end{aligned}$$

and analogously for the second and third term in the bracket, assertion (a) follows.

When applying Lemma 3.2.4 to (3.56), we obtain expressions of the form

$$\|(\widehat{m^{\varphi(s)}})^j \widetilde{U}_\varphi(s,0)\psi_0\|^2.$$

Using Proposition 3.2.3b and finally exploiting assumption A3 on the initial data, one computes

$$\begin{aligned}
\|\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|^2 &\overset{3.2.4}{\lesssim} N^{2+d\beta}\|(\widehat{m^{\varphi(s)}})^3 \widetilde{U}_\varphi(s,0)\psi_0\|^2 \\
&\overset{3.2.3\mathrm{b}}{\lesssim} N^{2+d\beta}\sum_{n=0}^3 N^{n(-1+d\beta)}\|(\widehat{m^{\varphi_0}})^{3-n}\psi_0\|^2
\end{aligned}$$

$$\overset{\mathrm{A3}}{\lesssim} \quad N^{2+d\beta} \sum_{n=0}^{3} N^{n(-1+d\beta+\gamma)-3\gamma} .$$

For the sake of readability, we dropped the time-dependent pre-factors, and we maintain this for the remainder of this section. As above, the size of γ determines the leading order term in the sum: for $\gamma \geq 1 - d\beta$, the dominant contribution issues from $n = 3$, whereas otherwise the addend corresponding to $n = 0$ is of leading order. Consequently,

$$\|\mathcal{C}^{\varphi(s)} \widetilde{U}_{\varphi}(s,0)\psi_0\|^2 \lesssim \begin{cases} N^{-1+4d\beta} & \text{for } 1 - d\beta \leq \gamma \leq 1\,, \\ N^{2+d\beta-3\gamma} & \text{for } \frac{2+d\beta}{3} < \gamma \leq 1 - d\beta\,. \end{cases} \tag{3.57}$$

To ensure that (3.57) converges to zero as $N \to \infty$, we restrict the range of parameters γ admitted by assumption A3 to $\gamma \in (\frac{2+d\beta}{3}, 1]$. Besides, in the first case, the bound is only small for $\beta < \frac{1}{4d}$, and the second case is anyway only possible for $\beta < \frac{1}{4d}$. This essentially causes the restriction of Theorem 3.2.2 to the parameter regime $\beta \in [0, \frac{1}{4d})$.

Analogously to (3.57), we obtain

$$\|\mathcal{Q}^{\varphi(s)} \widetilde{U}_{\varphi}(s,0)\psi_0\|^2 \lesssim \begin{cases} N^{-2+6d\beta} & \text{for } 1 - d\beta \leq \gamma \leq 1\,, \\ N^{2+2d\beta-4\gamma} & \text{for } \frac{2+d\beta}{3} < \gamma \leq 1 - d\beta\,. \end{cases} \tag{3.58}$$

Comparing (3.57) and (3.58), we conclude that the contribution with $\mathcal{C}^{\varphi(s)}$ dominates: since $\beta < \frac{1}{4d}$, it follows that $N^{-2+6d\beta} < N^{-1+4d\beta}$ and, for $\gamma > \frac{2+d\beta}{3} > d\beta$, that $N^{2+2d\beta-4\gamma} < N^{2+d\beta-3\gamma}$. This leads to the estimate

$$\|\psi(t) - \psi_{\varphi}^{(1)}(t)\|^2 \lesssim N^{-\delta(\beta,\gamma)} \tag{3.59}$$

with $\delta(\beta, \gamma)$ from (3.43), which is precisely (3.42) for $a = 1$.

Step 3.
Finally, the second element $\psi_{\varphi}^{(2)}(t)$ of the approximating sequence is constructed by adding to $\psi_{\varphi}^{(1)}(t)$ the leading order contribution in (3.55) with the true time evolution $U(t, s)$ replaced by $\widetilde{U}_{\varphi}(t, s)$. This yields

$$\psi_{\varphi}^{(2)}(t) := \widetilde{U}_{\varphi}(t,0)\psi_0 - \mathrm{i} \int_0^t \mathrm{d}s\, \widetilde{U}_{\varphi}(t,s) \mathcal{C}^{\varphi(s)} \widetilde{U}_{\varphi}(s,0)\psi_0\,,$$

which equals (3.40). In conclusion, the idea is to cancel the leading order contribution to (3.59) but for the difference between $U(t, s)$ and $\widetilde{U}_{\varphi}(t, s)$. Since this difference is evaluated on $\mathcal{C}^{\varphi(s)} \widetilde{U}_{\varphi}(s, 0)\psi_0$, which is small in norm, this improves the first order approximation $\psi_{\varphi}^{(1)}(t)$.

To prove (3.42) for $a > 1$ and to construct all higher elements $\psi_\varphi^{(a)}(t)$ of the sequence, one successively repeats the above three steps. In step three, one adds to $\psi_\varphi^{(a)}(t)$ as many terms of the expansion of $\psi(t) - \psi_\varphi^{(a)}(t)$ as needed to cancel the $\mathcal{O}(N^{-a\delta(\beta,\gamma)})$-contributions in the difference $\psi(t) - \psi_\varphi^{(a+1)}(t)$.

Let us demonstrate this scheme once more for $a = 2$. Using Duhamel's formula twice, we obtain

$$\begin{aligned}
&\psi(t) - \psi_\varphi^{(2)}(t) \\
&\quad = \quad -\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2\, U(t,s_2)\left(\mathcal{C}^{\varphi(s_2)} + \mathcal{Q}^{\varphi(s_2)}\right)\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0 \\
&\qquad\quad -\mathrm{i}\int_0^t U(t,s)\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\,\mathrm{d}s,
\end{aligned}$$

which implies

$$\begin{aligned}
\|\psi(t) - \psi_\varphi^{(2)}(t)\| \quad \leq \quad & \int_0^t \mathrm{d}s_1 \int_{s_2}^t \mathrm{d}s_2 \|\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\| \\
& + \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \|\mathcal{Q}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\| \qquad (3.60) \\
& + \int_0^t \mathrm{d}s \|\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|\,.
\end{aligned}$$

Combining Lemma 3.2.4 and Proposition 3.2.3b, the leading order term in (3.60) can be estimated as

$$\begin{aligned}
&\|\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2 \\
&\overset{3.2.4,\ 3.2.3\mathrm{b}}{\lesssim} \quad N^{2+d\beta}\sum_{n=0}^{3} N^{n(-1+d\beta)}\|(\widehat{m^{\varphi(s_1)}})^{3-n}\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2 \\
&\overset{3.2.4,\ 3.2.3\mathrm{b}}{\lesssim} \quad N^{4+2d\beta}\sum_{n=0}^{3}\sum_{l=0}^{6-n} N^{(n+l)(-1+d\beta)}\|(\widehat{m^{\varphi_0}})^{6-n-l}\psi_0\|^2 \\
&\overset{\mathrm{A3}}{\lesssim} \quad N^{-2+2d\beta}\sum_{n=0}^{3}\sum_{l=0}^{6-n} N^{(n+l)(-1+d\beta+\gamma)-6\gamma}\,.
\end{aligned}$$

As before, considering the two ranges of γ separately yields for sufficiently large N

$$\|\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2 \lesssim N^{-2\delta(\beta,\gamma)} \qquad (3.61)$$

with $\delta(\beta,\gamma)$ as in (3.43). Analogously, the second term is bounded by

$$\|\mathcal{Q}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2 \lesssim \begin{cases} N^{-3+10d\beta} & 1-d\beta\leq\gamma\leq 1\,, \\ N^{4+3d\beta-7\gamma} & \frac{2+d\beta}{3}<\gamma\leq 1-d\beta\,, \end{cases}$$

and the third term was already treated in (3.58). In summary, we obtain

$$\|\psi(t)-\psi_\varphi^{(2)}(t)\|^2 \lesssim N^{-2\delta(\beta,\gamma)}\,.$$

Finally, adding the two expressions (3.58) and (3.61) to $\psi_\varphi^{(2)}(t)$ after substituting the full time evolution $U(t,s)$ by $\widetilde{U}_\varphi(t,s)$ defines $\psi_\varphi^{(3)}(t)$ as given in (3.41).

Iterating this procedure a times proves Theorem 3.2.2 for any $a\in\mathbb{N}$. A key observation is that the leading term in every order is the expression containing exclusively a cubic terms $\mathcal{C}^{\varphi(t)}$, which can be shown by iteratively applying Lemma 3.2.4 and Proposition 3.2.3b.

3.2.3. Discussion

We begin the discussion with a review of results in the literature that are comparable to Theorem 3.2.2 and Proposition 3.2.3. Subsequently, we comment on our assumptions and discuss open questions and future perspectives.

Literature

To the best of our knowledge, the only existing result comparable to Theorem 3.2.2 is the work [142] by Paul and Pulvirenti. For the time evolution generated by the Hamiltonian $H_{N,\beta}$ with $\beta=0$ and $V^{\mathrm{ext}}=0$ and for factorised initial data, they derive higher order approximations of the reduced density matrices. More precisely, they construct a sequence $\{F_j^{N,n}(t)\}_{n\in\mathbb{N}}$ of trace class operators on $L^2(\mathbb{R}^{jd})$ which approximate the j-particle reduced density matrix $\gamma_N^{(j)}(t)$ for values of $j\lesssim\sqrt{N}$ with increasing accuracy. To compute the operator $F_j^{N,n}(t)$, a finite number of operations is required, which depends on j and n but not on N.

The work by Paul and Pulvirenti is based on the method of kinetic errors from [143] by Paul, Pulvirenti and Simonella. The j-particle reduced density matrix $\gamma_N^{(j)}(t)$ is characterised in terms of the operators $p^{\varphi(t)}$ and the so-called correlation errors $E_j^N(t)\in\mathcal{L}^1(L^2(\mathbb{R}^{jd}))$ as

$$\begin{aligned}&\gamma_N^{(j)}(t)(z_1,\dots,z_j)\\ &\quad=\sum_{k=0}^{j}\sum_{1\leq i_1<\dots<i_k\leq j}p^{\varphi(t)}(z_{i_1})\cdots p^{\varphi(t)}(z_{i_k})E_{j-k}^N(t)(z_1,\dots,z_j\setminus\{z_{i_1},\dots,z_{i_k}\})\,.\end{aligned} \tag{3.62}$$

Here, we abbreviated $z_l := (x_l; x_l')$, and denoted by $\gamma_N^j(t)(z_1, ..., z_j)$ and $p^{\varphi(t)}(z_l)$ the integral kernels of $\gamma_N^{(j)}(t)$ and $p^{\varphi(t)}$, respectively. Instead of considering the BBGKY-hierarchy for the reduced densities, the authors of [143] derive equations for the correlation errors, which, as in the BBGKY case, form an iterative hierarchy, in the sense that $E_j^N(t)$ depends on $E_{j+1}^N(t)$. In [142], these correlation errors are expanded as

$$E_j^N(t) = \sum_{k=0}^{\infty} \mathcal{E}_j^k(t) N^{-\frac{j+k}{2}} \,.$$

It is shown that the coefficients $\mathcal{E}_j^k(t)$ can be determined from the initial coefficients $\mathcal{E}_{j'}^{k'}(0)$ for $j' \leq j + k$, $k' \leq k$, in the following way:

- First, a two-parameter semigroup $U_j(t, s)$ on $\mathcal{L}(L^2(\mathbb{R}^{jd}))$ is constructed as a Dyson expansion in terms of the linearisation of the Hartree flow around $p^{\varphi(t)}$.
- The truncation of this Dyson series after $2n + 1$ steps yields the semigroup $U_j^n(t, s)$.
- Replacing $U_j(t, s)$ by $U_j^n(t, s)$ in the formula for $\mathcal{E}_j^k(t)$ yields the operators $\mathcal{E}_j^{k,n}(t)$.
- Adding all $\mathcal{E}_j^{k,n}(t)$ for $k = 1, ..., 2n$, one obtains the approximation $E_j^{N,n}(t)$ of the correlation errors $E_j^N(t)$.
- Finally, these $E_j^{N,n}(t)$ define $F_j^{N,n}(t)$ via (3.62).

In conclusion, the approximating operators $F_j^{N,n}(t)$ can be determined by an N-independent number of computations, needing as input only the initial data as well as the knowledge of the solution of the Hartree equation and its linearisation around this solution.

As a consequence of the very different approaches, it is not straightforward to compare our result with the construction of Paul and Pulvirenti. We note the following:

- While our approximations are on the level of the time-evolved N-body wave function, Paul and Pulvirenti derive higher order approximations of the reduced density matrices.
- In our perception, the construction of $\psi_\varphi^{(a)}(t)$ is more explicit than the operator-based scheme sketched above. In both results, an a-dependent, N-independent number of steps is required to obtain the a'th order approximation.
- The starting point in [142] is the time evolution $U_j(t, s)$, which is related to the linearisation of the Hartree flow around the solution of the Hartree equation. In contrast, we use the time evolution $\widetilde{U}_\varphi(t, s)$.

- While the analysis of Paul and Pulvirenti is restricted to factorised initial states, we cover a larger and more general class of initial data.
- Finally, in contrast to [142], our result includes small values of β beyond the Hartree scaling and admits possibly time-dependent external fields.

Let us also briefly comment on Proposition 3.2.3, where we estimate the growth of the first A moments of the number of excitations when the system evolves under the dynamics $U(t,s)$ or $\widetilde{U}_\varphi(t,s)$. Estimates of this kind are often needed to derive effective descriptions of the dynamics of interacting bosons, e.g., in [20, 28, 42, 135, 146, 158]. Our proof extends comparable statements for $\beta = 0$ and $d = 3$ obtained by Mitrouskas, Petrat and Pickl in [135, Lemma 2.1] and by Rodnianski and Schlein in [158, Proposition 3.3]. For Bose gases with large volume and large density, a similar estimate was derived by Petrat, Pickl and Soffer in [146, Corollary 4.2].

Assumptions on the potentials

Assumptions *A1* and *A2* are rather standard in the rigorous treatment of interacting many-boson systems. Note that we make no assumption on the sign of the potential or its scattering length but cover both repulsive and attractive interactions. Besides, we admit a large class of time-dependent external traps $V^{\text{ext}}(t,x)$, with the only constraints that they need to be bounded for fixed t and continuous for fixed x.

Assumption on the initial data

The simplest example of an N-body state satisfying A3 is the product state $\psi = {\varphi_0}^{\otimes N}$, which describes, e.g., the ground state of a non-interacting system. In contrast, the ground state as well as the lower excited states of interacting systems are not close to an exact product with respect to the $L^2(\mathbb{R}^{dN})$-norm due to the correlation structure related to the interactions.

Regarding interacting bosons, A3 is fulfilled for quasi-free states with subleading expected number of excitations, since it holds for any quasi-free state $\xi \in \mathcal{F}$ and any $\ell \geq 1$ that

$$\langle \xi, \mathcal{N}\xi\rangle_{\mathcal{F}} \lesssim N^{1-\gamma} \quad \Rightarrow \quad \left\langle \xi, \mathcal{N}^\ell \xi \right\rangle_{\mathcal{F}} \lesssim C_\ell (1 + N^{1-\gamma})^\ell$$

by (1.132). Note that we require a certain minimal size of γ, which is strictly greater than $\frac{2}{3}$. Since it follows from (1.142) and (1.74) that

$$N^{-1} \langle \xi, \mathcal{N}\xi\rangle_{\mathcal{F}} = \left\langle\!\left\langle \psi, \widehat{n^\varphi}\psi \right\rangle\!\right\rangle = \left\langle\!\left\langle \psi, q_1^\varphi \psi \right\rangle\!\right\rangle = 1 - \left\langle \varphi, \gamma_\psi^{(1)} \varphi \right\rangle_{L^2(\mathbb{R}^3)} \tag{3.63}$$

for $\xi = \mathfrak{U}_N^\varphi \psi$, the requirement that the expected number of excitations be bounded uniformly in N, which corresponds to $\gamma = 1$, is equivalent to BEC with optimal rate

N^{-1} (see Lemma 1.4.2). Note that it was shown by Lewin, Nam, Serfaty and Solovej in [114] that BEC with optimal rate is a sufficient condition for the validity of the Bogoliubov approximation (see Section 1.5.2). Besides, A3 with parameter $\gamma < 1$ is comparable to the assumption (1.128) made by Nam and Napiórkowski in [138] to obtain a norm approximation for the range $\beta \in [0, \frac{1}{2})$.

Finally, Mitrouskas showed in [134, Chapter 3] that assumption A3 with $\gamma = 1$ is fulfilled by the ground state and lower excited states of a homogeneous Bose gas on the d-dimensional torus for $\beta = 0$.

More precisely, let φ_0 be the minimiser of the Hartree functional on the torus corresponding to the ground state energy E_0, and let ψ_n denote the n'th excited eigenstate with energy E_n. Then the author proves that there exist constants $C, D > 0$ such that

$$\|P_a^{\varphi_0}\psi_n\|^2 \leq C\mathrm{e}^{-Da}$$

for all $(E_n - E_0) \leq a \leq N$ and with $P_a^{\varphi_0}$ as in Definition 1.4.1. As a corollary of this statement, it is shown that there exists $C_a > 0$ such that

$$\langle\!\langle \psi_n, q_1^{\varphi_0} \cdots q_a^{\varphi_0}\psi_n \rangle\!\rangle \leq N^{-a} C_a \left(1 + (E_n - E_0)^a\right) .$$

Due to the relation

$$\langle\!\langle \psi, q_1^{\varphi} \cdots q_a^{\varphi}\psi \rangle\!\rangle \leq \|\big(\widehat{m^{\varphi}}\big)^a \psi\|^2 \lesssim \sum_{j=1}^{a} N^{-a+j} \langle\!\langle \psi, q_1^{\varphi} \cdots q_a^{\varphi}\psi \rangle\!\rangle + N^{-a} , \tag{3.64}$$

which holds for any $\psi \in L^2_+(\mathbb{R}^d)$ ([34, Lemma 2.1a]), this implies that assumption A3 is satisfied.

Discussion of the result and perspectives

By construction, the first order correction $\psi_{\varphi}^{(1)}(t)$ coincides with the norm approximation found by Mitrouskas, Petrat and Pickl in [135] for the Hartree scaling. Recall that Theorem 3.2.2 establishes the approximation

$$\|\psi(t) - \psi_{\varphi}^{(1)}(t)\|^2 \lesssim \mathrm{e}^{c(a)\int_0^t \|\varphi(s)\|_{H^k(\mathbb{R}^d)}\,\mathrm{d}s} \begin{cases} N^{-(1-4d\beta)} & 1 - d\beta \leq \gamma \leq 1 \\ N^{-(3\gamma-2-d\beta)} & \frac{2+d\beta}{3} < \gamma \leq 1 - d\beta \,. \end{cases}$$

For $d = 3$ and $\beta = 0$, this reproduces the result (1.140) up to a different time dependent constant. Note that for $d = 3$, the exponent contains the $H^2(\mathbb{R}^3)$-norm, which depends on the choice of V^{ext}. For instance, in the homogeneous case without external field, this norm is preserved, which leads to the time dependence $\sim \mathrm{e}^{Ct}$. Hence, our result can be understood as an extension of (1.140) to arbitrary dimensions and to the range

$\beta \in [0, \frac{1}{4d})$.

In comparison to the results [137, 138, 139] by Nam and Napiórkowski, our error bounds for the first order correction are different. Making use of the equivalence (1.143), we note the following:

- For $d = 3$, our bound $\delta = 1 - 12\beta$ with assumption $\gamma \in [1 - 3\beta, 1]$ is worse than the result in [137]. Moreover, the analysis in [137] covers values of β up to $\frac{1}{3}$, and, with a different rate, is extended in [138] to the range $\beta \in [0, \frac{1}{2})$.

- In dimension $d = 2$, our bound $\delta = 1 - 8\beta$ under the assumption $\gamma \in [1 - 2\beta, 1]$ needs to be compared to the bound $\delta < \frac{1}{3}(1 - \beta)$ obtained in [139]. We conclude that our bound is better for $\beta < \frac{2}{23}$ and worse for larger β. Moreover, the analysis by Nam and Napiórkowski covers the range $\beta \in (0, 1)$, which is much larger than the regime $[0, \frac{1}{8})$ admitted by Theorem 3.2.2.

- For $d = 1$, our error bound under the assumption $\gamma \in [1 - \beta, 1]$ is given as $\delta = 1 - 4\beta$, while the respective parameter in [139] is $\delta = \frac{1}{2}$. Hence, our estimate is better for $\beta \in [0, \frac{1}{8})$ and worse for $\beta \in (\frac{1}{8}, \frac{1}{4})$. Besides, the result in [139] includes all $\beta > 0$, while our analysis is restricted to the range $\beta \in [0, \frac{1}{4})$.

To conclude this chapter, let us discuss the approximating functions $\psi_\varphi^{(a)}(t)$ from a physical point of view. Due to the inter-particle correlations, the full N-body time evolution $\psi(t)$ is an extremely complicated object: even if the system was initially in a factorised state, the interactions instantaneously correlate the particles in the sense of (1.20), making it very difficult to explicitly compute expectation values with respect to $\psi(t)$. In particular, the highly correlated dynamics $\psi(t)$ are practically inaccessible to any numerical analysis.

In this respect, the norm approximation provided by Nam and Napiórkowski in [137, 138, 139] provides a huge simplification. If the initial wave function is described by a quasi-free excitation vector ξ_{φ_0}, this property is preserved by the Bogoliubov time evolution. Hence, by the Wick property (1.131) of quasi-free state, all expectation values with respect to the time-evolved excitation vector $\xi_{\varphi(t)}$ can be computed from the one-body densities $(\gamma_{\xi_{\varphi(t)}}, \alpha_{\xi_{\varphi(t)}})$. Since these densities are determined by the system of equations (1.134) derived in [137], we conclude that every expectation value with respect to the approximating function can be obtained by solving the NLS equation for the condensate and the two equations (1.134) for the excitations. By unitarity of the time evolutions generated by $\mathcal{H}_{\text{Bog}}$ and $H_{N,\beta}$, this observation extends to initial states that are sufficiently close to quasi-free states.

At present, it remains an open question whether a comparable statement holds true for the first-quantised time evolution $\widetilde{U}_\varphi(t, 0)\psi_0$ and the higher order corrections $\psi_\varphi^{(a)}(t)$ for an appropriate class of initial states. The specific form of $\widetilde{H}^{\varphi(t)}$ suggests that the time evolved wave function $\widetilde{U}_\varphi(t,0)\psi_0$ should develop less correlations than the full dynamics $\psi(t)$, for which a heuristic argument is given in [146]: Since $\widetilde{H}^{\varphi(t)}$ contains exclusively terms of the form

$$p_1 q_2 v_{N,\beta}^{(12)} q_1 p_2 \,, \qquad p_1 p_2 v_{N,\beta}^{(12)} q_1 q_2 \,, \qquad q_1 q_2 v_{N,\beta}^{(12)} p_1 p_2 \,,$$

there are the following possibilities for the formation of correlations:

- If these operators are evaluated on a product state, the first two expressions yield zero, while the third one produces a pair correlation between particles 1 and 2.
- When acting on a state where particles 1 and 2 are correlated, the third term produces again such a state, while the first and second expression result in a state where particles 1 and 2 are uncorrelated.
- If we have pair correlations of particles 1 and 3 and of 2 and 4, respectively, then the first expression yields a state with a pair correlation of 2 and 3 and with particle 1 and 4 uncorrelated. The second expression produces a state with particles 1 and 2 in the condensate and particles 3 and 4 correlated, and the last term results in a state where particles 1 and 2 are correlated and particles 3 and 4 are uncorrelated.

In summary, none of the terms in $\widetilde{H}^{\varphi(t)}$ can lead to higher correlations than pairs, provided it acts on a state with at most pair correlations. Hence, it seems plausible that the time evolution $\widetilde{U}_\varphi(t,s)$ might preserve the property of having at most pair correlations. Naturally, this statement is quite vague and requires a precise formulation in mathematical terms, and above heuristics are far from a rigorous proof.

By construction of the second order correction $\psi_\varphi^{(2)}(t)$, a state evolving under $\widetilde{U}_\varphi(s,0)$ with at most pair correlations is acted upon by a cubic term $\mathcal{C}^{\varphi(s)}$, which contains three projectors q. By a similar reasoning as above, one can argue that the resulting state should have at most three-body correlations. This argument can be continued to the next order corrections $\psi_\varphi^{(a)}(t)$, where the length of the correlations grows with a. In conclusion, these heuristics can be understood as a hint that the approximating functions $\widetilde{U}_\varphi(t,0)\psi_0$ and $\psi_\varphi^{(a)}(t)$ should be simplifying and physically meaningful approximations of the highly correlated dynamics $\psi(t)$.

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A. Accepted Publications

A.1. Derivation of the 1d nonlinear Schrödinger equation from the 3d quantum many-body dynamics of strongly confined bosons

Derivation of the 1d nonlinear Schrödinger equation from the 3d quantum many-body dynamics of strongly confined bosons

Lea Boßmann*

Abstract

We consider the dynamics of N interacting bosons initially exhibiting Bose–Einstein condensation. Due to an external trapping potential, the bosons are strongly confined in two spatial directions, with the transverse extension of the trap being of order ε. The non-negative interaction potential is scaled such that its scattering length is positive and of order $(N/\varepsilon^2)^{-1}$ and the range of the interaction scales as $(N/\varepsilon^2)^{-\beta}$ for $\beta \in (0,1)$. We prove that in the simultaneous limit $N \to \infty$ and $\varepsilon \to 0$, the condensation is preserved by the dynamics and the time evolution is asymptotically described by a cubic defocusing nonlinear Schrödinger equation in one dimension, where the strength of the nonlinearity depends on the interaction and on the confining potential. This is the first derivation of a lower-dimensional effective evolution equation for singular potentials scaling with $\beta \geq \frac{1}{2}$ and lays the foundations for the derivation of the physically relevant one-dimensional Gross–Pitaevskii equation ($\beta = 1$). For our analysis, we adapt an approach by Pickl to the problem with strong confinement.

1 Introduction

We consider a system of N identical bosons in $\mathbb{R}^3$ interacting among each other through repulsive pair interactions. The bosons are trapped within a cigar-shaped trap, which effectively confines the particles to a region of length ε in two spatial directions. To describe this mathematically, let us first introduce the coordinates

$$z = (x, y) \in \mathbb{R}^{1+2}.$$

The cigar-shaped confinement is given by the scaled potential $\frac{1}{\varepsilon^2}V^\perp\left(\frac{y}{\varepsilon}\right)$ for $0 < \varepsilon \ll 1$ and $V^\perp : \mathbb{R}^2 \to \mathbb{R}$. The Hamiltonian of this system is

$$H_\beta(t) = \sum_{j=1}^{N}\left(-\Delta_j + \tfrac{1}{\varepsilon^2}V^\perp\left(\tfrac{y_j}{\varepsilon}\right) + V^{\parallel}(t, z_j)\right) + \sum_{1\leq i<j\leq N} w_\beta(z_i - z_j), \tag{1}$$

where Δ denotes the Laplace operator on $\mathbb{R}^3$ and $V^{\parallel}$ is a possibly time-dependent additional external potential. The units are chosen such that $\hbar = 1$ and $m = \frac{1}{2}$. In the

*Fachbereich Mathematik, Eberhard Karls Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: lea.bossmann@uni-tuebingen.de

limit $\varepsilon \to 0$, the system becomes effectively one-dimensional, in the sense that excitations in the transverse direction are energetically strongly suppressed.

The interaction between the particles is described by the potential w_β with scaling parameter $\beta \in (0,1)$. For the sake of this introduction, let us for the moment assume that

$$w_\beta(z) = \left(\tfrac{N}{\varepsilon^2}\right)^{-1+3\beta} w\left(\left(\tfrac{N}{\varepsilon^2}\right)^{\beta} z\right)$$

for some compactly supported, spherically symmetric, non-negative, bounded potential w. This scaling describes a dilute gas, where the scaling parameter β interpolates between the Hartree ($\beta = 0$) and the Gross–Pitaevskii ($\beta = 1$) regime. The proof of the physically relevant Gross–Pitaevskii regime relies essentially on the result for $\beta \in (0,1)$ and is given in [4]. An important parameter characterising the interaction w_β is its effective range,

$$\mu := \left(\tfrac{N}{\varepsilon^2}\right)^{-\beta}.$$

We study the dynamics of the system in the simultaneous limit $(N,\varepsilon) \to (\infty,0)$. The state $\psi^{N,\varepsilon}(t)$ of the system at time t is determined by the N-body Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi^{N,\varepsilon}(t) = H_\beta(t)\psi^{N,\varepsilon}(t) \tag{2}$$

with initial datum $\psi^{N,\varepsilon}(0) = \psi_0^{N,\varepsilon} \in L_s^2(\mathbb{R}^{3N}) := \otimes_{\mathrm{sym}}^N L^2(\mathbb{R}^3)$. We assume that the system initially exhibits Bose–Einstein condensation, i.e., that the one-particle reduced density matrix $\gamma^{(1)}_{\psi_0^{N,\varepsilon}}$ of $\psi_0^{N,\varepsilon}$,

$$\gamma^{(k)}_{\psi_0^{N,\varepsilon}} := \mathrm{Tr}_{k+1,\dots,N} \left|\psi_0^{N,\varepsilon}\right\rangle\left\langle\psi_0^{N,\varepsilon}\right| \tag{3}$$

for $k = 1$, is asymptotically, as $(N,\varepsilon) \to (\infty,0)$, close to the projection onto a one-body state $\varphi_0^\varepsilon \in L^2(\mathbb{R}^3)$. At low energies, the state factorises as a consequence of the strong confinement and is of the form $\varphi_0^\varepsilon(z) = \Phi_0(x)\chi^\varepsilon(y)$ (see Remark 1e). Here, Φ_0 denotes the wavefunction along the x-axis and χ^ε is the normalised ground state of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$ in the confined directions. Due to the rescaling by ε, χ^ε is given by

$$\chi^\varepsilon(y) = \tfrac{1}{\varepsilon}\chi(\tfrac{y}{\varepsilon}), \tag{4}$$

where χ is the normalised ground state of $-\Delta_y + V^\perp(y)$.

In Theorem 1, we show that if the state of the system is initially such a factorised Bose–Einstein condensate with condensate wavefunction $\varphi_0^\varepsilon = \Phi_0\chi^\varepsilon$, i.e., if

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}_{L^2(\mathbb{R}^3)} \left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}} - \left|\varphi_0^\varepsilon\right\rangle\left\langle\varphi_0^\varepsilon\right|\right| = 0,$$

where the limit $(N,\varepsilon) \to (\infty,0)$ is taken in an appropriate way, then the condensation of the system into a factorised state is preserved by the dynamics, i.e., for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}_{L^2(\mathbb{R}^{3k})} \left|\gamma^{(k)}_{\psi^{N,\varepsilon}(t)} - \left|\varphi^\varepsilon(t)\right\rangle\left\langle\varphi^\varepsilon(t)\right|^{\otimes k}\right| = 0.$$

The condensate wavefunction at time t is given by $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$, where $\Phi(t)$ is the solution of the one-dimensional nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\tfrac{\partial}{\partial t}\Phi(t,x) = \left(-\tfrac{\partial^2}{\partial x^2} + V^\parallel(t,(x,0)) + b_\beta|\Phi(t,x)|^2\right)\Phi(t,x) =: h(t)\Phi(t,x) \tag{5}$$

with $\Phi(0) = \Phi_0$ and coupling parameter $b_\beta = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4 \,\mathrm{d}y$.

To our knowledge, Theorem 1 is the first rigorous derivation of an effectively lower-dimensional evolution equation directly from the three-dimensional N-body dynamics for $\beta \geq \frac{1}{2}$. In [19], von Keler and Teufel consider a similar problem for $\beta \in (0, \frac{1}{3})$ and in [6] and [8], Chen and Holmer study interactions for different scaling regimes up to $\beta < \frac{1}{2}$. The extension to $\beta \in (0, 1)$ requires a non-trivial adaptation of methods used for the fully three-dimensional problem without strong confinement [33] to handle the additional limit $\varepsilon \to 0$ and the associated dimensional reduction. Not only is this an interesting mathematical problem on its own but it lays the foundations for the derivation of the physically relevant effectively one-dimensional Gross–Pitaevskii equation corresponding to the scaling $\beta = 1$ [4]. In fact, the main idea of the proof in [4] is to approximate the interaction $w_{\beta=1}$ by a softer scaling interaction which is covered by our Theorem 1, and to show that the remainders from this substitution vanish in the limit. The dimensional reduction occurs in the approximated interaction, hence the result for $\beta = 1$ relies essentially on the tools and results proven here.

Let us give a brief motivation of the effective equation (5). The N-body problem is interacting, hence the effective evolution is nonlinear and the strength of the linearity depends on the two-body scattering process. This process is to leading order described by the (s-wave) scattering length a_β of w_β, which scales as $(\frac{N}{\varepsilon^2})^{-1}$ for $\beta \in (0, 1]$ [9, Lemma A.1]. This implies that, for $\beta \in (0, 1)$, the length scale of the inter-particle correlations is small compared to the range $\mu = (\frac{N}{\varepsilon^2})^{-\beta}$of w_β. Hence, the correlations are negligible in the limit and the two-body scattering process is described by the first order Born approximation to the scattering length, $8\pi a_\beta \approx \int w_\beta(z)\,\mathrm{d}z$. The additional factor $\int_{\mathbb{R}^2} |\chi(y)|^4 \,\mathrm{d}y$ in the coupling parameter arises from integrating out the transverse degrees of freedom in the course of the dimensional reduction.

Quasi one-dimensional Bose gases in highly elongated traps have been studied experimentally [12, 15] and the dynamical behaviour of such systems is of great physical interest [11, 20, 28]. After the first proof of an effective Hartree evolution by Spohn [35], the first rigorous derivation of an NLS evolution for three-dimensional bosons is due to Erdős, Schlein and Yau [9]. The main tool of their proof is the convergence of the BBGKY hierarchy, a system of coupled equations determining the time evolution of all k-particle density matrices. Later, the authors adapted their proof to handle the Gross–Pitaevskii scaling of the interaction [10]. A different approach providing rates for the convergence of the reduced density matrices was proposed by Pickl [29, 32], who derived effective evolution equations for NLS and Gross–Pitaevskii scaling of the interaction, including time-dependent external potentials [33] as well as non-positive [31, 17] and singular interactions [22]. A third method for the Gross–Pitaevskii case, based on Bogoliubov transformations and coherent states on Fock space, was developed by Benedikter, De Oliveira and Schlein [3], and an optimal rate of convergence was recently proven by Brennecke and Schlein [5]. In [1, 7] and [21, 16, 18], effective equations in one and two dimensions were derived from the respective one and two dimensional quantum many-body dynamics.

Some authors have considered the problem of dimensional reduction for the NLS equation. In [27], Méhats and Raymond study the cubic NLS equation in a two-dimensional quantum waveguide, i.e., within a tube of width ε around a curve in $\mathbb{R}^2$. They show that in the limit $\varepsilon \to 0$, the nonlinear evolution is well approximated by a one-dimensional cubic NLS equation with an additional potential term due to the curvature. Ben Abdallah, Méhats, Schmeiser and Weishäupl consider in [2] an $(n+d)$-dimensional NLS equation subject to a strong confinement in d directions and derive an effective n-dimensional NLS equation with a modified nonlinearity.

As mentioned above, there are few results concerning the derivation of lower-dimensional NLS equations from the underlying three-dimensional N-body dynamics. Chen and Holmer consider three-dimensional bosons with pair interactions in a harmonic potential that is strongly confining in one [6] or two [8] directions. For a repulsive interaction scaling with $\beta \in (0, \frac{2}{5})$ in case of the disc-shaped and for an attractive interaction with $\beta \in (0, \frac{3}{7})$ in case of the cigar-shaped confinement, they prove that the dynamics are effectively described by a two- or respectively one-dimensional NLS equation. In [19], von Keler and Teufel study a Bose gas confined to a quantum waveguide with non-trivial geometry for scaling parameters $\beta \in (0, \frac{1}{3})$. They prove that the evolution is well captured by a one-dimensional NLS equation with additional potential terms arising from the twisting and bending of the waveguide.

The remainder of this paper is structured as follows: in Section 2, we specify our assumptions and present the result. Our proof follows an approach by Pickl, which is outlined in Section 3. This section also contains the proof of our main Theorem 1, relying essentially on two propositions. Finally, these propositions are proven in Section 4.

Notation. We will write $A \lesssim B$ to indicate that there exists some constant $C > 0$ independent of $\varepsilon, N, t, \psi_0^{N,\varepsilon}, \Phi_0$ such that $A \leq CB$. The constant may depend on the quantities fixed by the model, such as $V^\perp$, χ and $V^\parallel$. We will exclusively use the symbol $\widehat{\cdot}$ to denote the operators from Definition 3.3. Besides, we will use the abbreviations

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle := \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{3N})}, \quad \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^{3N})} \quad \text{and} \quad \|\cdot\|_{\text{op}} := \|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^{3N}))}.$$

2 Main Result

To study the effective behaviour of the many-body system in the simultaneous limit $(N, \varepsilon) \to (\infty, 0)$, let us consider families of initial data $\psi_0^{N,\varepsilon}$ along sequences $(N_n, \varepsilon_n) \to (\infty, 0)$ characterised as follows:

Definition 2.1. A sequence (N_n, ε_n) in $\mathbb{N} \times (0, 1)$ is called *admissible* if

$$\lim_{n\to\infty} (N_n, \varepsilon_n) = (\infty, 0) \quad \text{and} \quad \lim_{n\to\infty} \frac{\varepsilon_n^2}{\mu_n} = 0 \quad \text{for } \mu_n := \left(\frac{N_n}{\varepsilon_n^2}\right)^{-\beta}.$$

It is called *moderately confining* if

$$\lim_{n\to\infty} \frac{\mu_n}{\varepsilon_n} = 0.$$

Moderate confinement means that the extension ε of the confining potential shrinks to zero but is still large compared to the range of the interaction μ. This prevents the interaction from being supported mainly in a region that is quasi inaccessible to the particles due to the strong confinement. As $\mu/\varepsilon = N^{-\beta}\varepsilon^{2\beta-1}$, this condition is a restriction only for $\beta < \frac{1}{2}$.

The admissibility condition ensures that the energy gap above the transverse ground state, which is of order ε^{-2}, grows sufficiently fast compared to μ. In the proof, we will use this condition to control transverse excitations into states that are orthogonal to χ^ε (see also Remark 2d). Note that for $\delta > 0$, $\varepsilon^\delta/\mu = N^\beta \varepsilon^{\delta-2\beta}$, hence $\delta = 2$ is the smallest exponent for which $\varepsilon^\delta/\mu \to 0$ is possible for all $\beta \in (0,1)$. Both conditions are comparable to the assumptions in [8] for an attractive interaction scaling with $\beta \in (0, \frac{3}{7})$.[1]

We consider interactions of the following type:

Definition 2.2. Let $\beta \in (0,1)$ and $\eta > 0$. Define the set $\mathcal{W}_{\beta,\eta}$ as the set containing all families

$$w_\beta : \mathbb{N} \times (0,1) \to L^\infty(\mathbb{R}^3, \mathbb{R}), \quad (N,\varepsilon) \mapsto w_\beta((N,\varepsilon)),$$

such that for any $(N,\varepsilon) \in \mathbb{N} \times (0,1)$

$$\begin{cases} (a)\ \ \|w_\beta((N,\varepsilon))\|_{L^\infty(\mathbb{R}^3)} \lesssim \left(\frac{N}{\varepsilon^2}\right)^{-1+3\beta}, \\ (b)\ \ w_\beta((N,\varepsilon)) \text{ is non-negative and spherically symmetric,} \\ (c)\ \ \operatorname{supp} w_\beta((N,\varepsilon)) \subseteq \left\{ z \in \mathbb{R}^3 : |z| \lesssim \left(\frac{N}{\varepsilon^2}\right)^{-\beta} \right\}, \\ (d)\ \ \lim\limits_{(N,\varepsilon)\to(\infty,0)} \left(\frac{N}{\varepsilon^2}\right)^\eta |\, b_{N,\varepsilon}(w_\beta) - b_\beta(w_\beta)| = 0, \end{cases}$$

where

$$b_{N,\varepsilon}(w_\beta) := N \int\limits_{\mathbb{R}^3} w_\beta((N,\varepsilon),z)\,\mathrm{d}z \int\limits_{\mathbb{R}^2} |\chi^\varepsilon(y)|^4\,\mathrm{d}y = \frac{N}{\varepsilon^2} \int\limits_{\mathbb{R}^3} w_\beta((N,\varepsilon),z)\,\mathrm{d}z \int\limits_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y,$$

$$b_\beta(w_\beta) := \lim_{(N,\varepsilon)\to(\infty,0)} b_{N,\varepsilon}(w_\beta).$$

We will in the following abbreviate $w_\beta((N,\varepsilon)) \equiv w_\beta$, $b_{N,\varepsilon}(w_\beta) \equiv b_{N,\varepsilon}$ and $b_\beta(w_\beta) \equiv b_\beta$.

Condition (d) ensures that the (N,ε)-dependent parameter $b_{N,\varepsilon}$ converges sufficiently fast to its limit b_β. Clearly, the interaction $(\frac{N}{\varepsilon^2})^{-1+3\beta} w((\frac{N}{\varepsilon^2})^\beta z)$ from the introduction is contained in this set. In this case, $b_{N,\varepsilon} = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y = b_\beta$, hence (d) is true for any $\eta > 0$.

In order to formulate our main theorem, we will need two different notions of one-particle energies:

- The *"renormalised" energy per particle*: for $\psi \in \mathcal{D}(H_\beta(t)^{\frac{1}{2}})$,

$$E^\psi(t) := \tfrac{1}{N} \langle \psi, H_\beta(t)\psi \rangle_{L^2(\mathbb{R}^{3N})} - \tfrac{E_0}{\varepsilon^2}, \tag{6}$$

  where E_0 denotes the lowest eigenvalue of $-\Delta_y + V^\perp(y)$. By rescaling, $\frac{E_0}{\varepsilon^2}$ is the lowest eigenvalue of $-\Delta_y + \frac{1}{\varepsilon^2} V^\perp(\frac{y}{\varepsilon^2})$.

[1] In our notation, the assumptions in [8] are $N^{\nu_1(\beta)} \lesssim \varepsilon^{-2} \lesssim N^{\nu_2(\beta)}$, where ν_1 and ν_2 are given by $\nu_1(\beta) = \frac{\beta}{1-\beta}$ and $\nu_2 = \min\left\{ \frac{1-\beta}{\beta}, \frac{\frac{3}{5}-\beta}{\beta-\frac{1}{5}} \mathbb{1}_{\beta \geq \frac{1}{5}} + \infty \cdot \mathbb{1}_{\beta < \frac{1}{5}}, \frac{2\beta}{1-2\beta}, \frac{\frac{7}{8}-\beta}{\beta} \right\}$. Note that $N^{\nu_1(\beta)}\varepsilon^2 = (\frac{\varepsilon^2}{\mu})^{\frac{1}{1-\beta}}$ and $N^{\nu_2(\beta)}\varepsilon^2 \leq (\frac{\varepsilon}{\mu})^{\frac{2}{1-2\beta}}$ as $\nu_2(\beta) \leq \frac{2\beta}{1-2\beta}$, hence these conditions are comparable to our assumptions.

- The *effective energy per particle*: for $\Phi \in H^1(\mathbb{R})$,

$$\mathcal{E}^{\Phi}(t) := \left\langle \Phi, \left(-\tfrac{\partial^2}{\partial x^2} + V^{\parallel}(t,(x,0)) + \tfrac{b_\beta}{2}|\Phi|^2\right)\Phi \right\rangle_{L^2(\mathbb{R})}. \tag{7}$$

Further, we define the function $\mathfrak{e}:\mathbb{R}\to[1,\infty)$ by

$$\mathfrak{e}^2(t) := 1 + |E^{\psi_0^{N,\varepsilon}}(0)| + |\mathcal{E}^{\Phi_0}(0)| + \int_0^t \|\dot{V}^{\parallel}(s,\cdot)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s + \sup_{\substack{i,j\in\{0,1\}\\ k\in\{1,2\}}} \|\partial_t^i \partial_{y_k}^j V^{\parallel}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}. \tag{8}$$

Note that $\mathfrak{e}(t)$ is for each $t\in\mathbb{R}$ uniformly bounded in N and ε because we will assume that $E^{\psi_0^{N,\varepsilon}}(0) \to \mathcal{E}^{\Phi_0}(0)$ as $(N,\varepsilon)\to(\infty,0)$ (assumption A4 below) and boundedness of $\partial_t^i\partial_{y_k}^j V^{\parallel}$ (assumption A3 below). The function $\mathfrak{e}$ will be of use because, by the fundamental theorem of calculus,

$$\left|E^{\psi^{N,\varepsilon}(t)}(t)\right| \le \mathfrak{e}^2(t) - 1 \quad \text{and} \quad \left|\mathcal{E}^{\Phi(t)}(t)\right| \le \mathfrak{e}^2(t) - 1 \tag{9}$$

for any time $t\in\mathbb{R}$. Note that if the external field $V^{\parallel}$ is time-independent, $\mathfrak{e}^2(t) \lesssim 1$ for any t, hence in this case, $E^{\psi^{N,\varepsilon}(t)}(t)$ and $\mathcal{E}^{\Phi(t)}(t)$ are bounded uniformly in time.

Let us now state our assumptions:

A1 *Interaction.* Let the interaction $w_\beta \in \mathcal{W}_{\beta,\eta}$ for some $\eta > 0$.

A2 *Confining potential.* Let $V^{\perp}:\mathbb{R}^2\to\mathbb{R}$ such that $-\Delta_y + V^{\perp}$ is self-adjoint and has a non-degenerate ground state χ with energy $E_0 < \inf\sigma_{\mathrm{ess}}(-\Delta_y + V^{\perp})$. Assume that the negative part of $V^{\perp}$ is bounded and that $\chi \in \mathcal{C}_{\mathrm{b}}^1(\mathbb{R}^2)$, i.e., χ is bounded and continuously differentiable with bounded derivative. We choose χ normalised and real.

A3 *External field.* Let $V^{\parallel}:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ such that $V^{\parallel}(\cdot,z)\in\mathcal{C}^1(\mathbb{R})$. Further, assume that for each fixed $t\in\mathbb{R}$, $V^{\parallel}(t,(\cdot,0))\in H^4(\mathbb{R})$, $V^{\parallel}(t,\cdot), \dot{V}^{\parallel}(t,\cdot) \in L^\infty(\mathbb{R}^3)\cap\mathcal{C}^1(\mathbb{R}^3)$ and $\nabla_y V^{\parallel}(t,\cdot), \nabla_y \dot{V}^{\parallel}(t,\cdot) \in L^\infty(\mathbb{R}^3)$.

A4 *Initial data.* Assume that the family of initial data, $\psi_0^{N,\varepsilon} \in \mathcal{D}(H_\beta(0)) \cap L_s^2(\mathbb{R}^{3N})$ with $\|\psi_0^{N,\varepsilon}\|^2 = 1$, is close to a condensate with condensate wavefunction $\varphi_0^\varepsilon = \Phi_0\chi^\varepsilon$ for some normalised $\Phi_0 \in H^2(\mathbb{R})$ in the following sense: for some admissible, moderately confining sequence (N,ε), it holds that

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}_{L^2(\mathbb{R}^3)} \left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}} - |\Phi_0\chi^\varepsilon\rangle\langle\Phi_0\chi^\varepsilon|\right| = 0 \tag{10}$$

and

$$\lim_{(N,\varepsilon)\to(\infty,0)} \left|E^{\psi_0^{N,\varepsilon}}(0) - \mathcal{E}^{\Phi_0}(0)\right| = 0. \tag{11}$$

Remark 1. (a) Assumption A1 includes the interaction $w_\beta(z) = \left(\frac{N}{\varepsilon^2}\right)^{-1+3\beta} w\left((\frac{N}{\varepsilon^2})^\beta z\right)$ for $w:\mathbb{R}^3\to\mathbb{R}$ spherically symmetric, non-negative and with $\operatorname{supp} w \subseteq \overline{B_1(0)}$. We consider the larger class of interaction potentials $\mathcal{W}_{\beta,\eta}$ because due to this slight generalisation, one may immediately apply the result of Theorem 1 in [4].

(b) Assumption A2 is, for instance, fulfilled by a harmonic potential or by any bounded smooth potential with a bound state below the essential spectrum. Note that it is not necessary that the potential increases as $|y| \to \infty$. The confining effect of the potential is due to the rescaling by ε because the ground state of $-\Delta_y + V^\perp$ is exponentially localised [13, Theorem 1].

(c) The regularity condition on $V^\parallel(t,(\cdot,0))$ in A3 ensures the global existence of H^2-solutions of the NLS equation (5) (see Appendix A and Lemma 4.9). The further requirements for $V^\parallel, \dot{V}^\parallel, \nabla_y V^\parallel$ and $\nabla_y \dot{V}^\parallel$ are needed to control the one-particle energies and the interactions of bosons with the external field $V^\parallel$.

(d) Due to assumptions A1 – A3, the operators $H_\beta(t)$ are self-adjoint on the time-independent domain $\mathcal{D}(H_\beta)$. As $t \mapsto V^\parallel(t) \in \mathcal{L}(L^2(\mathbb{R}^3))$ is continuous, $H_\beta(t)$ generates a strongly continuous unitary evolution on $\mathcal{D}(H_\beta)$ [14].

(e) We assume in A4 that the system is initially given by a Bose–Einstein condensate with factorised condensate wavefunction. Both parts (10) and (11) of the assumption are standard when deriving effective evolution equations. For the scaling parameter $\beta = 1$ and a homogeneous external field $V^\parallel(z,0)$, it is shown in [26] that the ground state of $H_1(0)$ satisfies assumption A4. Note that for initial data in the ground state, it is important to admit a time-dependent external potential $V^\parallel$ to observe non-trivial dynamics. For related results without strong confinement, we refer to the review [25] for $\beta = 1$ and to [23] for $\beta < 1$.

Theorem 1. *Let $\beta \in (0,1)$ and assume that w_β, $V^\perp$ and $V^\parallel$ satisfy A1 – A3. Let $\psi_0^{N,\varepsilon}$ be a family of initial data satisfying A4, let $\psi^{N,\varepsilon}(t)$ denote the solution of the N-body Schrödinger equation* (2) *with initial datum $\psi^{N,\varepsilon}(0) = \psi_0^{N,\varepsilon}$ and let $\gamma^{(k)}_{\psi^{N,\varepsilon}(t)}$ denote its k-particle reduced density matrix as in* (3)*. Then for any $T \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[-T,T]} \operatorname{Tr}_{L^2(\mathbb{R}^{3k})} \left| \gamma^{(k)}_{\psi^{N,\varepsilon}(t)} - |\Phi(t)\chi^\varepsilon\rangle \langle \Phi(t)\chi^\varepsilon|^{\otimes k} \right| = 0 \tag{12}$$

and

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[-T,T]} \left| E^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}^{\Phi(t)}(t) \right| = 0, \tag{13}$$

where the limits are taken along the sequence from A4. $\Phi(t)$ is the solution of the NLS equation (5) *with initial datum $\Phi(0) = \Phi_0$ from A4, where the strength of the nonlinearity in* (5) *is given by b_β from Definition 2.2, namely*

$$b_\beta = \lim_{(N,\varepsilon)\to(\infty,0)} b_{N,\varepsilon} = \lim_{(N,\varepsilon)\to(\infty,0)} \tfrac{N}{\varepsilon^2} \int\limits_{\mathbb{R}^3} w_\beta(z)\,\mathrm{d}z \int\limits_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y. \tag{14}$$

Remark 2. (a) For the choice $w_\beta(z) = \left(\frac{N}{\varepsilon^2}\right)^{-1+3\beta} w\left(\left(\frac{N}{\varepsilon^2}\right)^\beta z\right)$, we obtain the coupling parameter $b_\beta = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y$.

(b) Our proof provides an estimate of the rate of the convergence of (12), which is explicitly stated in Corollary 3.9. The rate is not uniform in time but depends on it in terms of a double exponential. Note, however, that times of order one already correspond to long times on the microscopic scale.

(c) Let us comment on the difference of our work to the result of von Keler and Teufel [19], who consider $\beta \in (0, \frac{1}{3})$. The extension to $\beta \in (0,1)$ means a physically relevant improvement of the result: for $\beta < \frac{1}{3}$, the problem can still be considered as a mean-field problem since the mean inter-particle distance $\varrho^{-\frac{1}{3}} \sim (\frac{N}{\varepsilon^2})^{-\frac{1}{3}}$ is small compared to the range of the interaction $\mu = (\frac{N}{\varepsilon^2})^{-\beta}$. For $\beta > \frac{1}{3}$, the mean-field description breaks down and one must handle interactions which are too singular to be covered by the approach of [19]. We solve this by an integration by parts of the interaction, which comes at the price that one must control the kinetic energy of the N-particle wavefunction (Lemma 4.11 and Lemma 4.21). Also, note that our admissibility condition is weaker than the respective condition $\varepsilon^{\frac{4}{3}}/\mu \to 0$ in [19], which cannot be satisfied for $\beta > \frac{2}{3}$.
In [19], the bosons are trapped within a quantum waveguide with non-trivial geometry. The confinement is realised by means of Dirichlet boundary conditions, which restrict the system to a tube of width ε around some curve in $\mathbb{R}^3$. In our model, the confinement is by potentials. However, our result can easily be modified to a confinement via Dirichlet boundary conditions, corresponding to a straight and untwisted quantum waveguide. The main difference in the proof is the estimate of $\gamma_b^{(1)}$ (Section 4.4.2): one divides the expression (46) into an integral over those y sufficiently distant from the boundary that $\operatorname{supp} w_\beta((x,y)-\cdot)$ is completely contained in the waveguide, and into an integral over the rest, which is easily estimated.
In addition to moderately confining sequences, the authors of [19] consider sequences $(N,\varepsilon) \to (\infty, 0)$ with $\varepsilon/\mu \to 0$. This is possible for $\beta \in (0, \frac{1}{2})$ and leads to $b_\beta = 0$ in the effective equation because an essential part of the interaction is cut off such that the limiting effective equation becomes linear. We conjecture that the same effect occurs in our setup.

(d) Our analysis is restricted to sequences where $\varepsilon \ll N^{-\frac{\beta}{2(1-\beta)}}$ (Definition 2.1). As remarked before, similar conditions are needed in the comparable works [6, 8] whereas no analogue of this admissibility condition is required for the ground state result in [26]. In combination with the work on the confinement limit of the three-dimensional NLS equation in [2], this indicates that our dynamical result should in principle hold without any admissibility condition. For our strategy of proof, this condition is however indispensable to control the transverse excitations out of the transverse ground state χ^ε.

(e) In [8], Chen and Holmer study attractive interactions, i.e., $\int_{\mathbb{R}^3} w_\beta(z)\,\mathrm{d}z \leq 0$. In distinction from that work, we exclusively consider repulsive interactions with $w_\beta \geq 0$. However, as the condition $w_\beta \geq 0$ seems to be crucial only to the proofs of Lemma 4.11 and Lemma 4.21, it is likely that our result can be extended to include repulsive interactions with a certain negative part.

3 Proof of the main theorem

To prove Theorem 1, we need to show that the expressions in (12) and (13) vanish in the limit $(N,\varepsilon) \to (\infty, 0)$, given suitable initial data. Instead of estimating these differences directly, we follow the strategy by Pickl [29, 30, 31, 32, 33] and define a functional

$\alpha_\xi(\psi^{N,\varepsilon}(t), \varphi^\varepsilon(t))$ which provides a measure of the part of the N-particle wavefunction $\psi^{N,\varepsilon}$ that has not condensed into the single-particle orbital φ^ε. The functional is chosen such that $\alpha_\xi(\psi^{N,\varepsilon}(t), \varphi^\varepsilon(t)) \to 0$ is equivalent to (12) and (13). We follow in general [33]. However, the strongly asymmetric confinement requires a nontrivial modification of the formalism to treat the dimensional reduction and the more singular scaling of the interaction. For the construction of α_ξ, we need the following projections:

Definition 3.1. Let $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$, where $\Phi(t)$ is the solution of the NLS equation (5) with initial datum Φ_0 from A4 and with χ^ε as in (4). Let

$$p := |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)| ,$$

where we have dropped the time dependence of p in the notation. For $i \in \{1, \dots, N\}$, define the projection operators on $L^2(\mathbb{R}^{3N})$

$$p_j := \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes p \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \cdots \mathbb{1}}_{N-j} \quad \text{and} \quad q_j := \mathbb{1} - p_j.$$

Further, define the orthogonal projections on $L^2(\mathbb{R}^3)$

$$p^\Phi := |\Phi(t)\rangle \langle \Phi(t)| \otimes \mathbb{1}_{L^2(\mathbb{R}^2)}, \qquad q^\Phi := \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\Phi,$$
$$p^{\chi^\varepsilon} := \mathbb{1}_{L^2(\mathbb{R})} \otimes |\chi^\varepsilon\rangle \langle \chi^\varepsilon| , \qquad q^{\chi^\varepsilon} := \mathbb{1}_{L^2(\mathbb{R}^3)} - p^{\chi^\varepsilon},$$

and define p_j^Φ, q_j^Φ, $p_j^{\chi^\varepsilon}$ and $q_j^{\chi^\varepsilon}$ on $L^2(\mathbb{R}^{3N})$ analogously to p_j and q_j. Finally, for $0 \le k \le N$, define the many-body projections

$$P_k = \left(q_1 \cdots q_k p_{k+1} \cdots p_N\right)_{\text{sym}} := \sum_{\substack{J \subseteq \{1,\dots,N\} \\ |J|=k}} \prod_{j \in J} q_j \prod_{l \notin J} p_l$$

and $P_k = 0$ for $k < 0$ and $k > N$.

In the sequel, we will write $p_j = |\varphi^\varepsilon(t, z_j)\rangle \langle \varphi^\varepsilon(t, z_j)|$, $p_j^\Phi = |\Phi(t, x_j)\rangle \langle \Phi(t, x_j)|$ and $p_j^{\chi^\varepsilon} = |\chi^\varepsilon(y_j)\rangle \langle \chi^\varepsilon(y_j)|$. Some useful identities of the projections are listed in the following corollary:

Corollary 3.2. *For $0 \le k \le N$ and $1 \le j \le N$, it holds that*

(a) $\sum_{k=0}^{N} P_k = \mathbb{1}, \quad \sum_{j=1}^{N} q_j P_k = k P_k,$

(b) $p_j = p_j^\Phi p_j^{\chi^\varepsilon}, \quad p_j^\Phi q_j = p_j^\Phi q_j^{\chi^\varepsilon}, \quad p_j^{\chi^\varepsilon} q_j = p_j^{\chi^\varepsilon} q_j^\Phi$ *and*

$p_j^\sharp p_j = p_j, \quad q_j^\sharp q_j = q_j^\sharp, \quad q_j^\sharp p_j = 0$ *for* $\sharp \in \{\Phi, \chi^\varepsilon\}$,

(c) $q_j = q_j^\Phi p_j^{\chi^\varepsilon} + p_j^\Phi q_j^{\chi^\varepsilon} + q_j^\Phi q_j^{\chi^\varepsilon} = q_j^{\chi^\varepsilon} + q_j^\Phi p_j^{\chi^\varepsilon} = q_j^\Phi + p_j^\Phi q_j^{\chi^\varepsilon}.$

Proof. The first identity in (a) is due to the relation $p_j + q_j = \mathbb{1}$. The second identity follows from the fact that

$$\sum_{j=1}^{N} q_j = \sum_{j=1}^{N} q_j \sum_{k=0}^{N} P_k = \sum_{k=0}^{N} \sum_{j=1}^{N} q_j P_k = \sum_{k=0}^{N} k P_k$$

together with $P_k P_{k'} = \delta_{k,k'} P_k$. While part (b) is an immediate consequence of Definition 3.1, part (c) is implied by $q = 1 - p = (p^\Phi + q^\Phi)(p^{\chi^\varepsilon} + q^{\chi^\varepsilon}) - p^\Phi p^{\chi^\varepsilon} = p^\Phi q^{\chi^\varepsilon} + q^\Phi p^{\chi^\varepsilon} + q^\Phi q^{\chi^\varepsilon}$. □

Definition 3.3. For any function $f : \mathbb{N}_0 \to \mathbb{R}_0^+$, define the operator $\widehat{f} \in \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)$ by

$$\widehat{f} := \sum_{k=0}^{N} f(k) P_k$$

and, for any $d \in \mathbb{Z}$, the shifted operator $\widehat{f}_d \in \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)$ by

$$\widehat{f}_d := \sum_{j=-d}^{N-d} f(j+d) P_j.$$

We will in particular need the weight function n defined by $n(k) := \sqrt{\frac{k}{N}}$.

Definition 3.4. Define the functional $\alpha_f : L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}$ by

$$\alpha_f(\psi, \varphi^\varepsilon(t)) := \left\langle\!\left\langle \psi, \widehat{f}\psi \right\rangle\!\right\rangle = \sum_{k=0}^{N} f(k) \left\langle\!\left\langle \psi, P_k \psi \right\rangle\!\right\rangle .$$

The φ^ε-dependence of α_f is due to the φ^ε-dependence of the projectors P_k. As the operators P_k project onto states with exactly k particles outside the condensate, α_f is a measure of the relative number of such particles in the state ψ. We choose the weight f increasing and $f(0) \approx 0$, hence those parts of ψ with a larger "distance" to the condensate contribute more to $\alpha_f(\psi, \varphi^\varepsilon)$. On the other hand, $P_0\psi$ — the state where all particles are condensed into φ^ε — contributes hardly anything. The weight $\widehat{n}$ is in particular distinguished because for any symmetric wavefunction $\psi \in L_s^2(\mathbb{R}^{3N})$,

$$\alpha_{n^2}(\psi, \varphi^\varepsilon(t)) = \sum_{k=0}^{N} \tfrac{k}{N} \left\langle\!\left\langle \psi, P_k \psi \right\rangle\!\right\rangle = \sum_{k=0}^{N} \sum_{j=1}^{N} \tfrac{1}{N} \left\langle\!\left\langle \psi, q_j P_k \psi \right\rangle\!\right\rangle = \|q_1 \psi\|^2$$

by Corollary 3.2a.

Lemma 3.5. *Let $\psi^N \in L_s^2(\mathbb{R}^{3N})$ be a sequence of normalised N-particle wavefunctions and let $\gamma_N^{(k)}$ be the sequence of corresponding k-particle reduced density matrices for some fixed $k \in \mathbb{N}$. Let $t \in \mathbb{R}$. Then the following statements are equivalent:*

(a) $\lim\limits_{N\to\infty} \alpha_{n^a}(\psi^N, \varphi^\varepsilon(t)) = 0$ *for some* $a > 0$,

(b) $\lim\limits_{N\to\infty} \alpha_{n^a}(\psi^N, \varphi^\varepsilon(t)) = 0$ *for any* $a > 0$,

(c) $\lim\limits_{N\to\infty} \|\gamma_N^{(k)} - |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)|^{\otimes k}\|_{\mathcal{L}(L^2(\mathbb{R}^{3k}))} = 0$ *for all* $k \in \mathbb{N}$,

(d) $\lim\limits_{N\to\infty} \mathrm{Tr}_{L^2(\mathbb{R}^{3k})} \left|\gamma_N^{(k)} - |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)|^{\otimes k}\right| = 0$ *for all* $k \in \mathbb{N}$,

(e) $\lim\limits_{N\to\infty} \mathrm{Tr}_{L^2(\mathbb{R}^3)} \left|\gamma_N^{(1)} - |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)|\right| = 0.$

For the proof of this lemma, we refer to [19, Lemma 3.1] and to corresponding results in [22, 32, 33, 34]. We will in the following choose the weight function $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ with

$$m(k) := \begin{cases} n(k) & \text{for } k \geq N^{1-2\xi}, \\ \frac{1}{2}\left(N^{-1+\xi}k + N^{-\xi}\right) & \text{else} \end{cases}$$

for some $\xi \in (0, \frac{1}{2})$, i.e., m equals n with a smooth cut-off to soften the singularity of $\frac{\mathrm{d}n}{\mathrm{d}k}$ for small k. Clearly, $n(k) \leq m(k) \leq n(k) + \frac{1}{2}N^{-\xi}$ for all $k \geq 0$ and $\xi \in (0, \frac{1}{2})$, hence $\alpha_m(\psi, \varphi^\varepsilon(t)) \to 0$ is equivalent to $\alpha_n(\psi, \varphi^\varepsilon(t)) \to 0$ and thus to all cases in Lemma 3.5 for any choice of the parameter ξ. For the actual proof, we will consider a modified version of this functional, namely

$$\alpha_\xi(t) := \alpha_m(\psi^{N,\varepsilon}(t), \varphi^\varepsilon(t)) + \left|E^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}^{\Phi(t)}(t)\right|. \tag{15}$$

The convergence of $\alpha_\xi(t)$ to zero is equivalent to (12) and (13). Conversely, (10) and (11) imply $\alpha_\xi(0) \to 0$ as $(N, \varepsilon) \to (\infty, 0)$. The main idea of the proof is therefore to derive a bound for $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)|$ (Propositions 3.7 and 3.8), from which one obtains an estimate for $\alpha_\xi(t)$ by Grönwall's inequality. The propositions will be proven in Sections 4.3 and 4.4. The estimate of the rate of the convergence of $\alpha_\xi(t)$ gained from this procedure translates to a rate for the reduced density matrices:

Lemma 3.6. *For $\alpha_\xi(t)$ as in* (15)*, it holds that*

$$\operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| \leq \sqrt{8\alpha_\xi(t)},$$
$$\alpha_\xi(t) \leq \left|E^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}^{\Phi(t)}(t)\right| + \sqrt{\operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right|} + \tfrac{1}{2}N^{-\xi}.$$

Proof. Let us abbreviate $\psi^{N,\varepsilon}(t) \equiv \psi$ and drop all time dependencies. [22, Lemma 2.3] implies

$$\langle\!\langle \psi, \widehat{n}^2\psi\rangle\!\rangle \leq \operatorname{Tr}\left|\gamma^{(1)}_\psi - |\varphi^\varepsilon\rangle\langle\varphi^\varepsilon|\right| \leq \sqrt{8\,\langle\!\langle\psi, \widehat{n}^2\psi\rangle\!\rangle}.$$

The first inequality is thus immediately clear as $n(k)^2 \leq n(k) \leq m(k)$. For the second inequality, recall that $m(k) \leq n(k) + \frac{1}{2}N^{-\xi}$, hence

$$\langle\!\langle\psi, \widehat{m}\psi\rangle\!\rangle \leq \|\psi\|\|\widehat{n}\psi\| + \tfrac{1}{2}N^{-\xi} \leq \sqrt{\langle\!\langle\psi, \widehat{n}^2\psi\rangle\!\rangle} + \tfrac{1}{2}N^{-\xi} \leq \sqrt{\operatorname{Tr}\left|\gamma^{(1)}_\psi - |\varphi^\varepsilon\rangle\langle\varphi^\varepsilon|\right|} + \tfrac{1}{2}N^{-\xi}.$$

□

Proposition 3.7. *Under assumptions A1 – A4,*

$$\begin{aligned} \left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)\right| &\leq |\gamma_a(t)| + |\gamma_b(t)| \\ &\leq |\gamma_a(t)| + \left|\gamma_b^{(1)}(t)\right| + \left|\gamma_b^{(2)}(t)\right| + \left|\gamma_b^{(3)}(t)\right| \end{aligned}$$

for almost every $t \in \mathbb{R}$, where

$$\gamma_a(t) := \left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), \dot{V}^{\parallel}(t, z_1)\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle - \left\langle\Phi(t), \dot{V}^{\parallel}(t,(x,0))\Phi(t)\right\rangle_{L^2(\mathbb{R})}\right| \tag{16}$$
$$-2N\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), q_1\widehat{m}^a_{-1}\big(V^{\parallel}(t,z_1) - V^{\parallel}(t,(x_1,0))\big)p_1\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \tag{17}$$

$$\begin{aligned}\gamma_b(t) &:= -N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), Z_\beta^{(12)}\widehat{m}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle \quad (18)\\ &= \gamma_b^{(1)}(t)+\gamma_b^{(2)}(t)+\gamma_b^{(3)}(t),\end{aligned}$$

$$\gamma_b^{(1)}(t) := -2N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), q_1^{\Phi}\widehat{m}_{-1}^a p_1^{\chi^\varepsilon}p_2 Z_\beta^{(12)}p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \quad (19)$$

$$\begin{aligned}\gamma_b^{(2)}(t) &:= -N(N-1)\Im\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t), \left(2p_2\widehat{m}_{-1}^a+q_2(1+p_2^{\chi^\varepsilon})\widehat{m}_{-2}^b\right)w_\beta^{(12)}p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle \quad (20)\\ &\quad -2N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), (q_1^{\chi^\varepsilon}q_2+q_1^{\Phi}p_1^{\chi^\varepsilon}q_2^{\chi^\varepsilon})\widehat{m}_{-1}^a w_\beta^{(12)}p_1q_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle \quad (21)\\ &\quad -2N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), q_1^{\Phi}q_2^{\Phi}\widehat{m}_{-1}^a p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta^{(12)}p_1q_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \quad (22)\end{aligned}$$

$$\begin{aligned}\gamma_b^{(3)}(t) &:= -N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), q_1^{\Phi}q_2^{\Phi}\widehat{m}_{-2}^b p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta^{(12)}p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle \quad (23)\\ &\quad -2N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), q_1^{\Phi}q_2^{\Phi}\widehat{m}_{-1}^a p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta^{(12)}p_1p_2^{\chi^\varepsilon}q_2^{\Phi}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle \quad (24)\\ &\quad +2Nb_\beta\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t), q_1q_2\widehat{m}_{-1}^a|\Phi(t,x_1)|^2p_1q_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle. \quad (25)\end{aligned}$$

Here,

$$w_\beta^{(12)} := w_\beta(z_1-z_2) \quad \textit{and} \quad Z_\beta^{(12)} := w_\beta^{(12)} - \tfrac{b_\beta}{N-1}\left(|\Phi(t,x_1)|^2+|\Phi(t,x_2)|^2\right)$$

and $\widehat{m}^a$, $\widehat{m}^b$ denote the many-body operators corresponding to the weight functions

$$m^a(k) := m(k)-m(k+1) \quad \textit{and} \quad m^b(k) := m(k)-m(k+2).$$

The first term, γ_a, merely contains one-body contributions, i.e., interactions between the bosons and the external field $V^{\|}$, and is therefore the easiest to estimate. Note that (16) is small only if the system is in a state $\psi^{N,\varepsilon}$ close to the condensate with condensate wavefunction $\varphi^\varepsilon = \Phi\chi^\varepsilon$ (see Lemma 4.7). The term γ_b handles the two-body contributions, i.e., interactions among bosons. The expressions $\gamma_b^{(1)}$ and $\gamma_b^{(3)}$ contain the quasi one-dimensional interaction $\overline{w}(x_1-x_2)$ defined by $p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta(z_1-z_2)p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon} =: \overline{w}(x_1-x_2)p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}$ (see Definition 4.18), where the transverse degrees of freedom are integrated out. These terms are comparable to the corresponding three-dimensional terms in [33]. $\gamma_b^{(2)}$ has no equivalent in the situation without strong confinement as it collects the remainders that arise upon approximating the three-dimensional interaction w_β with the quasi one-dimensional interaction $\overline{w}$.

$\gamma_b^{(1)}$ is physically most relevant because it depends on the difference between the quasi one-dimensional interaction $\overline{w}$ and the one-dimensional effective potential $b_\beta|\Phi(t)|^2$. In other words, this term is small if and only if (5) is the right effective equation, in particular with the correct coupling parameter b_β. Note that for this term it is crucial that the sequence (N,ε) is moderately confining, i.e., that $\mu/\varepsilon \to 0$.

For $\gamma_b^{(2)}$ to be small, we require in particular the admissibility of the sequence (N,ε), i.e., that $\varepsilon^2/\mu \to 0$. The other key tool for the estimate is the observation that due to the strong confinement, it is unlikely that a particle is excited in the transverse directions. This implies in particular that $\|q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| = \mathcal{O}(\varepsilon)$ (Lemma 4.11).

The estimate of $\gamma_b^{(3)}$ relies on a bound for the kinetic energy of the part of $\psi^{N,\varepsilon}(t)$ with at least one particle not in $\Phi(t)$, i.e., a bound for $\|\partial_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|$ (Lemma 4.21). The proof of this bound again involves the splitting of the interaction w_β into a quasi one-dimensional part $\overline{w}$ and remainders. Hence for $\gamma_b^{(3)}$ to be small, we require both moderate confinement and the admissibility of the sequence (N,ε). The last line (25) is a remainder which is easily controlled.

Proposition 3.8. *Let μ be sufficiently small. Under assumptions A1 – A4, γ_a to $\gamma_b^{(3)}$ from Proposition 3.7 are bounded by*

$$\begin{aligned}
\big|\gamma_a(t)\big| &\lesssim \big(\langle\!\langle \psi^{N,\varepsilon}(t), \widehat{n}\psi^{N,\varepsilon}(t)\rangle\!\rangle + \varepsilon\big)\,\mathfrak{e}^3(t),\\
\big|\gamma_b^{(1)}(t)\big| &\lesssim \big(\tfrac{\mu}{\varepsilon} + N^{-1} + (\tfrac{N}{\varepsilon^2})^{-\eta}\big)\,\mathfrak{e}^2(t),\\
\big|\gamma_b^{(2)}(t)\big| &\lesssim \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}\mathfrak{e}^3(t),\\
\big|\gamma_b^{(3)}(t)\big| &\lesssim \bigg(\Big|E^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}^{\Phi(t)}(t)\Big| + \langle\!\langle \psi^{N,\varepsilon}(t), \widehat{n}\psi^{N,\varepsilon}(t)\rangle\!\rangle + \tfrac{\mu}{\varepsilon} + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\frac{\beta_1}{2}}\\
&\qquad + N^{-1+\beta_1+\xi} + (\tfrac{N}{\varepsilon^2})^{-\eta}\bigg)\mathfrak{e}(t)\exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}
\end{aligned}$$

for any $\xi \in (0, \frac{\beta}{4}]$, any $\beta_1 \in (0,\beta]$ and with η from Definition 2.2 and $\mathfrak{e}(t)$ as in (8).

The estimate of $\gamma_b^{(1)}$ is essentially the same as in the case $\beta \in (0, \frac{1}{3})$ in [19]. γ_a must be treated in a different way because the confinement is by a potential and not via Dirichlet boundary conditions. For the terms $\gamma_b^{(2)}$ and $\gamma_b^{(3)}$, the argument from [19] does not work because the interaction becomes too singular for $\beta > \frac{1}{3}$. To cope with this, we follow an idea from [33]: we identify a function h_ε such that $w_\beta = \Delta h_\varepsilon$ and integrate by parts. ∇h_ε is less singular, and the expressions resulting from ∇ acting on $\psi^{N,\varepsilon}(t)$ can be controlled with Lemma 4.11 (or the refined version, Lemma 4.21).

Our strategy differs from [33] in a relevant point: in [33], the interaction w_β is approximated by a potential U_{β_1} with softer but still singular scaling behaviour ($\beta_1 < \frac{1}{3}$). The author first proves bounds for $\beta < \frac{1}{3}$, the second step is to estimate the contribution from the difference $w_\beta - U_{\beta_1}$ using integration by parts. Instead of these two steps, we define h_ε as the solution of $\Delta h_\varepsilon = w_\beta$ on a ball with Dirichlet boundary conditions and integrate by parts on the ball. To prevent the emergence of boundary terms, we use smoothed step functions whose derivatives can be controlled. This mathematical trick enables us to avoid the separate estimate for $\beta < \frac{1}{3}$.

The control of the kinetic energy (Lemma 4.21) required for the integration by parts in $\gamma_b^{(3)}$ is also different from the corresponding Lemma 5.2 in [33]. Instead of following that path, we extend ideas from [19, Lemma 4.7] and [30, Lemma 4.6] and estimate the part of the kinetic energy in the free direction. Besides, we use with Lemma 4.8a a slightly sharpened version of [33, Lemma 4.3].

Proof of Theorem 1. From Propositions 3.7 and 3.8, we gather, for sufficiently small μ, that

$$\big|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)\big| \lesssim C(t)\big(\alpha_\xi(t) + R_{\xi,\beta_1,\eta}(N,\varepsilon)\big)$$

for almost every $t \in \mathbb{R}$, where

$$C(t) := \mathfrak{e}(t) \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}, \tag{26}$$

$$R_{\xi,\beta_1,\eta}(N,\varepsilon) := \tfrac{\mu}{\varepsilon} + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\frac{\beta_1}{2}} + N^{-1+\beta_1+\xi} + (\tfrac{N}{\varepsilon^2})^{-\eta}.$$

Recall that $\mathfrak{e}(t)$ is for each $t \in \mathbb{R}$ bounded uniformly in N and ε by assumption A4. The differential version of Grönwall's inequality yields

$$\alpha_\xi(t) \le \left(\alpha_\xi(0) + R_{\xi,\beta_1,\eta}(N,\varepsilon)\right) \exp\left\{2\int_0^t C(s)\,\mathrm{d}s\right\}$$

for all $t \in \mathbb{R}$. Due to assumption A4 and by Lemma 3.5, $\lim_{(N,\varepsilon)\to(\infty,0)} \alpha_\xi(0) = 0$ and $R_{\xi,\beta_1,\eta}(N,\varepsilon)$ vanishes in the limit $(N,\varepsilon) \to (\infty,0)$ for $\beta_1 \in (0,\beta]$ and $\xi \in (0,\frac{\beta}{4}]$, $\xi < 1-\beta_1$, because the sequence (N,ε) is by assumption A4 admissible and moderately confining. Again by Lemma 3.5, this implies (12) and (13) for any $t \in \mathbb{R}$. □

Corollary 3.9. *Let $t \in \mathbb{R}$. Then*

$$\operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| \lesssim \left(A(0) + \tfrac{\mu}{\varepsilon} + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\frac{\beta}{4}} + (\tfrac{N}{\varepsilon^2})^{-\eta}\right)^{\frac{1}{2}} \times \\ \times \exp\left\{\int_0^t C(s)\,\mathrm{d}s\right\}$$

for $C(t)$ as in (26) *and where*

$$A(0) := \left|E^{\psi_0^{N,\varepsilon}}(0) - \mathcal{E}^{\Phi_0}(0)\right| + \sqrt{\operatorname{Tr}\left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}} - |\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|\right|}.$$

Proof. This follows from Lemma 3.6 after optimisation over ξ and β_1. □

Remark 3. In the case without external field, i.e. $V^\parallel = 0$, we have $\|\Phi(t)\|_{H^2(\mathbb{R})} \lesssim C(\|\Phi_0\|_{H^2(\mathbb{R})})$ uniformly in t, where $C(\|\Phi_0\|_{H^2(\mathbb{R})})$ is some expression depending on $\|\Phi_0\|_{H^2(\mathbb{R})}$ [36, Exercise 3.36][2]. Defining $\widetilde{\mathfrak{e}} := 1 + |E^{\psi_0^{N,\varepsilon}}(0)| + |\mathcal{E}^{\Phi_0}(0)| + (C(\|\Phi_0\|_{H^2(\mathbb{R})}))^2$ in analogy to (8), this yields

$$\operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| \lesssim \left(A(0) + \tfrac{\mu}{\varepsilon} + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\frac{\beta}{4}} + (\tfrac{N}{\varepsilon^2})^{-\eta}\right)^{\frac{1}{2}} \exp\left(\widetilde{\mathfrak{e}}\, t\right),$$

where the growth in time is an exponential instead of a double exponential.

4 Proofs of the propositions

4.1 Preliminaries

In this section, we prove several lemmata which are needed for the proofs of the propositions. The first ones establish several properties of the weighted operators $\widehat{f}$, Lemma 4.7

[2] To show this, one observes that $E_2(\Phi) := \int_{\mathbb{R}} \left(|\partial_x^2\Phi|^2 + c_1|\partial_x\Phi|^2|\Phi|^2 + c_2\Re((\overline{\Phi}\partial_x\Phi)^2) + c_3|\Phi|^6\right)\mathrm{d}x$ is conserved for solutions of (5) with $V^\parallel = 0$, with c_1, c_2 and c_3 some absolute constants.

and Lemma 4.8 contain some useful estimates for scalar products, and the remainder of the section covers properties of the condensate wavefunction $\varphi^\varepsilon(t)$. In the following, we will always assume that assumptions A1 – A4 are satisfied.

Lemma 4.1. *Let* $f : \mathbb{N}_0 \to \mathbb{R}_0^+$, $d \in \mathbb{Z}$ *and define*

$$\widehat{l} := N \max\{\widehat{m}^a_{-1}, \widehat{m}^b_{-2}\},$$

where the max *is to be understood in the sense of inequalities between operators, i.e.,* $\widehat{l} = N\widehat{m}^a_{-1}$ *if* $\widehat{m}^a_{-1} - \widehat{m}^b_{-2}$ *is a positive operator and vice versa. Then*

(a) $\|\widehat{f}\|_{\text{op}} = \|\widehat{f}_d\|_{\text{op}} = \|\widehat{f}^{\frac{1}{2}}\|^2_{\text{op}} = \sup\limits_{0\leq k\leq N} f(k)$,

(b) $\|\widehat{l}\,\widehat{n}\|_{\text{op}} \lesssim 1, \quad \|\widehat{l}\|_{\text{op}} \leq N^\xi$.

Proof. Part (a) is obvious. For part (b), note that

$$\widehat{m}^a_{-1}\widehat{n} = \sum_{k=1}^N (m(k-1) - m(k))\, n(k) P_k, \qquad \widehat{m}^b_{-2} = \sum_{k=2}^N (m(k-2) - m(k))\, n(k) P_k.$$

The derivative of m with respect to k, where k is for the moment understood as real variable, is given by

$$m'(k) \equiv \tfrac{\mathrm{d}}{\mathrm{d}k} m(k) = \begin{cases} \frac{1}{2\sqrt{kN}} = \frac{1}{2} N^{-1} n(k)^{-1} & \text{for } k \geq N^{1-2\xi}, \\ \frac{1}{2} N^{-1+\xi} & \text{else.} \end{cases}$$

By the mean value theorem, $|m(k) - m(k-j)| = j|m'(\kappa)|$ for $j \in \{1, 2\}$ and $\kappa \in (k-j, k)$. For $\kappa \geq N^{1-2\xi}$, $|m'(\kappa)| = \frac{1}{2}N^{-1}n(\kappa)^{-1}$. For $\kappa < N^{1-2\xi}$, we obtain $|m'(\kappa)| = \frac{1}{2}N^{-1+\xi} < \frac{1}{2}\frac{1}{\sqrt{\kappa N}} = \frac{1}{2}N^{-1}n(\kappa)^{-1}$. Consequently,

$$\sum_{k=j}^N \big|m(k-j) - m(k)\big| n(k) P_k \leq \tfrac{1}{2} N^{-1} j \sum_{k=j}^N \sqrt{\tfrac{k}{\kappa}} P_k \lesssim N^{-1}\mathbb{1}$$

in the sense of operators. This proves the first part of (b). For the second identity, observe that $|m'(k)| \leq \frac{1}{2}N^{-1+\xi}$ uniformly in $k \geq 0$. □

Lemma 4.2. *Let* $f, g : \mathbb{N}_0 \to \mathbb{R}_0^+$ *be any weights and* $i, j \in \{1, ..., N\}$.

(a) For $k \in \{0, ..., N\}$,

$$\widehat{f}\,\widehat{g} = \widehat{fg} = \widehat{g}\widehat{f}, \qquad \widehat{f} p_j = p_j \widehat{f}, \qquad \widehat{f} q_j = q_j \widehat{f}, \qquad \widehat{f} P_k = P_k \widehat{f}.$$

(b) Define $Q_0 := p_j$, $Q_1 := q_j$, $\widetilde{Q}_0 := p_i p_j$, $\widetilde{Q}_1 \in \{p_i q_j, q_i p_j\}$ *and* $\widetilde{Q}_2 := q_i q_j$. *Let* S_j *be an operator acting only on factor* j *in the tensor product and* T_{ij} *acting only on* i *and* j. *Then for* $\mu, \nu \in \{0, 1, 2\}$

$$Q_\mu \widehat{f} S_j Q_\nu = Q_\mu S_j \widehat{f}_{\mu-\nu} Q_\nu \quad \text{and} \quad \widetilde{Q}_\mu \widehat{f} T_{ij} \widetilde{Q}_\nu = \widetilde{Q}_\mu T_{ij} \widehat{f}_{\mu-\nu} \widetilde{Q}_\nu.$$

(c) Let S_{x_j} be an operator acting only on the x-component of factor j. Then

$$q_j^\Phi \widehat{f} S_{x_j} p_j^\Phi = q_j^\Phi S_{x_j} (\widehat{f} q_j^{\chi^\varepsilon} + \widehat{f}_1 p_j^{\chi^\varepsilon}) p_j^\Phi \quad \textit{and} \quad q_j^\Phi \widehat{f} S_{x_j} q_j^\Phi = q_j^\Phi S_{x_j} \widehat{f} q_j^\Phi.$$

(d)

$$[T_{12}, \widehat{f}] = [T_{12}, p_1 p_2 (\widehat{f} - \widehat{f}_2) + (p_1 q_2 + q_1 p_2)(\widehat{f} - \widehat{f}_1)].$$

We will apply parts (b) and (c) to unbounded operators, for instance to $S_j \equiv \nabla_j$ and $S_{x_j} \equiv \partial_{x_j}$. In this case, the respective equality holds on the intersection of the domains of the operators on both sides of the equation.

Proof. Part (a) follows immediately from $P_k P_l = \delta_{k,l} P_k$. For assertion (b), note that for $j = 1$,

$$\begin{aligned} Q_\mu P_k S_1 Q_\nu &= Q_\mu \left(\sum_{\substack{J \subseteq \{2,\dots,N\} \\ |J| = k - \mu}} \prod_{j \in J} q_j \prod_{l \notin J} p_l \right) S_1 Q_\nu \\ &= Q_\mu S_1 \left(\sum_{\substack{J \subseteq \{2,\dots,N\} \\ |J| = k - \mu}} \prod_{j \in J} q_j \prod_{l \notin J} p_l \right) Q_\nu = Q_\mu S_1 P_{k-\mu+\nu} Q_\nu, \end{aligned}$$

hence

$$Q_\mu \widehat{f} S_1 Q_\nu = Q_\mu S_1 \left(\sum_{k=-(\mu-\nu)}^{N-(\mu-\nu)} f(k+\mu-\nu) P_k \right) Q_\nu = Q_\mu S_1 \widehat{f}_{\mu-\nu} Q_\nu.$$

Assertion (c) is a consequence of part (b) and Corollary 3.2b, for example

$$q_j^\Phi \widehat{f} S_{x_j} p_j^\Phi = q_j^\Phi \left(q_j \widehat{f} S_{x_j} (p_j + q_j) \right) p_j^\Phi = q_j^\Phi S_{x_j} (\widehat{f}_1 p_j^{\chi^\varepsilon} + \widehat{f} q_j^{\chi^\varepsilon}) p_j^\Phi.$$

Finally, observe that

$$\begin{aligned} &[T_{12}, p_1 p_2 (\widehat{f} - \widehat{f}_2) + (p_1 q_2 + q_1 p_2)(\widehat{f} - \widehat{f}_1)] \\ &\qquad = [T_{12}, \widehat{f}] - [T_{12}, q_1 q_2 \widehat{f} + (p_1 q_2 + q_1 p_2) \widehat{f}_1 + p_1 p_2 \widehat{f}_2]. \end{aligned}$$

The second commutator equals zero, which can be seen by inserting $1 = p_1 p_2 + (p_1 q_2 + q_1 p_2) + q_1 q_2$ in front of the commutator and applying part (c). □

For the next lemma, recall that the operators P_k (Definition 3.1), and thus also the weighted operators $\widehat{f}$ (Definition 3.3), depend on the real variable t due to the time dependence of the projections p and q.

Lemma 4.3. *Let $f : \mathbb{N}_0 \to \mathbb{R}_0^+$.*

(a) The operators P_k and $\widehat{f}$ are continuously differentiable as functions of time, i.e.,

$$P_k,\ \widehat{f} \in \mathcal{C}^1\big(\mathbb{R}, \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)\big)$$

for $0 \le k \le N$*. Moreover,*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{f} = \mathrm{i}\Big[\widehat{f}, \sum_{j=1}^{N} h_j(t)\Big],$$

where $h_j(t)$ *denotes the one-particle operator corresponding to* $h(t)$ *from* (5) *acting on the* j^{th} *coordinate.*

(b) $\Big[-\Delta_{y_j} + \tfrac{1}{\varepsilon^2} V^{\perp}(\tfrac{y_j}{\varepsilon}), \widehat{f}\Big] = 0 \quad$ *for* $1 \le j \le N$*.*

Proof. The first part of (a) is clear as $\varphi^\varepsilon \in \mathcal{C}^1\left(\mathbb{R}, L^2(\mathbb{R}^3)\right)$. For the second part, note that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}p = \tfrac{\mathrm{d}}{\mathrm{d}t}\left|\Phi(t)\chi^\varepsilon\right\rangle\left\langle\Phi(t)\chi^\varepsilon\right| = \mathrm{i}\left[\left|\Phi(t)\chi^\varepsilon\right\rangle\left\langle\Phi(t)\chi^\varepsilon\right|, h(t)\right] = \mathrm{i}[p, h(t)] \quad \text{and} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}q = \mathrm{i}[q, h(t)]$$

as $\Phi(t)$ is a solution of (5). Assertion (b) is due to the fact that $-\Delta_{y_j} + \frac{1}{\varepsilon^2}V^{\perp}(\frac{y_j}{\varepsilon})$ commutes with its spectral projection $p_j^{\chi^\varepsilon}$. □

We will consider functions which are symmetric only in the variables of a subset of $\{1, ..., N\}$, for instance the expressions $q_1\psi$ and $w_\beta^{(12)}\psi$ for $\psi \in L^2_s(\mathbb{R}^{3N})$.

Definition 4.4. Let $\mathcal{M} \subseteq \{1, \dots, N\}$. Define $\mathcal{H}_\mathcal{M} \subseteq L^2(\mathbb{R}^{3N})$ as the space of functions which are symmetric in all variables in $\mathcal{M}$, i.e., for $\psi \in \mathcal{H}_\mathcal{M}$,

$$\psi(z_1, ..., z_j, ..., z_k, ..., z_N) = \psi(z_1, ..., z_k, ..., z_j, ..., z_N) \qquad \forall\, j, k \in \mathcal{M}.$$

Lemma 4.5. *Let* $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ *and* $\mathcal{M}_1, \mathcal{M}_{1,2} \subseteq \{1, 2, ..., N\}$ *with* $1 \in \mathcal{M}_1$ *and* $1, 2 \in \mathcal{M}_{1,2}$*.*

(a) $\widehat{n}^2 = \frac{1}{N}\sum\limits_{j=1}^{N} q_j$,

(b) $\|\widehat{f}q_1\psi\|^2 \le \frac{N}{|\mathcal{M}_1|}\|\widehat{f}\widehat{n}\psi\|^2 \qquad \forall\psi \in \mathcal{H}_{\mathcal{M}_1}$,

(c) $\|\widehat{f}q_1q_2\psi\|^2 \le \frac{N^2}{|\mathcal{M}_{1,2}|(|\mathcal{M}_{1,2}|-1)}\|\widehat{f}\,\widehat{n}^2\psi\|^2 \qquad \forall\psi \in \mathcal{H}_{\mathcal{M}_{1,2}}$.

Proof. Part (a) follows immediately from Corollary 3.2a. Consequently, for $\psi \in \mathcal{H}_{\mathcal{M}_1}$,

$$\|\widehat{f}\,\widehat{n}\psi\|^2 = \frac{1}{N}\sum_{j=1}^{N}\left\langle\!\!\left\langle\psi, \widehat{f}^2 q_j\psi\right\rangle\!\!\right\rangle \ge \frac{1}{N}\sum_{j\in\mathcal{M}_1}\left\langle\!\!\left\langle\psi, \widehat{f}^2 q_j\psi\right\rangle\!\!\right\rangle = \frac{|\mathcal{M}_1|}{N}\|\widehat{f}q_1\psi\|^2$$

and analogously for $\psi \in \mathcal{H}_{\mathcal{M}_{1,2}}$,

$$\|\widehat{f}\widehat{n}^2\psi\| \ge \frac{1}{N^2}\sum_{j,k\in\mathcal{M}_{1,2}}\left\langle\!\!\left\langle\psi, \widehat{f}^2 q_j q_k\psi\right\rangle\!\!\right\rangle \ge \frac{|\mathcal{M}_{1,2}|(|\mathcal{M}_{1,2}|-1)}{N^2}\|\widehat{f}q_1q_2\psi\|^2.$$

□

Corollary 4.6. *Let $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and $\mathcal{H}_{\mathcal{M}_1}$, $\mathcal{H}_{\mathcal{M}_{1,2}}$ as in Lemma 4.5.*

(a) For $\psi \in \mathcal{H}_{\mathcal{M}_1}$,

$$\|\nabla_1 \widehat{f} q_1 \psi\| \lesssim \|\widehat{f}\|_{\mathrm{op}} \|\nabla_1 q_1 \psi\| \quad \text{and} \quad \|\partial_{x_1} \widehat{f} q_1^\Phi \psi\| \lesssim \|\widehat{f}\|_{\mathrm{op}} \|\partial_{x_1} q_1^\Phi \psi\|.$$

(b) For $\psi \in \mathcal{H}_{\mathcal{M}_{1,2}}$,

$$\|\nabla_1 \widehat{f} q_1 q_2 \psi\| \lesssim \|\widehat{f}\,\widehat{n}\|_{\mathrm{op}} \|\nabla_1 q_1 \psi\| \quad \text{and} \quad \|\partial_{x_1} \widehat{f} q_1^\Phi q_2^\Phi \psi\| \lesssim \|\widehat{f}\,\widehat{n}\|_{\mathrm{op}} \|\partial_{x_1} q_1^\Phi \psi\|.$$

Proof. Insertion of $1 = p_1 + q_1$ in front of ∇_1 yields with Lemma 4.2b

$$\|\nabla_1 \widehat{f} q_1 \psi\| \le (\|\widehat{f}\|_{\mathrm{op}} + \|\widehat{f}_1\|_{\mathrm{op}}) \|\nabla_1 q_1 \psi\| \overset{4.1}{\lesssim} \|\widehat{f}\|_{\mathrm{op}} \|\nabla_1 q_1 \psi\|$$

and

$$\|\nabla_1 \widehat{f} q_1 q_2 \psi\| \le \|\widehat{f}_1 q_2 \nabla_1 q_1 \psi\| + \|\widehat{f} q_2 \nabla_1 q_1 \psi\| \lesssim \left(\|\widehat{f}_1 \widehat{n}\|_{\mathrm{op}} + \|\widehat{f}\widehat{n}\|_{\mathrm{op}}\right) \|\nabla_1 q_1 \psi\|$$

by Lemma 4.5b as $\nabla_1 q_1 \psi \in \mathcal{H}_{\{2,\dots,N\}}$. As $n(k) \le n(k+1)$, $\|\widehat{f}_1 \widehat{n}\|_{\mathrm{op}} \le \|\widehat{f n_1}\|_{\mathrm{op}} = \|\widehat{f}\,\widehat{n}\|_{\mathrm{op}}$ by Lemma 4.1a. The respective second identities are shown analogously, using that $q^\Phi q = q^\Phi$ and that $\partial_{x_1} q_1^\Phi \psi \in \mathcal{H}_{\{2,\dots,N\}}$. □

The next lemma provides an estimate of the difference between expectation values with respect to a symmetric N-body wavefunction ψ and with respect to $\Phi(t)$.

Lemma 4.7. *Let $\psi \in L^2_s(\mathbb{R}^{3N})$ be normalised and $f \in L^\infty(\mathbb{R})$. Then*

$$\left| \langle\!\langle \psi, f(x_1)\psi \rangle\!\rangle - \langle \Phi(t), f\Phi(t) \rangle_{L^2(\mathbb{R})} \right| \lesssim \|f\|_{L^\infty(\mathbb{R})} \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle .$$

Proof. We drop the time dependence of Φ. Inserting $1 = p_1 + q_1$ on both sides of $f(x_1)$ yields

$$\begin{aligned} \left| \langle\!\langle \psi, f(x_1)\psi \rangle\!\rangle - \langle \Phi, f\Phi \rangle_{L^2(\mathbb{R})} \right| \le& \left| \langle\!\langle \psi, p_1 f(x_1) p_1 \psi \rangle\!\rangle - \langle \Phi, f\Phi \rangle_{L^2(\mathbb{R})} \right| \\ &+ | \langle\!\langle q_1\psi, f(x_1) q_1 \psi \rangle\!\rangle | + 2 | \langle\!\langle \psi, p_1 f(x_1) q_1 \psi \rangle\!\rangle |. \end{aligned}$$

We estimate the first term as

$$\begin{aligned} &\left| \left\langle\!\!\left\langle \psi, p_1^{\chi^\varepsilon} |\Phi(x_1)\rangle \langle \Phi(x_1)| f(x_1) |\Phi(x_1)\rangle \langle \Phi(x_1)| p_1^{\chi^\varepsilon} \psi \right\rangle\!\!\right\rangle - \langle \Phi, f\Phi \rangle_{L^2(\mathbb{R})} \right| \\ &\qquad \le | \langle \Phi, f\Phi \rangle_{L^2(\mathbb{R})} \langle\!\langle \psi, q_1 \psi \rangle\!\rangle | \; \le \; \|f\|_{L^\infty(\mathbb{R})} \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle \end{aligned}$$

by Lemma 4.5a and as $\widehat{n}^2 \le \widehat{n}$. The second term is bounded by

$$| \langle\!\langle q_1\psi, f(x_1) q_1 \psi \rangle\!\rangle | \le \|f\|_{L^\infty(\mathbb{R})} \|q_1\psi\|^2 \le \|f\|_{L^\infty(\mathbb{R})} \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle .$$

For the third term, we compute

$$\begin{aligned} \left| \left\langle\!\!\left\langle \psi, p_1 f(x_1) \widehat{n}^{\frac{1}{2}} q_1 \widehat{n}^{-\frac{1}{2}} \psi \right\rangle\!\!\right\rangle \right| &\overset{4.2b}{=} \left| \left\langle\!\!\left\langle \widehat{n}_1^{\frac{1}{2}} p_1 \psi, f(x_1) \widehat{n}^{-\frac{1}{2}} q_1 \psi \right\rangle\!\!\right\rangle \right| \\ &\le \; \|f\|_{L^\infty(\mathbb{R})} \|\widehat{n}_1^{\frac{1}{2}} \psi\| \|\widehat{n}^{-\frac{1}{2}} q_1 \psi\| \overset{4.5b}{\lesssim} \|f\|_{L^\infty(\mathbb{R})} \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle , \end{aligned}$$

where we have used that $\sqrt{k+1} \le \sqrt{k} + 1$, hence $n_1(k) \le n(k) + N^{-\frac{1}{2}} \le 2n(k) \lesssim n(k)$. □

In the following lemma, we estimate two particular scalar products.

Lemma 4.8. *Let $O_{j,k}$ be an operator that acts nontrivially only on the j^{th} and k^{th} coordinate and let $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^d$ for $d \in \mathbb{N}$.*

(a) Let $\Gamma, \Lambda \in \mathcal{H}_{\mathcal{M}}$ for some $\mathcal{M}$ such that $j \notin \mathcal{M}$ and $k, l \in \mathcal{M}$. Then

$$|\langle\!\langle \Gamma, O_{j,k}\Lambda \rangle\!\rangle| \leq \|\Gamma\| \Big(|\langle\!\langle O_{j,k}\Lambda, O_{j,l}\Lambda \rangle\!\rangle| + |\mathcal{M}|^{-1} \|O_{j,k}\Lambda\|^2 \Big)^{\frac{1}{2}}.$$

(b) Let r_k, s_k and t_j denote operators acting only on the factors k and j of the tensor product, respectively. Then for $j \neq k \neq l \neq j$,

$$|\langle\!\langle r_k F(z_j, z_k) s_k t_j \Gamma, r_l F(z_j, z_l) s_l t_j \Gamma \rangle\!\rangle| \leq \|s_k F(z_j, z_k) r_k t_j \Gamma\|^2.$$

Proof. Using the symmetry of Γ, Λ in all coordinates contained in $\mathcal{M}$, we find

$$\begin{aligned} |\langle\!\langle \Gamma, O_{j,k}\Lambda \rangle\!\rangle| &\leq \|\Gamma\| \frac{1}{|\mathcal{M}|} \Big\| \sum_{m \in \mathcal{M}} O_{j,m}\Lambda \Big\| \\ &\leq \|\Gamma\| \left(\frac{1}{|\mathcal{M}|^2} \Bigg(\sum_{\substack{n,m \in \mathcal{M} \\ n \neq m}} \langle\!\langle O_{j,m}\Lambda, O_{j,n}\Lambda \rangle\!\rangle + \sum_{m \in \mathcal{M}} \|O_{j,m}\Lambda\|^2 \Bigg) \right)^{\frac{1}{2}} \\ &= \|\Gamma\|^2 \left(\frac{|\mathcal{M}| - 1}{|\mathcal{M}|} \langle\!\langle O_{j,k}\Lambda, O_{j,l}\Lambda \rangle\!\rangle + \frac{1}{|\mathcal{M}|} \|O_{j,k}\Lambda\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

For part (b), we use that, for instance, r_l and $F(z_j, z_k)$ commute, hence

$$\begin{aligned} |\langle\!\langle t_j \Gamma, s_k F(z_j, z_k) r_k r_l F(z_j, z_l) s_l t_j \Gamma \rangle\!\rangle| &= |\langle\!\langle r_l t_j \Gamma, s_k F(z_j, z_l) F(z_j, z_k) s_l r_k t_j \Gamma \rangle\!\rangle| \\ &= |\langle\!\langle r_l t_j \Gamma, F(z_j, z_l) s_l s_k F(z_j, z_k) r_k t_j \Gamma \rangle\!\rangle| \\ &\leq \|s_k F(z_j, z_k) r_k t_j \Gamma\|^2. \end{aligned}$$

□

The next lemma collects estimates for the time evolved condensate wavefunction.

Lemma 4.9. *$H^2(\mathbb{R})$ solutions of the NLS equation* (5) *exist globally, i.e., for initial data $\Phi_0 \in H^2(\mathbb{R})$ it holds that $\Phi(t) \in H^2(\mathbb{R})$ for any $t \in \mathbb{R}$. Moreover, for sufficiently small ε,*

(a) $\|\Phi(t)\|_{L^2(\mathbb{R})} = 1$, $\quad \|\Phi(t)\|_{H^1(\mathbb{R})} \leq \mathfrak{e}(t)$,
$\|\Phi(t)\|_{L^\infty(\mathbb{R})} \leq \mathfrak{e}(t)$, $\quad \|\Phi(t)\|_{H^2(\mathbb{R})} \lesssim \exp\left\{ \mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s \right\}$,

(b) $\|\chi^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon^{-1}$, $\quad \|\nabla\chi^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim \varepsilon^{-2}$,
$\|\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mathfrak{e}(t)\varepsilon^{-1}$, $\quad \|\nabla\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mathfrak{e}(t)\varepsilon^{-2}$,
$\|\nabla|\varphi^\varepsilon(t)|^2\|_{L^2(\mathbb{R}^3)} \lesssim \mathfrak{e}(t)\varepsilon^{-2}$.

Proof. For $\frac{1}{2} < r \leq 4$ and $\Phi_0 \in H^r(\mathbb{R})$, (5) has a unique strong $H^r(\mathbb{R})$-solution $\Phi \in \mathcal{C}(\mathbb{R}; H^r(\mathbb{R}))$ depending continuously on the initial data. The proof of this is sketched in Appendix A. By assumption A4, $\Phi_0 \in H^2(\mathbb{R})$ and consequently $\Phi(t) \in H^2(\mathbb{R})$. This implies $\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi(t)\|^2_{L^2(\mathbb{R})} = 0$ and by definition of $\mathcal{E}^{\Phi(t)}$ and $\mathfrak{e}(t)$,

$$\|\Phi(t)\|^2_{H^1(\mathbb{R})} \leq \mathcal{E}^{\Phi(t)}(t) + \|V^{\|}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \leq \mathfrak{e}^2(t). \tag{27}$$

Besides, $\Phi(t) \in H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$, hence

$$\begin{aligned} |\Phi(t,x)|^2 &= \int_{-\infty}^{x} \left(\overline{\Phi'(t,\zeta)}\Phi(t,\zeta) + \overline{\Phi(t,\zeta)}\Phi'(t,\zeta)\right)\mathrm{d}\zeta \\ &\leq \int_{-\infty}^{x} \left(|\Phi'(t,\zeta)|^2 + |\Phi(t,\zeta)|^2\right)\mathrm{d}\zeta = \|\Phi(t)\|^2_{H^1(\mathbb{R})} \leq \mathfrak{e}^2(t), \\ \|\tfrac{\partial}{\partial x}|\Phi(t)|^2\|^2_{L^2(\mathbb{R})} &\leq 4\int_{\mathbb{R}} |\Phi'(t,x)|^2|\Phi(t,x)|^2\,\mathrm{d}x \leq 4\|\Phi(t)\|^2_{L^\infty(\mathbb{R})}\|\Phi(t)\|^2_{H^1(\mathbb{R})} \lesssim \mathfrak{e}^4(t). \end{aligned}$$

For $\Phi(t) \in H^4(\mathbb{R})$, we obtain

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\left(1 + \|\dot\Phi(t)\|^2_{L^2(\mathbb{R})}\right) \\ &\quad= -2\Im\left\langle \dot{V}^{\|}(t,(\cdot,0))\Phi(t), \dot\Phi(t)\right\rangle_{L^2(\mathbb{R})} - 2b_\beta\Im\left\langle \Phi(t)^2, \dot\Phi(t)^2\right\rangle_{L^2(\mathbb{R})} \\ &\quad\leq 2\|\dot{V}^{\|}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}(1 + \|\dot\Phi(t)\|^2_{L^2(\mathbb{R})}) + 2b_\beta\|\Phi(t)\|^2_{L^\infty(\mathbb{R})}\|\dot\Phi(t)\|^2_{L^2(\mathbb{R})}, \end{aligned}$$

hence by Grönwall's inequality and as $\|\Phi(t)\|_{L^\infty(\mathbb{R})} \leq \mathfrak{e}(t)$,

$$\begin{aligned} \|\dot\Phi(t)\|^2_{L^2(\mathbb{R})} &\leq \left(1 + \|\dot\Phi(0)\|^2_{L^2(\mathbb{R})}\right)\exp\left\{2\int_0^t \left(\|\dot{V}^{\|}(s,\cdot)\|_{L^\infty(\mathbb{R}^3)} + b_\beta\mathfrak{e}^2(s)\right)\mathrm{d}s\right\} \\ &\lesssim \exp\left\{2\mathfrak{e}^2(t) + 2\int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}. \end{aligned}$$

This implies a bound for $\|\Phi(t)\|_{H^2(\mathbb{R})}$ because

$$\|\dot\Phi(t)\|_{L^2(\mathbb{R})} \geq \|\Phi''(t)\|_{L^2(\mathbb{R})} - b_\beta\|\Phi(t)\|^2_{L^\infty(\mathbb{R})} - \|V^{\|}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} \gtrsim \|\Phi''(t)\|_{L^2(\mathbb{R})} - \mathfrak{e}^2(t)$$

and consequently

$$\begin{aligned} \|\Phi(t)\|_{H^2(\mathbb{R})} &\leq \|\Phi''(t)\|_{L^2(\mathbb{R})} + 2\|\Phi(t)\|_{H^1(\mathbb{R})} \\ &\lesssim \mathfrak{e}^2(t) + \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\} \lesssim \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}. \end{aligned}$$

By continuity of the solution map, this bound extends to $\Phi(t) \in H^2(\mathbb{R})$. If the solution $\Phi(t) \in H^3(\mathbb{R}) \subset \mathcal{C}^2(\mathbb{R})$, we find further

$$|\Phi'(x)|^2 = \int_{-\infty}^{x} \left(\overline{\Phi'(\zeta)}\Phi''(\zeta) + \overline{\Phi''(\zeta)}\Phi'(\zeta)\right)\mathrm{d}\zeta \leq \|\Phi\|^2_{H^2(\mathbb{R})},$$

which extends to $\Phi(t) \in H^2(\mathbb{R})$ by continuity of the solution map. For part (b), recall that $\chi^\varepsilon(y) = \frac{1}{\varepsilon}\chi(\frac{y}{\varepsilon})$, hence $\|\chi^\varepsilon\|_{L^\infty(\mathbb{R}^2)} = \frac{1}{\varepsilon}\|\chi\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{\varepsilon}$ and analogously $\|\nabla\chi^\varepsilon\|_{L^\infty(\mathbb{R}^2)} \lesssim \frac{1}{\varepsilon^2}$. Together with (a), this implies the bounds for $\|\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)}$ and $\|\nabla\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)}$ as

$$\begin{aligned} |\nabla\varphi^\varepsilon(t,z)|^2 &\leq |\Phi'(t,x)|^2|\chi^\varepsilon(y)|^2 + |\Phi(t,x)|^2|\nabla\chi^\varepsilon(y)|^2 \\ &\lesssim \|\Phi(t)\|^2_{H^2(\mathbb{R})}\varepsilon^{-2} + \mathfrak{e}^2(t)\varepsilon^{-4} \lesssim \mathfrak{e}^2(t)\varepsilon^{-4} \end{aligned}$$

for any fixed time t and ε small enough. Finally,

$$\begin{aligned} &\|\nabla|\varphi^\varepsilon(t)|^2\|^2_{L^2(\mathbb{R}^3)} \\ &\quad= \|\tfrac{\partial}{\partial x}|\Phi(t)|^2\|^2_{L^2(\mathbb{R})}\int\limits_{\mathbb{R}^2}|\chi^\varepsilon(y)|^4\,\mathrm{d}y + \int\limits_{\mathbb{R}}|\Phi(t,x)|^4\,\mathrm{d}x\int\limits_{\mathbb{R}^2}|\nabla_y|\chi^\varepsilon(y)|^2|^2\,\mathrm{d}y \\ &\quad\lesssim \mathfrak{e}^4(t)\varepsilon^{-2} + 4\mathfrak{e}^2(t)\int\limits_{\mathbb{R}^2}|\nabla_y\chi^\varepsilon(y)|^2|\chi^\varepsilon(y)|^2\,\mathrm{d}y \lesssim \mathfrak{e}^2(t)\varepsilon^{-4}. \end{aligned}$$

□

Now we prove some elementary facts enabling us to estimate one- and two-body potentials.

Lemma 4.10. *Let $t \in \mathbb{R}$ be fixed and let $j,k \in \{1,\ldots,N\}$. Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a measurable function such that $|g(z_j,z_k)| \leq G(z_k - z_j)$ almost everywhere for some $G : \mathbb{R}^3 \to \mathbb{R}$.*

(a) For $G \in L^1(\mathbb{R}^3)$,
$$\|p_j g(z_j,z_k)p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}^2(t)\varepsilon^{-2}\|G\|_{L^1(\mathbb{R}^3)}.$$

(b) For $G \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$,
$$\|g(z_j,z_k)p_j\|_{\mathrm{op}} = \|p_j g(z_j,z_k)\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-1}\|G\|_{L^2(\mathbb{R}^3)}.$$

(c) For $G \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$,
$$\|g(z_j,z_k)\nabla_j p_j\|_{\mathrm{op}} = \||\varphi^\varepsilon(t,z_j)\rangle\langle\nabla\varphi^\varepsilon(t,z_j)|\,g(z_j,z_k)\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-2}\|G\|_{L^2(\mathbb{R}^3)}.$$

(d) Now let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a measurable function such that $|g(x_j,x_k)| \leq G(x_k - x_j)$ almost everywhere for some $G \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then
$$\begin{aligned} \|g(x_j,x_k)p_j^\Phi\|_{\mathrm{op}} &= \|p_j^\Phi g(x_j,x_k)\|_{\mathrm{op}} \leq \mathfrak{e}(t)\|G\|_{L^2(\mathbb{R})}, \\ \|g(x_j,x_k)\partial_{x_j}p_j^\Phi\|_{\mathrm{op}} &= \||\Phi(t,x_j)\rangle\langle\partial_{x_j}\Phi(t,x_j)|\,g(x_j,x_k)\|_{\mathrm{op}} \leq \|\Phi\|_{H^2(\mathbb{R})}\|G\|_{L^2(\mathbb{R})}. \end{aligned}$$

Proof. Let $\psi \in L^2(\mathbb{R}^{3N})$ and drop the time dependence of φ^ε and Φ in the notation. Then

$$\begin{aligned} \|p_j g(z_j,z_k)p_j\psi\| &= \||\varphi^\varepsilon(z_j)\rangle\langle\varphi^\varepsilon(z_j)|\,g(z_j,z_k)\,|\varphi^\varepsilon(z_j)\rangle\langle\varphi^\varepsilon(z_j)|\,\psi\| \\ &\leq \int_{\mathbb{R}^3}|\varphi^\varepsilon(z_j)|^2|g(z_j,z_k)|\,\mathrm{d}z_j\;\|p_j\psi\| \end{aligned}$$

$$\leq \quad \|\varphi^\varepsilon\|^2_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |G(z_j - z_k)| \, \mathrm{d}z_j \; \|\psi\|.$$

The multiplication operators corresponding to G and g as well as p_j, $\nabla_j p_j$ and $\partial_{x_j} p_j^\Phi$ are bounded. This implies the first equalities in (b) to (d). The second equalities follow from

$$\begin{aligned}
\|g(z_j,z_k)p_j\|^2_{\mathrm{op}} &= \sup_{\substack{\psi\in L^2(\mathbb{R}^{3N})\\ \|\psi\|=1}} \langle\!\langle \psi, p_j |g(z_j,z_k)|^2 p_j \psi \rangle\!\rangle \leq \|p_j |g(z_j,z_k)|^2 p_j\|_{\mathrm{op}} \\
&\overset{\text{(a)}}{\lesssim} \|G\|^2_{L^2(\mathbb{R}^3)} \mathfrak{e}^2(t) \varepsilon^{-2}, \\
\|G(x_j)p_j^\Phi\|^2_{\mathrm{op}} &\leq \|p_j^\Phi |G(x_j)|^2 p_j^\Phi\|_{\mathrm{op}} \leq \|G\|^2_{L^2(\mathbb{R})} \|\Phi\|^2_{L^\infty(\mathbb{R})}, \\
\|g(z_j,z_k)\nabla_j p_j\|^2_{\mathrm{op}} &= \sup_{\substack{\psi\in L^2(\mathbb{R}^{3N})\\ \|\psi\|=1}} \langle\!\langle \psi, |\varphi^\varepsilon(z_j)\rangle \langle \nabla_j \varphi^\varepsilon(z_j)| \, g(z_j,z_k)|^2 \, |\nabla_j \varphi^\varepsilon(z_j)\rangle \langle \varphi^\varepsilon(z_j)| \, \psi \rangle\!\rangle \\
&\leq \int_{\mathbb{R}^3} |\nabla \varphi^\varepsilon(z_j)|^2 G(z_k - z_j)^2 \, \mathrm{d}z_j \; \|p_j\|^2_{\mathrm{op}} \leq \|\nabla\varphi^\varepsilon\|^2_{L^\infty(\mathbb{R}^3)} \|G\|^2_{L^2(\mathbb{R}^3)}
\end{aligned}$$

and analogously for the second part of (d). □

4.2 A priori estimate of the kinetic energy

In this section, we prove estimates for the kinetic energy $\|\nabla_j \psi^{N,\varepsilon}(t)\|$ and related quantities, which follow from the fact that the renormalised energy per particle $E^{\psi^{N,\varepsilon}(t)}(t)$ is bounded by $\mathfrak{e}(t)$. Particularly meaningful is assertion (a) of the following lemma: it states that the part of the wavefunction with one particle excited in the confined directions is of order ε. The lemma provides a sufficient estimate for most of the terms in Proposition 3.7. To bound (24), we require a better estimate (see Section 4.5).

Lemma 4.11. *Let ε be small enough and $t \in \mathbb{R}$ be fixed. Then*

(a) $\|q_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}(t)\varepsilon$, $\quad \|\widehat{l} q_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}(t) N^\xi \varepsilon$,

(b) $\|\partial_{x_1} p_1^\Phi\|_{\mathrm{op}} \leq \mathfrak{e}(t)$, $\quad \|\partial^2_{x_1} p_1^\Phi\|_{\mathrm{op}} \leq \|\Phi(t)\|_{H^2(\mathbb{R})}$,
$\|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\mathrm{op}} \lesssim \varepsilon^{-1}$, $\quad \|\nabla_1 p_1\|_{\mathrm{op}} \lesssim \varepsilon^{-1}$,

(c) $\|\partial_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)$, $\quad \|\nabla_1 q_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)$,
$\|\nabla_1 \widehat{l} q_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \lesssim N^\xi \mathfrak{e}(t)$,

(d) $\|\partial_{x_1} \psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}(t)$, $\quad \|\nabla_{y_1} \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1}$, $\quad \|\nabla_1 \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1}$,

(e) $\|\nabla_1 \widehat{l} p_1^{\chi^\varepsilon} q_1^\Phi q_2^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1}$, $\quad \|\nabla_1 p_1^{\chi^\varepsilon} q_1^\Phi q_2^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)$.

Proof. Abbreviating $\psi^{N,\varepsilon}(t) \equiv \psi$, we compute

$$E^\psi(t) \;=\; \tfrac{1}{N} \langle\!\langle \psi, H_\beta(t)\psi \rangle\!\rangle - \tfrac{E_0}{\varepsilon^2}$$

$$
\begin{aligned}
&= \left\langle\!\!\left\langle \psi, \tfrac{1}{N}\Bigg(\sum_{j=1}^{N} \Big(-\partial_{x_j}^2 + \Big(-\Delta_{y_j} + \tfrac{1}{\varepsilon^2} V^{\perp}(\tfrac{y_j}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2} \Big) + V^{\parallel}(t, z_j) \Big) \right.\right. \\
&\qquad\qquad \left.\left. + \sum_{i<j} w_\beta(z_i - z_j) \Bigg) \psi \right\rangle\!\!\right\rangle \\
&\geq \|\partial_{x_1}\psi\|^2 + \left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, \Big(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2} V^{\perp}(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2} \Big) q_1^{\chi^\varepsilon}\psi \right\rangle\!\!\right\rangle - \|V^{\parallel}(t)\|_{L^\infty(\mathbb{R}^3)}
\end{aligned}
$$

since $w_\beta \in \mathcal{W}_{\beta,\eta}$ is non-negative and $\big(-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2}\big)\chi^\varepsilon(y_1) = 0$. $\frac{E_0}{\varepsilon^2}$ is the smallest eigenvalue of $-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon})$ and as a consequence of the rescaling by ε, the spectral gap to the next eigenvalue is of order ε^{-2}. Hence

$$
\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, \Big(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2} V^{\perp}(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2} \Big) q_1^{\chi^\varepsilon}\psi \right\rangle\!\!\right\rangle \gtrsim \tfrac{1}{\varepsilon^2} \left\langle\!\!\left\langle \psi, q_1^{\chi^\varepsilon}\psi \right\rangle\!\!\right\rangle,
$$

which implies

$$
\|\partial_{x_1}\psi\|^2 + \tfrac{1}{\varepsilon^2}\|q_1^{\chi^\varepsilon}\psi\|^2 \leq \|V^{\parallel}(t)\|_{L^\infty(\mathbb{R}^3)} + |E^\psi(t)| \leq \mathfrak{e}^2(t). \tag{28}
$$

Besides, by assumption A2, $\|(V^\perp - E_0)_-\|_{L^\infty(\mathbb{R}^2)} \lesssim 1$, hence

$$
\begin{aligned}
\mathfrak{e}^2(t) &\geq \left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, \Big(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2} V^{\perp}(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2} \Big) q_1^{\chi^\varepsilon}\psi \right\rangle\!\!\right\rangle \\
&= \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 + \tfrac{1}{\varepsilon^2} \left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, \Big(V^{\perp}(\tfrac{y_1}{\varepsilon}) - E_0 \Big)_+ q_1^{\chi^\varepsilon}\psi \right\rangle\!\!\right\rangle \\
&\qquad - \tfrac{1}{\varepsilon^2} \left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, \Big(V^{\perp}(\tfrac{y_1}{\varepsilon}) - E_0 \Big)_- q_1^{\chi^\varepsilon}\psi \right\rangle\!\!\right\rangle \\
&\geq \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 - \tfrac{1}{\varepsilon^2}\|(V^\perp - E_0)_-\|_{L^\infty(\mathbb{R}^2)} \|q_1^{\chi^\varepsilon}\psi\|^2 \gtrsim \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 - \mathfrak{e}^2(t)
\end{aligned}
$$

and consequently $\|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 \lesssim \mathfrak{e}^2(t)$. The remaining inequalities of (a) to (d) follow by Lemma 4.1b, Lemma 4.2b, by using that $q_1^{(\Phi)} = 1 - p_1^{(\Phi)}$ and from $\|\partial_{x_1} p_1\|_{\text{op}} \leq \|\partial_{x_1} p_1^\Phi\|_{\text{op}} \leq \|\Phi'(t)\|_{L^2(\mathbb{R})}$ and $\|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\text{op}} \leq \|\nabla\chi^\varepsilon\|_{L^2(\mathbb{R}^2)} \lesssim \varepsilon^{-1}$. For the second part of (d), note that

$$
\|\nabla_{y_1}\psi\| \leq \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\| + \|\nabla_{y_1} p_1^{\chi^\varepsilon}\psi\| \lesssim \mathfrak{e}(t) + \varepsilon^{-1} \lesssim \varepsilon^{-1}
$$

for sufficiently small ε and fixed $t \in \mathbb{R}$. Assertion (e) is a consequence of parts (a) to (d) and Corollary 4.6, Lemma 4.1 and Lemma 4.5:

$$
\begin{aligned}
\|\nabla_1 \widehat{l} p_1^{\chi^\varepsilon} q_1^\Phi q_2^\Phi \psi\|^2 &\leq \|\partial_{x_1} \widehat{l} q_1^\Phi q_2^\Phi \psi\|^2 + \|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\text{op}}^2 \|\widehat{l} q_1^\Phi q_2^\Phi \psi\|^2 \lesssim \mathfrak{e}^2(t) + \varepsilon^{-2}\|\widehat{n}\psi\|^2, \\
\|\nabla_1 p_1^{\chi^\varepsilon} q_1^\Phi q_2^{\chi^\varepsilon} \psi\|^2 &\leq \|\partial_{x_1} q_1^\Phi \psi\|^2 + \|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\text{op}}^2 \|q_2^{\chi^\varepsilon}\psi\|^2 \lesssim \mathfrak{e}^2(t).
\end{aligned}
$$

□

For the last lemma in this section, we make use of Lemma 4.11a to prove an estimate which is crucial for the control of $\gamma_a(t)$.

Lemma 4.12. *Let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(t) \in \mathcal{C}^1(\mathbb{R}^3)$ and $\nabla_y f(t) \in L^\infty(\mathbb{R}^3)$ for any $t \in \mathbb{R}$. Then*

(a) $\|(f(t, z_1) - f(t, (x_1, 0)) p_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \leq \varepsilon \|\nabla_y f(t)\|_{L^\infty(\mathbb{R}^3)}$,

(b) $\|(f(t,z_1)-f(t,(x_1,0))\psi^{N,\varepsilon}(t)\| \le \varepsilon\left(\mathfrak{e}(t)\|f(t)\|_{L^\infty(\mathbb{R}^3)}+\|\nabla_y f(t)\|_{L^\infty(\mathbb{R}^3)}\right)$.

Proof. For the first part, we expand $f(t,(x_1,\cdot))$ around $y=0$, which yields

$$
\begin{aligned}
&\|(\,f(t,z_1)-f(t,(x_1,0))p_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\|^2\\
&= \|p_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\|^2\int_{\mathbb{R}^2}\mathrm{d}y_1|\chi^\varepsilon(y_1)|^2\left(f(t,z_1)-f(t,(x_1,0))\right)^2\\
&\le \tfrac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\mathrm{d}y_1|\chi(\tfrac{y_1}{\varepsilon})|^2\left(\int_0^1\mathrm{d}s\nabla_y f(x_1,sy_1)\cdot y_1\right)^2\\
&\le \varepsilon^2\int_{\mathbb{R}^2}\mathrm{d}y|y|^2|\chi(y)|^2\|\nabla_y f(t)\|^2_{L^\infty(\mathbb{R}^3)}\lesssim \varepsilon^2\|\nabla_y f(t)\|^2_{L^\infty(\mathbb{R}^3)}.
\end{aligned}
$$

The last step follows because χ decays exponentially by [13, Theorem 1] since $E_0 < \sigma_{\mathrm{ess}}(\Delta_y+V^\perp)$ (A2). To prove the second part, we insert $1=q_1^{\chi^\varepsilon}+p_1^{\chi^\varepsilon}$ and apply Lemma 4.11a. □

4.3 Proof of Proposition 3.7

Let us from now on drop the time dependence of Φ, φ^ε and $\psi^{N,\varepsilon}$ in the notation and further abbreviate $\psi^{N,\varepsilon}\equiv\psi$. The time derivative of $\alpha_\xi(t)$ is bounded by

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)\right|\le\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle\psi,\widehat{m}\psi\rangle\!\rangle\right|+\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left|E^\psi(t)-\mathcal{E}^\Phi(t)\right|\right|. \tag{29}$$

For the second term in (29), we compute first

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left(E^\psi(t)-\mathcal{E}^\Phi(t)\right)\right|=\left|\left\langle\!\!\left\langle\psi,\dot{V}^{\|}(t,z_1)\psi\right\rangle\!\!\right\rangle-\left\langle\Phi,\dot{V}^{\|}\left(t,(x,0)\right)\Phi\right\rangle_{L^2(\mathbb{R})}\right|. \tag{30}$$

By [24, Theorem 6.17], $\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left|E^\psi(t)-\mathcal{E}^\Phi(t)\right|\right|=\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left(E^\psi(t)-\mathcal{E}^\Phi(t)\right)\right|$ for almost every $t\in\mathbb{R}$ because $t\mapsto\tfrac{\mathrm{d}}{\mathrm{d}t}\left(E^\psi(t)-\mathcal{E}^\Phi(t)\right)$ is continuous due to assumption A3. The first term in (29) yields

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle\psi,\widehat{m}\psi\rangle\!\rangle\\
&\overset{4.3\mathrm{b}}{=} \mathrm{i}\left\langle\!\!\left\langle\psi,\left[H_\beta(t)-\sum_{j=1}^N h_j(t),\widehat{m}\right]\psi\right\rangle\!\!\right\rangle\\
&\overset{4.3\mathrm{b}}{=} \mathrm{i}N\left\langle\!\!\left\langle\psi,\left[V^{\|}(t,z_1)-V^{\|}\left(t,(x_1,0)\right),\widehat{m}\right]\psi\right\rangle\!\!\right\rangle+i\tfrac{N(N-1)}{2}\left\langle\!\!\left\langle\psi,\left[Z_\beta^{(12)},\widehat{m}\right]\psi\right\rangle\!\!\right\rangle\\
&\overset{4.2\mathrm{d}}{=} \mathrm{i}N\left\langle\!\!\left\langle\psi,\left[V^{\|}(t,z_1)-V^{\|}\left(t,(x_1,0)\right),\widehat{m}\right]\psi\right\rangle\!\!\right\rangle && (31)\\
&\quad+\mathrm{i}\tfrac{N(N-1)}{2}\left\langle\!\!\left\langle\psi,\left[Z_\beta^{(12)},Q_0(\widehat{m}-\widehat{m}_2)+Q_1(\widehat{m}-\widehat{m}_1)\right]\psi\right\rangle\!\!\right\rangle, && (32)
\end{aligned}
$$

where $Q_0:=p_1p_2$, $Q_1:=p_1q_2+q_1p_2$ and $Q_2:=q_1q_2$. To expand (32), we write the commutator explicitly and insert $1=Q_0+Q_1+Q_2$ appropriately before or after $Z_\beta^{(12)}$. Terms with the same Q_μ on both sides cancel as a consequence of Lemma 4.2b. Hence

$$\begin{aligned}
&[N(N-1)]^{-1}\,(32)\\
&= \tfrac{\mathrm{i}}{2}\left\langle\!\!\left\langle \psi, \Big((Q_1+Q_2)Z_\beta^{(12)}(\widehat{m}-\widehat{m}_2)Q_0 - Q_0(\widehat{m}-\widehat{m}_2)Z_\beta^{(12)}(Q_1+Q_2)\Big)\psi \right\rangle\!\!\right\rangle\\
&\quad +\tfrac{\mathrm{i}}{2}\left\langle\!\!\left\langle \psi, \Big((Q_0+Q_2)Z_\beta^{(12)}(\widehat{m}-\widehat{m}_1)Q_1 - Q_1(\widehat{m}-\widehat{m}_1)Z_\beta^{(12)}(Q_0+Q_2)\Big)\psi \right\rangle\!\!\right\rangle\\
&= \tfrac{\mathrm{i}}{2}\left\langle\!\!\left\langle \psi, \Big(Q_1(\widehat{m}_{-1}-\widehat{m}_1)Z_\beta^{(12)}Q_0 + Q_2(\widehat{m}_{-2}-\widehat{m})Z_\beta^{(12)}Q_0\Big)\psi \right\rangle\!\!\right\rangle\\
&\quad -\tfrac{\mathrm{i}}{2}\left\langle\!\!\left\langle \psi, \Big(Q_0Z_\beta^{(12)}(\widehat{m}_{-1}-\widehat{m}_1)Q_1 + Q_0Z_\beta^{(12)}(\widehat{m}_{-2}-\widehat{m})Q_2\Big)\psi \right\rangle\!\!\right\rangle\\
&\quad +\tfrac{\mathrm{i}}{2}\left\langle\!\!\left\langle \psi, \Big(Q_0Z_\beta^{(12)}(\widehat{m}-\widehat{m}_1)Q_1 + Q_2(\widehat{m}_{-1}-\widehat{m})Z_\beta^{(12)}Q_1\Big)\psi \right\rangle\!\!\right\rangle\\
&\quad -\tfrac{\mathrm{i}}{2}\left\langle\!\!\left\langle \psi, \Big(Q_1(\widehat{m}-\widehat{m}_1)Z_\beta^{(12)}Q_0 + Q_1Z_\beta^{(12)}(\widehat{m}_{-1}-\widehat{m})Q_2\Big)\psi \right\rangle\!\!\right\rangle\\
&= \Im\left\langle\!\!\left\langle \psi, Q_1(\widehat{m}-\widehat{m}_{-1})Z_\beta^{(12)}Q_0\psi \right\rangle\!\!\right\rangle + \Im\left\langle\!\!\left\langle \psi, Q_2(\widehat{m}-\widehat{m}_{-2})Z_\beta^{(12)}Q_0\psi \right\rangle\!\!\right\rangle\\
&\quad +\Im\left\langle\!\!\left\langle \psi, Q_2(\widehat{m}-\widehat{m}_{-1})Z_\beta^{(12)}Q_1\psi \right\rangle\!\!\right\rangle.
\end{aligned}$$

To simplify this expression, note that

$$\begin{aligned}
\widehat{m}-\widehat{m}_{-1} &= \sum_{k=0}^{N} m(k)P_k - \sum_{k=1}^{N} m(k-1)P_k = \sum_{k=1}^{N}\left(m(k)-m(k-1)\right)P_k + m(0)P_0\\
&= -\widehat{m}^a_{-1} + m(0)P_0
\end{aligned}$$

and analogously

$$\widehat{m}-\widehat{m}_{-2} = -\widehat{m}^b_{-2} + m(0)P_0 + m(1)P_1.$$

Using that $Q_1P_0 = Q_2P_0 = Q_2P_1 = 0$, we consequently obtain

$$\begin{aligned}
\frac{(32)}{N(N-1)} &= -2\Im\left\langle\!\!\left\langle \psi, q_1p_2\widehat{m}^a_{-1}Z_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle && (33)\\
&\quad -\Im\left\langle\!\!\left\langle \psi, q_1q_2\widehat{m}^b_{-2}Z_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle && (34)\\
&\quad -2\Im\left\langle\!\!\left\langle \psi, q_1q_2\widehat{m}^a_{-1}Z_\beta^{(12)}p_1q_2\psi \right\rangle\!\!\right\rangle, && (35)
\end{aligned}$$

where we have in (33) and (35) exploited the symmetry of ψ in coordinates 1 and 2. According to Corollary 3.2c, $q = q^{\chi^\varepsilon} + q^\Phi p^{\chi^\varepsilon}$, hence

$$\begin{aligned}
(33) &= -2\Im\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, p_2\widehat{m}^a_{-1}w_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle && (36)\\
&\quad -2\Im\left\langle\!\!\left\langle \psi, q_1^\Phi\widehat{m}^a_{-1}p_1^{\chi^\varepsilon}p_2Z_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle. && (37)
\end{aligned}$$

In (36), we have used that the contribution of $|\Phi(x_1)|^2 + |\Phi(x_2)|^2$ vanishes because $q_1^{\chi^\varepsilon}|\Phi(x_1)|^2p_1^{\chi^\varepsilon} = q_1^{\chi^\varepsilon}|\Phi(x_2)|^2p_1^{\chi^\varepsilon} = 0$. Similarly, we expand (34) and (35) into terms containing q^{χ^ε} and terms containing $p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta^{(12)}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}$:

$$\begin{aligned}
(34) &= -\Im\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2\widehat{m}^b_{-2}w_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle - \Im\left\langle\!\!\left\langle q_2^{\chi^\varepsilon}\psi, q_1^\Phi p_1^{\chi^\varepsilon}\widehat{m}^b_{-2}w_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle\\
&\quad -\Im\left\langle\!\!\left\langle \psi, q_1^\Phi q_2^\Phi\widehat{m}^b_{-2}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta^{(12)}p_1p_2\psi \right\rangle\!\!\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -\Im \left\langle\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2(1+p_2^{\chi^\varepsilon})\widehat{m}^b_{-2} w_\beta^{(12)} p_1p_2\psi\right\rangle\!\right\rangle &(38)\\
&\quad -\Im \left\langle\!\left\langle \psi, q_1^\Phi q_2^\Phi \widehat{m}^b_{-2} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_\beta^{(12)} p_1p_2\psi\right\rangle\!\right\rangle &(39)
\end{aligned}
$$

and

$$
\begin{aligned}
(35) \;&=\; -2\Im \left\langle\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2\widehat{m}^a_{-1} w_\beta^{(12)} p_1q_2\psi\right\rangle\!\right\rangle &(40)\\
&\quad -2\Im \left\langle\!\left\langle q_2^{\chi^\varepsilon}\psi, q_1^\Phi p_1^{\chi^\varepsilon}\widehat{m}^a_{-1} w_\beta^{(12)} p_1q_2\psi\right\rangle\!\right\rangle &(41)\\
&\quad -2\Im \left\langle\!\left\langle \psi, q_1^\Phi q_2^\Phi \widehat{m}^a_{-1} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_\beta^{(12)} p_1 q_2^{\chi^\varepsilon}\psi\right\rangle\!\right\rangle &(42)\\
&\quad -2\Im \left\langle\!\left\langle \psi, q_1^\Phi q_2^\Phi \widehat{m}^a_{-1} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_\beta^{(12)} p_1 p_2^{\chi^\varepsilon} q_2^\Phi\psi\right\rangle\!\right\rangle &(43)\\
&\quad +\tfrac{2b_\beta}{N-1}\Im \left\langle\!\left\langle \psi, q_1q_2\widehat{m}^a_{-1}|\Phi(x_1)|^2 p_1q_2\psi\right\rangle\!\right\rangle . &(44)
\end{aligned}
$$

Finally, we insert $1 = p_1 + q_1$ on both sides of the commutator in (31) and apply Lemma 4.2b. Analogously to above, we obtain

$$
\begin{aligned}
(31) \;&=\; \mathrm{i}N \left\langle\!\left\langle \psi, (p_1+q_1)\big(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\big)\widehat{m}(p_1+q_1)\psi\right\rangle\!\right\rangle\\
&\quad -\mathrm{i}N \left\langle\!\left\langle \psi, (p_1+q_1)\widehat{m}\big(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\big)(p_1+q_1)\psi\right\rangle\!\right\rangle\\
&= -2N\Im \left\langle\!\left\langle \psi, q_1\widehat{m}^a_{-1}\big(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\big)p_1\psi\right\rangle\!\right\rangle . \qquad (45)
\end{aligned}
$$

Collecting and regrouping all terms arising from (29) yields $\gamma_a = (30) + (45)$, $\gamma_b = (32)$, $\gamma_b^{(1)} = N(N-1)$ (37), $\gamma_b^{(2)} = N(N-1)\big[\big((36)+(38)\big) + \big((40)+(41)\big) + (42)\big]$ and $\gamma_b^{(3)} = N(N-1)\big((39)+(43)+(44)\big)$. □

4.4 Proof of Proposition 3.8

4.4.1 Proof of the bound for $\gamma_a(t)$

As $2N\widehat{m}^a_{-1} \lesssim \widehat{l}$ for $\widehat{l}$ from Lemma 4.1, we obtain with Lemma 4.7, Lemma 4.12, Lemma 4.5b and Lemma 4.1b

$$
\begin{aligned}
|(16)| &\lesssim \left|\left\langle\!\left\langle \psi, \left(\dot{V}^\parallel(t,z_1) - \dot{V}^\parallel(t,(x_1,0))\right)\psi\right\rangle\!\right\rangle\right|\\
&\quad + \left|\left\langle\!\left\langle \psi, \dot{V}^\parallel(t,(x_1,0))\psi\right\rangle\!\right\rangle - \left\langle \Phi, \dot{V}^\parallel(t,(x,0))\Phi\right\rangle_{L^2(\mathbb{R})}\right|\\
&\lesssim \mathfrak{e}^3(t)\varepsilon + \mathfrak{e}(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle,\\
|(17)| &\lesssim \|\widehat{l}\widehat{n}\psi\|\|(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0)))p_1^{\chi^\varepsilon}\psi\| \lesssim \mathfrak{e}^2(t)\varepsilon.
\end{aligned}
$$

□

4.4.2 Proof of the bound for $\gamma_b^{(1)}(t)$

To estimate $\gamma_b^{(1)}$, we need to prove that Nw_β is close to the effective potential $b_\beta|\Phi|^2$. As $(N-1)\widehat{m}^a_{-1} \leq \widehat{l}$, we obtain

$$
\left|\gamma_b^{(1)}\right| \lesssim \left|\left\langle\!\left\langle \widehat{l}q_1^\Phi\psi, p_1^{\chi^\varepsilon}p_2\left(Nw_\beta^{(12)} - b_{N,\varepsilon}|\Phi(x_1)|^2 + (b_{N,\varepsilon} - \tfrac{N}{N-1}b_\beta)|\Phi(x_1)|^2\right)p_1p_2\psi\right\rangle\!\right\rangle\right|
$$

$$\overset{4.9}{\lesssim} \quad \left| \left\langle\!\!\left\langle \widehat{l} q_1^\Phi \psi, p_1^{\chi^\varepsilon} p_2 \left(N w_\beta^{(12)} - b_{N,\varepsilon} |\Phi(x_1)|^2 \right) p_1^{\chi^\varepsilon} p_2 p_1^\Phi \psi \right\rangle\!\!\right\rangle \right| + \left((\tfrac{N}{\varepsilon^2})^{-\eta} + N^{-1} \right) \mathfrak{e}^2(t)$$

for μ small enough and with η from Definition 2.2 since $w_\beta \in \mathcal{W}_{\beta,\eta}$ and as $\|\widehat{l} q_1^\Phi \psi\| \lesssim 1$ by Lemma 4.1a. Writing the action of the projectors explicitly, we obtain by definition of $b_{N,\varepsilon}$

$$\begin{aligned}
p_1^{\chi^\varepsilon} p_2 b_{N,\varepsilon} |\Phi(x_1)|^2 p_1^{\chi^\varepsilon} p_2 &= b_{N,\varepsilon} |\Phi(x_1)|^2 p_1^{\chi^\varepsilon} p_2 \\
&= N \Bigg(\int_{\mathbb{R}^2} \mathrm{d}y_1' |\chi^\varepsilon(y_1')|^4 |\Phi(x_1)|^2 \|w_\beta\|_{L^1(\mathbb{R}^3)} \Bigg) p_1^{\chi^\varepsilon} p_2, \\
p_1^{\chi^\varepsilon} p_2 N w_\beta^{(12)} p_1^{\chi^\varepsilon} p_2 &= N \Bigg(\int_{\mathbb{R}^2} \mathrm{d}y_1' |\chi^\varepsilon(y_1')|^2 \int_{\mathbb{R}^3} \mathrm{d}z_2' |\varphi^\varepsilon(z_2')|^2 w_\beta(z_1'' - z_2') \Bigg) p_1^{\chi^\varepsilon} p_2,
\end{aligned}$$

where $z_1'' := (x_1, y_1')$. The substitution $z_2' \mapsto z := z_1'' - z_2'$ and subtraction of both lines suggests the definition

$$\Gamma(x_1) := N \int_{\mathbb{R}^2} |\chi^\varepsilon(y_1')|^2 \,\mathrm{d}y_1' \Bigg(\int_{\mathbb{R}^3} |\varphi^\varepsilon(z_1'' - z)|^2 w_\beta(z)\,\mathrm{d}z - |\varphi^\varepsilon(z_1'')|^2 \|w_\beta\|_{L^1(\mathbb{R}^3)} \Bigg). \tag{46}$$

Let us first consider an analogous expression where $|\varphi^\varepsilon|^2$ is replaced by some $g \in C_0^\infty(\mathbb{R}^3)$. Expanding $g(z_1'' - \cdot)$ around z_1'' yields

$$\begin{aligned}
\int_{\mathbb{R}^3} g(z_1'' - z) w_\beta(z)\,\mathrm{d}z &= g(z_1'') \|w_\beta\|_{L^1(\mathbb{R}^3)} - \int_{\mathbb{R}^3} \mathrm{d}z \int_0^1 \nabla g(z_1'' - sz) \cdot z w_\beta(z)\,\mathrm{d}s \\
&=: g(z_1'') \|w_\beta\|_{L^1(\mathbb{R}^3)} + R(z_1''),
\end{aligned}$$

where

$$|R(z_1'')| \le \sup_{\substack{s\in[0,1] \\ z\in\mathbb{R}^3}} |\nabla g(z_1'' - sz)| \int_{\mathbb{R}^3} \mathrm{d}z |z| w_\beta(z).$$

Hence

$$\|R\|^2_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^4 N^{-2} \mu^2 \|\nabla g\|^2_{L^2(\mathbb{R}^3)}$$

because $|z| \lesssim \mu$ for $z \in \operatorname{supp} w_\beta$ and as $w_\beta \in \mathcal{W}_{\beta,\eta}$ implies

$$\int_{\mathbb{R}^3} w_\beta(z)\,\mathrm{d}z \lesssim \varepsilon^2 N^{-1} b_{N,\varepsilon} = \varepsilon^2 N^{-1} (b_{N,\varepsilon} - b_\beta) + \varepsilon^2 N^{-1} b_\beta \lesssim \varepsilon^2 N^{-1}. \tag{47}$$

Consequently,

$$\begin{aligned}
&\left\| N \int_{\mathbb{R}^2} |\chi^\varepsilon(y_1')|^2 \,\mathrm{d}y_1' \Bigg(\int_{\mathbb{R}^3} g(z_1'' - z) w_\beta(z)\,\mathrm{d}z - g(z_1'') \|w_\beta\|_{L^1(\mathbb{R}^3)} \Bigg) \right\|^2_{L^2(\mathbb{R})} \\
&\quad \le N^2 \int_{\mathbb{R}} \mathrm{d}x_1 \left| \int_{\mathbb{R}^2} \mathrm{d}y_1' |\chi^\varepsilon(y_1')|^2 R(z_1'') \right|^2 \le N^2 \||\chi^\varepsilon|^2\|^2_{L^2(\mathbb{R}^2)} \|R\|^2_{L^2(\mathbb{R}^3)} \lesssim \mu^2 \varepsilon^2 \|\nabla g\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

where we have in the second step used Hölder's inequality. By density, this bound extends to $g \in H^1(\mathbb{R}^3)$ and in particular to $g \equiv |\varphi^\varepsilon|^2$, hence

$$\|\Gamma\|_{L^2(\mathbb{R})} \lesssim \mu\varepsilon \|\nabla|\varphi^\varepsilon|^2\|_{L^2(\mathbb{R}^3)} \overset{4.9}{\lesssim} \tfrac{\mu}{\varepsilon}\mathfrak{e}(t) \tag{48}$$

and

$$\left|\gamma_b^{(1)}\right| \leq \|\widehat{l}q_1^\Phi\psi\|\|p_1^\Phi\Gamma(x_1)\|_{\mathrm{op}} + \left(N^{-1} + (\tfrac{N}{\varepsilon^2})^{-\eta}\right)\mathfrak{e}^2(t) \overset{4.10\mathrm{d}}{\lesssim} \left(\tfrac{\mu}{\varepsilon} + (\tfrac{N}{\varepsilon^2})^{-\eta} + N^{-1}\right)\mathfrak{e}^2(t).$$

□

4.4.3 Proof of the bound for $\gamma_b^{(2)}(t)$

Let us first define the functions needed for the integration by parts of the interaction.

Definition 4.13. Define $h_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ by

$$h_\varepsilon(z) := \begin{cases} \frac{1}{4\pi}\left(\int\limits_{\mathbb{R}^3} \frac{w_\beta(\zeta)}{|z-\zeta|}\mathrm{d}\zeta - \int\limits_{\mathbb{R}^3} \frac{\varepsilon}{|\zeta|}\frac{w_\beta(\zeta)}{|\zeta^*-z|}\mathrm{d}\zeta\right) & \text{for } |z|<\varepsilon, \\ 0 & \text{else} \end{cases}$$

where

$$\zeta^* := \tfrac{\varepsilon^2}{|\zeta|^2}\zeta.$$

We will abbreviate

$$h_\varepsilon^{(ij)} := h_\varepsilon(z_i - z_j).$$

Lemma 4.14. *Let $\mu \ll \varepsilon$. Then*

(a) h_ε solves the boundary value problem

$$\begin{cases} \Delta h_\varepsilon(z) = w_\beta(z) & \text{for } z \in B_\varepsilon(0), \\ h_\varepsilon(z) = 0 & \text{for } z \in \partial B_\varepsilon(0), \end{cases} \tag{49}$$

where $B_\varepsilon(0) := \{z \in \mathbb{R}^3 : |z| < \varepsilon\}$.

(b) $\|\nabla h_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim N^{-1}\mu^{-2}\varepsilon^2, \qquad \|\nabla h_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim N^{-1}\mu^{-\frac{1}{2}}\varepsilon^2.$

Proof. Green's function for the problem (49) is $G(z,\zeta) = \frac{1}{4\pi}\left(\frac{1}{|\zeta-z|} - \frac{\varepsilon}{|\zeta|}\frac{1}{|z-\zeta^*|}\right)$, hence $h_\varepsilon\big|_{\overline{B_\varepsilon(0)}}$ is the unique solution of (49). For part (b), define

$$h^{(1)}(z) := \begin{cases} \int\limits_{\mathbb{R}^3} \frac{w_\beta(\zeta)}{|z-\zeta|}\mathrm{d}\zeta & \text{for } |z|<\varepsilon, \\ 0 & \text{else,} \end{cases} \qquad h^{(2)}(z) := \begin{cases} \int\limits_{\mathbb{R}^3} \frac{\varepsilon}{|\zeta|}\frac{w_\beta(\zeta)}{|\zeta^*-z|}\mathrm{d}\zeta & \text{for } |z|<\varepsilon, \\ 0 & \text{else,} \end{cases}$$

hence $h_\varepsilon(z) =: \frac{1}{4\pi}\left(h^{(1)}(z) + h^{(2)}(z)\right)$. We estimate $h^{(1)}$ and $h^{(2)}$ separately.

Estimate of $|\nabla h^{(1)}|$. Define $R := \operatorname{diam} \operatorname{supp} w_\beta$. Let $|z| \leq 2R$ and substitute $\zeta \mapsto \zeta' := \zeta - z$. As $w_\beta \in \mathcal{W}_\eta^\beta$, $R \lesssim \mu$, hence $|\zeta'| \leq |\zeta| + |z| \leq 3R \lesssim \mu$ for $\zeta \in \operatorname{supp} w_\beta$ and consequently

$$|\nabla h^{(1)}(z)| \leq \|w_\beta\|_{L^\infty(\mathbb{R}^3)} \int\limits_{|\zeta| \leq R} \frac{1}{|z-\zeta|^2} \, \mathrm{d}\zeta \lesssim \left(\tfrac{N}{\varepsilon^2}\right)^{-1+3\beta} \int\limits_{|\zeta'| \leq 3R} \frac{1}{|\zeta'|^2} \, \mathrm{d}\zeta' \lesssim N^{-1}\varepsilon^2\mu^{-2}.$$

For $2R \leq |z| < \varepsilon$, note that $\zeta \in \operatorname{supp} w_\beta$ implies $|\zeta| \leq R \leq \frac{1}{2}|z|$, hence $|z - \zeta| \geq |z| - |\zeta| \geq \frac{1}{2}|z|$ and consequently

$$|\nabla h^{(1)}(z)| \leq \frac{4}{|z|^2} \int\limits_{\mathbb{R}^3} w_\beta(\zeta) \, \mathrm{d}\zeta \lesssim N^{-1}\varepsilon^2|z|^{-2} \lesssim N^{-1}\varepsilon^2\mu^{-2}$$

due to (47). Hence,

$$\int\limits_{\mathbb{R}^3} \left|\nabla h^{(1)}(z)\right|^2 \mathrm{d}z \lesssim \int\limits_{|z| \leq 2R} N^{-2}\varepsilon^4\mu^{-4} \, \mathrm{d}z + \int\limits_{2R \leq |z| \leq \varepsilon} N^{-2}\varepsilon^4 \frac{1}{|z|^4} \, \mathrm{d}z \lesssim N^{-2}\varepsilon^4\mu^{-1}.$$

Estimate of $|\nabla h^{(2)}|$. $\zeta \in \operatorname{supp} w_\beta$ implies $|\zeta| \leq R$, hence $|\zeta^*| = \frac{\varepsilon^2}{|\zeta|} \geq \frac{\varepsilon^2}{R}$. For $R \lesssim \mu$ sufficiently small that $\frac{\varepsilon}{R} > 2$, we observe $|z| < \varepsilon < \frac{1}{2}\frac{\varepsilon^2}{R} \leq \frac{1}{2}|\zeta^*|$ and consequently $|\zeta^* - z| \geq |\zeta^*| - |z| > \frac{1}{2}|\zeta^*| = \frac{1}{2}\frac{\varepsilon^2}{|\zeta|}$. This yields

$$\left|\nabla h^{(2)}(z)\right| = \int\limits_{\mathbb{R}^3} \frac{\varepsilon}{|\zeta|} \frac{w_\beta(\zeta)}{|\zeta^* - z|^2} \, \mathrm{d}\zeta \lesssim \varepsilon^{-3}\|w_\beta\|_{L^\infty(\mathbb{R}^3)} \int\limits_{|\zeta| \leq R} |\zeta| \, \mathrm{d}\zeta \lesssim N^{-1}\varepsilon^{-1}\mu < N^{-1}\varepsilon^2\mu^{-2}$$

and consequently $\int_{\mathbb{R}^3} \left|\nabla h^{(2)}(z)\right|^2 \mathrm{d}z \lesssim N^{-2}\mu^2\varepsilon < N^{-2}\varepsilon^4\mu^{-1}$. □

Besides, we need a smoothed step function to prevent contributions from the boundary when integrating by parts over the ball $\overline{B_\varepsilon(0)}$.

Definition 4.15. Let $R := \operatorname{diam} \operatorname{supp} w_\beta$. Define $\Theta_\varepsilon : \mathbb{R}^3 \to [0,1]$ by

$$\Theta_\varepsilon(z) := \begin{cases} 1 & \text{for } |z| \leq R, \\ \theta_\varepsilon(|z|) & \text{for } R < |z| < \varepsilon, \\ 0 & \text{for } |z| \geq \varepsilon, \end{cases}$$

where $\theta_\varepsilon : [R, \varepsilon] \to [0,1]$ is given by

$$\theta_\varepsilon(x) := \frac{\exp\left(-\frac{\varepsilon - R}{\varepsilon - x}\right)}{\exp\left(-\frac{\varepsilon - R}{\varepsilon - x}\right) + \exp\left(-\frac{\varepsilon - R}{x - R}\right)}. \tag{50}$$

Clearly, θ_ε is a smooth, decreasing function with $\theta_\varepsilon(R) = 1$ and $\theta_\varepsilon(\varepsilon) = 0$. We will write

$$\Theta_\varepsilon^{(ij)} := \Theta_\varepsilon(z_i - z_j).$$

Lemma 4.16. *Let $\mu \ll \varepsilon$. Then*

(a) $\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 1, \qquad \|\Theta_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{3}{2}},$

(b) $\|\nabla\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon^{-1}, \qquad \|\nabla\Theta_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{1}{2}}.$

Proof. Part (a) follows immediately from the definition of Θ_ε. For part (b), note that $R \lesssim \mu$ as $w_\beta \in \mathcal{W}_\eta^\beta$, hence $|\frac{\mathrm{d}}{\mathrm{d}x}\theta_\varepsilon(x)| \leq 2(\varepsilon - R)^{-1} \lesssim 2\varepsilon^{-1}(1-\frac{\mu}{\varepsilon})^{-1} \lesssim \varepsilon^{-1}$. □

Corollary 4.17. *Let $\mu \ll \varepsilon$ and $j \in \{1,2\}$. Then*

(a) $\|p_j(\nabla_1 h_\varepsilon^{(12)})\|_{\mathrm{op}} = \|\big(\nabla_1 h_\varepsilon^{(12)}\big)p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)N^{-1}\mu^{-\frac{1}{2}}\varepsilon,$

$\|\big(\nabla_1 h_\varepsilon^{(12)}\big)\cdot\nabla_j p_j\|_{\mathrm{op}} = \||\varphi^\varepsilon(z_j)\rangle\langle\nabla\varphi^\varepsilon(z_j)|\,(\nabla_1 h_\varepsilon^{(12)})\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)N^{-1}\mu^{-\frac{1}{2}},$

(b) $\|p_j\Theta_\varepsilon^{(12)}\|_{\mathrm{op}} = \|\Theta_\varepsilon^{(12)}p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{\frac{1}{2}},$

$\|p_j(\nabla_1\Theta_\varepsilon^{(12)})\|_{\mathrm{op}} = \|(\nabla_1\Theta_\varepsilon^{(12)})p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-\frac{1}{2}},$

$\|\Theta_\varepsilon^{(12)}\nabla_j p_j\|_{\mathrm{op}} = \||\varphi^\varepsilon(z_j)\rangle\langle\nabla\varphi^\varepsilon(z_j)|\,\Theta_\varepsilon^{(12)}\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-\frac{1}{2}}.$

Proof. This follows immediately from Lemma 4.10, Lemma 4.14 and Lemma 4.16. □

Making use of these preliminaries, let us now estimate the three terms (20), (21) and (22) that form $\gamma_b^{(2)}(t)$.

Estimate of (20). Define $t_2 := 2p_2 + q_2(1+p_2^{\chi^\varepsilon})$. Then we obtain with $\widehat{l}$ from Lemma 4.1

$$
\begin{aligned}
&|(20)|\\
&\lesssim N\left|\left\langle\!\left\langle \widehat{l}t_2 q_1^{\chi^\varepsilon}\psi, w_\beta^{(12)}p_1p_2\psi\right\rangle\!\right\rangle\right| = N\left|\left\langle\!\left\langle \widehat{l}t_2 q_1^{\chi^\varepsilon}\psi, \Theta_\varepsilon^{(12)}w_\beta^{(12)}p_1p_2\psi\right\rangle\!\right\rangle\right|\\
&= N\int\limits_{\mathbb{R}^{3(N-1)}}\mathrm{d}z^{N-1}\int\limits_{\overline{B_\varepsilon(z_2)}}\mathrm{d}z_1\overline{(\widehat{l}t_2 q_1^{\chi^\varepsilon}\psi)(z_1,\dots,z_N)}\Theta_\varepsilon(z_1-z_2)w_\beta(z_1-z_2)(p_2p_1\psi)(z_1,\dots,z_N)
\end{aligned}
$$

as $\Theta_\varepsilon(z_1-z_2) = 1$ for $z_1 - z_2 \in \operatorname{supp} w_\beta$ and $\operatorname{supp}\Theta_\varepsilon = \overline{B_\varepsilon(0)}$. Thus $w_\beta(z_1-z_2) = \Delta_1 h_\varepsilon(z_1-z_2)$ on the whole domain of integration in the $\mathrm{d}z_1$-integral. Integration by parts in z_1 yields

$$
\begin{aligned}
|(20)| \lesssim\; & N\left|\left\langle\!\left\langle \widehat{l}q_1^{\chi^\varepsilon}\psi, t_2\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1 p_1p_2\psi\right\rangle\!\right\rangle\right| && (51)\\
&+N\left|\left\langle\!\left\langle \widehat{l}q_1^{\chi^\varepsilon}\psi, t_2(\nabla_1\Theta_\varepsilon^{(12)})\cdot(\nabla_1 h_\varepsilon^{(12)})p_1p_2\psi\right\rangle\!\right\rangle\right| && (52)\\
&+N\left|\left\langle\!\left\langle \nabla_1\widehat{l}q_1^{\chi^\varepsilon}\psi, t_2\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_1p_2\psi\right\rangle\!\right\rangle\right|, && (53)
\end{aligned}
$$

where the boundary terms vanish because $\Theta_\varepsilon(|z|) = 0$ for $|z| = \varepsilon$. We estimate these expressions by application of Lemma 4.8. To this end, we write each term as $\langle\!\langle\Gamma, O_{1,2}\Lambda\rangle\!\rangle$, where Γ and Λ are symmetric in the coordinates $\{2,\dots,N\}$. Hence

$$
\begin{aligned}
|(51)| &\overset{4.8\text{a}}{\lesssim} N\|\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\Big|\Big\langle\!\Big\langle t_2\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_2\cdot\nabla_1 p_1\psi, t_3\Theta_\varepsilon^{(13)}(\nabla_1 h_\varepsilon^{(13)})p_3\cdot\nabla_1 p_1\psi\Big\rangle\!\Big\rangle\Big| \\
&\qquad\qquad +N^{-1}\|t_2\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_2\cdot\nabla_1 p_1\psi\|^2\Big)^{\frac{1}{2}} \\
&\overset{4.8\text{b}}{\leq} N\|\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})t_2\cdot\nabla_1 p_1\psi\|^2 \\
&\qquad\qquad +N^{-1}\|t_2\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_2\cdot\nabla_1 p_1\psi\|^2\Big)^{\frac{1}{2}} \\
&\leq N\|\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2\Theta_\varepsilon^{(12)}\|_{\text{op}}^2\|(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1 p_1\|_{\text{op}}^2 \\
&\qquad\qquad +N^{-1}\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}^2\|(\nabla_1 h_\varepsilon^{(12)})p_2\|_{\text{op}}^2\|\nabla_1 p_1\|_{\text{op}}^2\Big)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}^3(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}\left(\varepsilon+N^{-1}\right)^{\frac{1}{2}}N^\xi
\end{aligned}
$$

by Lemma 4.11, Lemma 4.16 and Corollary 4.17. Analogously,

$$
\begin{aligned}
|(52)| &\lesssim N\|\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2(\nabla_1 h_\varepsilon^{(12)})\|_{\text{op}}^2\|(\nabla_1\Theta_\varepsilon^{(12)})p_1\|_{\text{op}}^2 \\
&\qquad\qquad +N^{-1}\|\nabla\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}^2\|(\nabla_1 h_\varepsilon^{(12)})p_2\|_{\text{op}}^2\Big)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}^3(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}\left(\varepsilon+N^{-1}\right)^{\frac{1}{2}}N^\xi, \\
|(53)| &\lesssim N\|\nabla_1\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2\Theta_\varepsilon^{(12)}\|_{\text{op}}^2\|(\nabla_1 h_\varepsilon^{(12)})p_1\|_{\text{op}}^2 \\
&\qquad\qquad +N^{-1}\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}^2\|(\nabla_1 h_\varepsilon^{(12)})p_2\|_{\text{op}}^2\Big)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}^3(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}\left(\varepsilon+N^{-1}\right)^{\frac{1}{2}}N^\xi.
\end{aligned}
$$

Hence

$$
|(20)|\lesssim\mathfrak{e}^3(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}\left(\varepsilon+N^{-1}\right)^{\frac{1}{2}}N^\xi\lesssim\mathfrak{e}^3(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}
$$

because $N^{-\frac{1}{2}+\xi}<1$ as $\xi<\frac{1}{2}$ and $\varepsilon^{\frac{1}{2}}N^\xi=(\frac{\varepsilon^2}{\mu})^{\frac{1}{4}}N^{\xi-\frac{\beta}{4}}\varepsilon^{\frac{\beta}{2}}\lesssim 1$ for $\mu\ll\varepsilon$ as $\xi\leq\frac{\beta}{4}$.

Estimate of (21). Define $t_{12}:=q_1^\Phi p_1^{\chi^\varepsilon}q_2^{\chi^\varepsilon}+q_1^{\chi^\varepsilon}q_2$. Analogously to the estimate of (20),

$$
\begin{aligned}
|(21)| &\leq N\left|\Big\langle\!\Big\langle\widehat{l}t_{12}\psi, w_\beta^{(12)}p_1q_2\psi\Big\rangle\!\Big\rangle\right| = N\left|\Big\langle\!\Big\langle\widehat{l}t_{12}\psi,\Theta_\varepsilon^{(12)}\left(\Delta_1 h_\varepsilon^{(12)}\right)p_1q_2\psi\Big\rangle\!\Big\rangle\right| \\
&\leq N\left|\Big\langle\!\Big\langle\widehat{l}t_{12}\psi,\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1 p_1q_2\psi\Big\rangle\!\Big\rangle\right| && (54) \\
&\quad +N\left|\Big\langle\!\Big\langle\widehat{l}t_{12}\psi,(\nabla_1\Theta_\varepsilon^{(12)})\cdot(\nabla_1 h_\varepsilon^{(12)})p_1q_2\psi\Big\rangle\!\Big\rangle\right| && (55) \\
&\quad +N\left|\Big\langle\!\Big\langle\nabla_1\widehat{l}t_{12}\psi,\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_1q_2\psi\Big\rangle\!\Big\rangle\right|. && (56)
\end{aligned}
$$

To estimate (54) to (56), we apply first Lemma 4.2b to commute $\widehat{l}$ next to q_2 and use the fact that $\|\widehat{l}_1q_2\psi\|\lesssim 1$ by Lemma 4.1 and Lemma 4.5. Observing that $t_{12}=t_{12}q_1q_2$

and consequently $\|t_{12}\psi\| \leq \|q_1^{\chi^\varepsilon}\psi\| \leq \varepsilon\mathfrak{e}(t)$ by Lemma 4.11a, we obtain

$$\begin{aligned}
(54) &= N\left|\left\langle\!\!\left\langle t_{12}\psi, q_1q_2\widehat{l}\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot(p_1+q_1)q_2\nabla_1p_1\psi\right\rangle\!\!\right\rangle\right| \\
&= N\left|\left\langle\!\!\left\langle t_{12}\psi, \Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot(\widehat{l}_1p_1+\widehat{l}q_1)\nabla_1p_1q_2\psi\right\rangle\!\!\right\rangle\right| \\
&= N\left|\left\langle\!\!\left\langle t_{12}\psi, \Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1p_1\widehat{l}_1q_2\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N\|t_{12}\psi\|\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1p_1\|_{\text{op}}\|\widehat{l}_1q_2\psi\| \lesssim \mathfrak{e}^2(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}
\end{aligned}$$

and analogously

$$\begin{aligned}
(55) &= N\left|\left\langle\!\!\left\langle t_{12}\psi, (\nabla_1\Theta_\varepsilon^{(12)})\cdot(\nabla_1 h_\varepsilon^{(12)})p_1\widehat{l}_1q_2\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N\|t_{12}\psi\|\|\nabla\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|(\nabla_1 h_\varepsilon^{(12)})p_1\|_{\text{op}}\|\widehat{l}_1q_2\psi\| \lesssim \mathfrak{e}^2(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}, \\
(56) &= N\left|\left\langle\!\!\left\langle \nabla_1t_{12}\psi, \Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_1\widehat{l}_1q_2\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N\left(\|\nabla_1q_1^\Phi p_1^{\chi^\varepsilon}q_2^{\chi^\varepsilon}\psi\|+\|q_2\nabla_1q_1^{\chi^\varepsilon}\psi\|\right)\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|(\nabla_1 h_\varepsilon^{(12)})p_1\|_{\text{op}}\|\widehat{l}_1q_2\psi\| \\
&\lesssim \mathfrak{e}^2(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}
\end{aligned}$$

by Lemma 4.16, Corollary 4.17a and Lemma 4.11.

Estimate of (22). Analogously to before,

$$\begin{aligned}
|(22)| &\leq N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_\beta^{(12)}p_1q_2^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&= N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\Theta_\varepsilon^{(12)}\left(\Delta_1 h_\varepsilon^{(12)}\right)p_1q_2^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1p_1q_2^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&\quad+N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}(\nabla_1\Theta_\varepsilon^{(12)})\cdot(\nabla_1 h_\varepsilon^{(12)})p_1q_2^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&\quad+N\left|\left\langle\!\!\left\langle \nabla_1\widehat{l}p_1^{\chi^\varepsilon}q_1^\Phi q_2^\Phi\psi, p_2^{\chi^\varepsilon}\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_1q_2^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|\|q_2^{\chi^\varepsilon}\psi\|\Big(\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1p_1\|_{\text{op}} \\
&\qquad\qquad+\|\nabla\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|p_1(\nabla_1 h_\varepsilon^{(12)})\|_{\text{op}}\Big) \\
&\quad+N\|\nabla_1\widehat{l}p_1^{\chi^\varepsilon}q_1^\Phi q_2^\Phi\psi\|\|\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|p_1(\nabla_1 h_\varepsilon^{(12)})\|_{\text{op}}\|q_2^{\chi^\varepsilon}\psi\| \\
&\lesssim \mathfrak{e}^2(t)\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}
\end{aligned}$$

by Lemma 4.11, Lemma 4.16, Corollary 4.17 and Lemma 4.5. □

4.4.4 Proof of the bound for $\gamma_b^{(3)}(t)$

We estimate (25) as

$$|(25)| \lesssim \left|\left\langle\!\!\left\langle \widehat{l}q_1q_2\psi, |\Phi(x_1)|^2p_1q_2\psi\right\rangle\!\!\right\rangle\right| \lesssim \|\Phi\|^2_{L^\infty(\mathbb{R})}\|\widehat{l}q_1q_2\psi\|\|q_2\psi\| \lesssim \mathfrak{e}^2(t)\left\langle\!\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\!\right\rangle$$

by Lemma 4.9 and Lemma 4.5c. For (23) and (24), we proceed similarly as in Section 4.4.3 for the quasi one-dimensional interaction $\overline{w}$ instead of the three-dimensional interaction w_β.

Definition 4.18. Define

$$\overline{w}(x) := \int\limits_{\mathbb{R}^2} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \int\limits_{\mathbb{R}^2} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 w_\beta(x, y_1 - y_2). \tag{57}$$

Further, for $\beta_1 \in [0,1]$, define $\overline{h}_{\beta_1} : \mathbb{R} \to \mathbb{R}$ by

$$\overline{h}_{\beta_1}(x) := \begin{cases} \int\limits_{-N^{-\beta_1}}^{N^{-\beta_1}} G(x', x)\overline{w}(x')\,\mathrm{d}x' & \text{for } |x| \leq N^{-\beta_1}, \\ 0 & \text{else,} \end{cases} \tag{58}$$

where

$$G(x', x) := \tfrac{1}{2} N^{\beta_1} \begin{cases} \left(x' + N^{-\beta_1}\right)\left(x - N^{-\beta_1}\right) & \text{for } x' < x, \\ \left(x' - N^{-\beta_1}\right)\left(x + N^{-\beta_1}\right) & \text{for } x' > x. \end{cases} \tag{59}$$

Besides, let $R := \operatorname{diam}\operatorname{supp}\overline{w}$ and define

$$\overline{\Theta}_{\beta_1}(x) := \begin{cases} 1 & \text{for } |x| \leq R, \\ \theta_{\beta_1}(|x|) & \text{for } R < |x| < N^{-\beta_1}, \\ 0 & \text{for } |x| \geq N^{-\beta_1}, \end{cases} \tag{60}$$

where $\theta_{\beta_1} : [R, N^{-\beta_1}] \to [0,1]$ is a smooth decreasing function with $\theta_{\beta_1}(R) = 1$, $\theta_{\beta_1}(N^{-\beta_1}) = 0$ analogously to (50). As before, we will write

$$\overline{w}^{(ij)} := \overline{w}(x_i - x_j), \qquad \overline{h}^{(ij)}_{\beta_1} := \overline{h}_{\beta_1}(x_i - x_j), \qquad \overline{\Theta}^{(ij)}_{\beta_1} := \overline{\Theta}_{\beta_1}(x_i - x_j).$$

Lemma 4.19. *(a) $\overline{h}_{\beta_1}$ solves the boundary-value problem*

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\overline{h}_{\beta_1} = \overline{w} & \text{for } x \in [-N^{-\beta_1}, N^{-\beta_1}], \\ \overline{h}_{\beta_1} = 0 & \text{for } |x| = N^{-\beta_1}. \end{cases} \tag{61}$$

(b) $\|\frac{\mathrm{d}}{\mathrm{d}x}\overline{h}_{\beta_1}\|_{L^\infty(\mathbb{R})} \lesssim N^{-1}, \qquad \|\frac{\mathrm{d}}{\mathrm{d}x}\overline{h}_{\beta_1}\|_{L^2(\mathbb{R})} \lesssim N^{-1-\frac{\beta_1}{2}},$

(c) $\|\overline{\Theta}_{\beta_1}\|_{L^\infty(\mathbb{R})} \leq 1, \qquad \|\overline{\Theta}_{\beta_1}\|_{L^2(\mathbb{R})} \lesssim N^{-\frac{\beta_1}{2}},$

$\|\frac{\mathrm{d}}{\mathrm{d}x}\overline{\Theta}_{\beta_1}\|_{L^\infty(\mathbb{R})} \lesssim N^{\beta_1}, \qquad \|\frac{\mathrm{d}}{\mathrm{d}x}\overline{\Theta}_{\beta_1}\|_{L^2\mathbb{R})} \lesssim N^{\frac{\beta_1}{2}}.$

Proof. Part (a) is evident as $G(x', x)$ is Green's function for the problem (61). For part (b), we compute for $x \in [-N^{-\beta_1}, N^{-\beta_1}]$

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}x}\overline{h}_{\beta_1}(x)\right| = \tfrac{N^{\beta_1}}{2}\left| \int\limits_{-N^{-\beta_1}}^{x} (x' + N^{-\beta_1})\overline{w}(x')\,\mathrm{d}x' + \int\limits_{x}^{N^{-\beta_1}} (x' - N^{-\beta_1})\overline{w}(x')\,\mathrm{d}x' \right|$$

$$\lesssim \|\overline{w}\|_{L^1(\mathbb{R})} \lesssim N^{-1}$$

since

$$\begin{aligned}\|\overline{w}\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}^2} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \int_{\mathbb{R}^2} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 w_\beta(x, y_1 - y_2) \\ &\leq \|\chi^\varepsilon\|^2_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \|w_\beta\|_{L^1(\mathbb{R}^3)} \lesssim N^{-1} \qquad (62)\end{aligned}$$

by (47). The second inequality in (b) follows from this as $\operatorname{supp} \overline{h}_{\beta_1} = [-N^{-\beta_1}, N^{-\beta_1}]$. Part (c) is shown analogously to Lemma 4.16, noting that $R \lesssim \mu$. □

Corollary 4.20. *Let $j \in \{0,1\}$. Then*

(a) $\|p_j^\Phi(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1})\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{-1-\frac{\beta_1}{2}}, \qquad \|(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1})(\partial_{x_j} p_j^\Phi)\|_{\mathrm{op}} \lesssim \|\Phi(t)\|_{H^2(\mathbb{R})} N^{-1-\frac{\beta_1}{2}}$,

(b) $\|p_j^\Phi\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(12)}_{\beta_1}\right)\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{\frac{\beta_1}{2}}$.

Proof. This follows immediately from Lemma 4.10d and Lemma 4.19. □

Estimate of (23). Observing that $p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_\beta^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} = \overline{w}^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}$, we obtain analogously to the estimate of (20)

$$\begin{aligned}|(23)| &\lesssim N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{w}^{(12)} p_1 p_2 \psi \right\rangle\!\!\right\rangle\right| = N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}^{(12)}_{\beta_1}\left(\tfrac{\mathrm{d}^2}{\mathrm{d}x_1^2}\overline{h}^{(12)}_{\beta_1}\right) p_1 p_2 \psi \right\rangle\!\!\right\rangle\right| \\ &\leq N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}^{(12)}_{\beta_1}\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) \partial_{x_1} p_1^\Phi p_1^{\chi^\varepsilon} p_2 \psi \right\rangle\!\!\right\rangle\right| \qquad (63) \\ &\quad + N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(12)}_{\beta_1}\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_1^\Phi p_1^{\chi^\varepsilon} p_2 \psi \right\rangle\!\!\right\rangle\right| \qquad (64) \\ &\quad + N\left|\left\langle\!\!\left\langle \partial_{x_1} \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}^{(12)}_{\beta_1}\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_1^\Phi p_1^{\chi^\varepsilon} p_2 \psi \right\rangle\!\!\right\rangle\right|. \qquad (65)\end{aligned}$$

The boundary terms upon integration by parts vanish as $\overline{\Theta}_{\beta_1}(\pm N^{-\beta_1}) = 0$. With Lemmata 4.1b, 4.5c, 4.11, 4.19 and Corollary 4.20, we conclude

$$\begin{aligned}(63) &\leq N\|\widehat{l} q_1^\Phi q_2^\Phi \psi\| \|\overline{\Theta}_{\beta_1}\|_{L^\infty(\mathbb{R})} \|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_2^\Phi\|_{\mathrm{op}} \|\partial_{x_1} p_1^\Phi\|_{\mathrm{op}} \lesssim \mathfrak{e}^2(t) \left\langle\!\left\langle \psi, \widehat{n}\psi \right\rangle\!\right\rangle^{\frac{1}{2}} N^{-\frac{\beta_1}{2}}, \\ (64) &\overset{4.2\mathrm{b}}{=} N\left|\left\langle\!\!\left\langle \widehat{l}^{\frac{1}{2}} q_1^\Phi \psi, \left(q_2^\Phi \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(12)}_{\beta_1}\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_2\right) p_1 \widehat{l}_2^{\frac{1}{2}} \psi \right\rangle\!\!\right\rangle\right| \\ &\overset{4.8\mathrm{a}}{\lesssim} N\|\widehat{l}^{\frac{1}{2}} q_1^\Phi \psi\| \times \\ &\quad \times \Bigg(\left|\left\langle\!\!\left\langle q_2^\Phi \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(12)}_{\beta_1}\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_2 p_1 \widehat{l}_2^{\frac{1}{2}} \psi, q_3^\Phi \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(13)}_{\beta_1}\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(13)}_{\beta_1}\right) p_3 p_1 \widehat{l}_2^{\frac{1}{2}} \psi \right\rangle\!\!\right\rangle\right| \\ &\quad + N^{-1} \|q_2^\Phi \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(12)}_{\beta_1}\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_2 p_1 \widehat{l}_2^{\frac{1}{2}} \psi\|^2 \Bigg)^{\frac{1}{2}} \\ &\overset{4.8\mathrm{b}}{\leq} N\|\widehat{l}^{\frac{1}{2}} q_1 \psi\| \Big(\|p_2^\Phi \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}^{(12)}_{\beta_1}\right)\|^2_{\mathrm{op}} \|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_1^\Phi\|^2_{\mathrm{op}} \|\widehat{l}_2^{\frac{1}{2}} q_2 \psi\|^2 \\ &\quad + N^{-1} \|\tfrac{\mathrm{d}}{\mathrm{d}x}\overline{\Theta}_{\beta_1}\|^2_{L^\infty(\mathbb{R})} \|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\beta_1}\right) p_1^\Phi\|^2_{\mathrm{op}} \|\widehat{l}_2^{\frac{1}{2}}\|^2_{\mathrm{op}} \Big)^{\frac{1}{2}}\end{aligned}$$

$$\lesssim \quad \mathfrak{e}^2(t)\,\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle^{\frac{1}{2}} \left(\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + N^{-1+\beta_1+\xi}\right)^{\frac{1}{2}} \lesssim \mathfrak{e}^2(t)\left(\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + N^{-1+\beta_1+\xi}\right),$$

$$(65) \quad \leq \quad N\|\partial_{x_1}\widehat{l}q_1^\Phi q_2^\Phi\psi\|\|\overline{\Theta}_{\beta_1}\|_{L^\infty(\mathbb{R})}\|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_{\beta_1}^{(12)}\right)p_1^\Phi\|_{\mathrm{op}} \overset{4.6\mathrm{b}}{\lesssim} \mathfrak{e}^2(t)N^{-\frac{\beta_1}{2}}.$$

Hence

$$|(23)| \lesssim \mathfrak{e}^2(t)\left(\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + N^{-\frac{\beta_1}{2}} + N^{-1+\beta_1+\xi}\right).$$

Estimate of (24). For this term, we choose $\beta_1 = 0$. Analogously to the estimate of (23),

$$\begin{aligned}
|(24)| \quad &\lesssim \quad N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, \overline{\Theta}_0^{(12)}\left(\tfrac{\mathrm{d}^2}{\mathrm{d}x_1^2}\overline{h}_0^{(12)}\right)p_1 p_2^{\chi^\varepsilon} q_2^\Phi\psi\right\rangle\!\!\right\rangle\right| \\
&\leq \quad N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, \overline{\Theta}_0^{(12)}\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}\right)\partial_{x_1}p_1^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} q_2^\Phi\psi\right\rangle\!\!\right\rangle\right| \\
&\qquad + N\left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi q_2^\Phi\psi, \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_0^{(12)}\right)\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}\right)p_1^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} q_2^\Phi\psi\right\rangle\!\!\right\rangle\right| \\
&\qquad + N\left|\left\langle\!\!\left\langle \partial_{x_1}\widehat{l}q_1^\Phi q_2^\Phi\psi, \overline{\Theta}_0^{(12)}\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}\right)p_1^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} q_2^\Phi\psi\right\rangle\!\!\right\rangle\right| \\
&\leq \quad N\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|\|q_2^\Phi\psi\|\Big(\|\overline{\Theta}_0\|_{L^\infty(\mathbb{R})}\|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}\right)\partial_{x_1}p_1^\Phi\|_{\mathrm{op}} \\
&\qquad\qquad + \|\tfrac{\mathrm{d}}{\mathrm{d}x}\overline{\Theta}_0\|_{L^\infty(\mathbb{R})}\|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}\right)p_1^\Phi\|_{\mathrm{op}}\Big) \\
&\qquad + N\|q_2^\Phi\psi\|\|\overline{\Theta}_0\|_{L^\infty(\mathbb{R})}\|\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}\right)p_1^\Phi\|_{\mathrm{op}}\|\partial_{x_1}\widehat{l}q_1^\Phi q_2^\Phi\psi\| \\
&\overset{4.6\mathrm{b}}{\lesssim} \quad \|\Phi\|_{H^2(\mathbb{R})}\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + \mathfrak{e}(t)\,\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle^{\frac{1}{2}}\,\|\partial_{x_1}q_1^\Phi\psi\|.
\end{aligned}$$

The estimate $\|\partial_{x_1}q_1^\Phi\psi\| \lesssim \mathfrak{e}(t)$ (Lemma 4.11c) is not sharp enough to see that this expression is small. We need a better control of the kinetic energy, which is established in the following refined energy lemma:

Lemma 4.21. *Under assumptions A1–A4,*

$$\|\partial_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\| \lesssim \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}\Big(\left|E^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}^{\Phi(t)}(t)\right| + \langle\!\langle \psi^{N,\varepsilon}(t), \widehat{n}\psi^{N,\varepsilon}(t)\rangle\!\rangle + \tfrac{\mu}{\varepsilon} + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\beta} + (\tfrac{N}{\varepsilon^2})^{-\eta}\Big)^{\frac{1}{2}}.$$

The proof is given in the next section. As a consequence,

$$|(24)| \lesssim \mathfrak{e}(t)\exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}\Big(|E^\psi(t) - \mathcal{E}^\Phi(t)| + \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + \tfrac{\mu}{\varepsilon} + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\beta} + (\tfrac{N}{\varepsilon^2})^{-\eta}\Big).$$

□

Finally, this concludes the proof of Proposition 3.8.

4.5 Proof of Lemma 4.21.

We prove a refined bound for the kinetic energy. The basic idea of the proof is comparable to Lemma 4.11. However, we estimate the single terms in terms of $\alpha_\xi(t)$ instead of using $\mathfrak{e}^2(t)$. Abbreviating $\psi^{N,\varepsilon}(t) \equiv \psi$ and $\Phi(t) \equiv \Phi$, we obtain

$$\begin{aligned}
&E^\psi(t) - \mathcal{E}^\Phi(t) \\
&= \|\partial_{x_1}\psi\|^2 - \|\Phi'\|^2_{L^2(\mathbb{R})} + \left\langle\!\!\left\langle \psi, \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right)\psi \right\rangle\!\!\right\rangle \\
&\quad + \tfrac{N-1}{2}\left\langle\!\!\left\langle \psi, w_\beta^{(12)}\psi \right\rangle\!\!\right\rangle - \tfrac{b_\beta}{2}\left\langle\!\left\langle \psi, |\Phi(x_1)|^2\psi \right\rangle\!\right\rangle \\
&\quad + \tfrac{b_\beta}{2}\Big(\left\langle\!\left\langle \psi, |\Phi(x_1)|^2\psi \right\rangle\!\right\rangle - \left\langle \Phi, |\Phi|^2\Phi \right\rangle_{L^2(\mathbb{R})} \Big) \\
&\quad + \left\langle\!\!\left\langle \psi, V^\parallel(t, z_1)\psi \right\rangle\!\!\right\rangle - \left\langle \Phi, V^\parallel\big(t, (x,0)\big)\Phi \right\rangle_{L^2(\mathbb{R})} \\
&\geq \|\partial_{x_1}\psi\|^2 - \|\Phi'\|^2_{L^2(\mathbb{R})} + \tfrac{1}{2}\left\langle\!\!\left\langle \psi, \left((N-1)w_\beta^{(12)} - b_\beta|\Phi(x_1)|^2\right)\psi \right\rangle\!\!\right\rangle \\
&\quad - \tfrac{b_\beta}{2}\Big| \left\langle\!\left\langle \psi, |\Phi(x_1)|^2\psi \right\rangle\!\right\rangle - \left\langle \Phi, |\Phi|^2\Phi \right\rangle_{L^2(\mathbb{R})} \Big| \\
&\quad - \Big| \left\langle\!\!\left\langle \psi, V^\parallel(t, z_1)\psi \right\rangle\!\!\right\rangle - \left\langle \Phi, V^\parallel\big(t, (x,0)\big)\Phi \right\rangle_{L^2(\mathbb{R})} \Big| \\
&\gtrsim \|\partial_{x_1}\psi\|^2 - \|\Phi'\|^2_{L^2(\mathbb{R})} + \tfrac{1}{2}\left\langle\!\!\left\langle \psi, \left((N-1)w_\beta^{(12)} - b_\beta|\Phi(x_1)|^2\right)\psi \right\rangle\!\!\right\rangle \\
&\quad - \mathfrak{e}^2(t)\left\langle\!\left\langle \psi, \widehat{n}\psi \right\rangle\!\right\rangle - \mathfrak{e}^3(t)\varepsilon
\end{aligned} \tag{66}$$

as $\left\langle\!\left\langle \psi, \left(-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}\right)\psi \right\rangle\!\right\rangle \geq 0$. The last step follows by Lemma 4.7, Lemma 4.9 and Lemma 4.12, analogously to Section 4.4.1. Further, using that $\|\partial_{x_1}p_1^\Phi\psi\|^2 = \|\Phi'\|^2_{L^2(\mathbb{R})}\|p_1^\Phi\psi\|^2 = \|\Phi'\|^2_{L^2(\mathbb{R})}(1 - \|q_1^\Phi\psi\|^2)$, we obtain

$$\begin{aligned}
\|\partial_{x_1}\psi\|^2 &= \|\partial_{x_1}q_1^\Phi\psi\|^2 + \|\partial_{x_1}p_1^\Phi\psi\|^2 + \left(\left\langle\!\left\langle \partial_{x_1}q_1^\Phi\psi, \partial_{x_1}p_1^\Phi\psi \right\rangle\!\right\rangle + c.c.\right) \\
&\overset{4.2c}{\geq} \|\partial_{x_1}q_1^\Phi\psi\|^2 + \|\Phi'\|^2_{L^2(\mathbb{R})}\left(1 - \|q_1^\Phi\psi\|^2\right) \\
&\quad -2\left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}}q_1^\Phi\psi, \partial_{x_1}^2 p_1^\Phi(\widehat{n}^{\frac{1}{2}}q_1^{\chi^\varepsilon} + \widehat{n}_1^{\frac{1}{2}}p_1^{\chi^\varepsilon})\psi \right\rangle\!\!\right\rangle\right| \\
&\overset{4.11b}{\gtrsim} \|\partial_{x_1}q_1^\Phi\psi\|^2 + \|\Phi'\|^2_{L^2(\mathbb{R})} - \left\langle\!\left\langle \psi, \widehat{n}\psi \right\rangle\!\right\rangle\left(\mathfrak{e}^2(t) + \|\Phi\|_{H^2(\mathbb{R})}\right),
\end{aligned} \tag{67}$$

where we have used that $\widehat{n}_1 \lesssim \widehat{n}$ and Lemma 4.5b. (66) and (67) yield

$$\begin{aligned}
\|\partial_{x_1}q_1^\Phi\psi\|^2 \lesssim\; & |E^\psi(t) - \mathcal{E}^\Phi(t)| + \|\Phi\|_{H^2(\mathbb{R})}\left\langle\!\left\langle \psi, \widehat{n}\psi \right\rangle\!\right\rangle \\
& + \left\langle\!\!\left\langle \psi, \left(b_\beta|\Phi(x_1)|^2 - (N-1)w_\beta^{(12)}\right)\psi \right\rangle\!\!\right\rangle + \mathfrak{e}^3(t)\varepsilon.
\end{aligned} \tag{68}$$

We estimate the second term of (68) by inserting $1 = p_1p_2 + 1 - p_1p_2$ into both slots of the scalar product:

$$\begin{aligned}
&\left\langle\!\left\langle \psi, (p_1p_2 + 1 - p_1p_2)\left(b_\beta|\Phi(x_1)|^2 - (N-1)w_\beta^{(12)}\right)(p_1p_2 + 1 - p_1p_2)\psi \right\rangle\!\right\rangle \\
&\quad= \left\langle\!\!\left\langle \psi, p_1p_2\left(b_\beta|\Phi(x_1)|^2 - Nw_\beta^{(12)}\right)p_1p_2\psi \right\rangle\!\!\right\rangle + \|\sqrt{w_\beta^{(12)}}p_1p_2\psi\|^2 \qquad (69) \\
&\qquad + \left\langle\!\!\left\langle \psi, (1 - p_1p_2)b_\beta|\Phi(x_1)|^2(1 - p_1p_2)\psi \right\rangle\!\!\right\rangle - (N-1)\|\sqrt{w_\beta^{(12)}}(1 - p_1p_2)\psi\|^2 \qquad (70)
\end{aligned}$$

$$+ \left(\left\langle\!\left\langle \psi, p_1 p_2 b_\beta |\Phi(x_1)|^2 (1 - p_1 p_2)\psi \right\rangle\!\right\rangle + c.c. \right) \tag{71}$$

$$-(N-1) \left(\left\langle\!\left\langle \psi, p_1 p_2 w_\beta^{(12)} (1 - p_1 p_2)\psi \right\rangle\!\right\rangle + c.c. \right). \tag{72}$$

Making use of $\Gamma(x_1)$ from (46), the first term can be estimated as

$$\begin{aligned} (69) &= \left\langle\!\left\langle \psi, p_1^\Phi \Gamma(x_1) p_1 p_2 \psi \right\rangle\!\right\rangle + \left\langle\!\left\langle \psi, p_1 p_2 (b_{N,\varepsilon} - b_\beta) |\Phi(x_1)|^2 p_1 p_2 \psi \right\rangle\!\right\rangle + \| \sqrt{w_\beta^{(12)}} p_1 \|_{\mathrm{op}}^2 \\ &\overset{4.10\mathrm{b}}{\lesssim} \mathfrak{e}^2(t) \left(\tfrac{\mu}{\varepsilon} + N^{-1} + (\tfrac{N}{\varepsilon^2})^{-\eta} \right) \end{aligned}$$

by (48) and (47) with η from Definition 2.2. Note that at this point, it is crucial that $\beta < 1$. For the second and third term, note that $1 - p_1 p_2 = q_2 + q_1 p_2$ and $\| \sqrt{w_\beta^{(12)}} (1 - p_1 p_2) \|^2 \geq 0$. Hence

$$\begin{aligned} (70) &\leq \left\langle\!\left\langle \psi, q_2 b_\beta |\Phi(x_1)|^2 q_2 \psi \right\rangle\!\right\rangle + \left\langle\!\left\langle \psi, q_1 p_2 b_\beta |\Phi(x_1)|^2 q_1 p_2 \psi \right\rangle\!\right\rangle \lesssim \langle\!\langle \psi, \widehat{n} \psi \rangle\!\rangle \, \mathfrak{e}^2(t), \\ (71) &\leq 2 \left| \left\langle\!\left\langle \widehat{n}_1^{\frac{1}{2}} \psi, p_1 p_2 b_\beta |\Phi(x_1)|^2 p_2 q_1 \widehat{n}^{-\frac{1}{2}} \psi \right\rangle\!\right\rangle \right| \lesssim \mathfrak{e}^2(t) \, \langle\!\langle \psi, \widehat{n} \psi \rangle\!\rangle \end{aligned}$$

by Lemma 4.5a and Lemma 4.9. For the last term, observe that $1 - p_1 p_2 = p_1 q_2 + q_1 p_2 + q_1 q_2$, hence, by symmetry of ψ,

$$\begin{aligned} (72) &\leq 2N \left| \left\langle\!\left\langle \psi, p_1 q_2 w_\beta^{(12)} p_1 p_2 \psi \right\rangle\!\right\rangle \right| + N \left| \left\langle\!\left\langle \psi, q_1 q_2 w_\beta^{(12)} p_1 p_2 \psi \right\rangle\!\right\rangle \right| \\ &\lesssim N \left| \left\langle\!\left\langle \widehat{n}^{-\frac{1}{2}} q_2 \psi, p_1 w_\beta^{(12)} p_1 p_2 \widehat{n}_1^{\frac{1}{2}} \psi \right\rangle\!\right\rangle \right| && (73) \\ &\quad + N \left| \left\langle\!\left\langle q_1^{\chi^\varepsilon} \psi, q_2 (1 + p_2^{\chi^\varepsilon}) w_\beta^{(12)} p_1 p_2 \psi \right\rangle\!\right\rangle \right| && (74) \\ &\quad + N \left| \left\langle\!\left\langle \psi, q_1^\Phi q_2^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_\beta^{(12)} p_1 p_2 \psi \right\rangle\!\right\rangle \right| && (75) \end{aligned}$$

analogously to the decomposition of (35). Using (47), (73) is easily estimated as

$$(73) \overset{4.10\mathrm{a}}{\lesssim} \mathfrak{e}^2(t) \, \langle\!\langle \psi, \widehat{n} \psi \rangle\!\rangle .$$

For (74), we obtain with $t_2 := q_2(1 + p_2^{\chi^\varepsilon})$, similarly to the estimate of (20),

$$\begin{aligned} (74) &\leq N \left| \left\langle\!\left\langle q_1^{\chi^\varepsilon} \psi, t_2 \Theta_\varepsilon^{(12)} (\nabla_1 h_\varepsilon^{(12)}) \cdot \nabla_1 p_1 p_2 \psi \right\rangle\!\right\rangle \right| \\ &\quad + N \left| \left\langle\!\left\langle q_1^{\chi^\varepsilon} \psi, t_2 (\nabla_1 \Theta_\varepsilon^{(12)}) \cdot (\nabla_1 h_\varepsilon^{(12)}) p_1 p_2 \psi \right\rangle\!\right\rangle \right| \\ &\quad + N \left| \left\langle\!\left\langle \nabla_1 q_1^{\chi^\varepsilon} \psi, t_2 \Theta_\varepsilon^{(12)} (\nabla_1 h_\varepsilon^{(12)}) p_1 p_2 \psi \right\rangle\!\right\rangle \right| \\ &\leq N \| q_1^{\chi^\varepsilon} \psi \| \left(\| \Theta_\varepsilon \|_{L^\infty(\mathbb{R}^3)} \| (\nabla_1 h_\varepsilon^{(12)}) \cdot \nabla_1 p_1 \|_{\mathrm{op}} + \| p_1 (\nabla_1 h_\varepsilon^{(12)}) \|_{\mathrm{op}} \| \nabla \Theta_\varepsilon \|_{L^\infty(\mathbb{R}^3)} \right) \\ &\quad + N \| \nabla_1 q_1^{\chi^\varepsilon} \psi \| \| \Theta_\varepsilon \|_{L^\infty(\mathbb{R}^3)} \| p_1 (\nabla_1 h_\varepsilon^{(12)}) \|_{\mathrm{op}} \lesssim \mathfrak{e}^2(t) \left(\tfrac{\varepsilon^2}{\mu} \right)^{\frac{1}{2}} . \end{aligned}$$

(75) is of the same structure as (23). Choosing $\beta_1 = \beta$, one computes analogously to (63) to (65)

$$(75) = N \left| \left\langle\!\left\langle \widehat{n}^{-\frac{1}{2}} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}_\beta^{(12)} \left(\tfrac{\mathrm{d}^2}{\mathrm{d}x_1^2} \overline{h}_\beta^{(12)} \right) p_1^\Phi p_1^{\chi^\varepsilon} \widehat{n}_2^{\frac{1}{2}} p_2 \psi \right\rangle\!\right\rangle \right|$$

$$
\begin{aligned}
&\leq N\left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}_\beta^{(12)} \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_\beta^{(12)}\right) \partial_{x_1} p_1^\Phi \widehat{n}_2^{\frac{1}{2}} p_1^{\chi^\varepsilon} p_2 \psi \right\rangle\!\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\!\left\langle \partial_{x_1}\widehat{n}^{-\frac{1}{2}} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}_\beta^{(12)} \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_\beta^{(12)}\right) p_1 p_2 \widehat{n}_2^{\frac{1}{2}} \psi \right\rangle\!\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}} q_1^\Phi \psi, q_2^\Phi \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_\beta^{(12)}\right) \left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_\beta^{(12)}\right) p_2 \widehat{n}_2^{\frac{1}{2}} p_1 \psi \right\rangle\!\!\right\rangle\right| \\
&\overset{4.8}{\lesssim} \|\Phi\|_{H^2(\mathbb{R})} \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle N^{-\frac{\beta}{2}} + \mathfrak{e}^2(t) N^{-\frac{\beta}{2}} \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle^{\frac{1}{2}} \\
&\quad + N\|\widehat{n}^{\frac{1}{2}}\psi\| \Bigg(\|p_2 (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_\beta^{(12)})(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_\beta^{(12)}) q_2^\Phi \widehat{n}_2^{\frac{1}{2}} p_1 \psi\|^2 \\
&\qquad\qquad + N^{-1}\|q_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_\beta^{(12)})(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_\beta^{(12)}) p_2 \widehat{n}_2^{\frac{1}{2}} p_1 \psi\|^2 \Bigg)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}^2(t) \left(\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + N^{-\beta} \right),
\end{aligned}
$$

since $n_2(k) \lesssim n(k)$ and by Corollary 4.6b and Lemma 4.11c. Besides, we have used that $N^{-1+\beta} < 1$ and $\|\Phi\|_{H^2(\mathbb{R})} N^{-\frac{\beta}{2}} \lesssim \mathfrak{e}^2(t)$ for sufficiently large N at fixed time t. Thus,

$$
(72) \lesssim \mathfrak{e}^2(t) \left(\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\beta} + \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle \right). \tag{76}
$$

Finally, inserting the bounds for (69) to (72) into (68) yields

$$
\begin{aligned}
\|\partial_{x_1} q_1^\Phi \psi\|^2 &\lesssim |E^\psi(t) - \mathcal{E}^\Phi(t)| + \|\Phi\|_{H^2(\mathbb{R})} \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle \\
&\qquad + \mathfrak{e}^2(t) \left(\left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + \tfrac{\mu}{\varepsilon} + N^{-\beta} + (\tfrac{N}{\varepsilon^2})^{-\eta} \right) \\
&\lesssim \exp\left\{ 2\mathfrak{e}^2(t) + 2\int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s \right\} \Bigg(|E^\psi(t) - \mathcal{E}^\Phi(t)| + \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle + \tfrac{\mu}{\varepsilon} \\
&\qquad + \left(\tfrac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}} + N^{-\beta} + (\tfrac{N}{\varepsilon^2})^{-\eta} \Bigg)
\end{aligned}
$$

since $\varepsilon < \left(\frac{\varepsilon^2}{\mu}\right)^{\frac{1}{2}}$ and $\mathfrak{e}^2(t) \lesssim \exp\{2\mathfrak{e}^2(t)\}$. □

A Well-posedness of the effective equation

Let $\frac{1}{2} < r \leq 4$ and let the initial datum $\Phi_0 \in H^r(\mathbb{R})$. Local existence of H^r-solutions of (5) on the maximal time interval $t \in [0, T_r)$ follows from the usual contraction argument on the subset $K := \{u \in X : \|u\|_X \leq 2R\}$ of the Banach space $X := \mathcal{C}([0,T]; H^r(\mathbb{R}))$ for some $R > 0$ and $T < T_r$, where one uses that the map $f : u \mapsto b_\beta |u|^2 u + V^{\|}(t,\cdot)u$ is locally Lipschitz continuous on $H^r(\mathbb{R})$. To prove global existence, one shows first that $T_s = T_r$ for all $\frac{1}{2} < r, s \leq 4$ and concludes from an estimate of $\|\Phi(t)\|_{H^1(\mathbb{R})}$ that no blow-up can occur [37]:
Let $\frac{1}{2} < r < s \leq 4$ and $\Phi_0 \in H^s(\mathbb{R})$. Clearly, $T_s \leq T_r$. Assume now $T_s < T_r$. Then $C_{T_s} := \sup_{t\in[0,T_s]} \|\Phi(t)\|_{H^r(\mathbb{R})} < \infty$. Applying twice the inequality

$$
\|uv\|_{H^s(\mathbb{R})} \leq C \left(\|u\|_{H^s(\mathbb{R})} \|v\|_{H^r(\mathbb{R})} + \|u\|_{H^r(\mathbb{R})} \|v\|_{H^s(\mathbb{R})} \right)
$$

and using the fact that $H^s(\mathbb{R})$ is an algebra, one concludes that for $t \in [0, T_s]$

$$\begin{aligned}\|\Phi(t)\|_{H^s(\mathbb{R})} &\leq \|\Phi_0\|_{H^s(\mathbb{R})} + \int_0^t \|f(\Phi(s))\|_{H^s(\mathbb{R})}\,\mathrm{d}s \\ &\leq \|\Phi_0\|_{H^s(\mathbb{R})} + C\int_0^t \left(C_{T_s}^2 + \|V^{\|}(s,\cdot)\|_{H^s(\mathbb{R})}\right)\|\Phi(s)\|_{H^s(\mathbb{R})}\,\mathrm{d}s.\end{aligned}$$

Grönwall's inequality implies that $\|\Phi(t)\|_{H^s(\mathbb{R})}$ cannot blow up at $t = T_s$, which contradicts $[0, T_s)$ being the maximal time interval where H^s-solutions exist. Therefore $T_s = T_r =: T_{\max}$. Hence for $\Phi_0 \in H^2(\mathbb{R})$, $\Phi(t) \in H^2(\mathbb{R})$ for $t \in [0, T_{\max})$. Consequently, (27) implies that $\lim_{t\to T_{\max}}\|\Phi(t)\|_{H^1(\mathbb{R})} < \infty$, hence $T_1 = T_{\max} = \infty$.

Acknowledgments

I thank Stefan Teufel for helpful remarks and for his involvement in the closely related joint project [4]. Helpful discussions with Serena Cenatiempo, Maximilian Jeblick, Nikolai Leopold, Peter Pickl and Christof Sparber are gratefully acknowledged. This work was supported by the German Research Foundation within the Research Training Group 1838 "Spectral Theory and Dynamics of Quantum Systems".

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A.2. Derivation of the 1d Gross–Pitaevskii equation from the 3d quantum many-body dynamics of strongly confined bosons

Derivation of the 1d Gross–Pitaevskii equation from the 3d quantum many-body dynamics of strongly confined bosons

Lea Boßmann* and Stefan Teufel*

Abstract

We consider the dynamics of N interacting bosons initially forming a Bose–Einstein condensate. Due to an external trapping potential, the bosons are strongly confined in two dimensions, where the transverse extension of the trap is of order ε. The non-negative interaction potential is scaled such that its range and its scattering length are both of order $(N/\varepsilon^2)^{-1}$, corresponding to the Gross–Pitaevskii scaling of a dilute Bose gas. We show that in the simultaneous limit $N \to \infty$ and $\varepsilon \to 0$, the dynamics preserve condensation and the time evolution is asymptotically described by a Gross–Pitaevskii equation in one dimension. The strength of the nonlinearity is given by the scattering length of the unscaled interaction, multiplied with a factor depending on the shape of the confining potential. For our analysis, we adapt a method by Pickl [31] to the problem with dimensional reduction and rely on the derivation of the one-dimensional NLS equation for interactions with softer scaling behaviour in [4].

1 Introduction

We consider N identical bosons in $\mathbb{R}^3$ interacting through a repulsive pair interaction. The bosons are trapped within a cigar-shaped potential, which effectively confines the particles in two directions to a region of order ε. Using the coordinates

$$z = (x, y) \in \mathbb{R}^{1+2} ,$$

the confinement in the y-directions is generated by a scaled potential $\frac{1}{\varepsilon^2} V^\perp\left(\frac{y}{\varepsilon}\right)$, where $V^\perp : \mathbb{R}^2 \to \mathbb{R}$ and $0 < \varepsilon \ll 1$. The Hamiltonian describing the system is

$$H(t) = \sum_{j=1}^{N} \left(-\Delta_j + \frac{1}{\varepsilon^2} V^\perp \left(\frac{y_j}{\varepsilon}\right) + V^{\parallel}(t, z_j) \right) + \sum_{1 \le i < j \le N} w_\mu(z_i - z_j), \tag{1}$$

where Δ denotes the Laplace operator on $\mathbb{R}^3$ and $V^{\parallel}$ is an additional unscaled external potential. The units are chosen such that $\hbar = 1$ and $m = \frac{1}{2}$.

The interaction between the particles is described by the potential

$$w_\mu(z) = \mu^{-2}\, w\left(\frac{z}{\mu}\right) \qquad \text{with} \qquad \mu := \frac{\varepsilon^2}{N} \tag{2}$$

*Fachbereich Mathematik, Eberhard Karls Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: lea.bossmann@uni-tuebingen.de, stefan.teufel@uni-tuebingen.de

and for some compactly supported, spherically symmetric, non-negative potential w. This scaling of the interaction describes a dilute gas in the Gross–Pitaevskii regime, which will be explained in detail below.

We are interested in the dynamics of the system in the simultaneous limit $(N,\varepsilon)\to(\infty,0)$. The state $\psi^{N,\varepsilon}(t)$ of the system at time t is given as the solution of the N-body Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi^{N,\varepsilon}(t)=H(t)\psi^{N,\varepsilon}(t) \tag{3}$$

with initial datum $\psi^{N,\varepsilon}(0)=\psi_0^{N,\varepsilon}\in L^2_+(\mathbb{R}^{3N}):=\otimes_{\mathrm{sym}}^N L^2(\mathbb{R}^3)$. We assume that the bosons initially form a Bose–Einstein condensate. Mathematically, this means that the one-particle reduced density matrix $\gamma^{(1)}_{\psi_0^{N,\varepsilon}}$ of $\psi_0^{N,\varepsilon}$,

$$\gamma^{(k)}_{\psi_0^{N,\varepsilon}}:=\mathrm{Tr}_{k+1,\dots,N}|\psi_0^{N,\varepsilon}\rangle\langle\psi_0^{N,\varepsilon}| \tag{4}$$

for $k=1$, is asymptotically close to a projection $|\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|$ onto a one-body state φ_0^ε. Because of the strong confinement, this condensate state factorises at low energies and is of the form $\varphi_0^\varepsilon(z)=\Phi_0(x)\chi^\varepsilon(y)\in L^2(\mathbb{R}^3)$ (see Remark 1c). Here, Φ_0 denotes the wavefunction along the x-axis and χ^ε is the normalised ground state of $-\Delta_y+\frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$. Due to the rescaling by ε, χ^ε is given by

$$\chi^\varepsilon(y)=\tfrac{1}{\varepsilon}\chi(\tfrac{y}{\varepsilon}), \tag{5}$$

where χ is the normalised ground state of $-\Delta_y+V^\perp(y)$.

In Theorem 1, we show that if the system initially condenses into a factorised state, i.e.

$$\lim_{(N,\varepsilon)\to(\infty,0)}\mathrm{Tr}_{L^2(\mathbb{R}^3)}\left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}}-|\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|\right|=0$$

with $\varphi_0^\varepsilon=\Phi_0\chi^\varepsilon$ and $\Phi_0\in H^2(\mathbb{R})$ (where the limit $(N,\varepsilon)\to(\infty,0)$ is taken in an appropriate way), then the condensation into a factorised state is preserved by the dynamics, i.e. for all $t\in\mathbb{R}$ and $k\in\mathbb{N}$

$$\lim_{(N,\varepsilon)\to(\infty,0)}\mathrm{Tr}_{L^2(\mathbb{R}^{3k})}\left|\gamma^{(k)}_{\psi^{N,\varepsilon}(t)}-|\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|^{\otimes k}\right|=0$$

with $\varphi^\varepsilon(t)=\Phi(t)\chi^\varepsilon$. Moreover, $\Phi(t)$ is the solution of the one-dimensional Gross–Pitaevskii equation

$$\mathrm{i}\tfrac{\partial}{\partial t}\Phi(t,x)=\left(-\tfrac{\partial^2}{\partial x^2}+V^{\parallel}(t,(x,0))+b|\Phi(t,x)|^2\right)\Phi(t,x)=:h(t)\Phi(t,x) \tag{6}$$

with $\Phi(0)=\Phi_0$ and

$$b=8\pi a\int_{\mathbb{R}^2}|\chi(y)|^4\,\mathrm{d}y=8\pi a\,\varepsilon^2\int_{\mathbb{R}^2}|\chi^\varepsilon(y)|^4\,\mathrm{d}y,$$

where a denotes the scattering length of the unscaled potential w.

To prove Theorem 1, we follow the approach developed by Pickl for the problem without strong confinement [31], which is outlined in Section 3. To handle the singular scaling of the interaction, he first shows the convergence for interactions with softer (but still singular) scaling behaviour, and as a second step uses this result to prove the Gross–Pitaevskii case.

The derivation of the one-dimensional NLS equation for softer scalings of the interaction combined with dimensional reduction was done in [4]. In the present paper, we extend the result from [4] to treat the Gross–Pitaevskii regime. As in [4], the strong asymmetry of the problem requires non-trivial adjustments to the method by Pickl. A description of the differences between our proof and [31] is given in Remark 3.

In the remaining part of the introduction, we will first motivate the scaling (2) of the interaction. This scaling is physically relevant since, written in suitable coordinates, it describes an (N,ε)-independent interaction. Subsequently, we comment on related literature.

We wish to study N three-dimensional bosons in an asymmetric trap, which confines in two directions to a length scale $L^\perp$ that is much smaller then the length scale $L^\parallel$ of the remaining direction[1]. Hence, we have

$$L^\perp = \varepsilon L^\parallel$$

with $\varepsilon \ll 1$. The transverse confinement on the scale $L^\perp$ is achieved by the potential $\frac{1}{(L^\perp)^2}V^\perp(\frac{\cdot}{L^\perp})$, where $-\Delta + V^\perp$ is assumed to have a localised ground state. In the remaining direction, the system is assumed to be localised in a region of length $L^\parallel$. The particle density is thus

$$\varrho_{3\mathrm{d}} \sim \frac{N}{L^\parallel (L^\perp)^2} = \frac{N}{\varepsilon^2 (L^\parallel)^3}\,.$$

To observe Gross–Pitaevskii dynamics in the longitudinal direction in the limit $(N,\varepsilon) \to (\infty, 0)$, we require the kinetic energy per particle in this direction, $E_{\mathrm{kin,p.p.}} \sim (L^\parallel)^{-2}$, to remain comparable to the total internal energy per particle, i.e. the total energy without the contributions from the confinement. For a dilute gas, the latter is given by $E_{\mathrm{p.p.}} \sim A\varrho_{3\mathrm{d}}$ [24, Chapter 2], where A denotes the (s-wave) scattering length of the interaction. The physical significance of this parameter is the following: the scattering of a slow and sufficiently distant particle at some other particle is to leading order described by its scattering at a hard sphere with radius A. Consequently, the length scale determined by A is the relevant length scale for the two-body correlations. The condition $E_{\mathrm{kin,p.p.}} \sim E_{\mathrm{p.p.}}$ implies the scaling condition

$$\frac{A}{L^\parallel} \sim \frac{\varepsilon^2}{N}. \tag{7}$$

It seems physically reasonable to fix $A \sim 1$ since A describes the two-body scattering process and should therefore be independent of N and ε. We will call this choice the microscopic frame of reference. By (7), the length scales of the problem with respect to this frame are given by $L^\parallel = \frac{N}{\varepsilon^2}$ and $L^\perp = \frac{N}{\varepsilon}$, hence both tend to infinity as $(N,\varepsilon) \to (\infty, 0)$. $\varrho_{3\mathrm{d}}$ is of order $\varepsilon^4 N^{-2}$ and converges to zero, which shows that we indeed consider a dilute gas. A useful characterisation of the low density regime is the requirement that the mean (three-dimensional) inter-particle distance $\varrho_{3\mathrm{d}}^{-\frac{1}{3}}$ be much larger than the scattering length, i.e. $A^3\varrho_{3\mathrm{d}} \to 0$.

For the mathematical analysis, we follow the common practice to choose coordinates where the longitudinal length scale $L^\parallel = 1$ is fixed. Consequently, $L^\perp = \varepsilon$ and the

[1] In this paragraph, the capital letters $L^\parallel$, $L^\perp$ and A indicate length scales. In Theorem 1 and the remainder of the paper, we use units where $L^\parallel = 1$.

scattering length shrinks as $A = a\,\frac{\varepsilon^2}{N}$. This frame of reference arises from the microscopic frame by the coordinate rescaling $z \mapsto \frac{\varepsilon^2}{N} z$ and $t \mapsto (\frac{\varepsilon^2}{N})^2 t$ in the Schrödinger equation (3), which yields the rescaled interaction (2). Note that times of order one with respect to this frame correspond to extremely long times on the microscopic time scale, which relates to the low density of the gas.

We admit an external field $V^{\parallel}$ varying on the length scale $L^{\parallel}$. Consequently, it depends on (N,ε) with respect to the microscopic frame of reference and is (N,ε)-independent in our coordinates. As $L^{\parallel} \gg A$, the external potential is asymptotically constant on the scale of the interaction and therefore does not affect the scaling condition (7).

Due to this scaling condition, the system always remains within the second of the five regions defined by Lieb, Seiringer and Yngvason in [25]. In that paper, the authors prove that the ground state energy and density of a dilute Bose gas in a highly elongated trap can be obtained by minimising the energy functional corresponding to the Lieb–Liniger Hamiltonian with coupling constant $g = \frac{A}{\varepsilon^2}\int |\chi(y)|^4\,\mathrm{d}y$ [25, Theorem 1.1]. If $g\overline{\varrho}^{-1} \to 0$, where $\overline{\varrho}$ denotes the mean one-dimensional density, the system can be described as one-dimensional limit of a three-dimensional effective theory. In particular, if $g\overline{\varrho}^{-1} \sim N^{-2}$, which is true for our system due to (7), the ground state is described by a one-dimensional Gross–Pitaevskii energy functional [25, Theorem 2.2]. The other regions can be reached by scaling A differently.[2]

It is also instructive to consider softer scaling interactions of the form

$$w_\beta(z) := (\tfrac{N}{\varepsilon^2})^{-1+3\beta} w\left((\tfrac{N}{\varepsilon^2})^{\beta} z\right), \tag{8}$$

where the scaling parameter $\beta \in (0,1)$ interpolates between the Hartree ($\beta = 0$) and the Gross–Pitaevskii ($\beta = 1$) regime. In this case, the scattering length still scales as $(\frac{N}{\varepsilon^2})^{-1}$ [9, Lemma A.1] whereas the effective range of w_β is now of order $(\frac{N}{\varepsilon^2})^{-\beta}$. This means that as $(N,\varepsilon) \to (\infty, 0)$, the scattering length becomes negligible compared to the range of the interaction, i.e. the two-body correlations become invisible on the length scale of the interaction. Consequently, the scattering length is well approximated by the first order Born approximation and the corresponding effective equation is the one-dimensional NLS equation (6) with b replaced by $\|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y$ [4].

Quasi one-dimensional bosons in highly elongated traps have been experimentally probed [13, 15] and the dynamics of such systems are physically very interesting [11, 20, 27]. The first rigorous derivation of NLS and Gross–Pitaevskii equations for three-dimensional bosons using BBGKY hierarchies is due to Erdős, Schlein and Yau [9, 10]. A different approach was proposed by Pickl [28, 29, 31, 17], who also obtained rates for the convergence of the reduced density matrices. A third method for the Gross–Pitaevskii case, using Bogoliubov transformations and coherent states on Fock space, was proposed by Benedikter, De Oliveira and Schlein [3]. Extending this approach, Brennecke and Schlein [5] recently proved an optimal rate of the convergence. Several further results

[2] Let us assume that the external field $V^{\parallel}$ is given by a homogeneous function of degree $s > 0$ acting only in the x-direction. The ideal gas case (region 1) is then obtained by the scaling $A \ll \varepsilon^2 N^{-1}$ and the Thomas–Fermi case (region 3) by choosing $\varepsilon^2 N^{-1} \ll A \ll \varepsilon^2 N^{\frac{s}{s+2}}$. Also the truly one-dimensional regime can be reached: $A \sim \varepsilon^2 N^{\frac{s}{s+2}}$ corresponds to region 4 and $A \gg \varepsilon^2 N^{\frac{s}{s+2}}$ yields a Girardeau–Tonks gas (region 5).

concern bosons in one [1, 7] and two [21, 16, 18] dimensions. The problem of dimensional reduction for the NLS equation was treated by Méhats and Raymond [26], who study the cubic NLS equation in a quantum waveguide. In [2], Ben Abdallah, Méhats, Schmeiser and Weishäupl consider an $(n+d)$-dimensional NLS equation subject to a strong confinement in d directions and derive an effective n-dimensional NLS evolution.

There are few works on the derivation of lower-dimensional time-dependent NLS equations from the three-dimensional N-body dynamics. Chen and Holmer consider three-dimensional bosons with pair interactions in a strongly confining potential in one [6] and two [8] directions. For repulsive interactions scaling with $\beta \in (0, \frac{2}{5})$ in case of a disc-shaped and for attractive interactions with $\beta \in (0, \frac{3}{7})$ in case of a cigar-shaped confinement, they show that the dynamics are effectively described by two- and one-dimensional NLS equations. In [19], von Keler and Teufel prove this for a Bose gas which is confined to a quantum waveguide with non-trivial geometry for $\beta \in (0, \frac{1}{3})$. In [4], Boßmann considers bosons interacting through a potential scaling with $\beta \in (0,1)$, but apart from this in the same setting as here, and shows that the evolution of the system is well captured by a one-dimensional NLS equation.

Notation. We use the notation $A \lesssim B$ to indicate that there exists a constant $C > 0$ independent of $\varepsilon, N, t, \psi_0^{N,\varepsilon}, \Phi_0$ such that $A \leq CB$. This constant may, however, depend on the quantities fixed by the model, such as $V^\perp$, χ and $V^\parallel$. Besides, we will exclusively use the symbol $\widehat{\cdot}$ to denote the weighted many-body operators from Definition 3.2 (see also Remark 2) and use the abbreviations

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle := \langle \cdot,\cdot \rangle_{L^2(\mathbb{R}^{3N})}, \quad \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^{3N})} \quad \text{and} \quad \|\cdot\|_{\text{op}} := \|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^{3N}))}.$$

2 Main Result

To study the effective dynamics of the many-body system in the limit $(N,\varepsilon) \to (\infty, 0)$, we consider families of initial data $\psi_0^{N,\varepsilon}$ along the following sequences $(N_n, \varepsilon_n) \to (\infty, 0)$:

Definition 2.1. A sequence (N_n, ε_n) in $\mathbb{N} \times (0,1)$ is called *admissible* if

$$\lim_{n\to\infty} (N_n, \varepsilon_n) = (\infty, 0) \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\varepsilon_n^{2+\delta}}{\mu_n} = 0 \quad \text{for } \mu_n := \left(\frac{N_n}{\varepsilon_n^2}\right)^{-1}$$

for some $0 < \delta < \frac{2}{5}$.

The second condition ensures that the energy gap of order ε^{-2} above the transverse ground state χ^ε grows sufficiently fast. In the proof, this will be used to control transverse excitations into states orthogonal to χ^ε (see also Remark 1e). Since

$$\frac{\varepsilon^{2+\delta}}{\mu} = N\varepsilon^\delta \to 0,$$

δ must be strictly positive, otherwise $N\varepsilon^\delta \to 0$ would be impossible.

To formulate our main theorem, we need two different one-particle energies:

- The *"renormalised" energy per particle*: for $\psi \in \mathcal{D}(H(t)^{\frac{1}{2}})$,
$$E^\psi(t) := \tfrac{1}{N} \langle\!\langle \psi, H(t)\psi \rangle\!\rangle - \tfrac{E_0}{\varepsilon^2}, \tag{9}$$
where E_0 denotes the lowest eigenvalue of $-\Delta_y + V^\perp(y)$. By rescaling, the lowest eigenvalue of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$ is $\frac{E_0}{\varepsilon^2}$.

- The *effective energy per particle*: for $\Phi \in H^1(\mathbb{R})$,

$$\mathcal{E}^{\Phi}(t) := \left\langle \Phi, \left(-\tfrac{\partial^2}{\partial x^2} + V^{\parallel}(t,(x,0)) + \tfrac{b}{2}|\Phi|^2 \right) \Phi \right\rangle_{L^2(\mathbb{R})}. \tag{10}$$

Further, define the function $\mathfrak{e} : \mathbb{R} \to [1,\infty)$ by

$$\begin{aligned}\mathfrak{e}^2(t) := 1 + |E^{\psi_0^{N,\varepsilon}}(0)| + |\mathcal{E}^{\Phi_0}(0)| + \int_0^t \|\dot{V}^{\parallel}(s,\cdot)\|_{L^\infty(\mathbb{R}^3)}\, \mathrm{d}s \\ + \sup_{\substack{i,j\in\{0,1\}\\ k\in\{1,2\}}} \|\partial_t^i \partial_{y_k}^j V^{\parallel}(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}.\end{aligned} \tag{11}$$

Note that $\mathfrak{e}(t)$ is for each $t \in \mathbb{R}$ uniformly bounded in N and ε because we will assume that $E^{\psi_0^{N,\varepsilon}}(0) \to \mathcal{E}^{\Phi_0}(0)$ as $(N,\varepsilon) \to (\infty,0)$ (see assumption A4 below) and boundedness of $V^{\parallel}$ and its derivatives (see assumption A3). The function $\mathfrak{e}$ will be useful because, by the fundamental theorem of calculus,

$$\left|E^{\psi^{N,\varepsilon}(t)}(t)\right| \le \mathfrak{e}^2(t) - 1 \quad \text{and} \quad \left|\mathcal{E}^{\Phi(t)}(t)\right| \le \mathfrak{e}^2(t) - 1 \tag{12}$$

for any $t \in \mathbb{R}$. Note that for a time-independent external field $V^{\parallel}$, it follows that $\mathfrak{e}^2(t) \lesssim 1$ for any t, hence $E^{\psi^{N,\varepsilon}(t)}(t)$ and $\mathcal{E}^{\Phi(t)}(t)$ are in this case bounded uniformly in time.

Let us now state our assumptions.

A1 *Interaction.* Let the unscaled interaction $w \in L^\infty(\mathbb{R}^3,\mathbb{R})$ be spherically symmetric, non-negative and let $\operatorname{supp} w \subseteq \{z \in \mathbb{R}^3 : |z| \le 1\}$.

A2 *Confining potential.* Let $V^\perp : \mathbb{R}^2 \to \mathbb{R}$ such that $-\Delta_y + V^\perp$ is self-adjoint and has a non-degenerate ground state χ with energy $E_0 < \inf \sigma_{\mathrm{ess}}(-\Delta_y + V^\perp)$. Assume that the negative part of $V^\perp$ is bounded and that $\chi \in \mathcal{C}_{\mathrm{b}}^2(\mathbb{R}^2)$, i.e. χ is bounded and twice continuously differentiable with bounded derivatives. We choose χ normalised and real.

A3 *External field.* Let $V^{\parallel} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that for fixed $z \in \mathbb{R}^3$, $V^{\parallel}(\cdot,z) \in \mathcal{C}^1(\mathbb{R})$. Further, assume that for each fixed $t \in \mathbb{R}$, $V^{\parallel}(t,(\cdot,0)) \in H^4(\mathbb{R})$, $V^{\parallel}(t,\cdot), \dot{V}^{\parallel}(t,\cdot) \in L^\infty(\mathbb{R}^3) \cap \mathcal{C}^1(\mathbb{R}^3)$ and $\nabla_y V^{\parallel}(t,\cdot), \nabla_y \dot{V}^{\parallel}(t,\cdot) \in L^\infty(\mathbb{R}^3)$.

A4 *Initial data.* Assume that the family of initial data, $\psi_0^{N,\varepsilon} \in \mathcal{D}(H(0)) \cap L_+^2(\mathbb{R}^{3N})$ with $\|\psi_0^{N,\varepsilon}\|^2 = 1$, is close to a condensate with condensate wavefunction $\varphi_0^\varepsilon = \Phi_0 \chi^\varepsilon$ for some normalised $\Phi_0 \in H^2(\mathbb{R})$ in the following sense: for some admissible sequence (N,ε), it holds that

$$\lim_{(N,\varepsilon)\to(\infty,0)} \operatorname{Tr}_{L^2(\mathbb{R}^3)} \left| \gamma^{(1)}_{\psi_0^{N,\varepsilon}} - |\Phi_0\chi^\varepsilon\rangle\langle\Phi_0\chi^\varepsilon| \right| = 0 \tag{13}$$

and

$$\lim_{(N,\varepsilon)\to(\infty,0)} \left| E^{\psi_0^{N,\varepsilon}}(0) - \mathcal{E}^{\Phi_0}(0) \right| = 0. \tag{14}$$

Theorem 1. *Assume that w, $V^\perp$ and $V^\parallel$ satisfy A1 – A3. Let $\psi_0^{N,\varepsilon}$ be a family of initial data satisfying A4, let $\psi^{N,\varepsilon}(t)$ denote the solution of* (3) *with initial datum $\psi^{N,\varepsilon}(0) = \psi_0^{N,\varepsilon}$ and let $\gamma^{(k)}_{\psi^{N,\varepsilon}(t)}$ denote its k-particle reduced density matrix as in* (4)*. Then for any $T \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[-T,T]} \operatorname{Tr}_{L^2(\mathbb{R}^{3k})} \left| \gamma^{(k)}_{\psi^{N,\varepsilon}(t)} - |\Phi(t)\chi^\varepsilon\rangle\langle\Phi(t)\chi^\varepsilon|^{\otimes k} \right| = 0 \tag{15}$$

and

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[-T,T]} \left| E^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}^{\Phi(t)}(t) \right| = 0, \tag{16}$$

where $\Phi(t)$ is the solution of (6) *with initial datum $\Phi(0) = \Phi_0$ and with*

$$b = 8\pi a \int\limits_{\mathbb{R}^2} |\chi(y)|^4 \, \mathrm{d}y \,. \tag{17}$$

Here, a denotes the scattering length of w and the limits in (15) *and* (16) *are taken along the sequence from A4.*

Remark 1. (a) Assumption A4 differs from the corresponding statement in [4] in that we impose a weaker admissibility condition than the condition $\varepsilon^2/\mu \to 0$ from [4], which cannot hold for $\beta = 1$.

(b) A2 is fulfilled, e.g., by a harmonic potential or by any smooth potential with at least one bound state below the essential spectrum. According to [14, Theorem 1], A2 implies that the ground state χ of $-\Delta_y + V^\perp$ decays exponentially. Thus, χ^ε is indeed exponentially localised on a scale of order ε. The regularity condition on $V^\parallel(t, (\cdot, 0))$ is needed to ensure the global existence of H^2 solutions of (6) (see [4, Appendix A]). Due to assumptions A1–A3, the operators $H(t)$ are for any $t \in \mathbb{R}$ self-adjoint on the time-independent domain $\mathcal{D}(H)$ and generate a strongly continuous unitary evolution on $\mathcal{D}(H)$.

(c) In [25], it is shown that the ground state of $H(0)$ with a homogeneous external field $V^\parallel(z, 0)$ satisfies assumption A4 (Theorem 2.2 and Theorem 5.1). Note that to observe non-trivial dynamics in this case, it is important that we admit a time-dependent external potential $V^\parallel$.

(d) Our proof yields an estimate of the rate of convergence of (15), which is given in Corollary 3.5. This rate is not uniform in time but, contrarily, depends on it in form of a double exponential.

(e) Our result is restricted to sequences where $\varepsilon^\delta \ll N^{-1}$ for some $\delta \in (0, \frac{2}{5})$ (Assumption A4). Similar conditions appear also in comparable works [4, 6, 8] for $\beta < 1$. However, for the ground state analysis in [25], no analogue of this admissibility condition is required. On a formal level, together with the result of the strong confinement limit of the three-dimensional NLS in [2], this suggests that our dynamical result could be extended to hold without imposing a condition on the rate of convergence of ε. As remarked before, in our proof this condition is crucial to control the transverse excitations by an a priori energy estimate. A possible approach to weaken the condition might be to replace the transverse ground state χ^ε of the linear operator $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{\cdot}{\varepsilon})$ by the x-dependent ground state of the nonlinear functional

$$\left\langle \widetilde{\chi^\varepsilon}(x,\cdot), \left(-\Delta_y + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{\cdot}{\varepsilon}) + \varepsilon^2 8\pi a|\Phi(x)|^2|\widetilde{\chi^\varepsilon}(x,\cdot)|^2\right)\widetilde{\chi^\varepsilon}(x,\cdot)\right\rangle_{L^2(\mathbb{R}^2)}$$

and to prove the smallness of transverse excitations by adiabatic-type arguments.

(f) We expect that our proof can be extended to cover systems that are trapped to quantum waveguides with non-trivial geometry as in [19]. However, this is not straightforward as a Taylor expansion of the interaction was used in [19] and the kinetic term now includes an additional vector potential due to the twisting of the waveguide.

(g) Further, we expect the same strategy to be applicable to one-dimensional confining potentials resulting in effectively two-dimensional condensates. The solution of this problem is not obvious since many of our estimates depend on the dimension and cannot be directly transferred. For instance, Green's function is different in two dimensions and the ratio of N and ε changes (the corresponding effective range is $\mu_{2\mathrm{d}} = \varepsilon/N$), making some key estimates invalid.

3 Proof of the main theorem

To prove Theorem 1, we must show that the expressions in (15) and (16) vanish in the limit $(N,\varepsilon)\to(\infty,0)$ for suitable initial data. Instead of directly estimating these differences, we follow the approach of Pickl [28, 29, 30, 31]. As one crucial first step, we define a functional

$$\alpha_\xi^< : \mathbb{R}\times L^2(\mathbb{R}^{3N})\times L^2(\mathbb{R}^3)\to\mathbb{R}\,,\quad (t,\psi^{N,\varepsilon},\varphi^\varepsilon)\mapsto \alpha_\xi^<(t,\psi^{N,\varepsilon},\varphi^\varepsilon)$$

measuring the part of $\psi^{N,\varepsilon}$ which has not condensed into φ^ε. This functional is chosen in such a way that $\alpha_\xi^<(t,\psi^{N,\varepsilon}(t),\varphi^\varepsilon(t))\to 0$ is equivalent to (15) and (16). While we roughly follow [31], the strong asymmetry of the setup and the more singular scaling of the interaction require a non-trivial adaptation of the formalism. We also heavily rely on the result in [4] for the case $\beta\in(0,1)$. The functional $\alpha_\xi^<$ is constructed as follows:

Definition 3.1. Let $\varphi\in L^2(\mathbb{R}^3)$ be of the form $\varphi(z)=\Phi(x)\chi(y)$ for some $\Phi\in L^2(\mathbb{R})$ and $\chi\in L^2(\mathbb{R}^2)$ and let

$$p^\varphi := |\varphi\rangle\langle\varphi| \quad\text{and}\quad q^\varphi := \mathbb{1} - p^\varphi \quad \in \mathcal{L}\left(L^2(\mathbb{R}^3)\right).$$

Further, define the orthogonal projections on $L^2(\mathbb{R}^3)$

$$\begin{aligned} p^\Phi &:= |\Phi\rangle\langle\Phi|\otimes\mathbb{1}_{L^2(\mathbb{R}^2)}, & q^\Phi &:= \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\Phi,\\ p^\chi &:= \mathbb{1}_{L^2(\mathbb{R})}\otimes|\chi\rangle\langle\chi|, & q^\chi &:= \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\chi\,. \end{aligned}$$

Note that $p^\varphi = p^\Phi p^\chi$, $q^{\Phi/\chi}q^\varphi = q^{\Phi/\chi}$, $q^\varphi = q^\chi + q^\Phi p^\chi$ and $p^{\Phi/\chi}q^\varphi = p^{\Phi/\chi}q^{\chi/\Phi}$.

These one-body projections are lifted to many-body projections on $L^2(\mathbb{R}^{3N})$ by defining

$$p_j^\varphi := \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes p^\varphi \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \cdots \mathbb{1}}_{N-j} \quad \text{and} \quad q_j^\varphi := \mathbb{1} - p_j^\varphi \quad \text{for } j \in \{1, \ldots, N\},$$

and analogously p_j^Φ, q_j^Φ, p_j^χ and q_j^χ. We will also write $p_j^\varphi = |\varphi(z_j)\rangle\langle\varphi(z_j)|$.
Finally, for $0 \leq k \leq N$, define the symmetrised many-body projections

$$P_k^\varphi = \left(q_1^\varphi \cdots q_k^\varphi p_{k+1}^\varphi \cdots p_N^\varphi\right)_{\text{sym}} := \sum_{\substack{J \subseteq \{1,\ldots,N\} \\ |J|=k}} \prod_{j \in J} q_j^\varphi \prod_{l \notin J} p_l^\varphi$$

and $P_k^\varphi = 0$ for $k < 0$ and $k > N$.

Definition 3.2. Let $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and $d \in \mathbb{Z}$. Using the projections P_k^φ from Definition 3.1, we define the operators $\widehat{f}^\varphi, \widehat{f}_d^\varphi \in \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)$ by

$$\widehat{f}^\varphi := \sum_{k=0}^{N} f(k) P_k^\varphi, \qquad \widehat{f}_d^\varphi := \sum_{j=-d}^{N-d} f(j+d) P_j^\varphi.$$

Definition 3.3. For $\xi \in (0, \frac{1}{2})$, define the functional

$$\alpha_\xi^< : \mathbb{R} \times L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \supset \mathbb{R} \times \mathcal{D}(H^{\frac{1}{2}}) \times (H^1(\mathbb{R}) \times L^2(\mathbb{R}^2)) \to \mathbb{R}$$

by

$$\alpha_\xi^<(t, \psi, \varphi = \Phi\chi) := \langle\!\langle \psi, \widehat{m}^\varphi \psi \rangle\!\rangle + \left| E^\psi(t) - \mathcal{E}^\Phi(t) \right|,$$

where the weight function $m : \mathbb{N}_0 \to \mathbb{R}_0^+$ is given by

$$m(k) := \begin{cases} \sqrt{\frac{k}{N}} & \text{for } k \geq N^{1-2\xi}, \\ \frac{1}{2}\left(N^{-1+\xi} k + N^{-\xi}\right) & \text{else.} \end{cases}$$

For simplicity, we will not explicitly indicate the ξ-dependence of the weight m in the notation. For the proof of Theorem 1, we will choose some fixed ξ within a suitable range.

The operators P_k^φ project onto states with k particles outside the condensate described by φ. Consequently, $\langle\!\langle \psi, \widehat{m}^\varphi \psi \rangle\!\rangle$ is a weighted measure of the relative number of such particles in the state ψ. Note that the weight function m is increasing and $m(0) \approx 0$, hence only the parts of ψ outside the condensate contribute significantly to $\langle\!\langle \psi, \widehat{m}^\varphi \psi \rangle\!\rangle$. For a sequence $(\psi^N)_{N \in \mathbb{N}}$ of N-body wavefunctions, [4, Lemma 3.2][3] implies that $\left\langle\!\left\langle \psi^N, \widehat{m}^\varphi \psi^N \right\rangle\!\right\rangle \to 0$ as $N \to \infty$ is equivalent to the convergence of the one-particle reduced density matrix of ψ^N to $|\varphi\rangle\langle\varphi|$ in trace norm or in operator norm. Further, convergence of the one-particle reduced density matrix implies convergence of all k-particle reduced density matrices. This is summarised in the following lemma:

[3]Lemma 3.2 in [4] collects different statements somewhat scattered in the literature. The respective proofs can be found e.g. in [19, 22, 30, 31, 32].

Lemma 3.1. *Let $t \in \mathbb{R}$, $k \in \mathbb{N}$, $\varphi = \Phi\chi \in H^1(\mathbb{R}) \times L^2(\mathbb{R}^2)$ with Φ and χ normalised. Let $(\psi^N)_{N\in\mathbb{N}} \subset L^2(\mathbb{R}^{3N})$ be a sequence of normalised N-body wavefunctions and denote by $\gamma^{(k)}_{\psi^N}$ the k-particle reduced density matrix of ψ^N. Then the following statements are equivalent:*

(a) $\lim\limits_{N\to\infty} \alpha^<_\xi(t,\psi^N,\varphi) = 0$ *for some* $\xi \in (0,\frac{1}{2})$,

(b) $\lim\limits_{N\to\infty} \alpha^<_\xi(t,\psi^N,\varphi) = 0$ *for any* $\xi \in (0,\frac{1}{2})$,

(c) $\lim\limits_{N\to\infty} \mathrm{Tr}\left|\gamma^{(k)}_{\psi^N} - |\varphi\rangle\langle\varphi|^{\otimes k}\right| = 0$ *and* $\lim\limits_{N\to\infty}\left|E^{\psi^N}(t) - \mathcal{E}^\Phi(t)\right| = 0$ *for all* $k \in \mathbb{N}$,

(d) $\lim\limits_{N\to\infty} \mathrm{Tr}\left|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right| = 0$ *and* $\lim\limits_{N\to\infty}\left|E^{\psi^N}(t) - \mathcal{E}^\Phi(t)\right| = 0$.

The relation between the rates of convergence of $\alpha^<_\xi(t,\psi^N,\varphi)$ and $\gamma^{(1)}_{\psi^N}$ is

$$\mathrm{Tr}\left|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right| \leq \sqrt{8\alpha^<_\xi(t,\psi^N,\varphi)},$$

$$\alpha^<_\xi(t,\psi^N,\varphi) \leq \left|E^{\psi^N}(t) - \mathcal{E}^\Phi(t)\right| + \sqrt{\mathrm{Tr}\left|\gamma^{(1)}_{\psi^N} - |\varphi\rangle\langle\varphi|\right|} + \tfrac{1}{2}N^{-\xi}.$$

Proof. [4], Lemma 3.2 and Lemma 3.3. □

To prove Theorem 1, we evaluate the functional $\alpha^<_\xi$ on the solution $\psi^{N,\varepsilon}(t)$ of (3) with initial datum $\psi^{N,\varepsilon}_0$ given by assumption A4, the solution $\Phi(t)$ of the Gross–Pitaevskii equation (6) with initial datum Φ_0 from A4, and the ground state χ^ε of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$ from A2. For simplicity, we will abbreviate

$$\alpha^<_\xi(t) := \alpha^<_\xi\Big(t,\, \psi^{N,\varepsilon}(t),\, \varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon\Big).$$

Due to Lemma 3.1, $\alpha^<_\xi(t) \to 0$ is equivalent to (15) and (16); conversely, (13) and (14) imply $\alpha^<_\xi(0) \to 0$. Hence, to prove Theorem 1, it suffices to show the convergence of $\alpha^<_\xi(t) \to 0$ for all $t \in \mathbb{R}$.

In [4], the functional $\alpha^<_\xi(t)$ is used as counting measure for the interaction (8) scaling with $\beta \in (0,1)$. For the proof in that case, one first shows an estimate of the kind $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha^<_\xi(t)| \lesssim \alpha^<_\xi(t) + \mathcal{O}(1)$ and subsequently applies Grönwall's inequality, using that $\alpha^<_\xi(0) \to 0$.

For the Gross–Pitaevskii scaling of the interaction, we cannot simply estimate $\frac{\mathrm{d}}{\mathrm{d}t}\alpha^<_\xi(t)$ for $\beta = 1$ because this derivative is not controllable with the methods used in [4]. To understand why this is the case, let us first give a heuristic argument why the NLS equation with coupling parameter $b_\beta = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y$ is the right effective description for $\beta \in (0,1)$ but not for $\beta = 1$. To this end, we compute the renormalised energy per particle with respect to the trial state $\psi_{\mathrm{prod}}(t,z_1,...,z_N) = \varphi^\varepsilon(t,z_1)\varphi^\varepsilon(t,z_2)\cdots\varphi^\varepsilon(t,z_N)$, i.e. the state where all particles are condensed into the single-particle orbital $\varphi^\varepsilon(t)$. For simplicity, we will ignore the external potential $V^\parallel$ and drop the time-dependence of φ^ε in the notation. Making use of the fact that $\left(-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon}) - \frac{E_0}{\varepsilon^2}\right)\chi^\varepsilon(y) = 0$ and that φ^ε is normalised, we obtain

$$\tfrac{1}{N}\left\langle\!\left\langle \psi_{\mathrm{prod}}, H\psi_{\mathrm{prod}}\right\rangle\!\right\rangle - \tfrac{E_0}{\varepsilon^2}$$

$$= \left\langle \Phi(x_1), (-\partial_{x_1}^2)\Phi(x_1) \right\rangle + \tfrac{N-1}{2N} \int \mathrm{d}z_1 |\Phi(x_1)|^2 |\chi(y_1)|^2 \times$$
$$\times \int \mathrm{d}z |\Phi(x_1 - \mu^\beta x)|^2 |\chi(y_1 - \tfrac{\mu^\beta}{\varepsilon} y)|^2 w(z)$$
$$\to \left\langle \Phi(x_1), \left(-\partial_{x_1}^2 + \tfrac{1}{2} \left(\int |\chi(y_1)|^4 \, \mathrm{d}y_1 \int w(z) \, \mathrm{d}z \right) |\Phi(x_1)|^2 \right) \Phi(x_1) \right\rangle \;=\; \mathcal{E}^\Phi_{\beta \in (0,1)}$$

in the limit $(N, \varepsilon) \to (\infty, 0)$, where we have chosen the limiting sequence in such a way that $\frac{\mu^\beta}{\varepsilon} \to 0$.[4] Here, $\mathcal{E}^\Phi_{\beta \in (0,1)}$ is the effective energy per particle for $\beta \in (0, 1)$, i.e. it equals (10) with $V^\parallel = 0$ and b replaced by b_β.

For the Gross–Pitaevskii scaling $\beta = 1$, this very argument yields the same one-particle energy $\mathcal{E}^\Phi_{\beta \in (0,1)}$, which differs from the correct expression (10) by an error of $\mathcal{O}(1)$ as $b_\beta \neq b$. The reason for this error is that for $\beta = 1$, the scattering length a_μ of w_μ is of the same order as its range μ, i.e. the inter-particle correlations live on the scale of the interaction and thus decrease the energy per particle by an amount of $\mathcal{O}(1)$.

Hence, an initial state $\psi_0^{N,\varepsilon}$ that is a pure product state is excluded by assumption A4. This reasoning suggests to include the pair correlations in our trial function. To do so, let us first recall the definition of the scattering length: the zero energy scattering equation for the interaction $w_\mu = \mu^{-2} w(\cdot/\mu)$ is given by

$$\begin{cases} \left(-\Delta + \frac{1}{2} w_\mu(z)\right) j_\mu(z) = 0 & \text{for } |z| < \infty, \\ j_\mu(z) \to 1 & \text{as } |z| \to \infty. \end{cases} \tag{18}$$

By [24, Theorems C.1 and C.2], the unique solution $j_\mu \in \mathcal{C}^1(\mathbb{R}^3)$ of (18) is spherically symmetric, non-negative, non-decreasing in $|z|$ and

$$\begin{cases} j_\mu(z) = 1 - \frac{a_\mu}{|z|} & \text{for } |z| > \mu, \\ j_\mu(z) \geq 1 - \frac{a_\mu}{|z|} & \text{else.} \end{cases} \tag{19}$$

The number $a_\mu \in \mathbb{R}$ is by definition the scattering length of w_μ. Equivalently,

$$8\pi a_\mu = \int_{\mathbb{R}^3} w_\mu(z) j_\mu(z) \, \mathrm{d}z. \tag{20}$$

By the scaling behaviour of (18), we obtain

$$\mu^{-2} \left(-\Delta + \tfrac{1}{2} w(z)\right) j_\mu(\mu z) = 0$$

for $|z| < \infty$, hence $j_\mu(z) = j_1(z/\mu)$ and

$$a_\mu = \mu a, \tag{21}$$

where a denotes the scattering length of the unscaled interaction $w = w_1$. From (19) and (21), one immediately concludes that j_μ differs from one by an error of $\mathcal{O}(1)$ on

[4]This condition in [4], called *moderate confinement*, ensures that the extension ε is always large compared to the range $\mu^\beta = (\frac{N}{\varepsilon^2})^{-\beta}$ of the interaction w_β. As $\frac{\mu^\beta}{\varepsilon} = N^{-\beta} \varepsilon^{2\beta - 1}$, this is a restriction only for $\beta < \frac{1}{2}$; in particular, it is satisfied for $\beta = 1$.

$\operatorname{supp} w_\mu$. Hence, (20) implies that the first order Born approximation $\frac{1}{8\pi}\int w_\mu(z)\,\mathrm{d}z$ is no valid approximation to the scattering length a_μ in the Gross–Pitaevskii regime, whereas this approximation was justified for interactions w_β as in (8) with $\beta \in (0,1)$.

For practical reasons, we will in the following consider a function $f_{\widetilde{\beta}}$ which asymptotically coincides with j_μ on $\operatorname{supp} w_\mu$ but is defined in such a way that $f_{\widetilde{\beta}}(z) = 1$ for $|z|$ sufficiently large. This is achieved by constructing a potential $U_{\widetilde{\beta}}$ in such a way that the scattering length of $w_\mu - U_{\widetilde{\beta}}$ equals zero; $f_{\widetilde{\beta}}$ is then defined as the scattering solution of $w_\mu - U_{\widetilde{\beta}}$. The advantage of using $f_{\widetilde{\beta}}$ instead of j_μ is that $\nabla f_{\widetilde{\beta}}$ and $1 - f_{\widetilde{\beta}}$ have compact support, which is not true for j_μ.

Definition 3.4. Let $\widetilde{\beta} \in (\frac{1}{3}, 1)$. Define

$$U_{\widetilde{\beta}}(z) := \begin{cases} \mu^{1-3\widetilde{\beta}} a & \text{for } \mu^{\widetilde{\beta}} < |z| < R_{\widetilde{\beta}}, \\ 0 & \text{else,} \end{cases}$$

where $R_{\widetilde{\beta}}$ is the minimal value in $(\mu^{\widetilde{\beta}}, \infty]$ such that the scattering length of $w_\mu - U_{\widetilde{\beta}}$ equals zero.

In Section 4.2, we show by explicit construction that a suitable $R_{\widetilde{\beta}}$ exists and that it is of order $\mu^{\widetilde{\beta}}$. We will abbreviate

$$U_{\widetilde{\beta}}^{(ij)} := U_{\widetilde{\beta}}(z_i - z_j) \quad \text{and} \quad w_\mu^{(ij)} := w_\mu(z_i - z_j).$$

Definition 3.5. Let $f_{\widetilde{\beta}} \in \mathcal{C}^1(\mathbb{R}^3)$ be the solution of

$$\begin{cases} \Big(-\Delta + \frac{1}{2}\Big(w_\mu(z) - U_{\widetilde{\beta}}(z) \Big) \Big) f_{\widetilde{\beta}}(z) = 0 & \text{for } |z| < R_{\widetilde{\beta}}, \\ f_{\widetilde{\beta}}(z) = 1 & \text{for } |z| \geq R_{\widetilde{\beta}}. \end{cases} \tag{22}$$

Further, define

$$g_{\widetilde{\beta}} := 1 - f_{\widetilde{\beta}}.$$

We will in the sequel abbreviate

$$g_{\widetilde{\beta}}^{(ij)} := g_{\widetilde{\beta}}(z_i - z_j) \quad \text{and} \quad f_{\widetilde{\beta}}^{(ij)} := f_{\widetilde{\beta}}(z_i - z_j).$$

Definitions 3.4 and 3.5 imply in particular that

$$\int\limits_{\mathbb{R}^3} \Big(w_\mu(z) - U_{\widetilde{\beta}}(z) \Big) f_{\widetilde{\beta}}(z)\,\mathrm{d}z = 0. \tag{23}$$

We now repeat the above heuristic estimate for the renormalised energy per particle with the trial function[5] $\psi_{\mathrm{cor}}(z_1, \dots, z_N) := \prod_{k=1}^N \varphi^\varepsilon(z_k) \prod_{1 \leq l < m \leq N} f_{\widetilde{\beta}}(z_l - z_m)$, where the product state is overlaid with a microscopic structure characterised by $f_{\widetilde{\beta}}$. For $V^{\parallel} = 0$, this yields

$$\frac{1}{N} \langle\!\langle \psi_{\mathrm{cor}}, H\psi_{\mathrm{cor}} \rangle\!\rangle - \frac{E_0}{\varepsilon^2}$$

[5] Note that this trial function is not normalised. However, a reasoning similar to Lemma 4.10 leads to the estimate $0 \leq 1 - \|\psi_{\mathrm{cor}}\|^2 \lesssim N\mu^{2\widetilde{\beta}}$. As $\widetilde{\beta} > \frac{1}{3}$, the normalisation error is thus irrelevant for our heuristic argument.

$$
\begin{aligned}
&= \left\langle\!\!\left\langle \prod_{k\geq 1} \varphi^\varepsilon(z_k) \prod_{l<m} f_{\widetilde{\beta}}^{(lm)}, (-\partial_{x_1}^2 \varphi^\varepsilon(z_1)) \prod_{k'\geq 2} \varphi^\varepsilon(z'_k) \prod_{l'<m'} f_{\widetilde{\beta}}^{(l'm')} \right\rangle\!\!\right\rangle \\
&\quad +(N-1)\left\langle\!\!\left\langle \prod_{k\geq 1} \varphi^\varepsilon(z_k) \prod_{l<m} f_{\widetilde{\beta}}^{(lm)}, \left(-\Delta_1 f_{\widetilde{\beta}}^{(12)} + \tfrac{1}{2} w_\mu^{(12)} f_{\widetilde{\beta}}^{(12)}\right) \times \right.\right. \\
&\qquad\qquad \left.\left. \times \prod_{k'\geq 1} \varphi^\varepsilon(z'_k) \prod_{\substack{l'<m' \\ (l',m')\neq(1,2)}} f_{\widetilde{\beta}}^{(l'm')} \right\rangle\!\!\right\rangle \\
&\quad +2(N-1)\left\langle\!\!\left\langle \prod_{k\geq 1} \varphi^\varepsilon(z_k) \prod_{l<m} f_{\widetilde{\beta}}^{(lm)}, \left(\nabla_1 \varphi^\varepsilon(z_1) \cdot \nabla_1 f_{\widetilde{\beta}}^{(12)}\right) \right.\right. \\
&\qquad\qquad \left.\left. \times \prod_{k'\geq 2} \varphi^\varepsilon(z'_k) \prod_{\substack{l'<m' \\ (l',m')\neq(1,2)}} f_{\widetilde{\beta}}^{(l'm')} \right\rangle\!\!\right\rangle \\
&\quad +(N-1)(N-2)\left\langle\!\!\left\langle \prod_{k\geq 1} \varphi^\varepsilon(z_k) \prod_{l<m} f_{\widetilde{\beta}}^{(lm)}, \left(\nabla_1 f_{\widetilde{\beta}}^{(12)} \cdot \nabla_1 f_{\widetilde{\beta}}^{(13)}\right) \right.\right. \\
&\qquad\qquad \left.\left. \times \prod_{k'\geq 1} \varphi^\varepsilon(z'_k) \prod_{\substack{l'<m' \\ (l',m')\notin \\ \{(1,2),(1,3)\}}} f_{\widetilde{\beta}}^{(l'm')} \right\rangle\!\!\right\rangle .
\end{aligned}
$$

Very roughly speaking, we may substitute $f_{\widetilde{\beta}} \approx 1$ unless we integrate against w_μ, which is peaked on the set where $f_{\widetilde{\beta}} \neq 1$, or apply the Laplacian to $f_{\widetilde{\beta}}$. For the last line, also note that $\operatorname{supp} \nabla f_{\widetilde{\beta}} \subseteq B_{R_{\widetilde{\beta}}}(0)$ with $R_{\widetilde{\beta}} = \mathcal{O}(\mu^{\widetilde{\beta}})$ (Lemma 4.9), which is for $\widetilde{\beta} > \frac{1}{3}$ negligible compared to the mean inter-particle distance $\mu^{\frac{1}{3}}$. Thus, the measure of the set $\operatorname{supp} \nabla_1 f_{\widetilde{\beta}}(\cdot - z_2) \cap \operatorname{supp} \nabla_1 f_{\widetilde{\beta}}(\cdot - z_3)$ vanishes sufficiently fast in the limit $(N, \varepsilon) \to (\infty, 0)$. For the second line, note that (22) implies $-\Delta_1 f_{\widetilde{\beta}}^{(12)} + \frac{1}{2} w_\mu^{(12)} f_{\widetilde{\beta}}^{(12)} = \frac{1}{2} U_{\widetilde{\beta}}^{(12)} f_{\widetilde{\beta}}^{(12)}$. Besides, $1 \geq f_{\widetilde{\beta}} \geq 1 - a\mu^{1-\widetilde{\beta}}$ on the support of $U^{\widetilde{\beta}}$ and $f_{\widetilde{\beta}} \approx j_\mu$ on the support of w_μ (Lemma 4.9). Hence $\|f_{\widetilde{\beta}} U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)} \approx \|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)} \approx \int_{\mathbb{R}^3} w_\mu(z) j_\mu(z)\,\mathrm{d}z = 8\pi\mu a$ according to (23) and (21). Thus, the second line gives to leading order

$$
\begin{aligned}
&\tfrac{N-1}{2} \int \mathrm{d}z_1 |\varphi^\varepsilon(z_1)|^2 \int \mathrm{d}z |\varphi^\varepsilon(z_1 - z)|^2 U_{\widetilde{\beta}}(z) f_{\widetilde{\beta}}(z) \\
&\qquad\qquad \to 4\pi a \int \mathrm{d}x_1 |\Phi(x_1)|^4 \int \mathrm{d}y_1 \chi(y_1)|^4 ,
\end{aligned}
$$

and the renormalised energy per particle is consequently given by the correct expression

$$
\begin{aligned}
&\tfrac{1}{N} \langle\!\langle \psi_{\mathrm{cor}}, H \psi_{\mathrm{cor}} \rangle\!\rangle - \tfrac{E_0}{\varepsilon^2} \\
&\qquad \to \left\langle \Phi(x_1), \left(-\partial_{x_1}^2 + \tfrac{1}{2}\left(8\pi a \int |\chi(y)|^4 \,\mathrm{d}y\right) |\Phi(x_1)|^2\right) \Phi(x_1) \right\rangle .
\end{aligned}
$$

This heuristic argument indicates that the state of the system is asymptotically close to ψ_{cor}. We will therefore modify the counting functional such that $p_1^\varphi p_2^\varphi \cdots p_N^\varphi = |\psi_{\mathrm{prod}}\rangle\langle\psi_{\mathrm{prod}}|$ is replaced by $|\psi_{\mathrm{cor}}\rangle\langle\psi_{\mathrm{cor}}|$, i.e. P_0 is replaced by the projection onto the product state overlaid with a microscopic structure minimising the energy. We substitute in the first term of $\alpha_\xi^<(t)$

$$\begin{aligned}\langle\!\langle\psi,\widehat{m}^{\varphi^\varepsilon}\psi\rangle\!\rangle &\mapsto \left\langle\!\!\left\langle\psi,\prod_{k<l}f_{\widetilde{\beta}}^{(lk)}\,\widehat{m}^{\varphi^\varepsilon}\prod_{r<s}f_{\widetilde{\beta}}^{(rs)}\psi\right\rangle\!\!\right\rangle\\ &\approx \langle\!\langle\psi,\widehat{m}^{\varphi^\varepsilon}\psi\rangle\!\rangle - N(N-1)\Re\left\langle\!\!\left\langle\psi,g_{\widetilde{\beta}}^{(12)}\widehat{m}^{\varphi^\varepsilon}\psi\right\rangle\!\!\right\rangle, \end{aligned}\tag{24}$$

where we have used the symmetry of $\psi^{N,\varepsilon}(t)\equiv\psi$ and expanded the products by writing $f_{\widetilde{\beta}}=1-g_{\widetilde{\beta}}$ and keeping only the terms which are at most linear in $g_{\widetilde{\beta}}$.

This correction in the functional effectively leads to the replacement of w_μ by $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$ in the time derivative of the new functional. The underlying physical idea is that the low energy scattering is essentially described by the s-wave scattering length, hence the scattering at w_μ is to leading order equivalent to the scattering at $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$. The terms containing $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$ can be controlled by the result from [4]; the remainders from this substitution must be estimated additionally. To understand how the substitution works, let us for simplicity consider the case $N=2$ with $V^{\parallel}=0$. The full argument is given in Section 4.4. Abbreviating $Z^{(12)}:=w_\mu^{(12)}-b(|\Phi(x_1)|^2+|\Phi(x_2)|^2)$, we obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\!\langle\psi,\widehat{m}^{\varphi^\varepsilon}\psi\rangle\!\rangle &= \mathrm{i}\left\langle\!\!\left\langle\psi,[Z^{(12)},\widehat{m}^{\varphi^\varepsilon}]\psi\right\rangle\!\!\right\rangle = -2\Im\left\langle\!\!\left\langle\psi,Z^{(12)}\widehat{m}^{\varphi^\varepsilon}\psi\right\rangle\!\!\right\rangle,\\ -2\tfrac{\mathrm{d}}{\mathrm{d}t}\Re\left\langle\!\!\left\langle\psi,g_{\widetilde{\beta}}^{(12)}\widehat{m}^{\varphi^\varepsilon}\psi\right\rangle\!\!\right\rangle &= 2\Im\left\langle\!\!\left\langle\psi,\Big(g_{\widetilde{\beta}}^{(12)}[Z^{(12)},\widehat{m}^{\varphi^\varepsilon}]+(w_\mu^{(12)}-U_{\widetilde{\beta}}^{(12)})f_{\widetilde{\beta}}^{(12)}\widehat{m}^{\varphi^\varepsilon}\right.\\ &\qquad\left.+4\nabla_1 f_{\widetilde{\beta}}^{(12)}\cdot\nabla_1\widehat{m}^{\varphi^\varepsilon}\Big)\psi\right\rangle\!\!\right\rangle.\end{aligned}$$

Adding these expressions and using that $g_{\widetilde{\beta}}=1-f_{\widetilde{\beta}}$, we observe that the term

$$\left\langle\!\!\left\langle\psi,Z^{(12)}\widehat{m}^{\varphi^\varepsilon}\psi\right\rangle\!\!\right\rangle$$

cancels. It remains, among other contributions,

$$-2\Im\left\langle\!\!\left\langle\psi,\Big(U_{\widetilde{\beta}}^{(12)}f_{\widetilde{\beta}}^{(12)}-b_{\widetilde{\beta}}(|\Phi(x_1)|^2+|\Phi(x_2)|^2)\Big)\,\widehat{m}^{\varphi^\varepsilon}\psi\right\rangle\!\!\right\rangle,$$

where w_μ is replaced by $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$.

Remark 2. To simplify the notation, we will in the following drop the index φ in all projections and (weighted) many-body operators from Definitions 3.1 and 3.2. From now on, $p=p^\Phi p^{\chi^\varepsilon}$ always projects onto $\varphi^\varepsilon(t)=\Phi(t)\chi^\varepsilon$, where $\Phi(t)$ is the solution of the Gross–Pitaevskii equation (6) with initial datum Φ_0 from A4, and χ^ε is the ground state of $-\Delta_y+\frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$ from A2.

In our proof, we will use a slightly modified variant of the correction term in (24). The reason for the modification is that Lemma 3.1 establishes the equivalence of (15) and (16) with $\alpha_\xi^<(t)\to 0$, hence we must ensure that the correction term converges to zero in the limit $(N,\varepsilon)\to(\infty,0)$. To make the correction term in (24) controllable, we replace $\widehat{m}$ by the weighted many-body operator $\widehat{r}$, which is defined as follows:

Definition 3.6. Define the weight functions

$$\begin{aligned} m^a(k) &:= m(k)-m(k+1), & m^b(k) &:= m(k)-m(k+2),\\ m^c(k) &:= m^a(k)-m^a(k+1), & m^d(k) &:= m^a(k)-m^a(k+2),\\ m^e(k) &:= m^b(k)-m^b(k+1), & m^f(k) &:= m^b(k)-m^b(k+2).\end{aligned}$$

The corresponding weighted many-body operators are denoted by $\widehat{m}^\sharp$, $\sharp \in \{a,b,c,d,e,f\}$. Further, define

$$\widehat{r} := \widehat{m}^b p_1 p_2 + \widehat{m}^a (p_1 q_2 + q_1 p_2).$$

Note that the weight functions $m^\sharp$ correspond to discrete derivatives of m, which appear in the computations when taking commutators with two-body operators such as $[Z^{(12)}, \widehat{m}]$.

When replacing $\widehat{m}$ by $\widehat{r}$ in (24), we gain an additional projection p_1, which allows us to estimate $g_{\widetilde{\beta}}^{(12)} p_1$ instead of $g_{\widetilde{\beta}}^{(12)}$ (Lemma 4.10b). This change does not affect the replacement of w_μ by $U_{\widetilde{\beta}}$ because $[Z^{(12)}, \widehat{m}] = [Z^{(12)}, \widehat{r}]$ by Lemma 4.2c. The modified functional is now defined as follows:

Definition 3.7.

$$\alpha_\xi(t) := \alpha_\xi^<(t) - N(N-1)\Re \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\, \widehat{r}\, \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle.$$

In Proposition 3.2, the time derivative of the new functional $\alpha_\xi(t)$ is explicitly calculated, following essentially the steps sketched for $N = 2$.

Proposition 3.2. *Under assumptions A1 – A4,*

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)\right| \leq \left|\gamma^<(t)\right| + \left|\gamma_a(t)\right| + \left|\gamma_b(t)\right| + \left|\gamma_c(t)\right| + \left|\gamma_d(t)\right| + \left|\gamma_e(t)\right| + \left|\gamma_f(t)\right|$$

for almost every $t \in \mathbb{R}$, where

$$\begin{aligned}
\gamma^<(t) \;&:=\; \left| \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \dot{V}^{\|}(t,z_1)\psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle - \left\langle \Phi(t), \dot{V}^{\|}(t,(x,0))\Phi(t) \right\rangle_{L^2(\mathbb{R})} \right| && (25)\\
&\quad -2N\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1 \widehat{m}^a_{-1}\big(V^{\|}(t,z_1) - V^{\|}(t,(x_1,0))\big) p_1 \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle && (26)\\
&\quad -N(N-1)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), \widetilde{Z}^{(12)}\widehat{m}\psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle, && (27)
\end{aligned}$$

$$\gamma_a(t) \;:=\; N^2(N-1)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)} \left[V^{\|}(t,z_1) - V^{\|}(t,(x_1,0)), \widehat{r}\right] \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle, \qquad (28)$$

$$\begin{aligned}
\gamma_b(t) \;&:=\; -N\Im \left\langle\!\!\left\langle \psi, b(|\Phi(x_1)|^2 + |\Phi(x_2)|^2) g_{\widetilde{\beta}}^{(12)}\, \widehat{r}\, \psi \right\rangle\!\!\right\rangle && (29)\\
&\quad -N\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (b_{\widetilde{\beta}} - b)(|\Phi(x_1)|^2 + |\Phi(x_2)|^2)\, \widehat{r}\, \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle && (30)\\
&\quad -N(N-1)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\, \widehat{r}\, Z^{(12)} \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle, && (31)
\end{aligned}$$

$$\gamma_c(t) \;:=\; -4N(N-1)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), (\nabla_1 g_{\widetilde{\beta}}^{(12)}) \cdot \nabla_1 \widehat{r}\, \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle, \qquad (32)$$

$$\begin{aligned}
\gamma_d(t) \;&:=\; -N(N-1)(N-2)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)} \left[b|\Phi(x_3)|^2, \widehat{r}\right] \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle && (33)\\
&\quad +2N(N-1)(N-2)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)} \left[w_\mu^{(13)}, \widehat{r}\right] \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle, && (34)
\end{aligned}$$

$$\gamma_e(t) \;:=\; \tfrac{1}{2}N(N-1)(N-2)(N-3)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)} \left[w_\mu^{(34)}, \widehat{r}\right] \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle, \qquad (35)$$

$$\gamma_f(t) \;:=\; -2N(N-2)\Im\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g^{(12)}_{\widetilde{\beta}}\left[b|\Phi(x_1)|^2,\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle. \tag{36}$$

Here, we have used the abbreviations

$$\begin{aligned} Z^{(ij)} &:= w^{(ij)}_\mu - \tfrac{b}{N-1}\left(|\Phi(x_i)|^2 + |\Phi(x_j)|^2\right), \\ \widetilde{Z}^{(ij)} &:= U^{(ij)}_{\widetilde{\beta}} f^{(ij)}_{\widetilde{\beta}} - \tfrac{b_{\widetilde{\beta}}}{N-1}(|\Phi(x_i)|^2 + |\Phi(x_j)|^2), \end{aligned}$$

where

$$b_{\widetilde{\beta}} := \lim_{(N,\varepsilon)\to(\infty,0)} \mu^{-1} \int_{\mathbb{R}^3} U_{\widetilde{\beta}}(z) f_{\widetilde{\beta}}(z)\,\mathrm{d}z \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y.$$

The first expression $\gamma^<$ equals $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha^<_\xi(t)|$ with w_μ replaced by the interaction $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$. The terms γ_a to γ_f collect all remainders resulting from this replacement. Whereas γ_a arises from the strong confinement, γ_b to γ_f are comparable to the corresponding terms from the problem without strong confinement in [31].

Proposition 3.3. *Let μ be sufficiently small and let assumptions A1 – A4 be satisfied. Then there exist $\frac{5}{6} < d < \widetilde{\beta} < \frac{2}{2+\delta}$ and $0 < \xi < \min\{1-\widetilde{\beta}, \frac{\widetilde{\beta}}{6}\}$ such that for any $t \in \mathbb{R}$*

$$\begin{aligned} \left|\gamma^<(t)\right| &\lesssim \mathfrak{e}(t)\exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}\Big(\alpha^<_\xi(t) + (N\varepsilon^\delta)^{1-\widetilde{\beta}} + N^{-1+\widetilde{\beta}+\xi} \\ &\qquad + \mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}}\Big), \\ \left|\gamma_a(t)\right| &\lesssim \mathfrak{e}^3(t)\,\varepsilon^2, \\ \left|\gamma_b(t)\right| &\lesssim \mathfrak{e}^3(t)\left(\varepsilon^{1+\widetilde{\beta}} + N^{-1+\widetilde{\beta}+\xi}\right), \\ \left|\gamma_c(t)\right| &\lesssim \mathfrak{e}^2(t)\,N^{-1+\widetilde{\beta}+\xi}, \\ \left|\gamma_d(t)\right| &\lesssim \mathfrak{e}^3(t)\left(\varepsilon^{1+\widetilde{\beta}} + (N\varepsilon^\delta)^{1-\widetilde{\beta}+\xi}\right), \\ \left|\gamma_e(t)\right| &\lesssim \mathfrak{e}^3(t)\,\varepsilon^{1+\widetilde{\beta}}, \\ \left|\gamma_f(t)\right| &\lesssim \mathfrak{e}^2(t)\,\varepsilon. \end{aligned}$$

To control $\gamma^<$, we first prove that the interaction $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ is of the kind considered in [4] and subsequently apply [4, Proposition 3.5]. This provides a bound of $|\gamma^<(t)|$ in terms of

$$\langle\!\langle \psi^{N,\varepsilon}(t), \widehat{m}\psi^{N,\varepsilon}(t)\rangle\!\rangle + \left|E^{\psi^{N,\varepsilon}(t)}_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - \mathcal{E}^{\Phi(t)}_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)\right|,$$

where $E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$ and $\mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$ denote the quantities corresponding to (9) and (10), respectively, but where w_μ is replaced by $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ and b by

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mu^{-1}\|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)} \int |\chi(y)|^4\,\mathrm{d}y.$$

The potential $U_{\widetilde{\beta}}$ is chosen in such a way that $\lim_{(N,\varepsilon)\to(\infty,0)}\|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)} = 8\pi a$, hence $\mathcal{E}^\Phi(t) = \mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$ but

$$\left|E^{\psi_0^{N,\varepsilon}}_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(0) - E^{\psi_0^{N,\varepsilon}}(0)\right| \sim \mathcal{O}(1). \tag{37}$$

To explain why one expects the energy difference (37) to be of order one, let us again consider the trial function ψ_{cor}. Following the same heuristic reasoning as before (i.e. $f_{\widetilde{\beta}} \approx 1$ unless we integrate against w_μ or apply the Laplacian, $f_{\widetilde{\beta}} \approx j_\mu$ on $\operatorname{supp} w_\mu$, and $f_{\widetilde{\beta}} \approx 1$ on $\operatorname{supp} U_{\widetilde{\beta}}$), this difference is to leading order given by

$$\begin{aligned}
&\tfrac{N-1}{2}\left|\left\langle\!\!\left\langle \psi_{\mathrm{cor}}, \left(w_\mu^{(12)} - (U_{\widetilde{\beta}} f_{\widetilde{\beta}})^{(12)}\right)\psi_{\mathrm{cor}}\right\rangle\!\!\right\rangle\right| \\
&\sim \quad N\left|\int \mathrm{d}z_1 |\varphi^\varepsilon(z_1)|^4 \int \mathrm{d}z\, f_{\widetilde{\beta}}(z)^2 (w_\mu(z) - U_{\widetilde{\beta}}(z))\right| \\
&\overset{(23)}{\sim} \quad \mu^{-1}\int \mathrm{d}z\, g_{\widetilde{\beta}}(z) w_\mu(z) f_{\widetilde{\beta}}(z) \\
&\geq \quad \mu^{-1} g_{\widetilde{\beta}}(\mu)\int \mathrm{d}z\, w_\mu(z) f_{\widetilde{\beta}}(z) \overset{(20)}{\sim} 8\pi a^2 \sim \mathcal{O}(1).
\end{aligned}$$

In the first line, we have substituted $z_2 \mapsto z := z_2 - z_1$, approximated $\varphi^\varepsilon(z_1+z) \approx \varphi^\varepsilon(z_1)$ for $z \in \operatorname{supp}(w_\mu - U_{\widetilde{\beta}})$ and used the estimate $\|\varphi^\varepsilon\|^2_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon^{-2}$ (Lemma 4.5). Further, we have decomposed $f_{\widetilde{\beta}} = 1 - g_{\widetilde{\beta}}$ and used that $g_{\widetilde{\beta}}$ is decreasing in $|z|$, $g_{\widetilde{\beta}}(\mu) \sim a$ and $g_{\widetilde{\beta}} \approx 0$ on $\operatorname{supp} U_{\widetilde{\beta}}$. Note that by (22), this difference between the potential energies equals exactly the part of the kinetic energy $\langle\!\langle \psi_{\mathrm{cor}}, (-\Delta_1)\psi_{\mathrm{cor}}\rangle\!\rangle$ that is due to the correlations.

As a consequence of (37), [4, Proposition 3.5] does not immediately provide a bound of $|\gamma^<(t)|$ in terms of $\alpha^<_\xi(t)$. However, the energy difference enters merely in the single term in this proposition[6] whose control requires a bound of the kinetic energy $\|\partial_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|$. For interactions w_β scaling with $\beta \in (0,1)$, one shows that (neglecting some terms that vanish in the limit)

$$\begin{aligned}
|E^\psi_{w_\beta}(t) - \mathcal{E}^\Phi_{w_\beta}(t)| \quad &\gtrsim \quad \left\langle\!\!\left\langle \psi, (-\Delta_1 + \tfrac{1}{\varepsilon^2}(V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2})\psi\right\rangle\!\!\right\rangle - \|\Phi'\|^2 \\
&\gtrsim \quad \|\partial_{x_1} q_1^\Phi \psi\|^2 + (\|\partial_{x_1} p_1^\Phi \psi\|^2 - \|\Phi'\|^2) \\
&\geq \quad \|\partial_{x_1} q_1^\Phi \psi\|^2 - \|\Phi'\|^2 \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle\,. \qquad (38)
\end{aligned}$$

Hence, essentially $\|\partial_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|^2 \lesssim \alpha^<_\xi(t)$ [4, Lemma 4.17], which is why the energy difference enters the estimate of $|\gamma^<(t)|$.

Turning back to the Gross–Pitaevskii regime, let us apply (38) to the interaction $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$. Making use of the fact that $\mathcal{E}^\Phi(t) = \mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$, we obtain

$$\begin{aligned}
|E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - E^\psi(t)| + |E^\psi(t) - \mathcal{E}^\Phi(t)| \quad &\geq \quad |E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - \mathcal{E}^\Phi(t)| \\
&\gtrsim \quad \|\partial_{x_1} q_1^\Phi \psi\|^2 - \|\Phi'\|^2 \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle\,.
\end{aligned}$$

Since $|E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - E^\psi(t)| \sim \mathcal{O}(1)$ already at time zero by (37) and $|E^\psi(t) - \mathcal{E}^\Phi(t)| \leq \alpha^<_\xi(t)$, we expect

$$\|\partial_{x_1} q_1^\Phi \psi\|^2 \lesssim \alpha^<_\xi(t) + \mathcal{O}(1)$$

for the Gross–Pitaevskii scaling of the interaction. The additional $\mathcal{O}(1)$-contribution arises because one of the terms[7] we have neglegted in (38) is not small for $\beta = 1$.

[6] It enters in (24) in [4], which is a part of $\gamma_b^{(3)}$ in Proposition 3.4. The estimate is given in [4, Section 4.4.4].

[7] This is the term $\left\langle\!\!\left\langle \psi, ((N-1)w_\mu^{(12)} - b|\Phi(x_1)|^2)\psi\right\rangle\!\!\right\rangle$. In the proof of Lemma 4.12, we cope with this term essentially by adding and subtracting the potential $U_{\widetilde{\beta}}$. The term containing the difference $w_\mu - U_{\widetilde{\beta}}$ together with the part of the kinetic energy close around the scattering centers is non-negative (Lemma 4.9d). The terms containing $U_{\widetilde{\beta}}$ can be shown to vanish in the limit as in (38).

The part of the kinetic energy orthogonal to the condensate $\|\partial_{x_1} q_1^\Phi \psi\|$ is not small since the microscopic structure does not vanish in the limit but carries a kinetic energy of order $\mathcal{O}(1)$. This energy is the reason for the factor $8\pi a$ in the effective equation, which is $\mathcal{O}(1)$ different from the factor $\|w\|_{L^1(\mathbb{R}^3)}$ for scalings $\beta \in (0,1)$ with negligible microscopic structure.

To estimate the one problematic term in $\gamma^<(t)$, one notes that the predominant part of the kinetic energy is localised around the scattering centers, where the microscopic structure is non-trivial. Therefore, we define the set $\mathcal{A}_1$ (Definition 4.1) as $\mathbb{R}^{3N}$ where sufficiently large balls around the scattering centers are cut out, and show that $\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\|^2 \lesssim |E^\psi(t) - \mathcal{E}^\Phi(t)| + \langle\!\langle \psi^{N,\varepsilon}(t), \widehat{n}\psi^{N,\varepsilon}(t)\rangle\!\rangle$ plus some terms vanishing in the limit (Lemma 4.12). Subsequently, we adapt the estimate from [4, Proposition 3.5] to this new energy lemma, making use of the fact that the complement of $\mathcal{A}_1$ is very small.

The remainder of the proof consists of estimating the terms γ_a to γ_f arising from the effective replacement of w_μ by $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$. The key tool for this is our knowledge of the microscopic structure (Lemma 4.9 and Lemma 4.10).

Remark 3. In principle, we adjust the method from [31] to the situation with strong confinement and to the associated more singular scaling of the interaction. We give a new proof for Lemma 4.9a-c (concerning the microscopic structure) by exploiting the spherical symmetry of the scattering problem to reduce it to an ODE and explicitly construct its solution.

The proof of Lemma 4.12 (providing an estimate for the kinetic energy) becomes more involved due to the confinement, since one must show that the positive expression $\|\nabla_{y_1}\psi^{N,\varepsilon}(t)\|^2$ compensates not only for a sufficient share of the negative part of $\langle\!\langle \psi^{N,\varepsilon}(t), (w_\mu - U_{\widetilde{\beta}})\psi^{N,\varepsilon}(t)\rangle\!\rangle$ as in [31] but also for the large negative part of the expectation value $\frac{1}{\varepsilon^2}\left\langle\!\left\langle \psi^{N,\varepsilon}(t), (V^\perp(\frac{y_1}{\varepsilon}) - E_0)\psi^{N,\varepsilon}(t)\right\rangle\!\right\rangle$.

For the control of γ^d, we follow [16]. The estimate of γ_c is different from the problem without confinement because each ∇ contributes a factor ε^{-1}. To handle this, we prove a new Lemma 4.11 which provides estimates for $\nabla g_{\widetilde{\beta}}$, and combine this with the new estimate in Lemma 4.10e.

The last proposition ensures that the correction term converges to zero as $(N,\varepsilon) \to (\infty, 0)$, which is required for the Grönwall argument.

Proposition 3.4. *Under assumptions A1 – A4, the correction term in $\alpha_\xi(t)$ is for all $t \in \mathbb{R}$ bounded as*

$$\left| N(N-1)\Re \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\, \widehat{r}\, \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle \right| \lesssim \varepsilon^{1+\widetilde{\beta}} N^{\xi - \frac{\widetilde{\beta}}{2}}.$$

Proof of Theorem 1. From Propositions 3.2 and 3.3, we gather that for sufficiently small μ, there exist suitable $\widetilde{\beta}$, ξ and d such that

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)\right| \lesssim C(t)\left(\alpha_\xi^<(t) + (N\varepsilon^\delta)^{1-\widetilde{\beta}+\xi} + N^{-1+\widetilde{\beta}+\xi} + \mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}}\right)$$

for almost every $t \in \mathbb{R}$. We have simplified the expression by noting that $\varepsilon^{1+\widetilde{\beta}} < \varepsilon < (N\varepsilon^\delta)^{1+\xi-\widetilde{\beta}}$ because $\delta(1+\xi-\widetilde{\beta}) < \delta(1+\xi) < 1$ as $\delta < \frac{2}{5}$ and $\xi < 1 - \widetilde{\beta} < \frac{1}{6}$. Besides, we have used the abbreviation

$$C(t) \;:=\; \mathfrak{e}(t)\exp\left\{\mathfrak{e}^2(t)+\int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}. \tag{39}$$

Recall that $\mathfrak{e}(t)$ is for each $t\in\mathbb{R}$ bounded uniformly in N and ε by assumption A4. Let us introduce the abbreviations

$$\begin{aligned} R(t) \;&:=\; -N(N-1)\Re\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g_{\widetilde{\beta}}^{(12)}\,\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \\ B \;&:=\; (N\varepsilon^\delta)^{1-\widetilde{\beta}+\xi}+N^{-1+\widetilde{\beta}+\xi}+\mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}}. \end{aligned}$$

By Proposition 3.4, $|R(t)|<B$ uniformly in t. $\alpha_\xi(t)+B$ is thus non-negative and

$$\begin{aligned} \alpha_\xi^<(t) \;&=\; \alpha_\xi(t)-R(t) \;\le\; \alpha_\xi(t)+|R(t)| \;\lesssim\; \alpha_\xi(t)+B, \\ \alpha_\xi(t)+B \;&=\; \alpha_\xi^<(t)+R(t)+B \;\lesssim\; \alpha_\xi^<(t)+B, \end{aligned}$$

hence

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\alpha_\xi(t)+B) \;\lesssim\; C(t)\,(\alpha_\xi(t)+B)$$

for almost every $t\in\mathbb{R}$. By the differential form of Grönwall's inequality,

$$0 \;\le\; \alpha_\xi^<(t)\lesssim\alpha_\xi(t)+B\lesssim\left(\alpha_\xi^<(0)+B\right)\exp\left\{2\int_0^t C(s)\,\mathrm{d}s\right\}$$

for all $t\in\mathbb{R}$. The sequence (N,ε) is admissible and $\xi<1-\widetilde{\beta}$, hence

$$\lim_{(N,\varepsilon)\to(\infty,0)} B=0$$

and (13) and (14) imply by Lemma 3.1 that

$$0 \;\le\; \lim_{(N,\varepsilon)\to(\infty,0)}(\alpha_\xi(0)+B) \;\lesssim\; \lim_{(N,\varepsilon)\to(\infty,0)}\left(\alpha_\xi^<(0)+B\right) \overset{3.1}{=} 0,$$

which by Lemma 3.1 concludes the proof. □

Corollary 3.5. *Let $t\in\mathbb{R}$. Then for any $\rho\in(0,\frac{1}{12})$,*

$$\begin{aligned} &\mathrm{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)}-|\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| \\ &\qquad\lesssim\; \left(A(0)+N^{-\frac{1}{12}+\rho}+\left(N\varepsilon^\delta\right)^{\frac{3}{12}-3\rho}\right)^{\frac{1}{2}}\exp\left\{\int_0^t C(s)\,\mathrm{d}s\right\}, \end{aligned}$$

with $C(t)$ as in (39) and where

$$A(0) \;:=\; \left|E^{\psi_0^{N,\varepsilon}}(0)-\mathcal{E}^{\Phi_0}(0)\right|+\sqrt{\mathrm{Tr}\left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}}-|\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|\right|}.$$

Proof. Follows from Lemma 3.1 after optimisation over ξ, $\widetilde{\beta}$ and d. □

Remark 4. For $V^{\parallel}=0$, one obtains $\|\Phi(t)\|_{H^2(\mathbb{R})}\lesssim C(\|\Phi_0\|_{H^2(\mathbb{R})})$ uniformly in t, where $C(\|\Phi_0\|_{H^2(\mathbb{R})})$ denotes some expression depending only on $\|\Phi_0\|_{H^2(\mathbb{R})}$ [33, Exercise 3.36][8]. Defining $\widetilde{\mathfrak{e}}:=1+|E^{\psi_0^{N,\varepsilon}}(0)|+|\mathcal{E}^{\Phi_0}(0)|+(C(\|\Phi_0\|_{H^2(\mathbb{R})}))^2$ in analogy to (11), we obtain the rate

$$\operatorname{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)}-|\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right|\lesssim\left(A(0)+N^{-\frac{1}{12}+\rho}+\left(N\varepsilon^\delta\right)^{\frac{3}{12}-3\rho}\right)^{\frac12}\exp\{\widetilde{\mathfrak{e}}\,t\},$$

where the growth in time is exponential instead of doubly exponential.

4 Proofs of the propositions

4.1 Preliminaries

In this section, we collect some useful lemmata, which are for the most part taken from [4] and we refer to this work for the proofs. Lemma 4.7 contains additional statements following [31, Proposition A.2]. We will from now on always assume that assumptions A1 – A4 are satisfied.

Lemma 4.1. *Let $f:\mathbb{N}_0\to\mathbb{R}_0^+$, $d\in\mathbb{Z}$, $\rho\in\{a,b\}$ and $\nu\in\{c,d,e,f\}$. Then*

(a) $\|\widehat{f}\|_{\text{op}}=\|\widehat{f}_d\|_{\text{op}}=\|\widehat{f}^{\frac12}\|^2_{\text{op}}=\sup\limits_{0\leq k\leq N}f(k)$,

(b) $\|\widehat{m}^\rho\|_{\text{op}}\leq N^{-1+\xi}$, $\|\widehat{m}^\nu\|_{\text{op}}\lesssim N^{-2+3\xi}$ *and* $\|\widehat{r}\|_{\text{op}}\lesssim N^{-1+\xi}$,

(c) $\|\widehat{m}^\rho q_1\psi^{N,\varepsilon}(t)\|\lesssim N^{-1}$,

(d) $\|\widehat{f}q_1q_2\psi^{N,\varepsilon}(t)\|^2\lesssim\|\widehat{f}\,\widehat{n}^2\psi^{N,\varepsilon}(t)\|^2$.

Proof. Assertions (a), (c) and (d) are proven in [4], Lemma 4.1 and 4.4. For part (b), note that

$$m'(k)=\begin{cases}\frac{1}{2\sqrt{kN}} & \text{for } k\geq N^{1-2\xi},\\ \frac12 N^{-1+\xi} & \text{else}\end{cases}\quad\text{and}\quad m''(k)=\begin{cases}-\frac{1}{4\sqrt{k^3N}} & \text{for } k\geq N^{1-2\xi},\\ 0 & \text{else,}\end{cases}$$

where $'\equiv\frac{\mathrm{d}}{\mathrm{d}k}$. Hence $|m'(k)|\leq\frac12N^{-1+\xi}$ and $|m''(k)|\leq\frac14N^{-2+3\xi}$ for any $k\geq0$. By the mean value theorem, this implies e.g. $|m^a(k)|\lesssim N^{-1+\xi}$ and $|m^c(k)|\lesssim N^{-2+3\xi}$. The other expressions work analogously. □

Lemma 4.2. *Let $f,g:\mathbb{N}_0\to\mathbb{R}_0^+$ be any weights and $i,j\in\{1,\dots,N\}$.*

(a) For $k\in\{0,\dots,N\}$,

$$\widehat{f}\,\widehat{g}=\widehat{fg}=\widehat{g}\widehat{f},\qquad\widehat{f}p_j=p_j\widehat{f},\qquad\widehat{f}q_j=q_j\widehat{f},\qquad\widehat{f}P_k=P_k\widehat{f}.$$

[8] To prove this, one observes that the quantity $E_2(\Phi):=\int_{\mathbb{R}}\left(|\partial_x^2\Phi|^2+c_1|\partial_x\Phi|^2|\Phi|^2+c_2\Re((\overline{\Phi}\partial_x\Phi)^2)+c_3|\Phi|^6\right)\mathrm{d}x$ is conserved for solutions of (6) with $V^{\parallel}=0$, where c_1, c_2 and c_3 denote some absolute constants.

(b) *Define* $Q_0 := p_j$, $Q_1 := q_j$, $\widetilde{Q}_0 := p_ip_j$, $\widetilde{Q}_1 \in \{p_iq_j, q_ip_j\}$ *and* $\widetilde{Q}_2 := q_iq_j$. *Let* S_j *be an operator acting only on factor* j *in the tensor product and* T_{ij} *acting only on* i *and* j. *Then for* $\mu, \nu \in \{0,1,2\}$

$$Q_\mu \widehat{f} S_j Q_\nu = Q_\mu S_j \widehat{f}_{\mu-\nu} Q_\nu \quad \text{and} \quad \widetilde{Q}_\mu \widehat{f} T_{ij} \widetilde{Q}_\nu = \widetilde{Q}_\mu T_{ij} \widehat{f}_{\mu-\nu} \widetilde{Q}_\nu.$$

(c)

$$[T_{ij}, \widehat{f}] = [T_{ij}, p_ip_j(\widehat{f} - \widehat{f}_2) + (p_iq_j + q_ip_j)(\widehat{f} - \widehat{f}_1)].$$

Proof. [4], Lemma 4.2. □

Lemma 4.3. *Let* $f : \mathbb{N}_0 \to \mathbb{R}_0^+$. *Then*

(a) $P_k,\ \widehat{f} \in \mathcal{C}^1\big(\mathbb{R}, \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)\big)$ *for* $0 \le k \le N$,

(b) $\left[-\Delta_{y_j} + \frac{1}{\varepsilon^2} V^\perp(\frac{y_j}{\varepsilon}), \widehat{f}\right] = 0$ *for* $1 \le j \le N$,

(c) $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{f} = i\Big[\widehat{f}, \sum\limits_{j=1}^N h_j(t)\Big]$,
where $h_j(t)$ *denotes the one-particle operator corresponding to* $h(t)$ *from* (6) *acting on the* j^{th} *factor in* $L^2(\mathbb{R}^{3N})$.

Proof. [4], Lemma 4.3. □

Lemma 4.4. *Let* $\Gamma, \Lambda \in L^2(\mathbb{R}^{3N})$ *be symmetric in the coordinates* $\{z_2, ..., z_N\}$, *let* r_2 *and* s_2 *denote operators acting only on the second factor of the tensor product, and let* $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^d$ *for* $d \in \mathbb{N}$. *Then*

$$|\langle\!\langle \Gamma, r_2 F(z_1,z_2) s_2 \Lambda \rangle\!\rangle| \le \|\Gamma\| \left(\|s_2 F(z_1,z_2) r_2 \Lambda\|^2 + \tfrac{1}{N-1} \|r_2 F(z_1,z_2) s_2 \Lambda\|^2 \right)^{\frac{1}{2}}.$$

Proof. [4], Lemma 4.7. □

Lemma 4.5. *The nonlinear equation* (6) *is well-posed and* $H^2(\mathbb{R})$ *solutions exist globally, i.e. for any initial datum* $\Phi_0 \in H^2(\mathbb{R})$, *it follows that* $\Phi(t) \in H^2(\mathbb{R})$ *for any* $t \in \mathbb{R}$. *Besides, for sufficiently small* ε,

(a) $\|\Phi(t)\|_{L^2(\mathbb{R})} = 1, \qquad \|\Phi(t)\|_{H^1(\mathbb{R})} \le \mathfrak{e}(t), \qquad \|\Phi(t)\|_{L^\infty(\mathbb{R})} \le \mathfrak{e}(t),$
$\|\Phi'\|_{L^\infty(\mathbb{R})} \le \|\Phi(t)\|_{H^2(\mathbb{R})} \lesssim \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\},$

(b) $\|\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mathfrak{e}(t)\varepsilon^{-1}, \quad \|\nabla\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mathfrak{e}(t)\varepsilon^{-2}.$

Proof. [4], Lemma 4.8. □

Lemma 4.6. *Let* $t \in \mathbb{R}$ *be fixed and let* $j, k \in \{1, ..., N\}$. *Let* $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *and* $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be measurable functions such that* $|g(z_j, z_k)| \le G(z_k - z_j)$ *and* $|h(x_j, x_k)| \le H(x_k - x_j)$ *almost everywhere for some* $G : \mathbb{R}^3 \to \mathbb{R}$, $H : \mathbb{R} \to \mathbb{R}$. *Then*

(a) $\|p_j g(z_j, z_k) p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}^2(t)\varepsilon^{-2}\|G\|_{L^1(\mathbb{R}^3)}$ *for* $G \in L^1(\mathbb{R}^3)$,

(b) $\|g(z_j, z_k) p_j\|_{\mathrm{op}} = \|p_j g(z_j, z_k)\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-1}\|G\|_{L^2(\mathbb{R}^3)}$ *for* $G \in L^2 \cap L^\infty(\mathbb{R}^3)$,

(c) $\|g(z_j,z_k)\nabla_j p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-2}\|G\|_{L^2(\mathbb{R}^3)}$ *for* $G \in L^2(\mathbb{R}^3)$,

(d) $\|h(x_j,x_k)p_j^\Phi\|_{\mathrm{op}} = \|p_j^\Phi h(x_j,x_k)\|_{\mathrm{op}} \leq \mathfrak{e}(t)\|H\|_{L^2(\mathbb{R})}$ *for* $H \in L^2 \cap L^\infty(\mathbb{R})$.

Proof. [4], Lemma 4.9. □

Lemma 4.7. *Let* ε *be sufficiently small and* $t \in \mathbb{R}$ *be fixed. Then*

(a) $\|\partial_{x_1} p_1^\Phi\|_{\mathrm{op}} \leq \mathfrak{e}(t)$, $\quad \|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\mathrm{op}} \lesssim \varepsilon^{-1}$, $\quad \|\partial_{x_1}^2 p_1\|_{\mathrm{op}} \leq \|\Phi(t)\|_{H^2(\mathbb{R})}$,

$\|q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}(t)\varepsilon$, $\quad \|\partial_{x_1} q_1^\Phi \psi\| \lesssim \mathfrak{e}(t)$, $\quad \|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)$,

$\|\partial_{x_1}\psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}(t)$, $\quad \|\nabla_{y_1}\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1}$, $\quad \|\nabla_1\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1}$,

(b) $\left\|\sqrt{w_\mu^{(12)}}\psi^{N,\varepsilon}(t)\right\| \lesssim \mathfrak{e}(t)N^{-\frac{1}{2}}$,

(c) $\|w_\mu^{(12)}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)N^{\frac{1}{2}}\varepsilon^{-2}$,

(d) $\|p_1\mathbb{1}_{\operatorname{supp} w_\mu}(z_1-z_2)\|_{\mathrm{op}} = \|\mathbb{1}_{\operatorname{supp} w_\mu}(z_1-z_2)p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)N^{-\frac{3}{2}}\varepsilon^2$,

(e) $\|p_1 w_\mu^{(12)}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}^2(t)N^{-1}$,

(f) $\|\big(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\big)\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}^3(t)\varepsilon$.

Proof. Part (a) is proven in [4, Lemma 4.10.]. $\frac{E_0}{\varepsilon^2}$ is the smallest eigenvalue of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$, hence $\left\langle\!\left\langle \psi^{N,\varepsilon}(t), (-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2})\psi^{N,\varepsilon}(t)\right\rangle\!\right\rangle \geq 0$. This implies (b) as

$$
\begin{aligned}
\mathfrak{e}^2(t) \;\geq\; |E^{\psi^{N,\varepsilon}(t)}(t)| \;&\geq\; \tfrac{N-1}{2}\left\|\sqrt{w_\mu^{(12)}}\psi^{N,\varepsilon}(t)\right\|^2 - \|V^\parallel(t)\|_{L^\infty(\mathbb{R}^3)} \\
&\gtrsim\; N\|\sqrt{w_\mu^{(12)}}\psi^{N,\varepsilon}(t)\|^2 - \mathfrak{e}^2(t).
\end{aligned}
$$

For part (c), observe that

$$
\|w_\mu^{(12)}\psi^{N,\varepsilon}(t)\| \leq \|w_\mu\|_{L^\infty(\mathbb{R}^3)}^{\frac{1}{2}}\left\|\sqrt{w_\mu^{(12)}}\psi^{N,\varepsilon}(t)\right\| \lesssim \mu^{-1}\mathfrak{e}(t)N^{-\frac{1}{2}}.
$$

Assertion (d) follows from Lemma 4.6b because $\|\mathbb{1}_{\operatorname{supp} w_\mu}\|^2_{L^2(\mathbb{R}^3)} \lesssim \mu^3$. Part (e) is a consequence of

$$
\begin{aligned}
\|p_1 w_\mu^{(12)}\psi^{N,\varepsilon}(t)\| &= \|p_1\mathbb{1}_{\operatorname{supp} w_\mu}(z_1-z_2)w_\mu^{(12)}\psi^{N,\varepsilon}(t)\| \\
&\leq \|p_1\mathbb{1}_{\operatorname{supp} w_\mu}(z_1-z_2)\|_{\mathrm{op}}\|w_\mu^{(12)}\psi^{N,\varepsilon}(t)\|\,.
\end{aligned}
$$

Finally, (f) is proven in [4, Lemma 4.11]. □

Lemma 4.8. *Let* $\psi \in L^2_+(\mathbb{R}^{3N})$ *be normalised and* $f \in L^\infty(\mathbb{R})$. *Then*

$$
\Big|\langle\!\langle \psi, f(x_1)\psi\rangle\!\rangle - \langle \Phi(t), f\Phi(t)\rangle_{L^2(\mathbb{R})}\Big| \lesssim \|f\|_{L^\infty(\mathbb{R})}\,\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle\,.
$$

Proof. [4], Lemma 4.6. □

4.2 Microscopic structure

In this section, we prove some important properties of the solution $f_{\widetilde{\beta}}$ of the zero-energy scattering equation (22) and of its complement $g_{\widetilde{\beta}}$.

Lemma 4.9. *Let $f_{\widetilde{\beta}}$ as in Definition 3.5, j_μ as in (18) and $R_{\widetilde{\beta}}$ as in Definition 3.4. Then*

(a) $f_{\widetilde{\beta}}$ is a non-negative, non-decreasing function of $|z|$,

(b) $f_{\widetilde{\beta}}(z) \geq j_\mu(z)$ for all $z \in \mathbb{R}^3$ and there exists $\kappa_{\widetilde{\beta}} \in \big(1, \frac{\mu^{\widetilde{\beta}}}{\mu^{\widetilde{\beta}}-\mu a}\big)$ such that for $|z| \leq \mu^{\widetilde{\beta}}$, $f_{\widetilde{\beta}}(z) = \kappa_{\widetilde{\beta}} j_\mu(z)$,

(c) $R_{\widetilde{\beta}} \lesssim \mu^{\widetilde{\beta}}$.

(d) $\|\mathbb{1}_{|z_1-z_2|<R_{\widetilde{\beta}}} \nabla_1 \psi\|^2 + \frac{1}{2} \left\langle\!\!\left\langle \psi, (w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)})\psi \right\rangle\!\!\right\rangle \geq 0$ for any $\psi \in \mathcal{D}(\nabla_1)$.

Proof. We prove this Lemma by explicitly constructing a spherically symmetric, continuously differentiable solution $f_{\widetilde{\beta}}$ of (22). This solution is unique by [12, Chapter 2.2, Theorem 16]. Consider $\widetilde{f} : \mathbb{R}_0^+ \to \mathbb{R}$ with

$$\widetilde{f}(r) := r f_{\widetilde{\beta}}(r), \tag{40}$$

where $r := |z|$. $f_{\widetilde{\beta}} \in \mathcal{C}^1(\mathbb{R}^3)$ solves (22) precisely if $\widetilde{f}$ solves the corresponding ODE

$$\begin{cases} \widetilde{f}''(r) = \frac{1}{2}\left(w_\mu(r) - U_{\widetilde{\beta}}(r)\right)\widetilde{f}(r) & \text{for } 0 < r < R_{\widetilde{\beta}}, \\ \widetilde{f}(r) = r & \text{for } r \geq R_{\widetilde{\beta}}, \\ \widetilde{f}(r) = 0 & \text{for } r = 0, \end{cases} \tag{41}$$

where $' \equiv \frac{\mathrm{d}}{\mathrm{d}r}$. Analogously, (18) is equivalent to

$$\begin{cases} \widetilde{j}''(r) = \frac{1}{2} w_\mu(r)\widetilde{j}(r) & \text{for } 0 < r < \mu, \\ \widetilde{j}(r) = r - \mu a & \text{for } r \geq \mu, \\ \widetilde{j}(r) = 0 & \text{for } r = 0, \end{cases} \tag{42}$$

where $\widetilde{j} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is defined as $\widetilde{j}(r) := r j_\mu(r)$ and depicted in Figure 1.

For $0 \leq r \leq \mu^{\widetilde{\beta}}$, $\widetilde{f}''(r) = \frac{1}{2} w_\mu(r)\widetilde{f}(r)$ and $\widetilde{f}(0) = 0$. Clearly, both conditions are fulfilled by the choice $\widetilde{f}_\kappa(r) = \kappa\,\widetilde{j}(r)$ for some $\kappa \geq 1$. Consequently,

$$\widetilde{f}_\kappa(\mu^{\widetilde{\beta}}) = \kappa(\mu^{\widetilde{\beta}} - \mu a) \quad \text{and} \quad \widetilde{f}_\kappa'(\mu^{\widetilde{\beta}}) = \kappa. \tag{43}$$

For $\mu^{\widetilde{\beta}} < r < R_{\widetilde{\beta}}$, $\widetilde{f}_\kappa$ solves $\widetilde{f}_\kappa''(r) = -\frac{1}{2} U_{\widetilde{\beta}}(r)\widetilde{f}_\kappa(r)$ and is subject to the boundary conditions (43). As $U_{\widetilde{\beta}}$ is constant over this region, the solution for $\mu^{\widetilde{\beta}} < r < R_{\widetilde{\beta}}$ is

$$\widetilde{f}_\kappa(r) = \kappa\Big[A\sin(ur) + B\cos(ur)\Big],$$

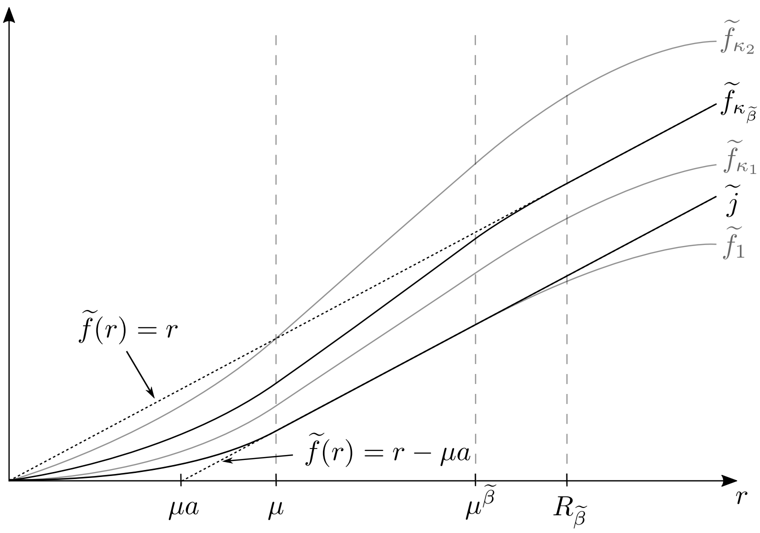

Figure 1: Construction of the solution $\widetilde{f}_{\kappa_{\widetilde{\beta}}}$ of (41). The lower black curve represents the solution $\widetilde{j}$ of (42), the dashed graphs mark the straight lines r and $r-\mu a$. The functions $\widetilde{f}_1$ and $\widetilde{f}_{\kappa_2}$ drawn in grey are exemplary members of the one-parameter family $\{\widetilde{f}_\kappa\}_{\kappa\geq 1}$ with $1 < \kappa_1 < \kappa_2$. For $0 < r < \mu^{\widetilde{\beta}}$, $\widetilde{f}_\kappa(r) = \kappa\widetilde{j}(r)$ is a multiple of $\widetilde{j}(r)$. This implies in particular that $\widetilde{f}_\kappa(r)$ is a straight line with slope κ for $\mu < r < \mu^{\widetilde{\beta}}$. In the region $r > \mu^{\widetilde{\beta}}$, $\widetilde{f}_\kappa$ is concave. The solution to (41) must become tangential to the straight line r at some point $r > \mu^{\widetilde{\beta}}$, which will be called $R_{\widetilde{\beta}}$. It is clear that $\widetilde{f}_1$ and $\widetilde{f}_{\kappa_1}$ will not touch the straight line r (at least not before they decrease and increase again). Contrarily, $\widetilde{f}_{\kappa_2}$ already intersects r at μ and is therefore ruled out as well. As the family is strictly increasing in κ, there must be a curve in between $\widetilde{f}_1$ and $\widetilde{f}_{\kappa_2}$ that is tangential to r at some point. This is the solution $\widetilde{f}_{\kappa_{\widetilde{\beta}}}$ of (41), drawn in black.

where $u := \sqrt{\frac{1}{2}a\mu^{1-3\widetilde{\beta}}}$ and

$$\begin{aligned} A &:= \left((\mu^{\widetilde{\beta}} - \mu a)\sin(\mu^{\widetilde{\beta}}u) + u^{-1}\cos(\mu^{\widetilde{\beta}}u)\right), \\ B &:= \left((\mu^{\widetilde{\beta}} - \mu a)\cos(\mu^{\widetilde{\beta}}u) - u^{-1}\sin(\mu^{\widetilde{\beta}}u)\right), \end{aligned}$$

i.e. A and B depend on the quantities μ, a and $\mu^{\widetilde{\beta}}$ but are independent of κ. The two parameters κ and $R_{\widetilde{\beta}}$ must be chosen such that

$$\widetilde{f}_\kappa(R_{\widetilde{\beta}}) = R_{\widetilde{\beta}} \quad \text{and} \quad \widetilde{f}'_\kappa(R_{\widetilde{\beta}}) = 1. \tag{44}$$

Denote the position of the first maximum of $\widetilde{f}_\kappa$ by $r_{\max}$. Clearly, $r_{\max}$ is independent of κ. $R_{\widetilde{\beta}}$ is defined as the minimal value where the scattering length of $w_\mu - U_{\widetilde{\beta}}$ equals zero. This means

$$R_{\widetilde{\beta}} := \min\{r \in (\mu^{\widetilde{\beta}}, r_{\max}] : \widetilde{f}_\kappa(r) = r \;\text{ and }\; \widetilde{f}'_\kappa(r) = 1\},$$

i.e. $R_{\widetilde{\beta}}$ is defined as the first value of r where $\widetilde{f}_\kappa$ is tangential to the straight line $\widetilde{f}(r) = r$. This implies in particular that $\widetilde{f}_\kappa$ is increasing. Clearly, $R_{\widetilde{\beta}}$ depends on κ, hence it remains to prove that suitable κ, $R_{\widetilde{\beta}}$ exist. To this end, consider the one-parameter family $\{\widetilde{f}_\kappa\}_{\kappa\geq 1}$.

- For $\kappa = 1$, we have $\widetilde{f}_1(r) = \widetilde{j}(r) \leq \widetilde{j}(\mu^{\widetilde{\beta}}) = \mu^{\widetilde{\beta}} - \mu a$ for $r \leq \mu^{\widetilde{\beta}}$. As $\widetilde{f}_1$ is concave for $\mu^{\widetilde{\beta}} < r < R_{\widetilde{\beta}}$, this implies $\widetilde{f}_1(r) < r$ for all $r \in (\mu^{\widetilde{\beta}}, r_{\max}]$. Consequently, the choice $\kappa = 1$ cannot be a solution of (41).
- On the other hand, $\kappa = \frac{\mu^{\widetilde{\beta}}}{\mu^{\widetilde{\beta}}-\mu a} > 1$ can neither yield a solution because in this case, $\widetilde{f}_\kappa(\mu^{\widetilde{\beta}}) = \mu^{\widetilde{\beta}}$ and $\widetilde{f}'_\kappa(\mu^{\widetilde{\beta}}) > 0$, hence $\widetilde{f}_\kappa > r$ for all $r \in (\mu^{\widetilde{\beta}}, r_{\max}]$.
- Since $\widetilde{f}_\kappa(r) = \kappa \widetilde{f}_1(r)$, the one-parameter family is strictly increasing in κ. Together with $\widetilde{f}_\kappa(r) < r$ for $\kappa = 1$ and $\widetilde{f}_\kappa(r) > r$ for $\kappa \geq \frac{\mu^{\widetilde{\beta}}}{\mu^{\widetilde{\beta}}-\mu a}$, this implies that there must be a unique $\kappa_{\widetilde{\beta}} \in (1, \frac{\mu^{\widetilde{\beta}}}{\mu^{\widetilde{\beta}}-\mu a})$ such that $\widetilde{f}_{\kappa_{\widetilde{\beta}}}$ satisfies (44).

To obtain an upper bound for $R_{\widetilde{\beta}}$, recall that $\widetilde{f}_{\kappa_{\widetilde{\beta}}}$ is increasing and, by construction, $\mathcal{C}^2$ in $[\mu^{\widetilde{\beta}}, R_{\widetilde{\beta}}]$, hence

$$\begin{aligned} \kappa_{\widetilde{\beta}} - 1 &= \widetilde{f}'_{\kappa_{\widetilde{\beta}}}(\mu^{\widetilde{\beta}}) - \widetilde{f}'_{\kappa_{\widetilde{\beta}}}(R_{\widetilde{\beta}}) = -\int_{\mu^{\widetilde{\beta}}}^{R_{\widetilde{\beta}}} \widetilde{f}''_{\kappa_{\widetilde{\beta}}}(r)\,\mathrm{d}r = \tfrac{1}{2} a \mu^{1-3\widetilde{\beta}} \int_{\mu^{\widetilde{\beta}}}^{R_{\widetilde{\beta}}} \widetilde{f}_{\kappa_{\widetilde{\beta}}}(r)\,\mathrm{d}r \\ &\geq \tfrac{1}{2} a \mu^{1-3\widetilde{\beta}} \widetilde{f}_{\kappa_{\widetilde{\beta}}}(\mu^{\widetilde{\beta}})(R_{\widetilde{\beta}} - \mu^{\widetilde{\beta}}) \gtrsim \kappa_{\widetilde{\beta}} \mu^{1-3\widetilde{\beta}}(\mu^{\widetilde{\beta}} - \mu a)(R_{\widetilde{\beta}} - \mu^{\widetilde{\beta}}). \end{aligned}$$

With $\frac{\kappa_{\widetilde{\beta}}-1}{\kappa_{\widetilde{\beta}}} < \frac{\mu a}{\mu^{\widetilde{\beta}}} \lesssim \mu^{1-\widetilde{\beta}}$, this yields

$$R_{\widetilde{\beta}} - \mu^{\widetilde{\beta}} \lesssim \frac{\kappa_{\widetilde{\beta}} - 1}{\kappa_{\widetilde{\beta}}(\mu^{\widetilde{\beta}} - \mu a)} \mu^{-1+3\widetilde{\beta}} \lesssim \frac{\mu^{2\widetilde{\beta}}}{\mu^{\widetilde{\beta}} - \mu a} = \frac{\mu^{\widetilde{\beta}}}{1 - \frac{\mu a}{\mu^{\widetilde{\beta}}}} \lesssim \mu^{\widetilde{\beta}}$$

for sufficiently small μ. Due to the respective properties of $\widetilde{f}_{\kappa_{\widetilde{\beta}}}$, it is immediately clear that $f_{\widetilde{\beta}}$ is non-negative, that $f_{\widetilde{\beta}} \geq j_\mu$ and that $f_{\widetilde{\beta}}(z) = \kappa_{\widetilde{\beta}} j_\mu(z)$ for $|z| \leq \mu^{\widetilde{\beta}}$. To see that $f_{\widetilde{\beta}}$ is non-decreasing, observe that for $\mu^{\widetilde{\beta}} \leq r \leq R_{\widetilde{\beta}}$, $\widetilde{f}'_{\kappa_{\widetilde{\beta}}}(R_{\widetilde{\beta}}) \leq \widetilde{f}'_{\kappa_{\widetilde{\beta}}}(r)$ as $\widetilde{f}_{\kappa_{\widetilde{\beta}}}$ is concave, hence

$$1 = \widetilde{f}'_{\kappa_{\widetilde{\beta}}}(R_{\widetilde{\beta}}) \leq \widetilde{f}'_{\kappa_{\widetilde{\beta}}}(r) = r(f_{\widetilde{\beta}})'(r) + f_{\widetilde{\beta}}(r) \leq r(f_{\widetilde{\beta}})'(r) + 1$$

for $\mu^{\widetilde{\beta}} \leq r \leq R_{\widetilde{\beta}}$ as $f_{\widetilde{\beta}}(r) = r^{-1}\widetilde{f}_{\kappa_{\widetilde{\beta}}}(r) \leq 1$. Thus $(f_{\widetilde{\beta}})'(r) \geq 0$ for all $r \geq 0$.

Finally, for the proof of part (d), we refer to [31, Lemma 5.1(3)] and the analogous two-dimensional statement in [16, Lemma 7.10]. The idea of the proof is the following: one shows first that the one-particle operator $H^{Z_n} := -\Delta + \frac{1}{2}\sum_{z_k \in Z_n}(w_\mu - U_{\widetilde{\beta}})(\cdot - z_k)$ is for each $n \in \mathbb{N}$ a positive operator, where Z_n is an n-elemental subset of $\mathbb{R}^3$ such that $B_{R_{\widetilde{\beta}}}(z_k)$ are pairwise disjoint for any two $z_k \in Z_n$. This first assertion follows from the definition of $f_{\widetilde{\beta}}$ and from the fact that if the ground state energy of H^{Z_n}

was negative, the ground state would be strictly positive. The next step is to prove that the quadratic form $Q(\psi) := \|\mathbb{1}_{|\cdot|\leq R_{\widetilde{\beta}}}\nabla\psi\|^2 + \frac{1}{2}\left\langle\psi, (w_\mu - U_{\widetilde{\beta}})\psi\right\rangle$ for $\psi \in H^1(\mathbb{R}^3)$ is nonnegative. Assuming that there exists a $\widetilde{\psi}$ such that $Q(\widetilde{\psi}) < 0$, one constructs a set Z_n and a function $\chi_R \in H^1(\mathbb{R}^3)$ such that $\langle\chi_R, H^{Z_n}\chi_R\rangle < 0$ for some n, contradicting the positivity of H^{Z_n} which holds for all $n \in \mathbb{N}$. The function χ_R is constructed in such a way that the part of $\langle\chi_R, H^{Z_n}\chi_R\rangle$ inside a ball with radius R containing a sufficiently large neighbourhood of Z_n equals $nQ(\widetilde{\psi}) < 0$. The decay of χ_R outside the ball is chosen such that its positive kinetic energy is not large enough to cancel this negative term for sufficiently large n. □

The next two lemmata provide estimates for expressions containing $g_{\widetilde{\beta}}$ or $\nabla g_{\widetilde{\beta}}$.

Lemma 4.10. *For $g_{\widetilde{\beta}}$ as in Definition 3.5 and sufficiently small ε,*

(a) $|g_{\widetilde{\beta}}(z)| \lesssim \frac{\mu}{|z|}$,

(b) $\|g_{\widetilde{\beta}}\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{2+\widetilde{\beta}}N^{-1-\frac{\widetilde{\beta}}{2}}, \quad \|p_1 g_{\widetilde{\beta}}^{(12)}\|_{\mathrm{op}} = \|g_{\widetilde{\beta}}^{(12)}p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{1+\widetilde{\beta}}N^{-1-\frac{\widetilde{\beta}}{2}}$,

(c) $\|g_{\widetilde{\beta}}^{(12)}\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon N^{-1}$,

(d) $\|p_1\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\|_{\mathrm{op}} = \|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\varepsilon^{-1+3\widetilde{\beta}}N^{-\frac{3}{2}\widetilde{\beta}}$,

(e) $\|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)\varepsilon^{2\widetilde{\beta}-\frac{2}{3}}N^{-\widetilde{\beta}}$.

Proof. By Lemma 4.9b, $f_{\widetilde{\beta}}(z) \geq j_\mu(z)$, hence

$$g_{\widetilde{\beta}}(z) = 1 - f_{\widetilde{\beta}}(z) \leq 1 - j_\mu(z) \leq \frac{\mu a}{|z|}$$

and, since $\operatorname{supp} g_{\widetilde{\beta}} \subseteq \{z \in \mathbb{R}^3 : |z| \leq R_{\widetilde{\beta}} \lesssim \mu^{\widetilde{\beta}}\}$,

$$\|g_{\widetilde{\beta}}\|^2_{L^2(\mathbb{R}^3)} = \int\limits_{|z|\leq R_{\widetilde{\beta}}} |g_{\widetilde{\beta}}(z)|^2\,\mathrm{d}z \lesssim \mu^2 \int\limits_{|z|\lesssim\mu^{\widetilde{\beta}}} \frac{1}{|z|^2}\,\mathrm{d}z \lesssim \mu^{2+\widetilde{\beta}}.$$

The second part of (b) then follows immediately from Lemma 4.6b. For part (c), observe that $\|g_{\widetilde{\beta}}^{(12)}\psi\| \lesssim \mu\|\frac{1}{|z_1-z_2|}\psi\|$ and

$$\begin{aligned}\left\|\tfrac{1}{|z_1-z_2|}\psi\right\|^2 &= \int\limits_{\mathbb{R}^{3(N-1)}} \mathrm{d}z_N\cdots\mathrm{d}z_2 \int\limits_{\mathbb{R}^3} \mathrm{d}z_1\overline{\psi(z_1,...,z_N)}\left(\nabla_1\cdot\tfrac{z_1-z_2}{|z_1-z_2|^2}\right)\psi(z_1,...,z_N)\\ &= -2\Re\left\langle\!\!\left\langle\nabla_1\psi, \tfrac{z_1-z_2}{|z_1-z_2|^2}\psi\right\rangle\!\!\right\rangle \leq 2\|\nabla_1\psi\|\left\|\tfrac{1}{|z_1-z_2|}\psi\right\|.\end{aligned}$$

Consequently,

$$\|g_{\widetilde{\beta}}^{(12)}\psi^{N,\varepsilon}(t)\| \lesssim \mu\|\nabla_1\psi^{N,\varepsilon}(t)\| \overset{4.7a}{\lesssim} \mu\varepsilon^{-1}.$$

The proof of (d) works analogously to the proof of Lemma 4.7d. Finally, using Hölder's inequality with $p = 3$, $q = \frac{3}{2}$ in the $\mathrm{d}z_1$-integration, we obtain for (e)

$$\|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\psi\|^2$$

$$\begin{aligned} &= \int \mathrm{d}z_N \cdots \mathrm{d}z_2 \int \mathrm{d}z_1 \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)|\psi(z_1, \dots, z_N)|^2 \\ &\leq \int \mathrm{d}z_N \cdots \mathrm{d}z_2 \left(\int \mathrm{d}z_1 \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\right)^{\frac{2}{3}} \left(\int \mathrm{d}z_1 |\psi(z_1, \dots, z_N)|^6\right)^{\frac{2}{6}} \\ &\lesssim \mu^{2\widetilde{\beta}} \int \mathrm{d}z_N \cdots \mathrm{d}z_2 \left(\int \mathrm{d}z_1 |\psi(z_1, \dots, z_N)|^6\right)^{\frac{2}{6}}. \end{aligned}$$

Substituting $z_1 \mapsto \widetilde{z}_1 = (x_1, \frac{y_1}{\varepsilon})$ and using Sobolev's inequality in the $\mathrm{d}\widetilde{z}_1$-integral, we obtain

$$\begin{aligned} &\left(\int \mathrm{d}z_1 |\psi(z_1, \dots, z_N)|^6\right)^{\frac{2}{6}} \\ &\quad = \left(\varepsilon^2 \int \mathrm{d}\widetilde{z}_1 |\psi((x_1, \varepsilon\widetilde{y}_1), z_2, \dots, z_N)|^6\right)^{\frac{2}{6}} \\ &\quad \lesssim \varepsilon^{\frac{2}{3}} \int \mathrm{d}\widetilde{z}_1 |\nabla_{\widetilde{z}_1} \psi((x_1, \varepsilon\widetilde{y}_1), z_2, \dots, z_N)|^2 \\ &\quad = \varepsilon^{-\frac{4}{3}} \int \mathrm{d}z_1 \left(|\partial_{x_1} \psi(z_1, \dots, z_N)|^2 + \varepsilon^2 |\nabla_{y_1} \psi(z_1, \dots, z_N)|^2\right) \end{aligned}$$

as $\nabla_{\widetilde{z}_1} = (\partial_{x_1}, \varepsilon\nabla_{y_1})$ and $\mathrm{d}z_1 = \varepsilon^2 \, \mathrm{d}\widetilde{z}_1$. Hence by Lemma 4.7a,

$$\begin{aligned} \|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\psi^{N,\varepsilon}(t)\|^2 &\lesssim \mu^{2\widetilde{\beta}} \varepsilon^{-\frac{4}{3}} \left(\|\partial_{x_1} \psi^{N,\varepsilon}(t)\|^2 + \varepsilon^2 \|\nabla_{y_1} \psi^{N,\varepsilon}(t)\|^2\right) \\ &\lesssim \mu^{2\widetilde{\beta}} \varepsilon^{-\frac{4}{3}} \mathfrak{e}^2(t). \end{aligned}$$

□

Lemma 4.11. *For $g_{\widetilde{\beta}}$ as in Definition 3.5, it holds that*

(a) $\|\nabla g_{\widetilde{\beta}}\|_{L^2(\mathbb{R}^3)} \lesssim N^{-\frac{1}{2}} \varepsilon$,

(b) $\|(\nabla_1 g_{\widetilde{\beta}}^{(12)}) p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{-\frac{1}{2}}$,

(c) $\|(\nabla_1 g_{\widetilde{\beta}}^{(12)}) \cdot \nabla_1 p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{-\frac{1}{2}} \varepsilon^{-1}$.

Proof. Denote $r \equiv |z|$ and $' \equiv \frac{\mathrm{d}}{\mathrm{d}r}$. As $g_{\widetilde{\beta}}$ is spherically symmetric, we define $\widetilde{g}(r) := r g_{\widetilde{\beta}}(r)$. Consequently,

$$|\nabla g_{\widetilde{\beta}}(r)| = |g_{\widetilde{\beta}}{}'(r)| = \tfrac{|\widetilde{g}'(r) - g_{\widetilde{\beta}}(r)|}{r} \leq \tfrac{|\widetilde{g}'(r)|}{r} + \tfrac{|g_{\widetilde{\beta}}(r)|}{r},$$

$\widetilde{g}'(r) = 1 - \widetilde{f}'(r)$ and $\widetilde{g}''(r) = -\widetilde{f}''(r)$ with $\widetilde{f}$ from (40). Hence $\widetilde{g}'(R_{\widetilde{\beta}}) = 0$ by (44) and

$$|\widetilde{g}'(r)| = |\widetilde{g}'(r) - \widetilde{g}'(R_{\widetilde{\beta}})| = \left|\int_r^{R_{\widetilde{\beta}}} \widetilde{f}''(\rho) \, \mathrm{d}\rho\right| \leq \tfrac{1}{2} \int_r^{R_{\widetilde{\beta}}} w_\mu(\rho)\rho \, \mathrm{d}\rho + \tfrac{1}{2} \int_r^{R_{\widetilde{\beta}}} U_{\widetilde{\beta}}(\rho)\rho \, \mathrm{d}\rho \qquad (45)$$

by (41) and as $\widetilde{f}(\rho) \leq \rho$. For $0 \leq r \leq \mu$,

$$|\widetilde{g}'(r)| \leq \tfrac{1}{2}\|w_\mu\|_{L^\infty(\mathbb{R}^3)} \int\limits_0^\mu \rho \,\mathrm{d}\rho + \tfrac{1}{2}a\mu^{1-3\widetilde{\beta}} \int\limits_{\mu^{\widetilde{\beta}}}^{R_{\widetilde{\beta}}} \rho \,\mathrm{d}\rho \overset{4.9c}{\lesssim} 1 + \mu^{1-\widetilde{\beta}} \lesssim 1$$

and $|g_{\widetilde{\beta}}(r)| \leq 1$, hence $|g_{\widetilde{\beta}}{}'(r)| \lesssim \frac{1}{r}$. For $\mu \leq r \leq R_{\widetilde{\beta}}$, the first term in (45) equals zero, hence $|g_{\widetilde{\beta}}{}'(r)| \lesssim \frac{\mu^{1-\widetilde{\beta}}}{r} + \frac{\mu}{r^2}$ by Lemma 4.10a. Thus

$$\|\nabla g_{\widetilde{\beta}}\|^2_{L^2(\mathbb{R}^3)} = \int\limits_0^\mu |g_{\widetilde{\beta}}{}'(r)|^2 r^2 \,\mathrm{d}r + \int\limits_\mu^{R_{\widetilde{\beta}}} |g_{\widetilde{\beta}}{}'(r)|^2 r^2 \,\mathrm{d}r \lesssim \mu + \mu^{2-\widetilde{\beta}} \lesssim \mu.$$

The two remaining inequalities follow by Lemma 4.6. □

4.3 Estimate of the kinetic energy

In this section, we provide a bound for the kinetic energy of $q_1^\Phi \psi^{N,\varepsilon}(t)$. The main part of the kinetic energy results from the microscopic structure, which is localised around the scattering centres (on the sets $\overline{\mathcal{C}}_j$ in Definition 4.1 below). We show that the kinetic energy in regions where sufficiently large neighbourhoods around these centres (the sets $\overline{\mathcal{A}}_j \supset \overline{\mathcal{C}}_j$) are cut out is of lower order. To prove this, we will also need the sets $\mathcal{B}_j$, which consist of all N-particle configurations where at most two particles interact (one of which is particle j).

Definition 4.1. Let $d \in (\frac{5}{6}, \widetilde{\beta})$, $j,k \in \{1,...,N\}$ and define

$$\begin{aligned} a_{j,k} &:= \left\{(z_1,...,z_N) : |z_j - z_k| < \mu^d\right\}, \\ c_{j,k} &:= \left\{(z_1,...,z_N) : |z_j - z_k| < R_{\widetilde{\beta}}\right\}, \\ a^x_{j,k} &:= \left\{(z_1,...,z_N) : |x_j - x_k| < \mu^d\right\}. \end{aligned}$$

Then the subsets $\overline{\mathcal{A}}_j$, $\overline{\mathcal{B}}_j$, $\overline{\mathcal{C}}_j$ and $\overline{\mathcal{A}}^x_j$ of $\mathbb{R}^{3N}$ are defined as

$$\overline{\mathcal{A}}_j := \bigcup_{k\neq j} a_{j,k}, \qquad \overline{\mathcal{B}}_j := \bigcup_{k,l\neq j} a_{k,l}, \qquad \overline{\mathcal{C}}_j := \bigcup_{k\neq j} c_{j,k}, \qquad \overline{\mathcal{A}}^x_j := \bigcup_{k\neq j} a^x_{j,k}$$

and their complements are denoted by $\mathcal{A}_j$, $\mathcal{B}_j$, $\mathcal{C}_j$ and $\mathcal{A}^x_j$, i.e. $\mathcal{A}_j := \mathbb{R}^{3N} \setminus \overline{\mathcal{A}}_j$ etc.

Note that the characteristic functions $\mathbb{1}_{\mathcal{B}_1}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$ do not depend on z_1, and $\mathbb{1}_{\overline{\mathcal{A}}^x_1}$ and $\mathbb{1}_{\mathcal{A}^x_1}$ do not depend on any y-coordinate. Hence, $\mathbb{1}_{\mathcal{B}_1}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$ commute with all operators acting exclusively on the first slot of the tensor product, and $\mathbb{1}_{\overline{\mathcal{A}}^x_1}$ and $\mathbb{1}_{\mathcal{A}^x_1}$ commute with all operators acting only the y-coordinates. The main result of this section is given by the following lemma:

Lemma 4.12.

$$\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi^{N,\varepsilon}(t)\| \lesssim \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}\left(\alpha^<_\xi(t) + (N\varepsilon^\delta)^{1-\widetilde{\beta}} + N^{-1+\widetilde{\beta}}\right)^{\frac{1}{2}}.$$

To prove Lemma 4.12, we need several estimates on the cutoff functions $\mathbb{1}_{\overline{\mathcal{A}}_1}$, $\mathbb{1}_{\overline{\mathcal{A}}_1^x}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$.

Lemma 4.13. *Let $\overline{\mathcal{A}}_1$, $\overline{\mathcal{A}}_1^x$ and $\overline{\mathcal{B}}_1$ as in Definition 4.1. Then*

(a) $\|\mathbb{1}_{\overline{\mathcal{A}}_1} p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}(t)\mu^{\frac{3d}{2}-\frac{1}{2}}$, $\qquad \|\mathbb{1}_{\overline{\mathcal{A}}_1}\partial_{x_1}p_1\|_{\mathrm{op}} \lesssim \|\Phi(t)\|_{H^2(\mathbb{R})}\,\mu^{\frac{3d}{2}-\frac{1}{2}}$,

(b) $\|\mathbb{1}_{\overline{\mathcal{A}}_1}\psi\| \lesssim \mu^{d-\frac{1}{3}}\left(\|\partial_{x_1}\psi\| + \varepsilon\|\nabla_{y_1}\psi\|\right)$ *for any* $\psi \in L^2(\mathbb{R}^{3N})$,

(c) $\|\mathbb{1}_{\overline{\mathcal{A}}_1}\nabla_{y_1}p_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}(t)N^{-\frac{1}{2}}$,

(d) $\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\| \lesssim \mu^{d-\frac{1}{3}}\left(\sum\limits_{k=2}^{N}\left(\|\partial_{x_k}\psi\|^2 + \varepsilon^2\|\nabla_{y_k}\psi\|^2\right)\right)^{\frac{1}{2}}$ *for any* $\psi \in L^2(\mathbb{R}^{3N})$,

(e) $\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon\mathfrak{e}(t)$,

(f) $\|\mathbb{1}_{\overline{\mathcal{A}}_1^x} q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\|^2 \lesssim \mathfrak{e}^2(t)\varepsilon^2(N\varepsilon^\delta)^{1-\widetilde{\beta}}$.

Proof. In the sense of operators, $\mathbb{1}_{\overline{\mathcal{A}}_1} = \mathbb{1}_{\bigcup\limits_{k\geq 2} a_{1,k}} \leq \sum\limits_{k=2}^{N}\mathbb{1}_{a_{1,k}}$. Hence, for any $\psi \in L^2(\mathbb{R}^{3N})$

$$\begin{aligned}\|\mathbb{1}_{\overline{\mathcal{A}}_1}p_1\psi\|^2 &\leq \sum_{k=2}^{N}\left\langle\!\!\left\langle \psi\,|\varphi^\varepsilon(z_1)\rangle\left(\int_{\mathbb{R}^3}\mathrm{d}z_1|\varphi^\varepsilon(z_1)|^2\mathbb{1}_{a_{1,k}}(z_1,z_k)\right)\langle\varphi^\varepsilon(z_1)|\,\psi\right\rangle\!\!\right\rangle\\ &\overset{4.5\mathrm{b}}{\lesssim} \mathfrak{e}^2(t)N\varepsilon^{-2}\mu^{3d}\|p_1\psi\|^2\end{aligned}$$

and the second part of assertion (a) follows analogously with Lemma 4.5a. Part (b) is proven analogously to Lemma 4.10e, noting that $\left(\int_{\mathbb{R}^3}\mathrm{d}z_1\mathbb{1}_{\overline{\mathcal{A}}_1}(z_1,...,z_N)\right)^{\frac{2}{3}} \lesssim N^{\frac{2}{3}}\mu^{2d}$. Part (c) follows from this with $\frac{1}{3}-d < -\frac{1}{2}$ and $2d-\frac{5}{3} > 0$ and as $\|\nabla_{y_1}p_1^{\chi^\varepsilon}\partial_{x_1}\psi\|^2 \lesssim \mathfrak{e}^2(t)\varepsilon^{-2}$ by Lemma 4.7a and

$$\|\nabla_{y_1}\partial_{y_1^{(1)}}p_1^{\chi^\varepsilon}\psi\|^2 + \|\nabla_{y_1}\partial_{y_1^{(2)}}p_1^{\chi^\varepsilon}\psi\|^2 \lesssim \varepsilon^{-4},$$

where we have put $y_1 = (y_1^{(1)}, y_1^{(2)})$. For assertion (d), note that $\mathbb{1}_{\overline{\mathcal{B}}_1} \leq \sum_{k=2}^N \mathbb{1}_{\overline{\mathcal{A}}_k}$, hence $\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|^2 \leq \sum_{k=2}^N\|\mathbb{1}_{\overline{\mathcal{A}}_k}\psi\|^2$, and (e) follows from Lemma 4.7a and since $d > \frac{5}{6}$. Finally,

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1^x}q_1^{\chi^\varepsilon}\psi\|^2 \leq \int\limits_{\mathbb{R}^{3N-1}}\mathrm{d}z_N\cdots\mathrm{d}y_1\left(\int\limits_{\mathbb{R}}\mathrm{d}x_1\mathbb{1}_{\overline{\mathcal{A}}_1^x}(x_1,...,x_N)\right)\left(\sup_{x_1\in\mathbb{R}}|q_1^{\chi^\varepsilon}\psi(z_1,...,z_N)|^2\right).$$

Note that $\int_{\mathbb{R}}\mathrm{d}x_1\mathbb{1}_{\overline{\mathcal{A}}_1^x}(x_1,...,x_N) \lesssim N\mu^d$ analogously to above. For the second factor in the integral, the one-dimensional Gagliardo-Nirenberg-Sobolev inequality [23, Theorem 8.5],

$$\sup_{x\in\mathbb{R}}|f(x)|^2 \leq \|f'\|_{L^2(\mathbb{R})}\|f\|_{L^2(\mathbb{R})} \qquad \text{for } f \in H^1(\mathbb{R}),$$

implies

$$\sup_{x_1\in\mathbb{R}}|q_1^{\chi^\varepsilon}\psi(z_1,...,z_N)|^2 \leq \|q_1^{\chi^\varepsilon}\partial_{x_1}\psi(\cdot,y_1,...,z_N)\|_{L^2(\mathbb{R})}\|q_1^{\chi^\varepsilon}\psi(\cdot,y_1,...,z_N)\|_{L^2(\mathbb{R})}.$$

Using Cauchy-Schwarz in the $\mathrm{d}y_1 \cdots \mathrm{d}z_N$-integration, we obtain

$$\begin{aligned}\|\mathbb{1}_{\overline{\mathcal{A}}_1^x} q_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\|^2 &\lesssim N\mu^d \|q_1^{\chi^\varepsilon} \partial_{x_1} \psi^{N,\varepsilon}(t)\| \|q_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \\ &\overset{4.7a}{\lesssim} \mathfrak{e}^2(t) N^{1-d} \varepsilon^{2d+1} = \mathfrak{e}^2(t)(N\varepsilon^\delta)^{1-d} \varepsilon^2 \varepsilon^{2d-1+\delta(d-1)}.\end{aligned}$$

Assertion (f) follows from this because $d < \widetilde{\beta}$ and since the last exponent is positive as $0 < \delta < \frac{2}{5}$ and $d > \frac{5}{6}$. □

We will use some techniques and intermediate results from [4], which are listed in Lemma 4.14 below. In [4], one considers a class of interaction potentials $\mathcal{W}_{\widetilde{\beta},\eta}$ ([4, Definition 2.2]), which, recalling that $\mu(N,\varepsilon) = \frac{\varepsilon^2}{N}$, can be characterised in the following way:

Definition 4.2. Let $\eta > 0$. The set $\mathcal{W}_{\widetilde{\beta},\eta}$ is defined as the set containing all families of interaction potentials

$$v : (0,1) \to L^\infty(\mathbb{R}^3, \mathbb{R}), \quad \mu \mapsto v(\mu),$$

such that it holds for all $\mu \in (0,1)$ that $\|v(\mu)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mu^{1-3\widetilde{\beta}}$, $v(\mu)$ is non-negative and spherically symmetric, $\operatorname{supp} v(\mu) \subseteq \left\{z \in \mathbb{R}^3 : |z| \lesssim \mu^{\widetilde{\beta}}\right\}$ and

$$\lim_{\mu \to 0} \mu^{-\eta} \left| b(\mu, v) - b(v) \right| = 0,$$

where

$$b(\mu, v) := \mu^{-1} \int_{\mathbb{R}^3} v(\mu, z)\, \mathrm{d}z \int_{\mathbb{R}^2} |\chi(y)|^4\, \mathrm{d}y \quad \text{and} \quad b(v) := \lim_{\mu \to 0} b(\mu, v).$$

Lemma 4.14. *Let $v \in \mathcal{W}_{\widetilde{\beta},\eta}$ for some $\eta > 0$.*

(a) Let $h_\varepsilon : \{z \in \mathbb{R}^3 : |z| \le \varepsilon\} \to \mathbb{R}$ be the unique solution of $\Delta h_\varepsilon = v(\mu)$ with boundary condition $h_\varepsilon\big|_{|z|=\varepsilon} = 0$ and denote $h_\varepsilon^{(ij)} := h_\varepsilon(z_i - z_j)$. Then

$$\|p_1(\nabla_1 h_\varepsilon^{(12)})\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{-1} \mu^{-\frac{\widetilde{\beta}}{2}} \varepsilon.$$

(b) Let $R \lesssim \mu^{\widetilde{\beta}}$ such that $\operatorname{supp} v(\mu) \subseteq \{z \in \mathbb{R}^3 : |z| \le R\}$. Let $\Theta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^3, [0,1])$ be spherically symmetric such that $\Theta_\varepsilon(z) = 1$ for $|z| \le R$, $\Theta_\varepsilon(z) = 0$ for $|z| \ge \varepsilon$, and Θ_ε is decreasing for $R < |z| < \varepsilon$. Denote $\Theta_\varepsilon^{(ij)} := \Theta_\varepsilon(z_i - z_j)$. Then

$$\|\nabla \Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon^{-1}.$$

(c) Let $\beta_1 \in [0, \widetilde{\beta}]$. Define

$$\overline{v(\mu, x)} := \int_{\mathbb{R}^2} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \int_{\mathbb{R}^2} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 v(\mu, (x, y_1 - y_2))$$

and let $\overline{h}_{\beta_1} : [-N^{-\beta_1}, N^{-\beta_1}] \to \mathbb{R}$ *be the unique solution of* $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\overline{h}_{\beta_1} = \overline{v(\mu)}$ *with boundary condition* $\overline{h}_{\beta_1}(\pm N^{-\beta_1}) = 0$. *Then*

$$\|p_1^\Phi(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_{\beta_1}^{(12)})\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{-1-\frac{\beta_1}{2}}.$$

(d) *Let* $R \lesssim \mu^{\widetilde{\beta}}$ *such that* $\operatorname{supp} v(\mu) \subseteq \{z \in \mathbb{R}^3 : |z| \le R\}$. *For* $\beta_1 \in [0, \widetilde{\beta}]$, *let* $\overline{\Theta}_{\beta_1} \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ *be an even function such that* $\overline{\Theta}_{\beta_1}(x) = 1$ *for* $|x| \le R$, $\overline{\Theta}_{\beta_1}(x) = 0$ *for* $|x| \ge N^{-\beta_1}$ *and* $\overline{\Theta}_{\beta_1}$ *is decreasing for* $R < |x| < N^{-\beta_1}$. *Denote* $\overline{\Theta}_{\beta_1}^{(ij)} := \overline{\Theta}_{\beta_1}(x_i - x_j)$. *Then*

$$\|\tfrac{\mathrm{d}}{\mathrm{d}x}\overline{\Theta}_{\beta_1}\|_{L^\infty(\mathbb{R})} \lesssim N^{\beta_1}, \qquad \|p_1^\Phi\left(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_{\beta_1}^{(12)}\right)\|_{\mathrm{op}} \lesssim \mathfrak{e}(t) N^{\frac{\beta_1}{2}}.$$

(e) *Let* $\psi \in L^2(\mathbb{R}^{3N})$ *be symmetric in* $\{z_1, ..., z_N\}$. *Then*

$$\left|\langle\!\langle \psi, p_1p_2\left((N-1)v(\mu, z_1 - z_2)\right)p_1p_2\psi\rangle\!\rangle - \langle\!\langle \psi, b(v)|\Phi(x_1)|^2\psi\rangle\!\rangle\right| \lesssim \mathfrak{e}^2(t)\left(\tfrac{\mu^{\widetilde{\beta}}}{\varepsilon} + N^{-1} + \mu^\eta + \langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle\right).$$

(f) *Let* $\psi, \widetilde{\psi} \in L^2(\mathbb{R}^{3N})$ *and* $t_2 \in \{q_2, q_2^\Phi p_2^{\chi^\varepsilon}\}$. *Then*

$$N\left|\left\langle\!\!\left\langle \psi, q_1^{\chi^\varepsilon} t_2 v(\mu, z_1 - z_2)p_1p_2\widetilde{\psi}\right\rangle\!\!\right\rangle\right| \lesssim \mathfrak{e}(t)\mu^{-\frac{\widetilde{\beta}}{2}}(\|q_1^{\chi^\varepsilon}\psi\| + \varepsilon\|\nabla_1 q_1^{\chi^\varepsilon}\psi\|)\|\widetilde{\psi}\|.$$

(g) *Let* $\psi \in L^2(\mathbb{R}^{3N})$ *be symmetric in* $\{z_1, ..., z_N\}$. *Then*

$$N\left|\left\langle\!\!\left\langle \psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}q_1^\Phi q_2^\Phi(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_{\beta_1}^{(12)})(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_{\widetilde{\beta}}^{(12)})p_1p_2\psi\right\rangle\!\!\right\rangle\right| \lesssim \mathfrak{e}^2(t)\,\langle\!\langle \psi, \widehat{n}\psi\rangle\!\rangle\,.$$

Proof. Parts (a) and (b) follow from Lemma 4.12, Lemma 4.13 and Corollary 4.14 in [4] and assertions (c) and (d) are taken from Lemma 4.15 and Corollary 4.16 in [4]. Parts (e) and (f) are (69)-(71) and (74) in [4], and (g) follows from the estimate of (75) in [4]. □

Lemma 4.15. *Let* $\eta > 0$. *Then the family* $U_{\widetilde{\beta}}$ *is contained in* $\mathcal{W}_{\widetilde{\beta},\eta}$.

Proof. Note that $\mu^{-1}\int_{\mathbb{R}^3} U_{\widetilde{\beta}}(z)\,\mathrm{d}z = \frac{4\pi}{3}a(R_{\widetilde{\beta}}^3\mu^{-3\widetilde{\beta}} - 1) = \frac{4\pi}{3}ac$ for some $c > 0$ by Lemma 4.9c, hence $b(\mu, U_{\widetilde{\beta}}) = b(U_{\widetilde{\beta}})$. The remaining requirements are easily verified. □

Lemma 4.16. *Let* $0 < \eta < 1 - \widetilde{\beta}$. *Then the family* $U_{\widetilde{\beta}}f_{\widetilde{\beta}}$ *is contained in* $\mathcal{W}_{\widetilde{\beta},\eta}$.

Proof. We drop the μ-dependence of the family members and write $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ instead of $(U_{\widetilde{\beta}} f_{\widetilde{\beta}})(\mu)$. By Lemma 4.9, $f_{\widetilde{\beta}}$ is spherically symmetric, $0 \leq f_{\widetilde{\beta}}(z) \leq 1$ and $R_{\widetilde{\beta}} \lesssim \mu^{\widetilde{\beta}}$, hence $\|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^\infty(\mathbb{R}^3)} \lesssim \mu^{1-3\widetilde{\beta}}$ and $\operatorname{supp} U_{\widetilde{\beta}} f_{\widetilde{\beta}} \subseteq \{z \in \mathbb{R}^3 : |z| \lesssim \mu^{\widetilde{\beta}}\}$ by Definition 3.4 of $U_{\widetilde{\beta}}$. Further,

$$\mu^{-1} \int\limits_{\mathbb{R}^3} U_{\widetilde{\beta}}(z) f_{\widetilde{\beta}}(z) \,\mathrm{d}z \overset{(23)}{=} \mu^{-1} \int\limits_{B_\mu(0)} w_\mu(z) f_{\widetilde{\beta}}(z)$$
$$\overset{4.9\text{b}}{=} \mu^{-1} \kappa_{\widetilde{\beta}} \int\limits_{B_\mu(0)} w_\mu(z) j_\mu(z) \overset{(20)}{=} \kappa_{\widetilde{\beta}} 8\pi a,$$

which yields $b(\mu, U_{\widetilde{\beta}} f_{\widetilde{\beta}}) = \kappa_{\widetilde{\beta}} 8\pi a \int_{\mathbb{R}^2} |\chi(y)|^4 \,\mathrm{d}y$ and consequently

$$b(U_{\widetilde{\beta}} f_{\widetilde{\beta}}) = \lim_{\mu \to 0} b(\mu, U_{\widetilde{\beta}} f_{\widetilde{\beta}}) = 8\pi a \int\limits_{\mathbb{R}^2} |\chi(y)|^4 \,\mathrm{d}y = b \tag{46}$$

by Lemma 4.9b. This implies

$$|b(\mu, U_{\widetilde{\beta}} f_{\widetilde{\beta}}) - b(U_{\widetilde{\beta}} f_{\widetilde{\beta}})| = 8\pi a (\kappa_{\widetilde{\beta}} - 1) \int\limits_{\mathbb{R}^2} |\chi(y)|^4 \,\mathrm{d}y \lesssim \frac{\mu a}{\mu^{\widetilde{\beta}} - \mu a} \overset{4.9\text{b}}{\lesssim} \mu^{1-\widetilde{\beta}}.$$

□

Proof of Lemma 4.12. In the following, we abbreviate $\psi^{N,\varepsilon}(t) \equiv \psi$ and $\Phi(t) \equiv \Phi$.

$$\begin{aligned}
&E^\psi(t) - \mathcal{E}^\Phi(t) \\
&= \|\mathbb{1}_{\mathcal{A}_1} \partial_{x_1} q_1 \psi\|^2 + \|\mathbb{1}_{\mathcal{A}_1} \partial_{x_1} p_1 \psi\|^2 + 2\Re \langle\!\langle \partial_{x_1} p_1 \psi, \mathbb{1}_{\mathcal{A}_1} \partial_{x_1} q_1 \psi \rangle\!\rangle + \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\overline{\mathcal{B}}_1} \partial_{x_1} \psi\|^2 \\
&\quad + \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\mathcal{B}_1} \partial_{x_1} \psi\|^2 + \left\langle\!\!\left\langle \psi, (-\Delta_{y_1} + \tfrac{1}{\varepsilon^2} V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}) \psi \right\rangle\!\!\right\rangle + \tfrac{N-1}{2} \left\| \mathbb{1}_{\overline{\mathcal{B}}_1} \sqrt{w_\mu^{(12)}} \psi \right\|^2 \\
&\quad + \tfrac{N-1}{2} \left\langle\!\!\left\langle \psi, \mathbb{1}_{\mathcal{B}_1} \left(w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)} \right) \psi \right\rangle\!\!\right\rangle + \tfrac{N-1}{2} \left\langle\!\!\left\langle \psi, \mathbb{1}_{\mathcal{B}_1} p_1 p_2 U_{\widetilde{\beta}}^{(12)} p_1 p_2 \mathbb{1}_{\mathcal{B}_1} \psi \right\rangle\!\!\right\rangle \\
&\quad + \tfrac{N-1}{2} \left\langle\!\!\left\langle \psi, \mathbb{1}_{\mathcal{B}_1} (1 - p_1 p_2) U_{\widetilde{\beta}}^{(12)} (1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1} \psi \right\rangle\!\!\right\rangle \\
&\quad + (N-1) \Re \left\langle\!\!\left\langle \psi, \mathbb{1}_{\mathcal{B}_1} p_1 p_2 U_{\widetilde{\beta}}^{(12)} (1 - p_1 p_2) \mathbb{1}_{\mathcal{B}_1} \psi \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \psi, V^\parallel(t, z_1) \psi \right\rangle\!\!\right\rangle \\
&\quad - \|\Phi'\|^2_{L^2(\mathbb{R})} - \left\langle \Phi, \tfrac{b}{2} |\Phi|^2 \Phi \right\rangle - \left\langle \Phi, V^\parallel(t, (x, 0)) \Phi \right\rangle \\
&\geq \|\mathbb{1}_{\mathcal{A}_1} \partial_{x_1} q_1 \psi\|^2 \\
&\quad + \|\mathbb{1}_{\overline{\mathcal{A}}_1} \mathbb{1}_{\mathcal{B}_1} \partial_{x_1} \psi\|^2 + \left\langle\!\!\left\langle \psi, (-\Delta_{y_1} + \tfrac{1}{\varepsilon^2} V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2}) \psi \right\rangle\!\!\right\rangle \\
&\qquad + \tfrac{N-1}{2} \left\langle\!\!\left\langle \psi, \mathbb{1}_{\mathcal{B}_1} \left(w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)} \right) \psi \right\rangle\!\!\right\rangle && (47) \\
&\quad + 2\Re \langle\!\langle \partial_{x_1} p_1 \psi, \mathbb{1}_{\mathcal{A}_1} \partial_{x_1} q_1 \psi \rangle\!\rangle && (48) \\
&\quad + \|\mathbb{1}_{\mathcal{A}_1} \partial_{x_1} p_1 \psi\|^2 - \|\Phi'\|^2_{L^2(\mathbb{R})} && (49) \\
&\quad + \tfrac{b}{2} \left(\langle\!\langle \psi, |\Phi(x_1)|^2 \psi \rangle\!\rangle - \langle \Phi, |\Phi|^2 \Phi \rangle \right) \\
&\qquad + \left\langle\!\!\left\langle \psi, V^\parallel(t, z_1) \psi \right\rangle\!\!\right\rangle - \left\langle \Phi, V^\parallel(t, (x, 0)) \Phi \right\rangle && (50)
\end{aligned}$$

$$+\tfrac{N-1}{2}\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1} p_1 p_2 U^{(12)}_{\widetilde{\beta}} p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle - \tfrac{b}{2}\left\langle\!\left\langle\psi, |\Phi(x_1)|^2\psi\right\rangle\!\right\rangle \tag{51}$$

$$+(N-1)\Re\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1}(p_1 q_2 + q_1 p_2) U^{(12)}_{\widetilde{\beta}} p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle \tag{52}$$

$$+(N-1)\Re\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1} q_1 q_2 U^{(12)}_{\widetilde{\beta}} p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle. \tag{53}$$

We will now estimate these expressions separately. For (47), recall that χ^ε is the ground state of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$ with eigenvalue $\frac{E_0}{\varepsilon^2}$, hence $(-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2})p_1^{\chi^\varepsilon} = 0$ and $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon}) - \frac{E_0}{\varepsilon^2} \geq 0$ as operator. Using further that $\mathbb{1}_{\overline{\mathcal{A}}_1^x} = (\mathbb{1}_{\overline{\mathcal{A}}_1^x})^2$, $\mathbb{1}_{\overline{\mathcal{B}}_1} = (\mathbb{1}_{\overline{\mathcal{B}}_1})^2$ and their complements commute with $-\Delta_{y_1} + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2}$ and with $q_1^{\chi^\varepsilon}$, we conclude

$$\begin{aligned}
&\left\langle\!\left\langle\psi, (-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2})\psi\right\rangle\!\right\rangle \\
&\quad\geq \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1} q_1^{\chi^\varepsilon}\psi, (-\Delta_{y_1} + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2})\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1} q_1^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle \\
&\quad\geq \|\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 \\
&\qquad -\tfrac{1}{\varepsilon^2}\|(V^\perp - E_0)_-\|_{L^\infty(\mathbb{R}^2)}\|\mathbb{1}_{\overline{\mathcal{A}}_1^x} q_1^{\chi^\varepsilon}\psi\|^2 \\
&\quad\overset{4.13f}{\gtrsim} \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 - \mathfrak{e}^2(t)(N\varepsilon^\delta)^{1-\widetilde{\beta}}
\end{aligned}$$

because $\mathbb{1}_{\overline{\mathcal{A}}_1^x} \geq \mathbb{1}_{\overline{\mathcal{A}}_1}$ in the sense of operators since $\overline{\mathcal{A}}_1^x \supset \overline{\mathcal{A}}_1$. Further,

$$\begin{aligned}
\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1}\psi\|^2 &\leq \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1} p_1^{\chi^\varepsilon}\psi\|^2 + \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 \\
&\quad +2\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1} p_1^{\chi^\varepsilon}\psi\|\|\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\| \\
&\lesssim \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{y_1} q_1^{\chi^\varepsilon}\psi\|^2 + \mathfrak{e}^2(t)N^{-\frac{1}{2}}
\end{aligned}$$

by Lemma 4.7a and Lemma 4.13c. Together, this implies

$$\begin{aligned}
(47) &\gtrsim \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_1\psi\|^2 + \tfrac{N-1}{2}\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1}\left(w_\mu^{(12)} - U^{(12)}_{\widetilde{\beta}}\right)\psi\right\rangle\!\!\right\rangle \\
&\qquad -\mathfrak{e}^2(t)\left(N^{-\frac{1}{2}} + (N\varepsilon^\delta)^{1-\widetilde{\beta}}\right).
\end{aligned}$$

As $d < \widetilde{\beta}$, it follows that $R_{\widetilde{\beta}} < 2R_{\widetilde{\beta}} < \mu^d$ for sufficiently small μ, and consequently $\overline{\mathcal{C}}_1 \subset \overline{\mathcal{A}}_1$ and $(c_{1,k}\cap\mathcal{B}_1)\cap(c_{1,l}\cap\mathcal{B}_1) = \emptyset$ for $k,l \neq 1$, $l \neq k$. Hence,

$$\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1} \geq \mathbb{1}_{\overline{\mathcal{C}}_1}\mathbb{1}_{\mathcal{B}_1} = \mathbb{1}_{\bigcup\limits_{k\geq 2} c_{1,k}\cap\mathcal{B}_1} = \sum_{k=2}^N \mathbb{1}_{c_{1,k}\cap\mathcal{B}_1} = \mathbb{1}_{\mathcal{B}_1}\sum_{k=2}^N \mathbb{1}_{c_{1,k}}$$

in the sense of operators, which implies

$$\begin{aligned}
(47) &\gtrsim (N-1)\|\mathbb{1}_{c_{1,2}}\nabla_1\mathbb{1}_{\mathcal{B}_1}\psi\|^2 + \tfrac{N-1}{2}\left\langle\!\!\left\langle\mathbb{1}_{\mathcal{B}_1}\psi, \left(w_\mu^{(12)} - U^{(12)}_{\widetilde{\beta}}\right)\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle \\
&\quad -\mathfrak{e}^2(t)\left(N^{-\frac{1}{2}} + (N\varepsilon^\delta)^{1-\widetilde{\beta}}\right) \\
&\gtrsim -\mathfrak{e}^2(t)\left(N^{-\frac{1}{2}} + (N\varepsilon^\delta)^{1-\widetilde{\beta}}\right)
\end{aligned}$$

by Lemma 4.9d because $\mathbb{1}_{\mathcal{B}_1}\psi \in \mathcal{D}(\nabla_1)$ and as $\mathbb{1}_{c_{1,2}} = \mathbb{1}_{|z_1-z_2|<R_{\widetilde{\beta}}}$. Next, observe that

$$\begin{aligned} |(48)| &\leq |\langle\!\langle \partial_{x_1} q_1\psi, \partial_{x_1} p_1\psi\rangle\!\rangle| + \left|\left\langle\!\!\left\langle \partial_{x_1} q_1\psi, \mathbb{1}_{\overline{\mathcal{A}}_1}\partial_{x_1} p_1\psi\right\rangle\!\!\right\rangle\right| \\ &\overset{4.2\text{b}}{\leq} \left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}} q_1\psi, \partial_{x_1}^2 p_1 \widehat{n}_1^{\frac{1}{2}}\psi\right\rangle\!\!\right\rangle\right| + \|\mathbb{1}_{\overline{\mathcal{A}}_1}\partial_{x_1} p_1\|_{\text{op}}\|\partial_{x_1} q_1\psi\| \\ &\lesssim \|\Phi\|_{H^2(\mathbb{R})}\left(\langle\!\langle\psi, \widehat{n}\psi\rangle\!\rangle + \mathfrak{e}(t)\mu^{-\frac{1}{2}+\frac{3d}{2}}\right) \end{aligned}$$

by Lemma 4.7a and Lemma 4.13a. Due to $\|\partial_{x_1} p_1\psi\|^2 = \|\Phi'\|^2_{L^2(\mathbb{R})}\|p_1\psi\|^2$,

$$\begin{aligned} |(49)| &= \left|-\|\mathbb{1}_{\overline{\mathcal{A}}_1}\partial_{x_1} p_1\psi\|^2 + \|\partial_{x_1} p_1\psi\|^2 - \|\Phi'\|^2_{L^2(\mathbb{R})}\right| \\ &\lesssim \|\Phi\|^2_{H^2(\mathbb{R})}\mu^{-1+3d} + \mathfrak{e}^2(t)\|q_1\psi\|^2. \end{aligned}$$

Applying Lemma 4.8 and Lemma 4.7f to (50) yields $|(50)| \lesssim \mathfrak{e}^2(t)\langle\!\langle\psi, \widehat{n}\psi\rangle\!\rangle + \mathfrak{e}^3(t)\varepsilon$. Using the identity $f_{\widetilde{\beta}} + g_{\widetilde{\beta}} = 1$ and decomposing $\mathbb{1}_{\mathcal{B}_1} = \mathbb{1} - \mathbb{1}_{\overline{\mathcal{B}}_1}$, we estimate (51) as

$$\begin{aligned} |(51)| &\leq \tfrac{1}{2}\left|\left\langle\!\!\left\langle\psi, p_1p_2\left((N-1)(U_{\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}\right)p_1p_2\psi\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle\psi, b|\Phi(x_1)|^2\psi\right\rangle\!\!\right\rangle\right| \\ &\quad + \tfrac{N-1}{2}\left|\left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\psi, p_1p_2(U_{\widetilde{\beta}}g_{\widetilde{\beta}})^{(12)}p_1p_2\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle\right| \\ &\quad + \tfrac{N-1}{2}\left|\left\langle\!\!\left\langle\psi, \mathbb{1}_{\overline{\mathcal{B}}_1}p_1p_2(U_{\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}p_1p_2\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\right\rangle\!\!\right\rangle\right| \\ &\quad + (N-1)\left|\left\langle\!\!\left\langle\psi, \mathbb{1}_{\overline{\mathcal{B}}_1}p_1p_2(U_{\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}p_1p_2\psi\right\rangle\!\!\right\rangle\right| \\ &\overset{4.14\text{e}}{\lesssim} \mathfrak{e}^2(t)\left(\tfrac{\mu^{\widetilde{\beta}}}{\varepsilon} + N^{-1} + \langle\!\langle\psi, \widehat{n}\psi\rangle\!\rangle\right) + N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\|p_1(U_{\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}p_1\|_{\text{op}} \\ &\quad + N\|p_1(U_{\widetilde{\beta}}g_{\widetilde{\beta}})^{(12)}p_1\|_{\text{op}} \\ &\lesssim \mathfrak{e}^2(t)\left(\tfrac{\mu^{\widetilde{\beta}}}{\varepsilon} + \langle\!\langle\psi, \widehat{n}\psi\rangle\!\rangle + \mathfrak{e}(t)\varepsilon + \mu^{1-\widetilde{\beta}} + \mu^{\eta}\right) \end{aligned}$$

for any $\eta < 1-\widetilde{\beta}$ by Lemma 4.13e and Lemma 4.6a. Here, we have used that $U_{\widetilde{\beta}}f_{\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$ for $\eta < 1-\widetilde{\beta}$ by Lemma 4.16, $\|U_{\widetilde{\beta}}f_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)} \lesssim \mu$ and

$$\|U_{\widetilde{\beta}}g_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)} = a\mu^{1-3\widetilde{\beta}}\int_{\operatorname{supp}U_{\widetilde{\beta}}} \mathrm{d}z|g_{\widetilde{\beta}}(z)| \lesssim \mu^{2-\widetilde{\beta}}$$

because $|g_{\widetilde{\beta}}(z)| \leq g_{\widetilde{\beta}}(\mu^{\widetilde{\beta}}) \leq \kappa_{\widetilde{\beta}}a\mu^{1-\widetilde{\beta}}$ on $\operatorname{supp}U_{\widetilde{\beta}}$ by Lemma 4.9b and (19). Decomposing $\mathbb{1}_{\mathcal{B}_1}$ as before and abbreviating $Q_0 := p_1p_2$ and $Q_1 := p_1q_2 + q_1p_2$, we find

$$\begin{aligned} |(52)| &\lesssim N\left|\left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, Q_1U^{(12)}_{\widetilde{\beta}}Q_0\psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle\psi, Q_1U^{(12)}_{\widetilde{\beta}}Q_0\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\right\rangle\!\!\right\rangle\right| \\ &\quad + N\left|\left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, Q_1U^{(12)}_{\widetilde{\beta}}Q_0\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle\psi, Q_1U^{(12)}_{\widetilde{\beta}}Q_0\psi\right\rangle\!\!\right\rangle\right| \\ &\overset{4.2\text{b}}{\lesssim} N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\|p_1U^{(12)}_{\widetilde{\beta}}p_1\|_{\text{op}} + N\left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}}q_2\psi, p_1U^{(12)}_{\widetilde{\beta}}p_1p_2\widehat{n}_1^{\frac{1}{2}}\psi\right\rangle\!\!\right\rangle\right| \\ &\lesssim \mathfrak{e}^2(t)\left(\mathfrak{e}(t)\varepsilon + \langle\!\langle\psi, \widehat{n}\psi\rangle\!\rangle\right) \end{aligned}$$

by Lemma 4.1d and Lemma 4.13e. For the last term, we decompose $q = q^{\chi^\varepsilon} + p^{\chi^\varepsilon} q^\Phi$, hence

$$\begin{aligned}
&|(53)| \\
&\lesssim N \left| \left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\psi, q_1^{\chi^\varepsilon} q_2 U_{\widetilde{\beta}}^{(12)} p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| + N \left| \left\langle\!\!\left\langle \psi, q_1^{\chi^\varepsilon} q_2^\Phi p_2^{\chi^\varepsilon} U_{\widetilde{\beta}}^{(12)} p_1 p_2 \mathbb{1}_{\mathcal{B}_2}\psi \right\rangle\!\!\right\rangle \right| \quad (54)\\
&\quad + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, q_2^{\chi^\varepsilon} q_1^\Phi p_1^{\chi^\varepsilon} U_{\widetilde{\beta}}^{(12)} p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| \quad (55)\\
&\quad + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\psi, q_1^\Phi q_2^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} U_{\widetilde{\beta}}^{(12)} p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right|, \quad (56)
\end{aligned}$$

where we have exchanged $1 \leftrightarrow 2$ in the second term of (54). As $\mathbb{1}_{\mathcal{B}_1}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$ are functions of $(z_2, ..., z_N)$ but not of z_1,

$$\|\nabla_1 q_1^{\chi^\varepsilon} \mathbb{1}_{\mathcal{B}_1}\psi\| = \|\mathbb{1}_{\mathcal{B}_1}\nabla_1 q_1^{\chi^\varepsilon}\psi\| \leq \|\nabla_1 q_1^{\chi^\varepsilon}\psi\| \lesssim \mathfrak{e}(t)$$

and analogously $\|q_1^{\chi^\varepsilon}\mathbb{1}_{\mathcal{B}_1}\psi\| \lesssim \mathfrak{e}(t)\varepsilon$ by Lemma 4.7a, hence Lemma 4.14f implies (54) $\lesssim \mathfrak{e}^2(t)\left(\frac{\varepsilon^2}{\mu^{\widetilde{\beta}}}\right)^{\frac{1}{2}}$. By Lemma 4.14a, $U_{\widetilde{\beta}}^{(12)} = \Theta_\varepsilon^{(12)}\Delta_1 h_\varepsilon^{(12)}$. Integrating by parts in z_1 yields

$$\begin{aligned}
(55) \;\leq\; & N \left| \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\nabla_1 p_1^{\chi^\varepsilon} q_1^\Phi\psi, q_2^{\chi^\varepsilon}\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)}) p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| \\
& + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, q_2^{\chi^\varepsilon} q_1^\Phi p_1^{\chi^\varepsilon}(\nabla_1\Theta_\varepsilon^{(12)})\cdot(\nabla_1 h_\varepsilon^{(12)}) p_1 p_2 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| \\
& + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, q_2^{\chi^\varepsilon} q_1^\Phi p_1^{\chi^\varepsilon}\Theta_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)}) p_2 \cdot \nabla_1 p_1 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| \\
\lesssim\; & N\|(\nabla_1 h_\varepsilon^{(12)})p_1\|_{\mathrm{op}}\Big(\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\left(\|\nabla\Theta_\varepsilon\|_{L^\infty(\mathbb{R}^3)} + \|\nabla_1 p_1\|_{\mathrm{op}}\right) \\
& \qquad + \|\mathbb{1}_{\overline{\mathcal{B}}_1}\nabla_1 p_1^{\chi^\varepsilon} q_1^\Phi\psi\|\Big) \\
\lesssim\; & \mathfrak{e}^2(t)\left(\frac{\varepsilon^2}{\mu^{\widetilde{\beta}}}\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used Lemmas 4.13e, 4.7a, 4.14b and 4.14c and the fact that

$$\begin{aligned}
\|\mathbb{1}_{\overline{\mathcal{B}}_1}\nabla_1 p_1^{\chi^\varepsilon} q_1^\Phi\psi\|^2 &= \|\mathbb{1}_{\overline{\mathcal{B}}_1} p_1^{\chi^\varepsilon}\partial_{x_1} q_1^\Phi\psi\|^2 + \|q_1^\Phi\nabla_{y_1} p_1^{\chi^\varepsilon}\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|^2 \\
&\leq \|\partial_{x_1} q_1^\Phi\psi\|^2 + \|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\mathrm{op}}^2\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|^2 \lesssim \mathfrak{e}^2(t).
\end{aligned}$$

Finally, choosing $\beta_1 = \widetilde{\beta}$ such that $p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} U_{\widetilde{\beta}}^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} = \Theta_{\widetilde{\beta}}^{(12)}(\frac{\mathrm{d}^2}{\mathrm{d}x_1^2}\overline{h}_{\widetilde{\beta}}^{(12)}) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}$ by Lemma 4.14c, we find with the abbreviations $Q_0 := p_1 p_2$ and $Q_2 := q_1^\Phi q_2^\Phi p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}$

$$\begin{aligned}
(56) \;\leq\; & N \left| \left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\partial_{x_1} Q_2\psi, \Theta_{\widetilde{\beta}}^{(12)}(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_{\widetilde{\beta}}^{(12)}) Q_0 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| \\
& + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\psi, Q_2\Theta_{\widetilde{\beta}}^{(12)}(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_{\widetilde{\beta}}^{(12)})\partial_{x_1} Q_0 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right| \\
& + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, Q_2(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\Theta_{\widetilde{\beta}}^{(12)})(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_{\widetilde{\beta}}^{(12)}) Q_0 \mathbb{1}_{\mathcal{B}_1}\psi \right\rangle\!\!\right\rangle \right|
\end{aligned}$$

$$
\begin{aligned}
&+N\left|\left\langle\!\!\left\langle\psi, Q_2(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\Theta^{(12)}_{\widetilde{\beta}})(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\widetilde{\beta}})Q_0\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\right\rangle\!\!\right\rangle\right| \\
&+N\left|\left\langle\!\!\left\langle\psi, Q_2(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\Theta^{(12)}_{\widetilde{\beta}})(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\widetilde{\beta}})Q_0\psi\right\rangle\!\!\right\rangle\right| \\
\overset{4.14g}{\leq}\;& N\|(\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}^{(12)}_{\widetilde{\beta}})p_1^\Phi\|_{\mathrm{op}}\left(\|\partial_{x_1}q_1^\Phi\psi\|+\|\partial_{x_1}p_1^\Phi\|_{\mathrm{op}}+\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\|\tfrac{\mathrm{d}}{\mathrm{d}x}\Theta_{\widetilde{\beta}}\|_{L^\infty(\mathbb{R})}\right) \\
&+\mathfrak{e}^2(t)\,\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle \\
\lesssim\;& \mathfrak{e}^2(t)\left(N^{-\frac{\widetilde{\beta}}{2}}+\varepsilon^{\widetilde{\beta}}\left(\tfrac{\varepsilon^2}{\mu^{\widetilde{\beta}}}\right)^{\frac{1}{2}}+\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\right).
\end{aligned}
$$

Thus, $|(53)|\lesssim\mathfrak{e}^2(t)\big(N^{-\frac{\widetilde{\beta}}{2}}+\left(\frac{\varepsilon^2}{\mu^{\widetilde{\beta}}}\right)^{\frac{1}{2}}+\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\big)$. The estimates for (47) to (53) imply

$$
\left|E^\psi(t)-\mathcal{E}^\Phi(t)\right|\gtrsim\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1}q_1\psi\|^2-\|\Phi\|^2_{H^2(\mathbb{R})}\left(\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle+(N\varepsilon^\delta)^{1-\widetilde{\beta}}+N^{-1+\widetilde{\beta}}\right)
$$

because $\mu^{\widetilde{\beta}}\varepsilon^{-1}<N^{-\widetilde{\beta}}, \varepsilon\mu^{-\frac{\widetilde{\beta}}{2}}<(N\varepsilon^\delta)^{\frac{\widetilde{\beta}}{2}}$, $\frac{\widetilde{\beta}}{2}>1-\widetilde{\beta}$ and $\mu^\eta<N^{-1+\widetilde{\beta}}$ for sufficiently large $\eta<1-\widetilde{\beta}$. As $\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1}q_1^\Phi\psi\|\leq\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1}q_1\psi\|+\|\partial_{x_1}p_1^\Phi\|_{\mathrm{op}}\|q_1^{\chi^\varepsilon}\psi\|\lesssim\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1}q_1\psi\|+\mathfrak{e}^2(t)\varepsilon$ by Lemma 4.7a, this proves the claim with Lemma 4.5a. □

4.4 Proof of Proposition 3.2

Also in this proof, we will abbreviate $\psi^{N,\varepsilon}\equiv\psi$ and $\Phi(t)\equiv\Phi$. We need to estimate

$$
\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)=\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi^<(t)-N(N-1)\Re\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle\psi,g^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle\right). \tag{57}
$$

Proposition 3.4 in [4] provides a bound for $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi^<(t)|$ for almost every $t\in\mathbb{R}$. This bound implies

$$
\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)\right|\leq|\gamma_a^<(t)|+\left|\gamma_b^<(t)-N(N-1)\Re\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle\psi,g^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle\right)\right|
$$

for almost every t, where we have added the superscript $^<$ to the notation to avoid confusion. The two first terms are given by

$$
\begin{aligned}
\gamma_a^<(t)\;:=\;&\left|\left\langle\!\!\left\langle\psi,\dot{V}^{\|}(t,z_1)\psi\right\rangle\!\!\right\rangle-\left\langle\Phi,\dot{V}^{\|}(t,(x,0))\Phi\right\rangle_{L^2(\mathbb{R})}\right| \\
&-2N\Im\left\langle\!\!\left\langle\psi,q_1\widehat{m}^a_{-1}\big(V^{\|}(t,z_1)-V^{\|}(t,(x_1,0))\big)p_1\psi\right\rangle\!\!\right\rangle,
\end{aligned} \tag{58}
$$

$$
\gamma_b^<(t)\;:=\;-N(N-1)\Im\left\langle\!\!\left\langle\psi,Z^{(12)}\widehat{m}\psi\right\rangle\!\!\right\rangle=-N(N-1)\Im\left\langle\!\!\left\langle\psi,Z^{(12)}\widehat{r}\psi\right\rangle\!\!\right\rangle. \tag{59}
$$

The last equality in (59) follows by Lemma 4.2c as

$$
\left[Z^{(12)},\widehat{m}\right]=\left[Z^{(12)},p_1p_2(\widehat{m}-\widehat{m}_2)+(p_1q_2+q_1p_2)(\widehat{m}-\widehat{m}_1)\right]=\left[Z^{(12)},\widehat{r}\right] \tag{60}
$$

since $p_1p_2P_{N-1}=p_1p_2P_N=(p_1q_2+q_1p_2)P_N=0$. For the second term in (57), we compute with the aid of Lemma 4.3c

$$
-N(N-1)\Re\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle\psi,g^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle\right)
$$

$$= N(N-1)\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[H(t)-\sum_{j=1}^{N}h_j(t),\widehat{r}\right]\psi\right\rangle\!\!\right\rangle \tag{61}$$

$$+N(N-1)\Im\left\langle\!\!\left\langle \psi, \left[H(t), g_{\widetilde{\beta}}^{(12)}\right]\widehat{r}\psi\right\rangle\!\!\right\rangle. \tag{62}$$

We expand the pair interaction in (61) as

$$\sum_{i<j} w_\mu^{(ij)} = w_\mu^{(12)} + \sum_{j=3}^{N}\left(w_\mu^{(1j)} + w_\mu^{(2j)}\right) + \sum_{3\le i<j\le N} w_\mu^{(ij)}$$

and use

$$w_\mu^{(12)} - b(|\Phi(x_1)|^2 + |\Phi(x_2)|^2) = Z^{(12)} - \tfrac{N-2}{N-1}b(|\Phi(x_1)|^2 + |\Phi(x_2)|^2),$$

hence by Lemma 4.3b and the symmetry of ψ,

$$(61) = N^2(N-1)\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[V^{\|}(t,z_1) - V^{\|}(t,(x_1,0)),\widehat{r}\right]\psi\right\rangle\!\!\right\rangle \tag{63}$$

$$+N(N-1)\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[Z^{(12)},\widehat{r}\right]\psi\right\rangle\!\!\right\rangle \tag{64}$$

$$-2N(N-2)\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[b|\Phi(x_1)|^2,\widehat{r}\right]\psi\right\rangle\!\!\right\rangle \tag{65}$$

$$+2N(N-1)(N-2)\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[w_\mu^{(13)},\widehat{r}\right]\psi\right\rangle\!\!\right\rangle \tag{66}$$

$$+\tfrac{1}{2}N(N-1)(N-2)(N-3))\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[w_\mu^{(34)},\widehat{r}\right]\psi\right\rangle\!\!\right\rangle \tag{67}$$

$$-N(N-1)(N-2)\Im\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[b|\Phi(x_3)|^2,\widehat{r}\right]\psi\right\rangle\!\!\right\rangle. \tag{68}$$

For (62), note that

$$\begin{aligned}
&\left[H(t), g_{\widetilde{\beta}}^{(12)}\right]\widehat{r}\psi \\
&\quad = -\left[H(t), f_{\widetilde{\beta}}^{(12)}\right]\widehat{r}\psi \\
&\quad = (\Delta_1 f_{\widetilde{\beta}}^{(12)} + \Delta_2 f_{\widetilde{\beta}}^{(12)})\widehat{r}\psi + 2(\nabla_1 f_{\widetilde{\beta}}^{(12)})\cdot\nabla_1\widehat{r}\psi + 2(\nabla_2 f_{\widetilde{\beta}}^{(12)})\cdot\nabla_2\widehat{r}\psi \\
&\quad = \left(w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)}\right) f_{\widetilde{\beta}}^{(12)}\widehat{r}\psi - 2(\nabla_1 g_{\widetilde{\beta}}^{(12)})\cdot\nabla_1\widehat{r}\psi - 2(\nabla_2 g_{\widetilde{\beta}}^{(12)})\cdot\nabla_2\widehat{r}\psi,
\end{aligned}$$

hence

$$(62) = -4N(N-1)\Im\left\langle\!\!\left\langle \psi, (\nabla_1 g_{\widetilde{\beta}}^{(12)})\cdot\nabla_1\widehat{r}\psi\right\rangle\!\!\right\rangle \tag{69}$$

$$+N(N-1)\Im\left\langle\!\!\left\langle \psi, \left(w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)}\right) f_{\widetilde{\beta}}^{(12)}\widehat{r}\psi\right\rangle\!\!\right\rangle. \tag{70}$$

We now identify some of the terms in $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_\xi(t)|$ with the expressions in Proposition 3.2: (63) = $\gamma_a(t)$, (69) = $\gamma_c(t)$, (66) + (68) = $\gamma_d(t)$, (67) = $\gamma_e(t)$ and (65) = $\gamma_f(t)$. The remaining terms are $\gamma_a^<(t)$, $\gamma_b^<(t)$, (64) and (70). The latter yield

$$\gamma_b^<(t) + (64) + (70)$$

$$
\begin{aligned}
&= N(N-1)\Im\Big(- \Big\langle\!\!\Big\langle \psi, Z^{(12)}\widehat{r}\psi \Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle \psi, (1-f_{\widetilde{\beta}}^{(12)}) \left[Z^{(12)}, \widehat{r}\right] \psi \Big\rangle\!\!\Big\rangle \\
&\qquad + \Big\langle\!\!\Big\langle \psi, (w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)}) f_{\widetilde{\beta}}^{(12)} \widehat{r}\psi \Big\rangle\!\!\Big\rangle \Big) \\
&= N(N-1)\Im\Big(- \Big\langle\!\!\Big\langle \psi, g_{\widetilde{\beta}}^{(12)}\widehat{r} Z^{(12)}\psi \Big\rangle\!\!\Big\rangle - \Big\langle\!\!\Big\langle Z^{(12)} f_{\widetilde{\beta}}^{(12)}\psi, \widehat{r}\psi \Big\rangle\!\!\Big\rangle \\
&\qquad + \Big\langle\!\!\Big\langle (w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)}) f_{\widetilde{\beta}}^{(12)}\psi, \widehat{r}\psi \Big\rangle\!\!\Big\rangle \Big).
\end{aligned}
$$

Observing that

$$
Z^{(12)} f_{\widetilde{\beta}}^{(12)} = \left(w_\mu^{(12)} - U_{\widetilde{\beta}}^{(12)}\right) f_{\widetilde{\beta}}^{(12)} + U_{\widetilde{\beta}}^{(12)} f_{\widetilde{\beta}}^{(12)} - \tfrac{b}{N-1}\left(|\Phi(x_1)|^2 + |\Phi(x_2)|^2\right) f_{\widetilde{\beta}}^{(12)},
$$

we conclude

$$
\begin{aligned}
&\gamma_b^<(t) + (64) + (70) \qquad (71)\\
&\quad = -N(N-1)\Im \Big\langle\!\!\Big\langle \psi, g_{\widetilde{\beta}}^{(12)}\widehat{r} Z^{(12)}\psi \Big\rangle\!\!\Big\rangle \\
&\qquad -N(N-1)\Im \Big\langle\!\!\Big\langle \psi, \left(U_{\widetilde{\beta}}^{(12)} - \tfrac{b}{N-1}\left(|\Phi(x_1)|^2 + |\Phi(x_2)|^2\right)\right)(1-g_{\widetilde{\beta}}^{(12)})\widehat{r}\psi \Big\rangle\!\!\Big\rangle \\
&\quad = -N(N-1)\Im \Big\langle\!\!\Big\langle \psi, g_{\widetilde{\beta}}^{(12)}\widehat{r} Z^{(12)}\psi \Big\rangle\!\!\Big\rangle \qquad (72)\\
&\qquad -N\Im \Big\langle\!\!\Big\langle \psi, b(|\Phi(x_1)|^2 + |\Phi(x_2)|^2) g_{\widetilde{\beta}}^{(12)}\widehat{r}\psi \Big\rangle\!\!\Big\rangle \qquad (73)\\
&\qquad -N\Im \Big\langle\!\!\Big\langle \psi, (b_{\widetilde{\beta}} - b)(|\Phi(x_1)|^2 + |\Phi(x_2)|^2)\widehat{r}\psi \Big\rangle\!\!\Big\rangle \qquad (74)\\
&\qquad -N(N-1)\Im \Big\langle\!\!\Big\langle \psi, \widetilde{Z}^{(12)}\widehat{m}\psi \Big\rangle\!\!\Big\rangle, \qquad (75)
\end{aligned}
$$

where we have used the fact that $\Im \Big\langle\!\!\Big\langle \psi, \widetilde{Z}^{(12)}\widehat{r}\psi \Big\rangle\!\!\Big\rangle = \Im \Big\langle\!\!\Big\langle \psi, \widetilde{Z}^{(12)}\widehat{m}\psi \Big\rangle\!\!\Big\rangle$ as in (60). Hence (72) + (73) + (74) = $\gamma_b(t)$ and $\gamma_a^<(t) + (75) = \gamma^<(t)$. □

4.5 Proof of Proposition 3.3

4.5.1 Proof of the bound for $\gamma^<(t)$

The main tool for the estimate of $\gamma^<(t)$ is Proposition 3.5 from [4], which we apply to the interaction potential $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ (which, given w, is completely determined by a choice for μ and $\widetilde{\beta}$, cf. Definitions 3.4 and 3.5). Let us therefore first verify that the assumptions of this proposition are fulfilled, i.e. that

(a) $\mu^{\widetilde{\beta}}/\varepsilon \to 0$, $\varepsilon^2/\mu^{\widetilde{\beta}} \to 0$ and $\xi \leq \frac{\widetilde{\beta}}{4}$ (for ξ from Definition 3.3),

(b) the family $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ is contained in $\mathcal{W}_{\widetilde{\beta},\eta}$ for some $\eta > 0$.

We will in the sequel drop the μ-dependence of the family members and simply write $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ instead of $(U_{\widetilde{\beta}} f_{\widetilde{\beta}})(\mu)$. Part (a) is satisfied since $\mu^{\widetilde{\beta}}/\varepsilon \to 0$ because $\widetilde{\beta} > \frac{5}{6} > \frac{1}{2}$. Further, $\varepsilon^2/\mu^{\widetilde{\beta}} = \left(N\varepsilon^\delta\right)^{\widetilde{\beta}} \varepsilon^{2-\widetilde{\beta}(2+\delta)} < \left(N\varepsilon^\delta\right)^{\widetilde{\beta}} \to 0$ because $\widetilde{\beta} \leq \frac{2}{2+\delta}$, and finally $\xi < \frac{\widetilde{\beta}}{6}$ by assumption. Part (b) is proven in Lemma 4.16.

Proposition 3.5 in [4] implies that for any $\beta_1 \in (0, \widetilde{\beta}]$, $\gamma_a^<(t)$ and $\gamma_b^<(t)$ are bounded by

$$\begin{aligned}|\gamma_a^<(t)| + |\gamma_b^<(t)| \lesssim \mathfrak{e}(t) \exp\left\{\mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s\right\}\Big(\big|E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - \mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)\big| \\ + \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle + \tfrac{\mu^{\widetilde{\beta}}}{\varepsilon} + \Big(\tfrac{\varepsilon^2}{\mu^{\widetilde{\beta}}}\Big)^{\frac12} + N^{-\frac{\beta_1}{2}} + N^{-1+\beta_1+\xi} + \mu^\eta\Big),\end{aligned} \tag{76}$$

where $E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$ and $\mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$ denote the respective quantities corresponding to (9) and (10) but with w_μ replaced by $U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ and b by $b(U_{\widetilde{\beta}} f_{\widetilde{\beta}})$. Note that the energy difference $\big|E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - \mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)\big|$ enters only in the estimate of $\gamma_b^<(t)$, exclusively in the term (24) in [4, Proposition 3.4], which is given by

$$-2N(N-1)\Im \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), q_1^\Phi q_2^\Phi \widehat{m}^a_{-1} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} (U_{\widetilde{\beta}} f_{\widetilde{\beta}})^{(12)} p_2 p_1^{\chi^\varepsilon} q_1^\Phi \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle. \tag{77}$$

To obtain a bound in terms of $|E^\psi(t) - \mathcal{E}^\Phi(t)|$ instead of $|E^\psi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t) - \mathcal{E}^\Phi_{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(t)|$, we need a new estimate of (77) by means of Lemma 4.12.

Define $\widehat{l} := N\widehat{m}^a_{-1}$. We apply Lemma 4.14c and 4.14d with the choice $\beta_1 = 0$, i.e. $\overline{\Theta}_0 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\overline{h}_0 = \overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}$, where $p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} (U_{\widetilde{\beta}} f_{\widetilde{\beta}})^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} = p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} \overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(x_1 - x_2)$. Integrating by parts and subsequently inserting the identity $\mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}}_1}$ before $\partial_{x_1} q_1^\Phi \psi$ yields

$$\begin{aligned}(77) &\lesssim N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}_0^{(12)} (\tfrac{\mathrm{d}^2}{\mathrm{d}x_1^2}\overline{h}_0^{(12)}) p_2 p_1^{\chi^\varepsilon} q_1^\Phi \psi \right\rangle\!\!\right\rangle\right| \\
&\le N\left|\left\langle\!\!\left\langle \mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi, q_2^\Phi \overline{\Theta}_0^{(12)} (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}) p_2 p_1^{\chi^\varepsilon} \widehat{l}_1 q_1^\Phi \psi \right\rangle\!\!\right\rangle\right| && (78)\\
&\quad + N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{\Theta}_0^{(12)} (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}) p_2 p_1^{\chi^\varepsilon} \mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi \right\rangle\!\!\right\rangle\right| && (79)\\
&\quad + N\left|\left\langle\!\!\left\langle \partial_{x_1} q_1^\Phi \psi, \mathbb{1}_{\overline{\mathcal{A}}_1} q_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}) \overline{\Theta}_0^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} p_2^\Phi \widehat{l}_1 q_1^\Phi \psi \right\rangle\!\!\right\rangle\right| && (80)\\
&\quad + N\left|\left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{A}}_1} p_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}) \overline{\Theta}_0^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} q_2^\Phi \widehat{l} q_1^\Phi \psi, \partial_{x_1} q_1^\Phi \psi \right\rangle\!\!\right\rangle\right| && (81)\\
&\quad + N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{\Theta}_0^{(12)}) (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)}) p_2 p_1^{\chi^\varepsilon} q_1^\Phi \psi \right\rangle\!\!\right\rangle\right|. && (82)\end{aligned}$$

To estimate (78), note that $\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi$ and $\widehat{l}_1 p_1^{\chi^\varepsilon} q_1^\Phi \psi$ are symmetric in $\{z_2, ..., z_N\}$, hence Lemma 4.4 implies

$$\begin{aligned}(78) &\lesssim N\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi\| \|p_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1}\overline{h}_0^{(12)})\|_{\mathrm{op}} \left(\|\widehat{l}_1 q_1^\Phi q_2^\Phi \psi\| + N^{-\frac12}\|\widehat{l}_1 q_1^\Phi \psi\|\right) \\
&\overset{4.14c}{\lesssim} \mathfrak{e}(t)\left(\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi\|^2 + \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle + N^{-\frac12}\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi\|\right)\end{aligned}$$

by Lemma 4.14c because $\|\widehat{l}_1 q_1^\Phi \psi\| \lesssim 1$ by Lemma 4.1c and $\|\widehat{l} q_1^\Phi q_2^\Phi \psi\| \lesssim \|\widehat{n}\psi\|$ by Lemma 4.1d. (79) is immediately controlled by

$$(79) \lesssim \mathfrak{e}(t)\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi\| \,\langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle^{\frac12} \lesssim \mathfrak{e}(t)\left(\|\mathbb{1}_{\mathcal{A}_1}\partial_{x_1} q_1^\Phi \psi\|^2 + \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle\right).$$

Similarly, (82) $\lesssim \mathfrak{e}(t)\,\langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle$. To estimate the two remaining terms, let

$$(s_2^\Phi, t_2^\Phi) \in \{(p_2^\Phi, q_2^\Phi), (q_2^\Phi, p_2^\Phi)\}$$

and $\widehat{l}_j \in \{\widehat{l}, \widehat{l}_1\}$. By Lemma 4.13b and Lemma 4.7a,

$$\begin{aligned}
&(80) + (81) \\
&\lesssim N\|\partial_{x_1} q_1^\Phi \psi\| \|\mathbb{1}_{\overline{\mathcal{A}}_1} s_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1} \overline{h}_0^{(12)}) \overline{\Theta}_0^{(12)} t_2^\Phi p_2^{\chi^\varepsilon} p_1^{\chi^\varepsilon} \widehat{l}_j q_1^\Phi \psi\| \\
&\lesssim N\mathfrak{e}(t)\mu^{d-\frac{1}{3}} \Big(\|s_2^\Phi (\tfrac{\mathrm{d}^2}{\mathrm{d}x_1^2} \overline{h}_0^{(12)}) \overline{\Theta}_0^{(12)} t_2^\Phi p_2^{\chi^\varepsilon} p_1^{\chi^\varepsilon} \widehat{l}_j q_1^\Phi \psi\| \\
&\qquad + \|s_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1} \overline{h}_0^{(12)}) (\tfrac{\mathrm{d}}{\mathrm{d}x_1} \overline{\Theta}_0^{(12)}) t_2^\Phi p_2^{\chi^\varepsilon} p_1^{\chi^\varepsilon} \widehat{l}_j q_1^\Phi \psi\| \\
&\qquad + \|s_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1} \overline{h}_0^{(12)}) \overline{\Theta}_0^{(12)} t_2^\Phi p_2^{\chi^\varepsilon} p_1^{\chi^\varepsilon} \partial_{x_1} \widehat{l}_j q_1^\Phi \psi\| \\
&\qquad + \varepsilon \|s_2^\Phi (\tfrac{\mathrm{d}}{\mathrm{d}x_1} \overline{h}_0^{(12)}) \overline{\Theta}_0^{(12)} t_2^\Phi p_2^{\chi^\varepsilon} \nabla_{y_1} p_1^{\chi^\varepsilon} \widehat{l}_j q_1^\Phi \psi\| \Big) \\
&\lesssim \mathfrak{e}(t) N \mu^{d-\frac{1}{3}} \|p_2^\Phi \overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(x_1 - x_2)\|_{\mathrm{op}} \\
&\quad + \mathfrak{e}(t) N \mu^{d-\frac{1}{3}} \|(\tfrac{\mathrm{d}}{\mathrm{d}x_1} \overline{h}_0^{(12)}) p_2^\Phi\|_{\mathrm{op}} \Big(\|\tfrac{\mathrm{d}}{\mathrm{d}x} \overline{\Theta}_0\|_{L^\infty(\mathbb{R})} + N^\xi \mathfrak{e}^2(t) + \varepsilon \|\nabla_{y_1} p_1^{\chi^\varepsilon}\|_{\mathrm{op}} \Big)
\end{aligned}$$

as $\|\partial_{x_1} \widehat{l}_j q_1^\Phi \psi\| \lesssim \|\widehat{l}_j\|_{\mathrm{op}} \|\partial_{x_1} q_1^\Phi \psi\| \lesssim N^\xi \mathfrak{e}(t)$ by Lemma 4.2b and Lemma 4.1. The last line is bounded by $\mathfrak{e}^3(t)\mu^{d-\frac{1}{3}} N^\xi$ by Lemma 4.14c and 4.14d and Lemma 4.7a. Finally, note that $|x_1 - x_2| < R_{\widetilde{\beta}} \lesssim \mu^{\widetilde{\beta}}$ for $(x_1 - x_2) \in \operatorname{supp} \overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}$, hence

$$\begin{aligned}
\|p_2^\Phi \overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(x_1 - x_2)\|_{\mathrm{op}} &= \|p_2^\Phi \mathbb{1}_{|\cdot| < R_{\widetilde{\beta}}}(x_1 - x_2)\|_{\mathrm{op}} \|\overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}\|_{L^\infty(\mathbb{R})} \\
&\overset{4.6\mathrm{d}}{\lesssim} \mathfrak{e}(t) \|\overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}\|_{L^\infty(\mathbb{R})} \|\mathbb{1}_{|\cdot| < R_{\widetilde{\beta}}}\|_{L^2(\mathbb{R})} \lesssim \mathfrak{e}(t) N^{-1} \mu^{-\frac{\widetilde{\beta}}{2}}.
\end{aligned}$$

The last bound follows since $\|\mathbb{1}_{|x_1 - x_2| < R_{\widetilde{\beta}}}\|_{L^2(\mathbb{R})} \lesssim \mu^{\frac{\widetilde{\beta}}{2}}$ and as

$$\begin{aligned}
\left|\overline{U_{\widetilde{\beta}} f_{\widetilde{\beta}}}(x)\right| &= \int_{\mathbb{R}^2} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \int_{\mathbb{R}^2} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 (U_{\widetilde{\beta}} f_{\widetilde{\beta}})(x, y_1 - y_2) \\
&\leq \varepsilon^{-2} \int_{\mathbb{R}^2} \mathrm{d}y_1 |\chi^\varepsilon(y)|^2 \int_{|y| < R_{\widetilde{\beta}}} \mathrm{d}y \|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon^{-2} \mu^{1-\widetilde{\beta}},
\end{aligned}$$

where we have used that $|y| < R_{\widetilde{\beta}}$ for $(x, y) \in \operatorname{supp} U_{\widetilde{\beta}} f_{\widetilde{\beta}}$ as above and that χ^ε is normalised and $\|U_{\widetilde{\beta}} f_{\widetilde{\beta}}\|_{L^\infty(\mathbb{R}^3)} \lesssim \mu^{1-3\widetilde{\beta}}$. Hence,

$$(77) \lesssim \mathfrak{e}(t) \exp\left\{ \mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s \right\} \left(\alpha_\xi^<(t) + (N\varepsilon^\delta)^{1-\widetilde{\beta}} + N^{-1+\widetilde{\beta}} + \mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}} \right), \quad (83)$$

where we have used Lemma 4.12 and the fact that $\mu^{d-\frac{1}{3}} N^\xi < \mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}}$ and $N^{-\frac{1}{2}} < \mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}}$.

Combining this new bound for (77) with the remaining estimates of [4, Proposition 3.5], we find

$$|\gamma^<(t)| \lesssim \mathfrak{e}(t) \exp\left\{ \mathfrak{e}^2(t) + \int_0^t \mathfrak{e}^2(s)\,\mathrm{d}s \right\} \times$$

$$\times \left(\alpha_\xi^<(t) + \left(N\varepsilon^\delta\right)^{1-\widetilde{\beta}} + N^{-1+\widetilde{\beta}+\xi} + \mu^{d-\frac{1}{3}-\frac{\widetilde{\beta}}{2}} \right),$$

where we have chosen $\beta_1 = \widetilde{\beta}$ and used that $-1 + \frac{3\widetilde{\beta}}{2} + \xi > 0$, $\mu^{1-\widetilde{\beta}} < N^{-1+\widetilde{\beta}+\xi}$, $\frac{\mu^{\widetilde{\beta}}}{\varepsilon} < N^{-\frac{\widetilde{\beta}}{2}} < N^{-1+\widetilde{\beta}+\xi}$ and $\varepsilon\mu^{-\frac{\widetilde{\beta}}{2}} < (N\varepsilon^\delta)^{\frac{\widetilde{\beta}}{2}} < (N\varepsilon^\delta)^{1-\widetilde{\beta}}$. □

4.5.2 Proof of the bound for $\gamma_a(t)$

By definition of $\widehat{r}$ and with Lemma 4.7f, Lemma 4.10b and Lemma 4.1b, we compute

$$\begin{aligned}
&|(28)| \\
&\lesssim N^3 \left| \left\langle\!\!\left\langle \left(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\right)\psi, g_{\widetilde{\beta}}^{(12)} \left(p_1p_2\widehat{m}^b + (p_1q_2+q_1p_2)\widehat{m}^a\right)\psi \right\rangle\!\!\right\rangle \right| \\
&\quad + N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} \left(p_1p_2\widehat{m}^b + (p_1q_2+q_1p_2)\widehat{m}^a\right) \left(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\right)\psi \right\rangle\!\!\right\rangle \right| \\
&\leq 2N^3 \|(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0)))\psi\| \|g_{\widetilde{\beta}}^{(12)} p_1\|_{\text{op}} \left(\|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}}\right) \\
&\lesssim \mathfrak{e}^3(t) N^{1+\xi-\frac{\widetilde{\beta}}{2}} \varepsilon^{2+\widetilde{\beta}} = \mathfrak{e}^3(t) \left(N\varepsilon^\delta\right)^{1+\xi-\frac{\widetilde{\beta}}{2}} \varepsilon^{2+\widetilde{\beta}-\delta(1+\xi-\frac{\widetilde{\beta}}{2})} < \mathfrak{e}^3(t)\varepsilon^2
\end{aligned}$$

as $\widetilde{\beta} - \delta(1+\xi-\frac{\widetilde{\beta}}{2}) > 0$ and since $1+\xi-\frac{\widetilde{\beta}}{2} > 0$. □

4.5.3 Proof of the bound for $\gamma_b(t)$

Estimate of (29). By Lemma 4.10b, Lemma 4.1b and Lemma 4.5a and as $-1-\frac{\widetilde{\beta}}{2}+\xi < 0$,

$$\begin{aligned}
|(29)| &\lesssim N\|\Phi\|^2_{L^\infty(\mathbb{R})} \|g_{\widetilde{\beta}}^{(12)} p_1\|_{\text{op}} \left(\|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}}\right) \\
&\lesssim \mathfrak{e}^3(t) N^{-1-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{1+\widetilde{\beta}} < \mathfrak{e}^3(t)\varepsilon^{1+\widetilde{\beta}}.
\end{aligned}$$

Estimate of (30). Note that $b_{\widetilde{\beta}} = b(U_{\widetilde{\beta}} f_{\widetilde{\beta}}) = b$ by (46), hence $(30) = 0$.

Estimate of (31). By definition of $\widehat{r}$ and due to the symmetry of ψ,

$$\begin{aligned}
|(31)| &\leq N^2 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_1 \widehat{m}^b p_2 Z^{(12)}\psi \right\rangle\!\!\right\rangle + 2\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_1 q_2 \widehat{m}^a p_1 Z^{(12)}\psi \right\rangle\!\!\right\rangle \right| \\
&\lesssim N^2 \|p_1 g_{\widetilde{\beta}}^{(12)}\|_{\text{op}} \left(\|\widehat{m}^a\|_{\text{op}} + \|\widehat{m}^b\|_{\text{op}}\right) \times \\
&\qquad\qquad \times \left\| p_1 \left(w_\mu^{(12)} - \tfrac{b}{N-1}(|\Phi(x_1)|^2 + |\Phi(x_2)|^2)\right)\psi \right\| \\
&\lesssim \mathfrak{e}(t) N^{-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{1+\widetilde{\beta}} \left(\|p_1 w_\mu^{(12)}\psi\| + N^{-1}\|\Phi\|^2_{L^\infty(\mathbb{R})}\right) \\
&\lesssim \mathfrak{e}^3(t) N^{-1-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{1+\widetilde{\beta}} < \mathfrak{e}^3(t)\varepsilon^{1+\widetilde{\beta}}
\end{aligned}$$

as a consequence of Lemma 4.10b, Lemma 4.1b, Lemma 4.7e and Lemma 4.5a.

4.5.4 Proof of the bound for $\gamma_c(t)$

$$
\begin{aligned}
&|(32)|\\
\lesssim\;& N^2\left|\left\langle\!\!\left\langle \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1-z_2)\psi, (\nabla_1 g_{\widetilde{\beta}}^{(12)})\cdot\left(p_2\nabla_1(p_1\widehat{m}^b+q_1\widehat{m}^a)\psi+\nabla_1 p_1 q_2\widehat{m}^a\psi\right)\right\rangle\!\!\right\rangle\right|\\
\le\;& N^2\|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1-z_2)\psi\|\Big(\|(\nabla_1 g_{\widetilde{\beta}}^{(12)})p_2\|_{\mathrm{op}}\|\nabla_1 p_1\|_{\mathrm{op}}\|\widehat{m}^b\|_{\mathrm{op}}\\
&\qquad+\|(\nabla_1 g_{\widetilde{\beta}}^{(12)})\nabla_1 p_1\|_{\mathrm{op}}\|\widehat{m}^a\|_{\mathrm{op}}+\|(\nabla_1 g_{\widetilde{\beta}}^{(12)})p_2\|_{\mathrm{op}}\|\nabla_1 q_1\widehat{m}^a\psi\|\Big)\\
\lesssim\;& \mathfrak{e}^2(t)\varepsilon^{2\widetilde{\beta}-\frac{5}{3}}N^{\frac{1}{2}+\xi-\widetilde{\beta}} \;<\; \mathfrak{e}^2(t)N^{\frac{1}{2}+\xi-\widetilde{\beta}} \;<\; \mathfrak{e}^2(t)N^{-1+\xi+\widetilde{\beta}}
\end{aligned}
$$

because $2\widetilde{\beta}-\frac{5}{3}>0$ and $\frac{1}{2}-\widetilde{\beta}<-1+\widetilde{\beta}$ as $\widetilde{\beta}>\frac{5}{6}$. In the third step, we have used Lemma 4.10e, Lemma 4.1b, Lemma 4.7a, Lemma 4.11 and the fact that

$$
\begin{aligned}
\|\nabla_1 q_1\widehat{m}^a\psi\| &\overset{4.2\mathrm{b}}{\le} \|p_1\widehat{m}_1^a\nabla_1(1-p_1)\psi\|+\|q_1\widehat{m}^a\nabla_1(1-p_1)\psi\|\\
&\overset{4.1\mathrm{a}}{\lesssim} \|\widehat{m}^a\|_{\mathrm{op}}\left(\|\nabla_1\psi\|+\|\nabla_1 p_1\psi\|\right) \overset{4.7\mathrm{a}}{\lesssim} N^{-1+\xi}\varepsilon^{-1}.
\end{aligned}
$$

□

4.5.5 Proof of the bound for $\gamma_d(t)$

Estimate of (33). With Lemma 4.10b, Lemma 4.1b and Lemma 4.5a,

$$
\begin{aligned}
|(33)| \;\lesssim\;& N^3\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}p_1p_2 b\left[|\Phi(x_3)|^2,\widehat{m}^b\right]\psi\right\rangle\!\!\right\rangle\right|\\
&+N^3\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}\left(p_1q_2+q_1p_2\right)b\left[|\Phi(x_3)|^2,\widehat{m}^a\right]\psi\right\rangle\!\!\right\rangle\right|\\
\lesssim\;& N^3\|g_{\widetilde{\beta}}^{(12)}p_1\|_{\mathrm{op}}\|\Phi\|^2_{L^\infty(\mathbb{R})}\left(\|\widehat{m}^a\|_{\mathrm{op}}+\|\widehat{m}^b\|_{\mathrm{op}}\right)\\
\lesssim\;& \mathfrak{e}^3(t)N^{1+\xi-\frac{\widetilde{\beta}}{2}}\varepsilon^{1+\widetilde{\beta}} < \mathfrak{e}^3(t)\left(N\varepsilon^{\delta}\right)^{1+\xi-\frac{\widetilde{\beta}}{2}}
\end{aligned}
$$

analogously to the estimate of $\gamma_a(t)$.

Estimate of (34). Observe first that

$$
\widehat{r}=\widehat{m}^b p_1p_2+\widehat{m}^a(p_1(1-p_2)+(1-p_1)p_2)=\widehat{m}^a(p_1+p_2)+(\widehat{m}^b-2\widehat{m}^a)p_1p_2.
$$

As a consequence,

$$
\begin{aligned}
|(34)| \;\lesssim\;& N^3\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}[w_\mu^{(13)},\widehat{r}]\psi\right\rangle\!\!\right\rangle\right|\\
\le\;& N^3\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}p_2[w_\mu^{(13)},\widehat{m}^a]\psi\right\rangle\!\!\right\rangle\right| && (84)\\
&+N^3\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}w_\mu^{(13)}p_1\widehat{m}^a\psi\right\rangle\!\!\right\rangle\right| && (85)\\
&+N^3\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}p_1(\widehat{m}^a+p_2(\widehat{m}^b-2\widehat{m}^a))p_1w_\mu^{(13)}\psi\right\rangle\!\!\right\rangle\right| && (86)
\end{aligned}
$$

$$+N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} w_\mu^{(13)} p_2 p_1 (\widehat{m}^b - 2\widehat{m}^a)\psi \right\rangle\!\!\right\rangle \right|. \tag{87}$$

We estimate (84) to (87) separately.

$$(84) = N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_2 \left[w_\mu^{(13)}, p_1 p_3 (\widehat{m}^a - \widehat{m}_2^a) + (p_1 q_3 + q_1 p_3)(\widehat{m}^a - \widehat{m}_1^a) \right] \psi \right\rangle\!\!\right\rangle \right|.$$

By definition of $\widehat{m}^c$ and $\widehat{m}^d$,

$$\begin{aligned} p_1 p_3 (\widehat{m}^a - \widehat{m}_2^a) &= p_1 p_3 \widehat{m}^d + p_1 p_3 \left(m^a (N+1) P_{N-1} + m(N+2) P_N \right) \\ &= p_1 p_3 \widehat{m}^d, \\ (p_1 q_3 + q_1 p_3)(\widehat{m}^a - \widehat{m}_1^a) &= (p_1 q_3 + q_1 p_3)\widehat{m}^c. \end{aligned}$$

This leads to

$$\begin{aligned} (84) \;\le\;& N^3 \left| \left\langle\!\!\left\langle w_\mu^{(13)} \psi, g_{\widetilde{\beta}}^{(12)} p_2 \mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3) \Big(p_1 p_3 \widehat{m}^d + (p_1 q_3 + q_1 p_3)\widehat{m}^c \Big) \psi \right\rangle\!\!\right\rangle \right| \\ & + N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_2 \Big(p_1 p_3 \widehat{m}^d + (p_1 q_3 + q_1 p_3)\widehat{m}^c \Big) w_\mu^{(13)} \psi \right\rangle\!\!\right\rangle \right| \\ \lesssim\;& N^3 \| g_{\widetilde{\beta}}^{(12)} p_2 \|_{\mathrm{op}} \left(\|\widehat{m}^d\|_{\mathrm{op}} + \|\widehat{m}^c\|_{\mathrm{op}} \right) \times \\ & \times \left(\|w_\mu^{(13)}\psi\| \|\mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3) p_1\|_{\mathrm{op}} + \|p_1 w_\mu^{(13)} \psi\| \right) \\ \lesssim\;& \mathfrak{e}^3(t) N^{-1+3\xi-\frac{\widetilde{\beta}}{2}} \varepsilon^{1+\widetilde{\beta}} \;<\; \mathfrak{e}^3(t) \varepsilon^{1+\widetilde{\beta}} \end{aligned}$$

by Lemma 4.7, Lemma 4.10b and Lemma 4.1b. In order to estimate (85), observe first that $g_{\widetilde{\beta}}^{(12)} w_\mu^{(13)} \neq 0$ implies $|z_2 - z_3| \lesssim R_{\widetilde{\beta}}$. This can be seen as follows: $g_{\widetilde{\beta}}^{(12)} \neq 0$ implies $|z_1 - z_2| \le R_{\widetilde{\beta}}$ and $w_\mu^{(13)} \neq 0$ implies $|z_1 - z_3| \le \mu$. Together, this yields

$$|z_2 - z_3| \le |z_1 - z_2| + |z_1 - z_3| \le R_{\widetilde{\beta}} + \mu \le 2R_{\widetilde{\beta}}.$$

Consequently, (85) can be written as

$$\begin{aligned} (85) \;=\;& N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} w_\mu^{(13)} \mathbb{1}_{B_{2R_{\widetilde{\beta}}}(0)}(z_2 - z_3) p_1 \widehat{m}^a \psi \right\rangle\!\!\right\rangle \right| \\ =\;& N^3 \left| \left\langle\!\!\left\langle p_1 \mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3) w_\mu^{(13)} g_{\widetilde{\beta}}^{(12)} \psi, \mathbb{1}_{B_{2R_{\widetilde{\beta}}}(0)}(z_2 - z_3) \widehat{m}^a \psi \right\rangle\!\!\right\rangle \right| \\ \le\;& N^3 \| p_1 \mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3)\|_{\mathrm{op}} \|g_{\widetilde{\beta}}\|_{L^\infty(\mathbb{R}^3)} \|w_\mu^{(13)}\psi\| \|\mathbb{1}_{B_{2R_{\widetilde{\beta}}}(0)}(z_2 - z_3)\widehat{m}^a \psi\| \\ \lesssim\;& \mathfrak{e}^3(t) N^{1+\xi-\widetilde{\beta}} \varepsilon^{2\widetilde{\beta}-\frac{2}{3}} \;<\; \mathfrak{e}^3(t) \left(N\varepsilon^\delta \right)^{1+\xi-\widetilde{\beta}} \end{aligned}$$

by Lemma 4.7 and as $2\widetilde{\beta} - \frac{2}{3} - \delta(1 + \xi - \widetilde{\beta}) > 0$. We have used that as in the proof of Lemma 4.10e,

$$\begin{aligned} \|\mathbb{1}_{B_{2R_{\widetilde{\beta}}}(0)}(z_2 - z_3)\widehat{m}^a \psi\|^2 \;\lesssim\;& \varepsilon^{-\frac{4}{3}} \mu^{2\widetilde{\beta}} (\|\partial_{x_1} \widehat{m}^a \psi\|^2 + \varepsilon^2 \|\nabla_{y_1} \widehat{m}^a \psi\|^2) \\ \lesssim\;& N^{-2+2\xi-2\widetilde{\beta}} \varepsilon^{4\widetilde{\beta}-\frac{4}{3}} \mathfrak{e}^2(t) \end{aligned}$$

because by Lemma 4.1b, Lemma 4.2b and Lemma 4.7,

$$\|\partial_{x_1}\widehat{m}^a\psi\| \lesssim \|\widehat{m}^a\|_{\mathrm{op}}\left(\|\partial_{x_1}p_1\psi\| + \|\partial_{x_1}(1-p_1)\psi\|\right) \lesssim N^{-1+\xi}\mathfrak{e}(t)$$

and analogously $\|\nabla_{y_1}\widehat{m}^a\psi\| \lesssim N^{-1+\xi}\varepsilon^{-1}$. The remaining two terms (86) and (87) can be estimated as

$$\begin{aligned}
(86) &\lesssim N^3\|g_{\widetilde{\beta}}^{(12)}p_1\|_{\mathrm{op}}\left(\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}}\right)\|p_1 w_\mu^{(13)}\psi\| \\
&\lesssim \mathfrak{e}^3(t)N^{-\frac{\widetilde{\beta}}{2}+\xi}\varepsilon^{1+\widetilde{\beta}} < \mathfrak{e}^3(t)\varepsilon^{1+\widetilde{\beta}}, \\
(87) &= N^3\left|\left\langle\!\!\left\langle w_\mu^{(13)}\psi, g_{\widetilde{\beta}}^{(12)}p_2\mathbb{1}_{\mathrm{supp}\,w_\mu}(z_1-z_3)p_1(\widehat{m}^b-2\widehat{m}^a)\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N^3\|w_\mu^{(13)}\psi\|\|g_{\widetilde{\beta}}^{(12)}p_2\|_{\mathrm{op}}\|\mathbb{1}_{\mathrm{supp}\,w_\mu}(z_1-z_3)p_1\|_{\mathrm{op}}\left(\|\widehat{m}^b\|_{\mathrm{op}} + 2\|\widehat{m}^a\|_{\mathrm{op}}\right) \\
&\lesssim \mathfrak{e}^3(t)N^{-\frac{\widetilde{\beta}}{2}+\xi}\varepsilon^{1+\widetilde{\beta}} \;<\; \mathfrak{e}^3(t)\varepsilon^{1+\widetilde{\beta}},
\end{aligned}$$

where we used that $\xi < \frac{\widetilde{\beta}}{6}$ as well as Lemma 4.7, Lemma 4.1b and Lemma 4.10b. □

4.5.6 Proof of the bound for $\gamma_e(t)$

Using again Lemma 4.2c, $|\gamma_e(t)|$ can be written as

$$|(35)| \lesssim N^4\left|\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}\left[w_\mu^{(34)}, p_3p_4(\widehat{r}-\widehat{r}_2) + (p_3q_4+q_3p_4)(\widehat{r}-\widehat{r}_1)\right]\psi\right\rangle\!\!\right\rangle\right|. \tag{88}$$

By definition of $\widehat{r}$ and $\widehat{m}^{c/d/e/f}$, we obtain

$$\begin{aligned}
&p_3p_4(\widehat{r}-\widehat{r}_2) + (p_3q_4+q_3p_4)(\widehat{r}-\widehat{r}_1) \\
&\quad= (p_1q_2+q_1p_2)(p_3q_4+q_3p_4)\widehat{m}^c + (p_1q_2+q_1p_2)p_3p_4\widehat{m}^d \\
&\qquad+ p_1p_2(p_3q_4+q_3p_4)\widehat{m}^e + p_1p_2p_3p_4\widehat{m}^f.
\end{aligned}$$

Due to the symmetry of (88) under the exchanges $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$, this yields

$$|(35)| \lesssim N^4\left|\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}p_1q_2\left[w_\mu^{(34)}, p_3q_4\widehat{m}^c + p_3p_4\widehat{m}^d\right]\psi\right\rangle\!\!\right\rangle\right| \tag{89}$$

$$+N^4\left|\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}p_1p_2\left[w_\mu^{(34)}, p_3q_4\widehat{m}^e + p_3p_4\widehat{m}^f\right]\psi\right\rangle\!\!\right\rangle\right|, \tag{90}$$

where by Lemma 4.7e, Lemma 4.10b and Lemma 4.1b,

$$\begin{aligned}
(89) &\leq N^4\left|\left\langle\!\!\left\langle \psi, w_\mu^{(34)}p_3g_{\widetilde{\beta}}^{(12)}p_1q_2(q_4\widehat{m}^c + p_4\widehat{m}^d)\psi\right\rangle\!\!\right\rangle\right| \\
&\quad+N^4\left|\left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)}p_1q_2(q_4\widehat{m}^c + p_4\widehat{m}^d)p_3w_\mu^{(34)}\psi\right\rangle\!\!\right\rangle\right| \\
&\lesssim N^4\|p_3w_\mu^{(34)}\psi\|\|g_{\widetilde{\beta}}^{(12)}p_1\|_{\mathrm{op}}\left(\|\widehat{m}^c\|_{\mathrm{op}} + \|\widehat{m}^d\|_{\mathrm{op}}\right) \\
&\lesssim \mathfrak{e}^3(t)N^{-\frac{\widetilde{\beta}}{2}+3\xi}\varepsilon^{1+\widetilde{\beta}} \;<\; \mathfrak{e}^3(t)\varepsilon^{1+\widetilde{\beta}}
\end{aligned}$$

as $\xi < \frac{\widetilde{\beta}}{6}$. Analogously, one derives the same bound for (90). □

4.5.7 Proof of the bound for $\gamma_f(t)$

Finally, as a consequence of Lemma 4.1, Lemma 4.5a and Lemma 4.10,

$$\begin{aligned} |(36)| &\lesssim N^2 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_2 \left[b|\Phi(x_1)|^2, \widehat{m}^b p_1 + \widehat{m}^a q_1 \right] \psi \right\rangle\!\!\right\rangle \right| \\ &\quad + N^2 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} \left[b|\Phi(x_1)|^2, p_1 \widehat{m}^a \right] q_2 \psi \right\rangle\!\!\right\rangle \right| \\ &\lesssim N^2 \|\Phi\|^2_{L^\infty(\mathbb{R})} \left(\|p_2 g_{\widetilde{\beta}}^{(12)}\|_{\mathrm{op}} \left(\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}} \right) + \|g_{\widetilde{\beta}}^{(12)} \psi\| \|q_2 \widehat{m}^a \psi\| \right) \\ &\lesssim \mathfrak{e}^3(t) N^{-\frac{\widetilde{\beta}}{2} + \xi} \varepsilon^{1+\widetilde{\beta}} + \mathfrak{e}^2(t)\varepsilon \ \lesssim \ \mathfrak{e}^2(t)\varepsilon. \end{aligned}$$

□

4.6 Proof of Proposition 3.4

Using Lemma 4.1b and Lemma 4.10b, we estimate

$$N(N-1)\Re \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} \widehat{r} \psi \right\rangle\!\!\right\rangle \lesssim N^2 \|g_{\widetilde{\beta}}^{(12)} p_1\|_{\mathrm{op}} \left(\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}} \right) \lesssim \mathfrak{e}(t) N^{\xi - \frac{\widetilde{\beta}}{2}} \varepsilon^{1+\widetilde{\beta}}.$$

□

Acknowledgments

We thank Serena Cenatiempo, Maximilian Jeblick, Nikolai Leopold and Peter Pickl for helpful discussions. This work was supported by the German Research Foundation within the Research Training Group 1838 "Spectral Theory and Dynamics of Quantum Systems".

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B. Submitted manuscripts

B.1. Derivation of the 2d Gross–Pitaevskii equation for strongly confined 3d bosons

Derivation of the 2d Gross–Pitaevskii equation for strongly confined 3d bosons

Lea Boßmann*

Abstract

We study the dynamics of a system of N interacting bosons in a disc-shaped trap, which is realised by an external potential that confines the bosons in one spatial dimension to a region of order ε. The interaction is non-negative and scaled in such a way that its scattering length is of order $(N/\varepsilon)^{-1}$, while its range is proportional to $(N/\varepsilon)^{-\beta}$ with scaling parameter $\beta \in (0, 1]$. The choice $\beta = 1$ corresponds to the physically relevant Gross–Pitaevskii regime.
We consider the simultaneous limit $(N, \varepsilon) \to (\infty, 0)$ and assume that the system initially exhibits Bose–Einstein condensation. We prove that condensation is preserved by the N-body dynamics, where the time-evolved condensate wave function is the solution of a two-dimensional non-linear equation. The strength of the non-linearity depends on the scaling parameter β. For $\beta \in (0, 1)$, we obtain a cubic defocusing non-linear Schrödinger equation, while the choice $\beta = 1$ yields a Gross–Pitaevskii equation featuring the scattering length of the interaction. In both cases, the coupling parameter depends on the confining potential.

1 Introduction

Since two decades, it has been experimentally possible to realise quasi-two dimensional Bose gases in disc-shaped traps [14, 31, 33]. The study of such systems is physically of particular interest since they permit the detection of inherently two-dimensional effects and serve as models for different statistical physics phenomena [17, 18, 35]. In this article, our aim is to contribute to the mathematically rigorous understanding of such systems. We consider a Bose–Einstein condensate of N identical, non-relativistic, interacting bosons in a disc-shaped trap, which effectively confines the particles in one spatial direction to an interval of length ε. We study the dynamics of this system in the simultaneous limit $(N, \varepsilon) \to (\infty, 0)$, where the Bose gas becomes quasi two-dimensional. To describe the N bosons, we use the coordinates

$$z = (x, y) \in \mathbb{R}^{2+1},$$

where x denotes the two longitudinal dimensions and y is the transverse dimension. The confinement in the y-direction is modelled by the scaled potential $\frac{1}{\varepsilon^2} V^{\perp}\left(\frac{y}{\varepsilon}\right)$ for

*Fachbereich Mathematik, Eberhard Karls Universität Tübingen
Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: lea.bossmann@uni-tuebingen.de

$0 < \varepsilon \ll 1$ and some $V^\perp : \mathbb{R} \to \mathbb{R}$. In units such that $\hbar = 1$ and $m = \frac{1}{2}$, the Hamiltonian is given by

$$H_{\mu,\beta}(t) = \sum_{j=1}^{N} \left(-\Delta_j + \frac{1}{\varepsilon^2} V^\perp \left(\frac{y_j}{\varepsilon} \right) + V^\parallel(t, z_j) \right) + \sum_{1 \le i < j \le N} w_{\mu,\beta}(z_i - z_j)\,, \tag{1}$$

where Δ denotes the Laplace operator on $\mathbb{R}^3$ and $V^\parallel : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is an additional external potential, which may depend on time. The interaction $w_{\mu,\beta}$ between the particles is purely repulsive and scaled in dependence of the parameters N and ε. In this paper, we consider two fundamentally different scaling regimes, corresponding to different choices of the scaling parameter $\beta \in \mathbb{R}$: $\beta \in (0, 1)$ yields the non-linear Schrödinger (NLS) regime, while $\beta = 1$ is known as the Gross–Pitaevskii regime. Making use of the parameter

$$\mu := \frac{\varepsilon}{N}\,,$$

the Gross–Pitaevskii regime is realised by scaling an interaction $w : \mathbb{R}^3 \to \mathbb{R}$, which is compactly supported, spherically symmetric and non-negative, as

$$w_\mu(z) = \frac{1}{\mu^2} w \left(\frac{z}{\mu} \right)\,. \tag{2}$$

For the NLS regime, we will consider a more generic form of the interaction (see Definition 2.2). For the length of this introduction, let us focus on the special case

$$w_{\mu,\beta}(z) = \mu^{1-3\beta}\, w \left(\mu^{-\beta} z \right) \tag{3}$$

with $\beta \in (0, 1)$. Clearly, (2) equals (3) with the choice $\beta = 1$. Both scaling regimes describe very dilute gases, and we comment on their physical relevance below.

The N-body wave function $\psi^{N,\varepsilon}(t) \in L^2(\mathbb{R}^{3N})$ at time $t \in \mathbb{R}$ is determined by the Schrödinger equation

$$\begin{cases} \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi^{N,\varepsilon}(t) = H_{\mu,\beta}(t)\psi^{N,\varepsilon}(t) \\ \psi^{N,\varepsilon}(0) = \psi_0^{N,\varepsilon} \end{cases} \tag{4}$$

with initial datum $\psi_0^{N,\varepsilon} \in L^2_+(\mathbb{R}^{3N}) := \otimes_{\mathrm{sym}}^N L^2(\mathbb{R}^3)$. We assume that this initial state exhibits Bose–Einstein condensation, i.e., that the one-particle reduced density matrix $\gamma^{(1)}_{\psi_0^{N,\varepsilon}}$ of $\psi_0^{N,\varepsilon}$,

$$\gamma^{(1)}_{\psi_0^{N,\varepsilon}} := \mathrm{Tr}_{2,\dots,N} |\psi_0^{N,\varepsilon}\rangle\langle\psi_0^{N,\varepsilon}|\,, \tag{5}$$

converges to a projection onto the so-called condensate wave function $\varphi_0^\varepsilon \in L^2(\mathbb{R}^3)$. At low energies, the strong confinement in the transverse direction causes the condensate wave function to factorise in the limit $\varepsilon \to 0$ into a longitudinal part $\Phi_0 \in L^2(\mathbb{R}^2)$ and a transverse part $\chi^\varepsilon \in L^2(\mathbb{R})$,

$$\varphi_0^\varepsilon(z) = \Phi_0(x)\chi^\varepsilon(y).$$

The transverse part χ^ε is given by the normalised ground state of $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$, which is defined by

$$\left(-\tfrac{\mathrm{d}^2}{\mathrm{d}y^2} + \tfrac{1}{\varepsilon^2}V^\perp\left(\tfrac{\cdot}{\varepsilon}\right)\right)\chi^\varepsilon = \tfrac{E_0}{\varepsilon^2}\chi^\varepsilon\,.$$

Here, E_0 denotes the minimal eigenvalue of the unscaled operator $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V^\perp$, corresponding to the normalised ground state χ. The relation of χ^ε and χ is

$$\chi^\varepsilon(y) := \tfrac{1}{\sqrt{\varepsilon}}\,\chi\left(\tfrac{y}{\varepsilon}\right)\,. \tag{6}$$

In this paper, we derive an effective description of the many-body dynamics $\psi^{N,\varepsilon}(t)$. We show that if the system initially forms a Bose–Einstein condensate with factorised condensate wave function, then the dynamics generated by $H_{\mu,\beta}(t)$ preserve this property. Under the assumption that

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}_{L^2(\mathbb{R}^3)}\left|\gamma^{(1)}_{\psi_0^{N,\varepsilon}} - |\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|\right| = 0\,,$$

where the limit $(N,\varepsilon)\to(\infty,0)$ is taken along a suitable sequence, we show that

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}_{L^2(\mathbb{R}^3)}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| = 0$$

with time-evolved condensate wave function $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$. While the transverse part of the condensate wave function remains in the ground state, merely undergoing phase oscillations, the longitudinal part is subject to a non-trivial time evolution. We show that this evolution is determined by the two-dimensional non-linear equation

$$\begin{cases} \mathrm{i}\frac{\partial}{\partial t}\Phi(t,x) = \left(-\Delta_x + V^\parallel(t,(x,0)) + b_\beta|\Phi(t,x)|^2\right)\Phi(t,x) =: h_\beta(t)\Phi(t,x) \\ \Phi(0) = \Phi_0\,. \end{cases} \tag{7}$$

The coupling parameter b_β in (7) depends on the scaling regime and is given by

$$b_\beta = \begin{cases} \|w\|_{L^1(\mathbb{R}^3)} \displaystyle\int_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y & \text{for } \beta\in(0,1), \\ 8\pi a \displaystyle\int_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y & \text{for } \beta = 1, \end{cases}$$

where a denotes the scattering length of w (see Section 3.2 for a definition). The evolution equation (7) provides an effective description of the dynamics. Since the N bosons interact, it contains an effective one-body potential, which is given by the probability density $N|\Phi(t)|^2$ times the two-body scattering process times a factor $\int_{\mathbb{R}}|\chi^\varepsilon(y)|^4\,\mathrm{d}y$ from the confinement. At low energies, the scattering is to leading order described by the s-wave scattering length $a_{\mu,\beta}$ of the interaction $w_{\mu,\beta}$, which scales as $a_{\mu,\beta}\sim\mu$ for the whole parameter range $\beta\in(0,1]$ (see [11, Lemma A.1]) and characterises the length scale of the inter-particle correlations.

For the regime $\beta\in(0,1)$, we find $a_{\mu,\beta}\ll\mu^\beta$, i.e., the scattering length is negligible compared to the range of the interaction in the limit $(N,\varepsilon)\to(\infty,0)$. In this situation, the first order Born approximation $8\pi a_{\mu,\beta}\approx\int_{\mathbb{R}^3} w_{\mu,\beta}(z)\,\mathrm{d}z$ is a valid description of the scattering length and yields above coupling parameter b_β for $\beta\in(0,1)$.

In the scaling regime $\beta = 1$, the first order Born approximation breaks down since $a_{\mu,1} \sim \mu$, which implies that the correlations are visible on the length scale μ of the interaction even in the limit $(N, \varepsilon) \to (\infty, 0)$. Consequently, the coupling parameter b_1 contains the full scattering length, which makes (7) a Gross–Pitaevskii equation.

Physically, the scaling $\beta = 1$ is relevant because it corresponds to an (N, ε)-independent interaction via a suitable coordinate transformation. The Gross–Pitaevskii regime is characterised by the requirement that the kinetic energy per particle (in the longitudinal directions) is of the same order of magnitude as the total energy per particle (without counting the energy from the confinement or the external potential). For N bosons which interact via a potential with scattering length A in a trap with longitudinal extension L and transverse size εL, the former scales as $E_{\text{kin}} \sim L^{-2}$. The latter can be computed as $E_{\text{total}} \sim A\varrho_{3d} \sim AN/(L^3\varepsilon)$, where ϱ_{3d} denotes the particle density. Both quantities being of the same order implies the scaling condition $A/L \sim \varepsilon/N$.

The choice $A \sim 1$ entails $L \sim N/\varepsilon$, which corresponds to an (N, ε)-independent interaction potential. Hence, to capture N bosons in a strongly asymmetric trap while remaining in the Gross–Pitaevskii regime, one must increase the longitudinal length scale of the trap as N/ε and the transverse scale as N. For our analysis, we choose to work instead in a setting where $L \sim 1$, thus we consider interactions with scattering length $A \sim \varepsilon/N$. Both choices are related by the coordinate transform $z \mapsto (\varepsilon/N)z$, which comes with the time rescaling $t \mapsto (\varepsilon/N)^2 t$ in the N-body Schrödinger equation (4).

For the scaling regime $\beta \in (0, 1)$, there is no such coordinate transform relating $w_{\mu,\beta}$ to a physically relevant (N, ε)-independent interaction. We consider this case mainly because the derivation of the Gross–Pitaevskii equation for $\beta = 1$ relies on the corresponding result for $\beta \in (0, 1)$. The central idea of the proof is to approximate the interaction w_μ by an appropriate potential with softer scaling behaviour covered by the result for $\beta \in (0, 1)$, and to control the remainders from this substitution. We follow the approach developed by Pickl in [30], which was adapted to the problem with strong confinement in [4] and [5], where an effectively one-dimensional NLS resp. Gross–Pitaevskii equation was derived for three-dimensional bosons in a cigar-shaped trap. The model considered in [4, 5] is analogous to our model (1) but with a two-dimensional confinement, i.e., where $(x, y) \in \mathbb{R}^{1+2}$. Since many estimates are sensitive to the dimension and need to be reconsidered, the adaptation to our problem with one-dimensional confinement is non-trivial. A detailed account of the new difficulties is given in Remarks 4 and 5.

To the best of our knowledge, the only existing derivation of a two-dimensional evolution equation from the three-dimensional N-body dynamics is by Chen and Holmer in [8]. Their analysis is restricted to the range $\beta \in (0, \frac{2}{5})$, which in particular does not include the physically relevant Gross–Pitaevskii case. In this paper, we extend their result to the full regime $\beta \in (0, 1]$ and include a larger class of confining traps as well as a possibly time-dependent external potential. We impose different conditions on the parameters N and ε, which are stronger than in [8] for small β but much less restrictive for larger β (see Remark 3). Related results for a cigar-shaped confinement were obtained in [4, 5, 9, 22].

Regarding the situation without strong confinement, the first mathematically rigorous justification of a three-dimensional NLS equation from the quantum many-body dynamics of three-dimensional bosons with repulsive interactions was by Erdős, Schlein and Yau in [11], who extended their analysis to the Gross-Pitaevskii regime in [12]. With a different approach, Pickl derived effective evolution equations for both regimes [30], providing also estimates of the rate of convergence. Benedikter, De Oliveira and Schlein proposed a third and again different strategy in [3], which was then adapted by Brennecke and Schlein in [6] to yield the optimal rate of convergence. For two-dimensional bosons, effective NLS dynamics of repulsively interacting bosons were first derived by Kirkpatrick, Schlein and Staffilani in [23]. This result was extended to more singular scalings of the interaction, including the Gross–Pitaevskii regime, by Leopold, Jeblick and Pickl in [20], and two-dimensional attractive interactions were covered in [10, 21, 24].

The dimensional reduction of non-linear one-body equations was studied in [2] by Ben Abdallah, Méhats, Schmeiser and Weishäupl, who consider an $n+d$-dimensional NLS equation with a d-dimensional quadratic confining potential. In the limit where the diameter of this confinement converges to zero, they obtain an effective n-dimensional NLS equation. A similar problem for a cubic NLS equation in a quantum waveguide, resulting in a limiting one-dimensional equation, was covered by Méhats and Raymond in [28].

The remainder of the paper is structured as follows: in Section 2, we state our assumptions and present the main result. The strategy of proof for the NLS scaling is explained in Section 3.1, while the Gross–Pitaevskii scaling is covered in Section 3.2. Section 3.3 contains the proof of our main result, which depends on five propositions. Section 4 collects some auxiliary estimates, which are used in Sections 5 and 6 to prove the propositions for $\beta \in (0,1)$ and $\beta = 1$, respectively.

Notation. We use the notations $A \lesssim B$, $A \gtrsim B$ and $A \sim B$ to indicate that there exists a constant $C > 0$ independent of $\varepsilon, N, t, \psi_0^{N,\varepsilon}, \Phi_0$ such that $A \leq CB$, $A \geq CB$ or $A = CB$, respectively. This constant may, however, depend on the quantities fixed by the model, such as $V^\perp$, χ and $V^\parallel$. Besides, we will exclusively use the symbol $\widehat{\cdot}$ to denote the weighted many-body operators from Definition 3.1 and use the abbreviations

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle := \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^{3N})}, \quad \|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^{3N})} \quad \text{and} \quad \|\cdot\|_{\mathrm{op}} := \|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^{3N}))}.$$

Finally, we write x^+ and x^- to denote $(x+\sigma)$ and $(x-\sigma)$ for any fixed $\sigma > 0$, which is to be understood in the following sense: Let the sequence $(N_n, \varepsilon_n)_{n\in\mathbb{N}} \to (\infty, 0)$. Then

$$\begin{aligned}
f(N,\varepsilon) \lesssim N^{-x^-} \quad &:\Leftrightarrow \quad \text{for any } \sigma > 0,\ f(N_n,\varepsilon_n) \lesssim N_n^{-x+\sigma} \ \text{ for sufficiently large } n\,, \\
f(N,\varepsilon) \lesssim \varepsilon^{x^-} \quad &:\Leftrightarrow \quad \text{for any } \sigma > 0,\ f(N_n,\varepsilon_n) \lesssim \varepsilon_n^{x-\sigma} \ \text{ for sufficiently large } n\,, \\
f(N,\varepsilon) \lesssim \mu^{x^-} \quad &:\Leftrightarrow \quad \text{for any } \sigma > 0,\ f(N_n,\varepsilon_n) \lesssim \mu_n^{x-\sigma} \ \text{ for sufficiently large } n\,.
\end{aligned}$$

Note that these statements concern fixed σ in the limit $(N,\varepsilon) \to (\infty, 0)$ and do in general not hold uniformly as $\sigma \to 0$.

2 Main result

Our aim is to derive an effective description of the dynamics $\psi^{N,\varepsilon}(t)$ in the simultaneous limit $(N,\varepsilon) \to (\infty, 0)$. To this end, we consider families of initial data $\psi_0^{N,\varepsilon}$ along sequences (N_n, ε_n) with the following two properties:

Definition 2.1. Let $\{(N_n, \varepsilon_n)\}_{n\in\mathbb{N}} \subset \mathbb{N} \times (0,1)$ such that $\lim_{n\to\infty}(N_n, \varepsilon_n) = (\infty, 0)$, and let $\mu_n := \varepsilon_n/N_n$. The sequence is called

- $(\Theta$-$)$*admissible*, if
$$\lim_{n\to\infty} \frac{\varepsilon_n^\Theta}{\mu_n} = N_n \varepsilon_n^{\Theta-1} = 0\,,$$
- $(\Gamma$-$)$*moderately confining*, if
$$\lim_{n\to\infty} \frac{\varepsilon_n^\Gamma}{\mu_n} = N_n \varepsilon_n^{\Gamma-1} = \infty\,.$$

Our result holds for sequences (N, ε) that are (Θ, Γ)-admissible with parameters

$$\begin{cases} \frac{1}{\beta} = \Gamma < \Theta < \frac{3}{\beta} & \beta \in (0,1)\,, \\ 1 < \Gamma < \Theta \leq 3 & \beta = 1\,. \end{cases} \tag{8}$$

To make a clear distinction between the cases $\beta \in (0,1)$ and $\beta = 1$, we use the following notation:

- $\beta \in (0,1)$: $\quad (\Theta, \Gamma)_\beta = \left(\frac{\delta}{\beta}, \frac{1}{\beta}\right)$. Hence, (8) implies $1 < \delta < 3$.
- $\beta = 1$: $\quad (\Theta, \Gamma)_1 = (\vartheta, \gamma)$. Here, (8) implies $1 < \gamma < \vartheta \leq 3$.

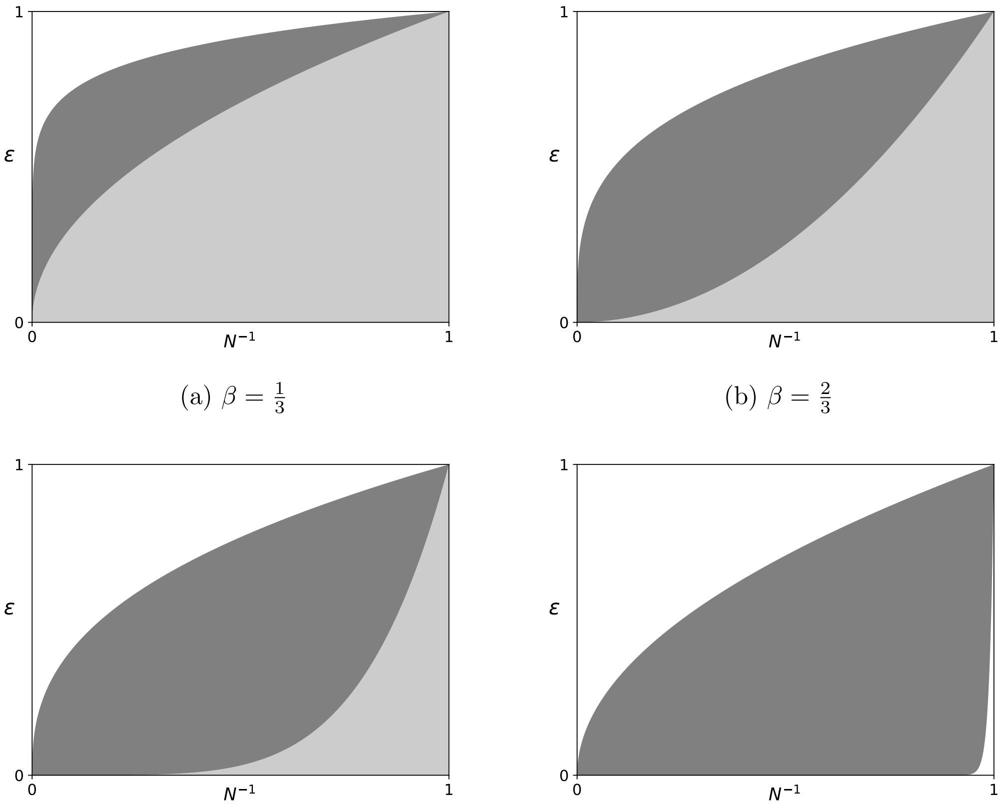
By imposing the admissibility condition, we ensure that the diameter ε of the confining potential does not shrink too slowly compared to the range μ^β of the interaction. Consequently, the energy gap above the transverse ground state, which scales as ε^{-2}, is always large enough to sufficiently suppress transverse excitations. Equivalently, the condition can be written as

$$\frac{\varepsilon^\Theta}{\mu} = N\varepsilon^{\Theta-1} \ll 1 \quad \Leftrightarrow \quad \begin{cases} \frac{\varepsilon^\delta}{\mu^\beta} \ll 1 & \beta \in (0,1) \\ \frac{\varepsilon^\vartheta}{\mu} \ll 1 & \beta = 1 \end{cases} \tag{9}$$

for sufficiently large N and small ε. Clearly, it is necessary to choose $\Theta > 1$, and the condition is weaker for larger Θ. In the proof, we require the admissibility condition to control the orthogonal excitations in the transverse direction (see Remark 4), which results in the respective upper bound for Θ. The threshold $\Theta = 3^+$ admits $N \sim \varepsilon^{-2}$, which has a physical implication: if the confinement is realised by a harmonic trap $V^\perp(y) = \omega^2 y^2$, the frequency ω_ε of the rescaled oscillator $\varepsilon^{-2}V^\perp(y/\varepsilon)$ scales as $\omega_\varepsilon = \omega\varepsilon^{-2}$. Hence, $\Theta = 3^+$ means that the frequency of the confining trap grows proportionally to N.

The moderate confinement condition implies that, for sufficiently large N and small ε,

$$\frac{\mu}{\varepsilon^\Gamma} = N^{-1}\varepsilon^{1-\Gamma} \ll 1 \quad \Leftrightarrow \quad \begin{cases} \frac{\mu^\beta}{\varepsilon} \ll 1 & \beta \in (0,1) \\ \frac{\mu}{\varepsilon^\gamma} \ll 1 & \beta = 1\,. \end{cases} \tag{10}$$



(a) $\beta = \frac{1}{3}$ (b) $\beta = \frac{2}{3}$

(c) $\beta = \frac{5}{6}$ (d) $\beta = 1$

Figure 1: Best possible coverage of the parameter space $\mathbb{N} \times [0, 1]$ for some exemplary choices of $\beta \in (0, 1)$ and for $\beta = 1$. We chose the least restrictive conditions satisfying Definition 2.1, i.e., $(\Theta, \Gamma)_\beta = (\frac{3}{\beta}^-, \frac{1}{\beta})$ and $(\Theta, \Gamma)_1 = (3, 1^+)$. To make the moderate confinement condition $\Gamma = 1^+$ for $\beta = 1$ visible, we implemented it as $\Gamma = 1.01$. Theorem 1 applies in the dark grey area, while the white region is excluded from our analysis. In the light grey part, we expect the dynamics to be effectively described by a free evolution equation. Plotted with Matplotlib [19].

Moderate confinement means that ε does not shrink too fast compared to μ^β. For $\beta \in (0, 1)$, it implies that the interaction is always supported well within trap. This is automatically true for $\beta = 1$ because $\mu/\varepsilon = N^{-1}$, but we require a somewhat stronger condition to handle the Gross–Pitaevskii scaling (see Remark 5). This leads to the additional moderate confinement condition for $\beta = 1$ with parameter $\gamma > 1$, which is clearly a weaker restriction for smaller γ. The upper bound $\Gamma < \Theta$ is necessary to ensure the mutual compatibility of admissibility and moderate confinement.

From a technical point of view, the moderate confinement condition allows us to compensate for certain powers of ε^{-1} in terms of powers of N^{-1}, while the admissibility condition admits the control of powers of N by powers of ε.

To visualise the restrictions due to admissibility and moderate confinement, we plot in Figure 1 the largest possible subset of the parameter space $\mathbb{N} \times [0,1]$ which can be covered by our analysis. A sequence $(N,\varepsilon) \to (\infty,0)$ passes through this space from the top right to the bottom left corner. The two boundaries correspond to the two-stage limits where first $N \to \infty$ at constant ε and subsequently $\varepsilon \to 0$, and vice versa. The edge cases are not contained in our model.

The sequences $(N,\varepsilon) \to (\infty,0)$ within the dark grey region in Figure 1 are covered by our analysis and yield an NLS or Gross–Pitaevskii equation, respectively. Naturally, these restrictions are meaningful only for sufficiently large N and small ε, which implies that mainly the section of the plot around the bottom left corner is of importance. The white region in figures (a) to (c) is excluded from our analysis by the admissibility condition. In figure (d), there is an additional prohibited region due to moderate confinement. Note that Chen and Holmer impose constraints which are weaker for small β and stronger for larger $\beta \in (0, \frac{2}{5})$, which are discussed in Remark 3 and plotted in Figure 2.

The light grey region in Figure 1, which is present for $\beta \in (0,1)$, is not contained in Theorem 1 as a consequence of the moderate confinement condition. We expect the dynamics in this region to be described by an effective equation with coupling parameter $b_\beta = 0$ since it corresponds to the condition $\varepsilon/\mu^\beta \ll 1$, implying that the the confinement shrinks much faster than the interaction. Consequently, the interaction is predominantly supported in a region that is essentially inaccessible to the bosons, which results in a free evolution equation. For $\beta < \frac{1}{3}$ and a cigar-shaped confinement by Dirichlet boundary conditions, this was shown in [22].

As mentioned above, we will consider interactions in the NLS scaling regime $\beta \in (0,1)$ which are of a more generic form than (3).

Definition 2.2. Let $\beta \in (0,1)$ and $\eta > 0$. Define the set $\mathcal{W}_{\beta,\eta}$ as the set containing all families

$$w_{\mu,\beta} : (0,1) \to L^\infty(\mathbb{R}^3,\mathbb{R}), \quad \mu \mapsto w_{\mu,\beta},$$

such that for any $\mu \in (0,1)$

$$\begin{cases} (a)\ \|w_{\mu,\beta}\|_{L^\infty(\mathbb{R}^3)} \lesssim \mu^{1-3\beta}, \\ (b)\ w_{\mu,\beta} \text{ is non-negative and spherically symmetric,} \\ (c)\ \varrho_\beta := \operatorname{diam}(\operatorname{supp} w_{\mu,\beta}) \sim \mu^\beta, \\ (d)\ \lim\limits_{(N,\varepsilon)\to(\infty,0)} \mu^{-\eta} \left| b_{\beta,N,\varepsilon}(w_{\mu,\beta}) - \lim\limits_{(N,\varepsilon)\to(\infty,0)} b_{\beta,N,\varepsilon}(w_{\mu,\beta}) \right| = 0, \end{cases}$$

where

$$b_{\beta,N,\varepsilon}(w_{\mu,\beta}) := N \int\limits_{\mathbb{R}^3} w_{\mu,\beta}(z)\,\mathrm{d}z \int\limits_{\mathbb{R}} |\chi^\varepsilon(y)|^4\,\mathrm{d}y = \mu^{-1} \int\limits_{\mathbb{R}^3} w_{\mu,\beta}(z)\,\mathrm{d}z \int\limits_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y\,.$$

In the sequel, we will abbreviate $b_{\beta,N,\varepsilon}(w_{\mu,\beta}) \equiv b_{\beta,N,\varepsilon}$.

Condition (d) in Definition 2.2 regulates how fast the (N,ε)-dependent coupling parameter $b_{\beta,N,\varepsilon}$ converges to its limit as $(N,\varepsilon) \to (\infty,0)$. For the special case (3), we find that $b_{\beta,N,\varepsilon} = \|w\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}} |\chi(y)|^4 \, \mathrm{d}y$ is independent of N and ε, hence this interaction is contained in $\mathcal{W}_{\beta,\eta}$ for any choice of $\eta > 0$.

Throughout the paper, we will use two notions of one-particle energies:

- The *"renormalised" energy per particle*: for $\psi \in \mathcal{D}(H_{\mu,\beta}(t)^{\frac{1}{2}})$,

$$E^{\psi}_{w_{\mu,\beta}}(t) := \tfrac{1}{N} \langle\!\langle \psi, H_{\mu,\beta}(t)\psi \rangle\!\rangle - \tfrac{E_0}{\varepsilon^2}, \tag{11}$$

  where E_0 denotes the lowest eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V^{\perp}(y)$. By rescaling, the lowest eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{\varepsilon^2} V^{\perp}(\frac{y}{\varepsilon})$ is given by $\frac{E_0}{\varepsilon^2}$.

- The *effective energy per particle*: for $\Phi \in H^1(\mathbb{R}^2)$ and $b \in \mathbb{R}$,

$$\mathcal{E}^{\Phi}_b(t) := \left\langle \Phi, \left(-\Delta_x + V^{\parallel}(t,(x,0)) + \tfrac{b}{2}|\Phi|^2 \right) \Phi \right\rangle_{L^2(\mathbb{R}^2)}. \tag{12}$$

We can now state our assumptions:

A1 *Interaction potential.*

- $\beta \in (0,1)$: Let $w_{\mu,\beta} \in \mathcal{W}_{\beta,\eta}$ for some $\eta > 0$.
- $\beta = 1$: Let w_μ be given by (2) with $w \in L^\infty(\mathbb{R}^3, \mathbb{R})$ spherically symmetric, non-negative and with $\operatorname{supp} w \subseteq \{z \in \mathbb{R}^3 : |z| \leq 1\}$.

A2 *Confining potential.* Let $V^{\perp} : \mathbb{R} \to \mathbb{R}$ such that $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V^{\perp}$ is self-adjoint and has a non-degenerate ground state χ with energy $E_0 < \inf \sigma_{\mathrm{ess}}(-\Delta_y + V^{\perp})$. Assume that the negative part of $V^{\perp}$ is bounded and that $\chi \in \mathcal{C}^2_{\mathrm{b}}(\mathbb{R})$, i.e., χ is bounded and twice continuously differentiable with bounded derivatives. We choose χ normalised and real.

A3 *External field.* Let $V^{\parallel} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that for fixed $z \in \mathbb{R}^3$, $V^{\parallel}(\cdot, z) \in \mathcal{C}^1(\mathbb{R})$. Further, assume that for each fixed $t \in \mathbb{R}$, $V^{\parallel}(t,\cdot), \dot{V}^{\parallel}(t,\cdot) \in L^\infty(\mathbb{R}^3) \cap \mathcal{C}^1(\mathbb{R}^3)$ and $\partial_y V^{\parallel}(t,\cdot)$, $\partial_y \dot{V}^{\parallel}(t,\cdot) \in L^\infty(\mathbb{R}^3)$.

A4 *Initial data.* Let $(N,\varepsilon) \to (\infty, 0)$ be admissible and moderately confining with parameters $(\Theta,\Gamma)_\beta$ as in (8). Assume that the family of initial data $\psi_0^{N,\varepsilon} \in \mathcal{D}(H_{\mu,\beta}(0)) \cap L^2_+(\mathbb{R}^{3N})$ with $\|\psi_0^{N,\varepsilon}\|^2 = 1$, is close to a condensate with condensate wave function $\varphi_0^\varepsilon = \Phi_0 \chi^\varepsilon$ for some normalised $\Phi_0 \in H^4(\mathbb{R}^2)$, i.e.,

$$\lim_{(N,\varepsilon)\to(\infty,0)} \operatorname{Tr}_{L^2(\mathbb{R}^3)} \left| \gamma^{(1)}_{\psi_0^{N,\varepsilon}} - |\Phi_0\chi^\varepsilon\rangle\langle\Phi_0\chi^\varepsilon| \right| = 0. \tag{13}$$

Further, let

$$\lim_{(N,\varepsilon)\to(\infty,0)} \left| E^{\psi_0^{N,\varepsilon}}_{w_{\mu,\beta}}(0) - \mathcal{E}^{\Phi_0}_{b_\beta}(0) \right| = 0. \tag{14}$$

In our main result, we prove the persistence of condensation in the state $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$ for initial data $\psi_0^{N,\varepsilon}$ from *A4*. Naturally, we are interested in times for which the condensate wave function $\Phi(t)$ exists, and, moreover, we require $H^4(\mathbb{R}^2)$-regularity of $\Phi(t)$ for the proof. Let us therefore introduce the maximal time of $H^4(\mathbb{R}^2)$-existence,

$$T^{\mathrm{ex}}_{V^\parallel} := \sup\left\{t \in \mathbb{R}^+_0 : \|\Phi(t)\|_{H^4(\mathbb{R}^2)} < \infty\right\}, \tag{15}$$

where $\Phi(t)$ is the solution of (7) with initial datum Φ_0 from *A4*.

Remark 1. The regularity of the initial data is for many choices of $V^\parallel$ propagated by the evolution (7). For several classes of external potentials, global existence in $H^4(\mathbb{R}^2)$-sense and explicit bounds on the growth of $\|\Phi(t)\|_{H^4(\mathbb{R}^2)}$ are known:

- The case without external field, $V^\parallel = 0$, was covered in [34, Corollary 1.3]: for initial data $\Phi_0 \in H^k(\mathbb{R}^2)$ with $k > 0$, there exists $C_k > 0$ depending on $\|\Phi_0\|_{H^k(\mathbb{R}^2)}$ such that
$$\|\Phi(t)\|_{H^k(\mathbb{R}^2)} \leq C_k(1+|t|)^{\frac{4}{7}k^+}\|\Phi_0\|_{H^k(\mathbb{R}^2)}$$
for all $t \in \mathbb{R}$. If the initial data are further restricted to the set
$$\Sigma^k := \left\{f \in L^2(\mathbb{R}^2) : \|f\|_{\Sigma^k} := \sum_{|\alpha|+|\beta|\leq k} \|x^\alpha \partial_x^\beta f\|_{L^2(\mathbb{R}^2)} < \infty\right\} \subset H^k(\mathbb{R}^2),$$
the bound is even uniform in $t \in \mathbb{R}$. This is, for $\Phi_0 \in \Sigma^k$, there exists $C > 0$ such that
$$\|\Phi(t)\|_{H^k(\mathbb{R}^2)} < C$$
for all $t \in \mathbb{R}$ [7, Section 1.2].

- For time-dependent external potentials $V^\parallel(t,(x,0))$ that are at most quadratic in x uniformly in time, global existence of $H^k(\mathbb{R}^2)$-solutions with double exponential growth was shown in [7, Corollary 1.4] for initial data $\Phi_0 \in \Sigma^k$:

  Assume that $V^\parallel(\cdot,(\cdot,0)) \in L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^2)$ is real-valued such that the map $x \mapsto V^\parallel(t,(x,0))$ is $\mathcal{C}^\infty(\mathbb{R}^2)$, the map $x \mapsto V(t,(x,0))$ is $\mathcal{C}^\infty(\mathbb{R}^2)$ for almost all $t \in \mathbb{R}$, and the map $t \mapsto \sup_{|x|\leq 1}|V^\parallel(t,(x,0))|$ is $L^\infty(\mathbb{R})$. Moreover, let $\partial_x^\alpha V^\parallel(\cdot,(\cdot,0)) \in L^\infty(\mathbb{R}\times\mathbb{R}^2)$ for all $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 2$. Let $\Phi_0 \in \Sigma^k(\mathbb{R}^2)$ with $k \geq 2$. Then there exists a constant $C > 0$ such that
$$\|\Phi(t)\|_{H^k(\mathbb{R}^2)} \leq C\mathrm{e}^{\mathrm{e}^{Ct}}$$
for all $t \in \mathbb{R}$. In case of a time-independent harmonic potential and initial data $\Phi_0 \in \Sigma^k$, this can be improved to an exponential rather than double exponential bound. Note, however, that unbounded potentials $V^\parallel(t,z)$ are excluded by assumption *A3*.

Theorem 1. *Let $\beta \in (0,1]$ and assume that the potentials $w_{\mu,\beta}$, $V^\perp$ and $V^\parallel$ satisfy* A1 – A3. *Let $\psi_0^{N,\varepsilon}$ be a family of initial data satisfying* A4, *let $\psi^{N,\varepsilon}(t)$ denote the solution of* (4) *with initial datum $\psi_0^{N,\varepsilon}$, and let $\gamma^{(1)}_{\psi^{N,\varepsilon}(t)}$ denote its one-particle reduced density matrix as in* (5). *Then for any $0 \leq T < T^{\mathrm{ex}}_{V^\parallel}$,*

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[0,T]} \mathrm{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\Phi(t)\chi^\varepsilon\rangle\langle\Phi(t)\chi^\varepsilon|\right| \quad = \quad 0, \tag{16}$$

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[0,T]} \left| E_{w_{\mu,\beta}}^{\psi^{N,\varepsilon}(t)}(t) - \mathcal{E}_{b_\beta}^{\Phi(t)}(t) \right| = 0\,, \tag{17}$$

where the limits are taken along the sequence from A4. *Here,* $\Phi(t)$ *is the solution of* (7) *with initial datum* $\Phi(0) = \Phi_0$ *from* A4 *and with coupling parameter*

$$b_\beta := \begin{cases} \lim\limits_{(N,\varepsilon)\to(\infty,0)} b_{\beta,N,\varepsilon} & \text{for } \beta \in (0,1)\,, \\ 8\pi a \displaystyle\int_{\mathbb{R}} |\chi(y)|^4 \,\mathrm{d}y & \text{for } \beta = 1 \end{cases} \tag{18}$$

with $b_{\beta,N,\varepsilon}$ *from Definition 2.2.*

Remark 2. (a) By [15, Theorem 1], the ground state χ^ε of $-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{\varepsilon^2}V^\perp(\frac{\cdot}{\varepsilon})$ is exponentially localised on a scale of order ε for any potential $V^\perp$ satisfying *A2*. Valid examples for $V^\perp$ are harmonic potentials or smooth, bounded potentials that admit at least one bound state below the essential spectrum.

(b) Due to assumptions *A1–A3*, the Hamiltonian $H_{\mu,\beta}(t)$ is for any $t \in \mathbb{R}$ self-adjoint on its time-independent domain $\mathcal{D}(H_{\mu,\beta})$. Since we assume continuity of $t \mapsto V^\parallel(t) \in \mathcal{L}(L^2(\mathbb{R}^3))$, [16] implies that the family $\{H_{\mu,\beta}(t)\}_{t\in\mathbb{R}}$ generates a unique, strongly continuous, unitary time evolution that leaves $\mathcal{D}(H_{\mu,\beta})$ invariant. By imposing the further assumptions on $V^\parallel$, we can control the growth of the one-particle energies and the interactions of the particles with the external potential. Note that it is physically important to include time-dependent external traps, since this admits non-trivial dynamics even if the system is initially prepared in an eigenstate.

(c) Assumption *A4* states that the system is initially a Bose–Einstein condensate which factorises in a longitudinal and a transverse part. In [32, Theorems 1.1 and 1.3], Schnee and Yngvason prove that both parts of the assumption are fulfilled by the ground state of $H_{\mu,\beta}(0)$ for $\beta = 1$ and $V^\parallel(0,z) = V(x)$ with V locally bounded and diverging as $|x| \to \infty$.

(d) The situation of a strong confinement in two directions is studied in [4, 5]. Our proof can be understood as an adaptation of these works, and we summarise the mathematical differences in Remarks 4 and 5.

(e) Our proof yields an estimate of the rate of the convergence (16). Since we did not focus on obtaining an optimal rate, we do not state it explicitly. However, it can be recovered from the bounds in Propositions 3.6 and 3.11 by optimising over the parameters.

Remark 3. The sequences $(N,\varepsilon) \to (\infty,0)$ covered by Theorem 1 are restricted by admissibility and moderate confinement condition (Definition 2.1 and (8)). To conclude this section, let us discuss these constraints:

- By (8), the weakest possible constraints are given by $(\Theta,\Gamma)_\beta = (\frac{3}{\beta}^-, \frac{1}{\beta})$ for $\beta \in (0,1)$ and $(\Theta,\Gamma)_1 = (3, 1^+)$ for $\beta = 1$. Instead of choosing these least restrictive values, we present Theorem 1 and all estimates in explicit dependence of the parameters Θ and Γ, making it more transparent where the conditions enter the proof. Moreover, the rate of convergence improves for more restrictive choices of the parameters Γ and Θ.

- In [8], Chen and Holmer prove Theorem 1 for the regime $\beta \in (0, \frac{2}{5})$ under different assumptions on the sequence (N, ε). The subset of the parameter range $\mathbb{N} \times [0, 1]$ covered by their analysis is visualised in Figure 2.
  While no admissibility condition is required for their proof, they impose a moderate confinement condition which is equivalent to our condition for $\beta \in (0, \frac{3}{11}]$. For larger $\beta \in (\frac{3}{11}, \frac{2}{5})$, they restrict the parameter range much stronger[1], and their condition becomes so restrictive with increasing β that it limitates the range of scaling parameters to $\beta \in (0, \frac{2}{5})$.

- No restriction comparable to the admissibility condition is needed for the ground state problem in [32]. Given the work [28] where the strong confinement limit of the three-dimensional NLS equation is taken, this suggests that our result should hold without any such restriction. However, for the present proof, the condition is indispensable (see Remarks 4 and 5).

- As argued above, the moderate confinement condition for $\beta \in (0, 1)$ is optimal, in the sense that we expect a free evolution equation if $\mu^\beta/\varepsilon \to \infty$. For $\beta = 1$, we require that $\mu/\varepsilon^\gamma \to 0$ for $\gamma > 1$. Note that the choice $\gamma = 1$ would mean no restriction at all because $\mu/\varepsilon = N^{-1}$. Our proof works for γ that are arbitrarily close to 1. However, since the estimates are not uniform in γ, the case $\gamma = 1$ is excluded.

- Although no moderate confinement condition is required to derive the one-dimensional Gross–Pitaevskii equation in the cigar-shaped case [5], our analysis covers a considerably larger subset of the parameter space $\mathbb{N} \times [0, 1]$ than is included in [5]. In that work, the admissibility condition is given as $N\varepsilon^{\frac{2}{5}^-} \to 0$, which is much more restrictive than our condition.

3 Proof of the main result

The proof of Theorem 1, both for the NLS scaling $\beta \in (0, 1)$ and the Gross–Pitaevskii case $\beta = 1$, follows the approach developed by Pickl in [30]. The main idea is to avoid a direct estimate of the differences in (16) and (17), but instead to define a functional

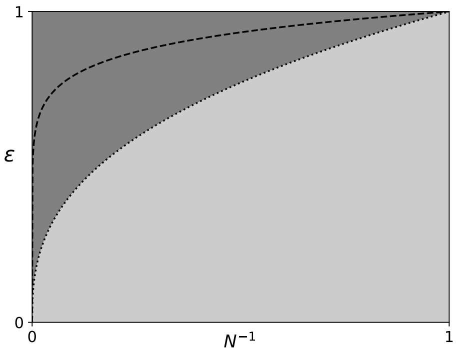
$$\alpha^<_{\xi, w_{\mu,\beta}} : \mathbb{R} \times L^2(\mathbb{R}^{3N}) \times L^2(\mathbb{R}^3) \to \mathbb{R}^+_0, \quad (t, \psi^{N,\varepsilon}(t), \varphi^\varepsilon(t)) \mapsto \alpha^<_{\xi, w_{\mu,\beta}}(t, \psi^{N,\varepsilon}(t), \varphi^\varepsilon(t))$$

in such a way that

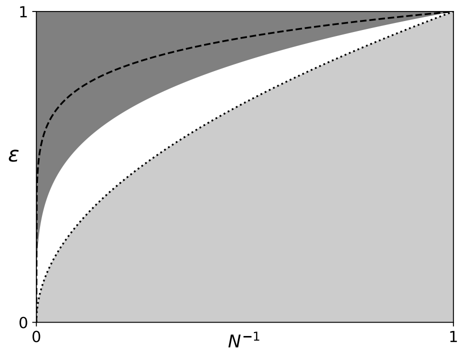
$$\lim_{(N,\varepsilon)\to(\infty,0)} \alpha^<_{\xi, w_{\mu,\beta}}(t, \psi^{N,\varepsilon}(t), \varphi^\varepsilon(t)) = 0 \quad \iff \quad (16) \wedge (17).$$

Physically, the functional $\alpha^<_{\xi, w_{\mu,\beta}}$ measures the part of the wave function $\psi^{N,\varepsilon}(t)$ that remains outside the condensed phase $\varphi^\varepsilon(t)$, and is therefore also referred to as a counting

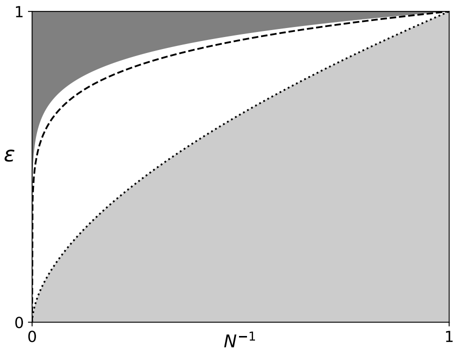
[1] More precisely, Chen and Holmer consider sequences (N, ε) such that $N \gg \varepsilon^{-2\nu(\beta)}$, where $\nu(\beta) := \max\left\{\frac{1-\beta}{2\beta}, \frac{5\beta/4-1/12}{1-5\beta/2}, \frac{\beta/2+5/6}{1-\beta}, \frac{\beta+1/3}{1-2\beta}\right\}$. For the regime $\beta \in (0, \frac{3}{11}]$, this implies $\nu(\beta) = \frac{1-\beta}{2\beta}$, which is equivalent to the choice $\Gamma = \frac{1}{\beta}$ and thus exactly our moderate confinement condition. For $\beta \in (\frac{3}{11}, \frac{1}{3}]$, one obtains $\nu(\beta) = \frac{\beta+1/3}{1-2\beta}$, which corresponds to the choice $\Gamma = \frac{5}{3-6\beta} > \frac{1}{\beta}$, and for $\beta \in (\frac{1}{3}, \frac{2}{5})$, one concludes $\nu(\beta) = \frac{5\beta/4-1/12}{1-5\beta/2}$, corresponding to $\Gamma = \frac{5}{6-15\beta} > \frac{1}{\beta}$. Since the moderate confinement condition is weaker for smaller Γ, we conclude that our condition is weaker for $\beta > \frac{3}{11}$.



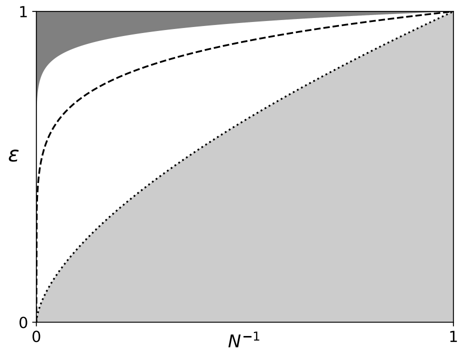
(a) $d = 2$, $\beta = \frac{3}{11}$



(b) $d = 2$, $\beta = \frac{1}{3}$



(c) $d = 2$, $\beta = \frac{11}{30}$



(d) $d = 2$, $\beta = \frac{23}{60}$

Figure 2: Coverage of the parameter space $\mathbb{N} \times [0, 1]$ for some exemplary choices of $\beta \in (0, \frac{2}{5})$. In [8], Chen and Holmer cover sequences within the dark grey region, while the white and light grey area are excluded. In comparison, Theorem 1 applies to all sequences enclosed between the black dashed line and the black dotted line, where the dashed line corresponds to the admissibility and the dotted line to the moderate confinement condition. Limiting sequences within the light grey region are expected to yield a free effective evolution equation. Plotted with Matplotlib [19].

functional. The index ξ is a parameter which is required for technical reasons and will be defined below. The index $w_{\mu,\beta}$ indicates that the evolutions of $\psi^{N,\varepsilon}(t)$ and $\varphi^\varepsilon(t)$ are generated by $H_{\mu,\beta}(t)$ and $h_\beta(t)$, which depend, directly or indirectly, on the interaction $w_{\mu,\beta}$. To define the functional $\alpha^<_{\xi, w_{\mu,\beta}}$, we recall the projectors onto the condensate wave function that were introduced in [29, 22]:

Definition 3.1. Let $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$, where $\Phi(t)$ is the solution of the NLS equation (7) with initial datum Φ_0 from *A4* and with χ^ε as in (6). Let

$$p := |\varphi^\varepsilon(t)\rangle \langle \varphi^\varepsilon(t)| \,,$$

where we drop the t- and ε-dependence of p in the notation. For $i \in \{1, \ldots, N\}$, define the projection operators on $L^2(\mathbb{R}^{3N})$

$$p_j := \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes p \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N-j} \quad \text{and} \quad q_j := \mathbb{1} - p_j.$$

Further, define the orthogonal projections on $L^2(\mathbb{R}^3)$

$$\begin{aligned} p^\Phi &:= |\Phi(t)\rangle \langle \Phi(t)| \otimes \mathbb{1}_{L^2(\mathbb{R})}, & q^\Phi &:= \mathbb{1}_{L^2(\mathbb{R}^3)} - p^\Phi, \\ p^{\chi^\varepsilon} &:= \mathbb{1}_{L^2(\mathbb{R}^2)} \otimes |\chi^\varepsilon\rangle \langle \chi^\varepsilon|, & q^{\chi^\varepsilon} &:= \mathbb{1}_{L^2(\mathbb{R}^3)} - p^{\chi^\varepsilon}, \end{aligned}$$

and define p_j^Φ, q_j^Φ, $p_j^{\chi^\varepsilon}$ and $q_j^{\chi^\varepsilon}$ on $L^2(\mathbb{R}^{3N})$ analogously to p_j and q_j. Finally, for $0 \leq k \leq N$, define the many-body projections

$$P_k = \big(q_1 \cdots q_k \, p_{k+1} \cdots p_N\big)_{\text{sym}} := \sum_{\substack{J \subseteq \{1,\dots,N\} \\ |J|=k}} \prod_{j \in J} q_j \prod_{l \notin J} p_l$$

and $P_k = 0$ for $k < 0$ and $k > N$. Further, for any function $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and $d \in \mathbb{Z}$, define the operators $\widehat{f}, \widehat{f}_d \in \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)$ by

$$\widehat{f} := \sum_{k=0}^{N} f(k) P_k \,, \qquad \widehat{f}_d := \sum_{j=-d}^{N-d} f(j+d) P_j \,.$$

Clearly, $\sum_{k=0}^N P_k = \mathbb{1}$. Besides, note the useful relations $p = p^\Phi p^{\chi^\varepsilon}$, $q^\Phi q = q^\Phi$, $q^{\chi^\varepsilon} q = q^{\chi^\varepsilon}$ and $q = q^{\chi^\varepsilon} + q^\Phi p^{\chi^\varepsilon} = q^\Phi + p^\Phi q^{\chi^\varepsilon}$. In the sequel, we will make use of the following weight functions:

Definition 3.2. Define

$$n : \mathbb{N}_0 \to \mathbb{R}_0^+, \qquad k \mapsto n(k) := \sqrt{\tfrac{k}{N}} \,,$$

and, for some $\xi \in (0, \frac{1}{2})$,

$$m : \mathbb{N} \to \mathbb{R}_0^+, \qquad m(k) := \begin{cases} n(k) & \text{for } k \geq N^{1-2\xi} \,, \\ \frac{1}{2}\left(N^{-1+\xi} k + N^{-\xi}\right) & \text{else} \,. \end{cases}$$

Further, define the weight functions $m^\sharp : \mathbb{N}_0 \to \mathbb{R}_0^+$, $\sharp \in \{a, b, c, d, e\}$, by

$$\begin{aligned} m^a(k) &:= m(k) - m(k+1), & m^b(k) &:= m(k) - m(k+2), \\ m^c(k) &:= m^a(k) - m^a(k+1), & m^d(k) &:= m^a(k) - m^a(k+2), \\ m^e(k) &:= m^b(k) - m^b(k+1), & m^f(k) &:= m^b(k) - m^b(k+2). \end{aligned}$$

The corresponding weighted many-body operators are denoted by $\widehat{m}^\sharp$. Finally, define

$$\widehat{r} := \widehat{m}^b p_1 p_2 + \widehat{m}^a (p_1 q_2 + q_1 p_2).$$

Note that m equals n with a smooth, ξ-dependent cut-off to soften the singularity of $\frac{\mathrm{d}n}{\mathrm{d}k}$ for small k.

Definition 3.3. For $\beta \in (0,1)$, define

$$\alpha^<_{\xi,\, w_{\mu,\beta}}(t) := \alpha^<_{\xi,\, w_{\mu,\beta}}(t, \psi^{N,\varepsilon}(t), \varphi^\varepsilon(t)) := \left\langle\!\left\langle \psi^{N,\varepsilon}(t), \widehat{m}\psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle + \left| E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t) - \mathcal{E}^{\Phi(t)}_{b_\beta}(t) \right| .$$

The expression $\langle\!\langle \psi^{N,\varepsilon}(t), \widehat{m}\psi^{N,\varepsilon}(t) \rangle\!\rangle$ is a suitably weighted sum of the expectation values of $P_k\psi^{N,\varepsilon}(t)$, i.e., of the parts of $\psi^{N,\varepsilon}(t)$ with k particles outside $\varphi^\varepsilon(t)$. As $m(0) \approx 0$ and m is increasing, $P_k\psi^{N,\varepsilon}(t)$ with larger k contribute more to $\alpha^<_{\xi,\, w_{\mu,\beta}}(t)$ than $P_k\psi^{N,\varepsilon}(t)$ with smaller k. It is well known that the convergence $\langle\!\langle \psi^{N,\varepsilon}(t), \widehat{m}\psi^{N,\varepsilon}(t) \rangle\!\rangle \to 0$ is equivalent to the convergence (16) of the one-particle reduced density matrix of $\psi^{N,\varepsilon}(t)$ to $|\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|$. Hence, the convergence $\alpha^<_{\xi,\, w_{\mu,\beta}}(t) \to 0$ is equivalent to (16) and (17). The relation between the respective rates of convergence is stated in the following lemma, whose proof is given in [4, Lemma 3.6]:

Lemma 3.4. *For any $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$, it holds that*

$$\mathrm{Tr}\left| \gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)| \right| \leq \sqrt{8\alpha^<_{\xi,\, w_{\mu,\beta}}(t)} ,$$

$$\alpha^<_{\xi,\, w_{\mu,\beta}}(t) \leq \left| E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t) - \mathcal{E}^{\Phi(t)}_{b_\beta}(t) \right| + \sqrt{\mathrm{Tr}\left| \gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)| \right|} + \tfrac{1}{2}N^{-\xi} .$$

3.1 The NLS case $\beta \in (0,1)$

The strategy of our proof is to derive a bound for $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha^<_{\xi,\, w_{\mu,\beta}}(t)|$, which leads to an estimate of $\alpha^<_{\xi,\, w_{\mu,\beta}}(t)$ by means of Grönwall's inequality. The first step is therefore to characterise the expressions arising from this derivative.

Proposition 3.5. *Assume* A1 – A4 *for $\beta \in (0,1)$. Let*

$$w^{(12)}_{\mu,\beta} := w_{\mu,\beta}(z_1 - z_2) \quad \textit{and} \quad Z^{(12)}_\beta := w^{(12)}_{\mu,\beta} - \tfrac{b_\beta}{N-1}\left(|\Phi(t,x_1)|^2 + |\Phi(t,x_2)|^2\right)$$

and define

$$\mathcal{L} := \left\{ N\widehat{m}^a_{-1},\, N\widehat{m}^b_{-2} \right\} . \tag{19}$$

Then

$$\left| \tfrac{\mathrm{d}}{\mathrm{d}t}\alpha^<_{\xi,\, w_{\mu,\beta}}(t) \right| \leq \left|\gamma_{a,<}(t)\right| + |\gamma_{b,<}(t)|$$

for almost every $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$, where

$$\begin{aligned} \gamma_{a,<}(t) \quad &:= \quad \left| \left\langle\!\left\langle \psi^{N,\varepsilon}(t), \dot{V}^\parallel(t,z_1)\psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle - \left\langle \Phi(t), \dot{V}^\parallel(t,(x,0))\Phi(t) \right\rangle_{L^2(\mathbb{R}^2)} \right| \quad (20)\\ &\quad -2N\Im\left\langle\!\left\langle \psi^{N,\varepsilon}(t), \widehat{m}^a_{-1}q_1\big(V^\parallel(t,z_1) - V^\parallel(t,(x_1,0))\big)p_1\psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle , \quad (21)\end{aligned}$$

$$\begin{aligned} \gamma_{b,<}(t) \quad &:= \quad -N(N-1)\Im\left\langle\!\left\langle \psi^{N,\varepsilon}(t), Z^{(12)}_\beta \widehat{m}\psi^{N,\varepsilon}(t) \right\rangle\!\right\rangle , \quad (22)\\ &=: \quad \gamma^{(1)}_{b,<}(t) + \gamma^{(2)}_{b,<}(t) + \gamma^{(3)}_{b,<}(t) + \gamma^{(4)}_{b,<}(t) ,\end{aligned}$$

with

$$\left|\gamma_{b,<}^{(1)}(t)\right| := N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),\widehat{l}q_1^{\Phi}p_1^{\chi^\varepsilon}p_2Z_\beta^{(12)}p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \tag{23}$$

$$\left|\gamma_{b,<}^{(2)}(t)\right| := N\max_{\widehat{l}\in\mathcal{L}}\max_{t_2\in\{p_2,\,q_2,\,q_2^{\Phi}p_2^{\chi^\varepsilon}\}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1^{\chi^\varepsilon}t_2\widehat{l}w_{\mu,\beta}^{(12)}p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \tag{24}$$

$$+N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1^{\chi^\varepsilon}q_2\widehat{l}w_{\mu,\beta}^{(12)}p_1q_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \tag{25}$$

$$+N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_2^{\chi^\varepsilon}q_1^{\Phi}p_1^{\chi^\varepsilon}\widehat{l}w_{\mu,\beta}^{(12)}p_1q_2^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \tag{26}$$

$$+N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1^{\chi^\varepsilon}q_2^{\chi^\varepsilon}\widehat{l}w_{\mu,\beta}^{(12)}p_1p_2^{\chi^\varepsilon}q_2^{\Phi}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \tag{27}$$

$$\left|\gamma_{b,<}^{(3)}(t)\right| := N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),(q_2^{\chi^\varepsilon}q_1^{\Phi}p_1^{\chi^\varepsilon}+q_1^{\chi^\varepsilon}q_2^{\Phi}p_2^{\chi^\varepsilon})\widehat{l}w_{\mu,\beta}^{(12)}p_1p_2^{\chi^\varepsilon}q_2^{\Phi}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \tag{28}$$

$$+N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1^{\Phi}q_2^{\Phi}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\widehat{l}w_{\mu,\beta}^{(12)}p_2q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|, \tag{29}$$

$$\left|\gamma_{b,<}^{(4)}(t)\right| := N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1^{\Phi}q_2^{\Phi}\widehat{l}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_{\mu,\beta}^{(12)}p_1p_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \tag{30}$$

$$+N\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1^{\Phi}q_2^{\Phi}\widehat{l}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_{\mu,\beta}^{(12)}p_1p_2^{\chi^\varepsilon}q_2^{\Phi}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right| \tag{31}$$

$$+b_\beta\max_{\widehat{l}\in\mathcal{L}}\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1q_2\widehat{l}|\Phi(t,x_1)|^2p_1q_2\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|. \tag{32}$$

The term $\gamma_{a,<}$ summarises all contributions from interactions between the particles and the external field $V^{\parallel}$, while $\gamma_{b,<}$ collects all contributions from the mutual interactions between the bosons. The latter can be subdivided into four parts:

- $\gamma_{b,<}^{(1)}$ and $\gamma_{b,<}^{(4)}$ contain the quasi two-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}(x_1-x_2)$ resulting from integrating out the transverse degrees of freedom in $w_{\mu,\beta}$, which is given as

$$p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}w_{\mu,\beta}(z_1-z_2)p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon} =: \overline{\overline{w_{\mu,\beta}}}(x_1-x_2)p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}$$

  (see Definition 5.4). Hence, $\gamma_{b,<}^{(1)}$ and $\gamma_{b,<}^{(4)}$ can be understood as two-dimensional analogue of the corresponding expressions in the three-dimensional problem without confinement [30, Lemma A.4], and the estimates are inspired by [30]. Note that $\gamma_{b,<}^{(1)}$ contains the difference between the quasi two-dimensional interaction potential $\overline{\overline{w_{\mu,\beta}}}$ and the effective one-body potential $b_\beta|\Phi(t)|^2$, which means that it vanishes in the limit $(N,\varepsilon)\to(\infty,0)$ only if (7) with coupling parameter b_β is the correct effective equation. The last line (32) of $\gamma_{b,<}^{(4)}$ contains merely the effective interaction potential $b_\beta|\Phi(t)|^2$ instead of the pair interaction $w_{\mu,\beta}$, hence, it is easily controlled.

- $\gamma_{b,<}^{(2)}$ and $\gamma_{b,<}^{(3)}$ are remainders from the replacement $w_{\mu,\beta}\to\overline{\overline{w_{\mu,\beta}}}$, hence they have no three-dimensional equivalent. They are comparable to the expression $\gamma_b^{(2)}$ in [4] from the analogous replacement of the originally three-dimensional interaction by its quasi one-dimensional counterpart.

The second step is to control $\gamma_{a,<}$ to $\gamma^{(4)}_{b,<}$ in terms of $\alpha^<_{\xi,w_{\mu,\beta}}(t)$ and by expressions that vanish in the limit $(N,\varepsilon)\to(\infty,0)$. To write the estimates in a more compact form, let us define the function $\mathfrak{e}_\beta:[0,T^{\mathrm{ex}}_{V^\parallel})\to[1,\infty)$ as

$$\begin{aligned}\mathfrak{e}^2_\beta(t):=\|\Phi(t)\|^2_{H^4(\mathbb{R}^2)}+|E^{\psi_0^{N,\varepsilon}}_{w_{\mu,\beta}}(0)|+|\mathcal{E}^{\Phi_0}_{b_\beta}(0)|+\int\limits_0^t\|\dot{V}^\parallel(s)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}s\\ +\sup_{i,j\in\{0,1\}}\|\partial_t^i\partial_y^jV^\parallel(t)\|_{L^\infty(\mathbb{R}^3)}\,,\end{aligned}\tag{33}$$

where $\Phi(t)$ denotes the solution of (7) with initial datum Φ_0 from *A4*. Note that $\mathfrak{e}_\beta(t)$ is bounded uniformly in N and ε because the only (N,ε)-dependent quantity $E^{\psi_0^{N,\varepsilon}}_{w_{\mu,\beta}}(0)$ converges to $\mathcal{E}^{\Phi_0}_{b_\beta}(0)$ as $(N,\varepsilon)\to(\infty,0)$ by *A4*. The function $\mathfrak{e}_\beta$ is particularly useful since

$$\left|E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t)\right|\le\mathfrak{e}^2_\beta(t)-1\quad\text{ and }\quad\left|\mathcal{E}^{\Phi(t)}_{b_\beta}(t)\right|\le\mathfrak{e}^2_\beta(t)-1$$

for any $t\in[0,T^{\mathrm{ex}}_{V^\parallel})$ by the fundamental theorem of calculus. Note that for a time-independent external field $V^\parallel$, $\mathfrak{e}^2_\beta(t)\lesssim1$ as a consequence of Remark 1, hence $E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t)$ and $\mathcal{E}^{\Phi(t)}_{b_\beta}(t)$ are in this case bounded uniformly in $t\in[0,T^{\mathrm{ex}}_{V^\parallel})$.

Proposition 3.6. *Let $\beta\in(0,1)$ and assume* A1 – A4 *with parameters β and η in* A1 *and $(\Theta,\Gamma)_\beta=(\frac{\delta}{\beta},\frac{1}{\beta})$ in* A4. *Let*

$$0<\xi<\min\left\{\tfrac13\,,\,\tfrac{1-\beta}{2}\,,\,\beta\,,\,\tfrac{\beta(3-\delta)}{2(\delta-\beta)}\right\},\quad 0<\sigma<\min\left\{\tfrac{1-3\xi}{4}\,,\,\beta-\xi\right\}.$$

Then, for sufficiently small μ, the terms $\gamma_{a,<}$ to $\gamma^{(4)}_{b,<}$ from Proposition 3.5 are bounded by

$$\begin{aligned}
\left|\gamma_{a,<}(t)\right| &\lesssim \mathfrak{e}^3_\beta(t)\,\varepsilon+\mathfrak{e}_\beta(t)\left\langle\!\left\langle\psi^{N,\varepsilon}(t),\widehat{n}\psi^{N,\varepsilon}(t)\right\rangle\!\right\rangle,\\
\left|\gamma^{(1)}_{b,<}(t)\right| &\lesssim \mathfrak{e}^2_\beta(t)\left(\tfrac{\mu^\beta}{\varepsilon}+N^{-1}+\mu^\eta\right),\\
\left|\gamma^{(2)}_{b,<}(t)\right| &\lesssim \mathfrak{e}^3_\beta(t)\left(\left(\tfrac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}+\frac12}+\varepsilon^{\frac{1-\beta}{2}}\right),\\
\left|\gamma^{(3)}_{b,<}(t)\right| &\lesssim \mathfrak{e}^3_\beta(t)\left(\left(\tfrac{\delta}{\beta}\right)^{\frac12}\left(\tfrac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}}+\left(\tfrac{1}{1-\beta}\right)^{\frac12}N^{-\frac{\beta}{2}}\right),\\
\left|\gamma^{(4)}_{b,<}(t)\right| &\lesssim \mathfrak{e}^3_\beta(t)\,\alpha^<_\xi(t)\,+\,\mathfrak{e}^3_\beta(t)\left(\tfrac{\mu^\beta}{\varepsilon}+\left(\tfrac{\varepsilon^3}{\mu^\beta}\right)^{\frac12}+N^{-\sigma}+\mu^\eta+\mu^{\frac{1-\beta}{2}}\right).
\end{aligned}$$

Remark 4. (a) The estimates of $\gamma_{a,<}$, $\gamma^{(1)}_{b,<}$ and $\gamma^{(2)}_{b,<}$ work analogously to the corresponding bounds in [4] and are briefly summarised in Sections 5.2.1 and 5.2.2. While $\gamma_{a,<}$ is easily bounded since it contains only one-body contributions, the key for the estimate of $\gamma^{(1)}_{b,<}$ is that for sufficiently large N and small ε,

$$\begin{aligned}&N\int\mathrm{d}y_2|\chi^\varepsilon(y_2)|^2\int\mathrm{d}z_1|\varphi^\varepsilon(z_1)|^2w_{\mu,\beta}(z_1-z_2)\\ &\qquad\approx N\left(\int\mathrm{d}y_2|\chi^\varepsilon(y_2)|^4\right)\|w_{\mu,\beta}\|_{L^1(\mathbb{R}^3)}|\Phi(x_2)|^2=b_{\beta,N,\varepsilon}|\Phi(x_2)|^2\end{aligned}$$

due to sufficient regularity of φ^ε and since the support of $w_{\mu,\beta}$ shrinks as μ^β. For this argument, it is crucial that the sequence (N,ε) is moderately confining.

The main idea to control $\gamma^{(2)}_{b,<}$ is an integration by parts, exploiting that the antiderivative of $w_{\mu,\beta}$ is less singular than $w_{\mu,\beta}$ and that $\nabla\psi^{N,\varepsilon}(t)$ can be controlled in terms of the energy $E^{\psi^{N,\varepsilon}(t)}_{w_{\mu,\beta}}(t)$. To this end, we define the function h_ε as the solution of the equation $\Delta h_\varepsilon = w_{\mu,\beta}$ on a three-dimensional ball with radius ε and Dirichlet boundary conditions and integrate by parts on that ball. To prevent contributions from the boundary, we insert a smoothed step function whose derivative can be controlled (Definition 5.1). To make up for the factors ε^{-1} from the derivative, one observes that all expressions in $\gamma^{(2)}_{b,<}$ contain at least one projection q^{χ^ε}. Since $\|q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| = \mathcal{O}(\varepsilon)$ (Lemma 4.9a), which follows since the spectral gap between ground state and excitation spectrum grows proportionally to ε^{-2}, the projections q^{χ^ε} provide the missing factors ε. The second main ingredient is the admissibility condition, which allows us to cancel small powers of N by powers of ε gained from q^{χ^ε}.

(b) For $\gamma^{(3)}_{b,<}$, this strategy of a three-dimensional integration by parts does not work: whereas q^{χ^ε} cancels the factor ε^{-1} from the derivative, we do not gain sufficient powers of ε to compensate for all positive powers of N. Note that this problem did not occur in [4], where the ratio of N and ε was different.[2]

To cope with $\gamma^{(3)}_{b,<}$, note that both (28) and (29) contain the expression $p_1^{\chi^\varepsilon} w^{(12)}_{\mu,\beta} p_1^{\chi^\varepsilon}$, which, analogously to $\overline{\overline{w_{\mu,\beta}}}$, defines a function $\overline{w_{\mu,\beta}}(x_1-x_2,y_2)$ where one of the y-variables is integrated out (Definition 5.4). We integrate by parts only in the x-variable, which has the advantages that ∇_x does not generate factors ε^{-1} and that the x-antiderivative of $\overline{w_{\mu,\beta}}(\cdot,y)$ diverges only logarithmically in μ^{-1} (Lemma 5.6b). Due to admissibility and moderate confinement condition, this can be cancelled by any positive power of ε or N^{-1}. In distinction to $\gamma^{(2)}_{b,<}$, we do not integrate by parts on a ball with Dirichlet boundary conditions but instead add and subtract suitable counter-terms as in [30] and integrate over $\mathbb{R}^2$. Note that one obtains the same result when choosing the other path, but in this way the estimates are easily transferable to $\gamma^{(4)}_{b,<}$ (see below).

More precisely, we construct $\overline{v}_\rho(\cdot,y)$ such that $\|\overline{w_{\mu,\beta}}(\cdot,y)\|_{L^1(\mathbb{R}^2)} = \|\overline{v}_\rho(\cdot,y)\|_{L^1(\mathbb{R}^2)}$ and that $\operatorname{supp}\overline{v}_\rho(\cdot,y)$ scales as $\rho\in(\mu^\beta,1]$ (Definition 5.4). As a consequence of Newton's theorem, the solution $\overline{h}_{\varrho_\beta,\rho}$ of $\Delta_x\overline{h}_{\varrho_\beta,\rho} = \overline{w_{\mu,\beta}} - \overline{v}_\rho$ is supported within a two-dimensional ball with radius ρ. We then write $\overline{w_{\mu,\beta}}(\cdot,y) = \Delta_x\overline{h}_{\varrho_\beta,\rho}(\cdot,y) + \overline{v}_\rho(\cdot,y)$, integrate the first term by parts in x, and choose ρ sufficiently large that the contributions from $\overline{v}_\rho$ can be controlled. The full argument is given in

[2] In the 3d → 1d case [4], the range of the interaction scales as $\mu^\beta_{1\mathrm{d}} = (\varepsilon^2/N)^\beta$, besides $\chi^\varepsilon_{1\mathrm{d}}(y) = \varepsilon^{-1}\chi_{1\mathrm{d}}(y/\varepsilon)$, and the admissibility condition reads $\varepsilon^2/\mu^\beta_{1\mathrm{d}} \to 0$. These slightly different formulas lead to the estimate $\|(\nabla_1 h^{1\mathrm{d}}_\varepsilon(z_1-z_2))p_1^{1\mathrm{d}}\|_{\mathrm{op}} \lesssim N^{-1+\frac{\beta}{2}}\varepsilon^{1-\beta}$, while we obtain in our case $\|(\nabla_1 h^{(12)}_\varepsilon)p_1\|_{\mathrm{op}} \lesssim N^{-1+\frac{\beta}{2}}\varepsilon^{\frac{1-\beta}{2}}$ (Lemma 5.2). Following the same path as in $\gamma^{(2)}_{b,<}$, e.g., for (28) (corresponding to (21) in [4]), we obtain in the 1d problem the estimate $\sim N^{\frac{\beta}{2}}\varepsilon^{1-\beta} = (\varepsilon^2/\mu^\beta_{1\mathrm{d}})^{\frac{1}{2}}$, which can be controlled by the respective admissibility condition. As opposed to this, we compute in our case that (28) $\sim N^{\frac{\beta}{2}}\varepsilon^{\frac{1-\beta}{2}} = (\varepsilon/\mu^\beta)^{\frac{1}{2}}$, which diverges due to moderate confinement.

Sections 5.2.3 and 5.2.4.

(c) Finally, to estimate $\gamma_{b,<}^{(4)}$ (Section 5.2.5), we define $\overline{\overline{w_{\mu,\beta}}}$ as above and integrate by parts in x, using an auxiliary potential $\overline{\overline{v}}_\rho$ analogously to $\overline{v}_\rho$ (Definition 5.4). To cope with the logarithmic divergences from the two-dimensional Green's function, we integrate by parts twice, following an idea from [30]. This is the reason why we defined $\overline{h}_{\varrho_\beta,\rho}$ and $\overline{\overline{h}}_{\varrho_\beta,\rho}$ on $\mathbb{R}^2$ and not on a ball, which would require the use of a smoothed step function. While the results are the same when integrating by parts only once, it turns out that the additional factors ρ^{-1} from a second derivative hitting the step function cannot be controlled sufficiently well.

For (31), the bound $\|\nabla_{x_1}\psi^{N,\varepsilon}(t)\|^2 \lesssim 1$ from *a priori* energy estimates is insufficient, comparable to the situation in [30] and [4]. Instead, we require an improved bound on the kinetic energy of the part of $\psi^{N,\varepsilon}(t)$ with at least one particle orthogonal to $\Phi(t)$, given by $\|\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2$. Essentially, one shows that

$$\begin{aligned}
&\left|E_{w_{\mu,\beta}}^{\psi^{N,\varepsilon}(t)} - \mathcal{E}_{b_\beta}^{\Phi(t)}(t)\right| \\
&\quad\gtrsim \|\nabla_{x_1}\psi^{N,\varepsilon}(t)\|^2 - \|\nabla_x\Phi(t)\|^2 - \mathcal{O}(1)\\
&\quad\gtrsim \|\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2 + (\|\nabla_{x_1}p_1^\Phi\psi^{N,\varepsilon}(t)\|^2 - \|\nabla_x\Phi(t)\|^2) - \mathcal{O}(1)\\
&\quad\geq \|\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2 - \|\nabla_x\Phi(t)\|^2 \left\langle\!\left\langle\psi^{N,\varepsilon}(t), \widehat{n}\psi^{N,\varepsilon}(t)\right\rangle\!\right\rangle - \mathcal{O}(1)\ ,
\end{aligned}$$

which implies

$$\|\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2 \lesssim \alpha_{\xi,\,w_{\mu,\beta}}^<(t) + \mathcal{O}(1)\ .$$

The rigorous proof of this bound (Lemma 5.7) is an adaptation of the corresponding Lemma 4.21 in [4] and requires the new strategies described above, as well as both moderate confinement and admissibility condition.

3.2 The Gross–Pitaevskii case $\beta = 1$

For an interaction w_μ in the Gross–Pitaevskii scaling regime, the previous strategy, i.e., deriving an estimate of the form $|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}^<(t)| \lesssim \alpha_{\xi,w_\mu}^<(t) + \mathcal{O}(1)$, cannot work. To understand this, let us analyse the term $\gamma_{b,<}^{(1)}$, which contains the difference between the quasi two-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$ and the effective potential $b_1|\Phi(t)|^2$. As pointed out in Remark 4a, the basic idea here is to expand $|\varphi^\varepsilon(z_1-z_2)|^2$ around z_2, which can be made rigorous for sufficiently regular φ^ε and yields

$$N\int \mathrm{d}y_2|\chi^\varepsilon(y_2)|^2\int\mathrm{d}z_1|\varphi^\varepsilon(z_1)|^2 w_\mu(z_1-z_2) \approx N\left(\int\mathrm{d}y|\chi^\varepsilon(y)|^4\right)\|w_\mu\|_{L^1(\mathbb{R}^3)}|\Phi(x_2)|^2\,. \tag{34}$$

Whereas this equals (at least asymptotically) the coupling parameter b_β for $\beta \in (0,1)$, the situation is now different since $b_1 = 8\pi a\int|\chi(y)|^4\,\mathrm{d}y$. In order to see that (34) and b_1 are not asymptotically equal, but actually differ by an error of $\mathcal{O}(1)$, let us briefly recall the definition of the scattering length and its scaling properties.

The zero energy scattering equation for the interaction $w_\mu = \mu^{-2}w(\cdot/\mu)$ is

$$\begin{cases}\left(-\Delta + \frac{1}{2}w_\mu(z)\right)j_\mu(z) = 0 & \text{for } |z| < \infty\,,\\ j_\mu(z)\to 1 & \text{as } |z|\to\infty\,.\end{cases} \tag{35}$$

By [27, Theorems C.1 and C.2], the unique solution $j_\mu \in \mathcal{C}^1(\mathbb{R}^3)$ of (35) is spherically symmetric, non-negative and non-decreasing in $|z|$, and satisfies

$$\begin{cases} j_\mu(z) = 1 - \frac{a_\mu}{|z|} & \text{for } |z| > \mu, \\ j_\mu(z) \geq 1 - \frac{a_\mu}{|z|} & \text{else.} \end{cases} \tag{36}$$

The parameter $a_\mu \in \mathbb{R}$ in (36) defines the scattering length of w_μ. Equivalently,

$$8\pi a_\mu = \int\limits_{\mathbb{R}^3} w_\mu(z) j_\mu(z)\,\mathrm{d}z\,. \tag{37}$$

From the scaling behaviour of (35), it is obvious that $j_\mu(z) = j_{\mu=1}(z/\mu)$ and that

$$a_\mu = \mu a\,, \tag{38}$$

where a denotes the scattering length of the unscaled interaction w. Returning to the original question, this implies that

$$b_1 = 8\pi a \int\limits_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y = N \int\limits_{\mathbb{R}} |\chi^\varepsilon(y)|^4\,\mathrm{d}y \int\limits_{\mathbb{R}^3} w_\mu(z) j_\mu(z)\,\mathrm{d}z\,,$$

and consequently

$$\begin{aligned} (34) - b_1|\Phi(x_2)|^2 \;&=\; N|\Phi(x_2)|^2 \int\limits_{\mathbb{R}} |\chi^\varepsilon(y)|^4\,\mathrm{d}y \int\limits_{\mathbb{R}^3} w_\mu(z)(1 - j_\mu(z)) \\ &\geq\; \mu^{-1}|\Phi(x_2)|^2 \int\limits_{\mathbb{R}} |\chi(y)|^4\,\mathrm{d}y\,(1 - j_\mu(\mu))\,\|w_\mu\|_{L^1(\mathbb{R}^3)} \;=\; \mathcal{O}(1)\,, \end{aligned}$$

where we have used that $\|w_\mu\|_{L^1(\mathbb{R}^3)} = \mu\|w\|_{L^1(\mathbb{R}^3)}$ and that $j_\mu(z)$ is continuous and non-decreasing, hence $j_\mu(z) \leq j_\mu(\mu)$ for $z \in \operatorname{supp} w_\mu$ and $1 - j_\mu(\mu) \approx a$. In conclusion, the contribution from $\gamma_{b,<}^{(1)}$ does not vanish if b_1 is the coupling parameter in [4]. Naturally, one could amend this by taking $\int |\chi(y)|^4\,\mathrm{d}y \|w\|_{L^1(\mathbb{R}^3)}$ instead of b_1 as parameter in the non-linear equation. However, for this choice, the contributions from $\gamma_{b,<}^{(2)}$ to $\gamma_{b,<}^{(4)}$ would not vanish in the limit $(N,\varepsilon) \to (\infty, 0)$, as can easily be seen by setting $\beta = 1$ in Proposition 3.6.

The physical reason why the Gross–Pitaevskii scaling is fundamentally different — and why it requires a different strategy of proof — is the fact that the length scale a_μ of the inter-particle correlations is of the same order as the range μ of the interaction. In contrast, for $\beta \in (0,1)$, the relation $a_{\mu,\beta} \ll \mu^\beta$ implies that $j_{\mu,\beta} \approx 1$ on the support of $w_{\mu,\beta}$, hence the first order Born approximation $8\pi a_{\mu,\beta} \approx \|w_{\mu,\beta}\|_{L^1(\mathbb{R}^3)}$ applies in this case.

Before explaining the strategy of proof for the Gross–Pitaevskii scaling, let us introduce the auxiliary function $f_{\widetilde{\beta}} \in \mathcal{C}^1(\mathbb{R}^3)$. This function will be defined in such a way that it asymptotically coincides with j_μ on $\operatorname{supp} w_\mu$ but, in contrast to j_μ, satisfies $f_{\widetilde{\beta}}(z) = 1$ for sufficiently large $|z|$, which has the benefit of $1 - f_{\widetilde{\beta}}$ and $\nabla f_{\widetilde{\beta}}$ being compactly supported. To construct $f_{\widetilde{\beta}}$, we define the potential $U_{\mu,\widetilde{\beta}}$ such that the scattering length of $w_\mu - U_{\mu,\widetilde{\beta}}$ equals zero, and define $f_{\widetilde{\beta}}$ as the solution of the corresponding zero energy scattering equation:

Definition 3.7. Let $\widetilde{\beta} \in (\frac{1}{3}, 1)$. Define

$$U_{\mu,\widetilde{\beta}}(z) := \begin{cases} \mu^{1-3\widetilde{\beta}} a & \text{for } \mu^{\widetilde{\beta}} < |z| < \varrho_{\widetilde{\beta}}, \\ 0 & \text{else,} \end{cases}$$

where $\varrho_{\widetilde{\beta}}$ is the minimal value in $(\mu^{\widetilde{\beta}}, \infty]$ such that the scattering length of $w_\mu - U_{\mu,\widetilde{\beta}}$ equals zero. Further, let $f_{\widetilde{\beta}} \in \mathcal{C}^1(\mathbb{R}^3)$ be the solution of

$$\begin{cases} \Big(-\Delta + \frac{1}{2} \Big(w_\mu(z) - U_{\mu,\widetilde{\beta}}(z) \Big) \Big) f_{\widetilde{\beta}}(z) = 0 & \text{for } |z| < \varrho_{\widetilde{\beta}}, \\ f_{\widetilde{\beta}}(z) = 1 & \text{for } |z| \geq \varrho_{\widetilde{\beta}}\,, \end{cases} \tag{39}$$

and define

$$g_{\widetilde{\beta}} := 1 - f_{\widetilde{\beta}}.$$

In the sequel, we will abbreviate

$$U^{(ij)}_{\mu,\widetilde{\beta}} := U_{\mu,\widetilde{\beta}}(z_i - z_j)\,, \quad g^{(ij)}_{\widetilde{\beta}} := g_{\widetilde{\beta}}(z_i - z_j) \quad \text{and} \quad f^{(ij)}_{\widetilde{\beta}} := f_{\widetilde{\beta}}(z_i - z_j).$$

In [5, Lemma 4.9], it is shown by explicit construction that a suitable $\varrho_{\widetilde{\beta}}$ exists and that it is of order $\mu^{\widetilde{\beta}}$. Note that Definition 3.7 implies in particular that

$$\int\limits_{\mathbb{R}^3} \Big(w_\mu(z) - U_{\mu,\widetilde{\beta}}(z) \Big) f_{\widetilde{\beta}}(z)\, \mathrm{d}z = 0\,, \tag{40}$$

which is an equivalent way of expressing that the scattering length of $w_\mu - U_{\mu,\widetilde{\beta}}$ equals zero. Heuristically, one may think of the condensed N-body state as a product state that is overlaid with a microscopic structure described by $f_{\widetilde{\beta}}$, i.e.,

$$\psi_{\mathrm{cor}}(t, z_1, ..., z_N) := \prod_{k=1}^{N} \varphi^\varepsilon(t, z_k) \prod_{1 \leq l < m \leq N} f_{\widetilde{\beta}}(z_l - z_m)\,. \tag{41}$$

For $\beta \in (0, 1)$, it holds that $f_{\widetilde{\beta}} \approx 1$, i.e., the condensate is approximately described by the product $(\varphi^\varepsilon)^{\otimes N}$ — which is precisely the state onto which the operator $P_0 = p_1 \cdots p_N$ projects. For the Gross–Pitaevskii scaling, however, $f_{\widetilde{\beta}}$ is not approximately constant, and the product state is no appropriate description of the condensed N-body wave function. The idea in [30] is to account for this in the counting functional by replacing the projection P_0 onto the product state by the projection onto the correlated state ψ_{cor}. In this spirit, one substitutes the expression $\langle\!\langle \psi, \widehat{m}\psi \rangle\!\rangle$ in $\alpha^<_{\xi, w_{\mu,\beta}}(t)$ by

$$\left\langle\!\!\left\langle \psi, \prod_{k<l} f^{(lk)}_{\widetilde{\beta}} \widehat{m} \prod_{r<s} f^{(rs)}_{\widetilde{\beta}} \psi \right\rangle\!\!\right\rangle \approx \langle\!\langle \psi, \widehat{m}\psi \rangle\!\rangle - N(N-1) \Re \left\langle\!\!\left\langle \psi, g^{(12)}_{\widetilde{\beta}} \widehat{m}\psi \right\rangle\!\!\right\rangle\,,$$

where we expanded $f_{\widetilde{\beta}} = 1 - g_{\widetilde{\beta}}$ and kept only the terms which are at most linear in $g_{\widetilde{\beta}}$. This leads to the following definition:

Definition 3.8.

$$\alpha_{\xi, w_\mu}(t) := \alpha^<_{\xi, w_\mu}(t) - N(N-1) \Re \left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g^{(12)}_{\widetilde{\beta}}\, \widehat{r}\, \psi^{N,\varepsilon}(t) \right\rangle\!\!\right\rangle.$$

The new functional $\alpha_{\xi,w_\mu}(t)$ equals $\alpha^<_{\xi,w_\mu}(t)$ up to a correction term. Since the convergence of $\alpha^<_{\xi,w_\mu}(t)$ is equivalent to (16) and (17), an estimate of $\alpha_{\xi,w_\mu}(t)$ is only meaningful if this correction converges to zero as $(N,\varepsilon)\to(\infty,0)$. This is the reason why we defined it using the operator $\widehat{r}$ (Definition 3.2) instead of $\widehat{m}$: as $\widehat{r}$ contains additional projections p_1 and p_2, we can use the estimate $\|g^{(12)}_{\widetilde{\beta}}p_1\|_{\mathrm{op}}\lesssim\varepsilon^{-\frac12}\mu^{1+\frac{\widetilde{\beta}}{2}}$ instead of $\|g_{\widetilde{\beta}}\|_\infty\lesssim 1$ (Lemma 6.2). In the following proposition, it is shown that this suffices for the correction term to vanish in the limit.

Proposition 3.9. *Assume* A1 – A4. *Then*

$$\left|N(N-1)\Re\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),g^{(12)}_{\widetilde{\beta}}\,\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle\right|\lesssim\varepsilon$$

for all $t\in[0,T^{\mathrm{ex}}_{V^\parallel})$.

By adding the correction term to $\alpha^<_{\xi,w_\mu}(t)$, we effectively replace w_μ by $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}$ in the time derivative of $\alpha^<_{\xi,w_\mu}(t)$. To explain what is meant by this statement, let us analyse the contributions to the time derivative of $\alpha_{\xi,w_\mu}(t)$, which are collected in the following proposition:

Proposition 3.10. *Assume* A1 – A4 *for* $\beta=1$. *Then*

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}(t)\right|\leq\left|\gamma^<(t)\right|+\left|\gamma_a(t)\right|+|\gamma_b(t)|+|\gamma_c(t)|+|\gamma_d(t)|+|\gamma_e(t)|+|\gamma_f(t)|$$

for almost every $t\in[0,T^{\mathrm{ex}}_{V^\parallel})$, *where*

$$\begin{aligned}\gamma^<(t)\;&:=\;\left|\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),\dot{V}^\parallel(t,z_1)\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle-\left\langle\Phi(t),\dot{V}^\parallel(t,(x,0))\Phi(t)\right\rangle_{L^2(\mathbb{R}^2)}\right| &(42)\\&\quad-2N\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),q_1\widehat{m}^a_{-1}\big(V^\parallel(t,z_1)-V^\parallel(t,(x_1,0))\big)p_1\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle &(43)\\&\quad-N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),\widetilde{Z}^{(12)}\widehat{m}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, &(44)\end{aligned}$$

$$\gamma_a(t)\;:=\;N^2(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),g^{(12)}_{\widetilde{\beta}}\left[V^\parallel(t,z_1)-V^\parallel(t,(x_1,0)),\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \quad(45)$$

$$\begin{aligned}\gamma_b(t)\;&:=\;-N\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),b_1(|\Phi(t,x_1)|^2+|\Phi(t,x_2)|^2)g^{(12)}_{\widetilde{\beta}}\,\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle &(46)\\&\quad-N\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),(b_{\widetilde{\beta}}-b_1)(|\Phi(t,x_1)|^2+|\Phi(t,x_2)|^2)\,\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle &(47)\\&\quad-N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),g^{(12)}_{\widetilde{\beta}}\,\widehat{r}\,Z^{(12)}\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, &(48)\end{aligned}$$

$$\gamma_c(t)\;:=\;-4N(N-1)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),(\nabla_1 g^{(12)}_{\widetilde{\beta}})\cdot\nabla_1\widehat{r}\,\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \quad(49)$$

$$\begin{aligned}\gamma_d(t)\;&:=\;-N(N-1)(N-2)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),g^{(12)}_{\widetilde{\beta}}\left[b_1|\Phi(t,x_3)|^2,\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle &(50)\\&\quad+2N(N-1)(N-2)\Im\left\langle\!\!\left\langle\psi^{N,\varepsilon}(t),g^{(12)}_{\widetilde{\beta}}\left[w^{(13)}_\mu,\widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, &(51)\end{aligned}$$

$$\gamma_e(t) \;:=\; \tfrac{1}{2}N(N-1)(N-2)(N-3)\Im\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g^{(12)}_{\widetilde{\beta}}\left[w^{(34)}_\mu, \widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle, \tag{52}$$

$$\gamma_f(t) \;:=\; -2N(N-2)\Im\left\langle\!\!\left\langle \psi^{N,\varepsilon}(t), g^{(12)}_{\widetilde{\beta}}\left[b_1|\Phi(t,x_1)|^2, \widehat{r}\right]\psi^{N,\varepsilon}(t)\right\rangle\!\!\right\rangle. \tag{53}$$

Here, we have used the abbreviations

$$Z^{(ij)} \;:=\; w^{(ij)}_\mu - \tfrac{b_1}{N-1}\left(|\Phi(t,x_i)|^2 + |\Phi(t,x_j)|^2\right),$$
$$\widetilde{Z}^{(ij)} \;:=\; U^{(ij)}_{\mu,\widetilde{\beta}} f^{(ij)}_{\widetilde{\beta}} - \tfrac{b_{\widetilde{\beta}}}{N-1}(|\Phi(t,x_i)|^2 + |\Phi(t,x_j)|^2),$$

where

$$b_{\widetilde{\beta}} := \lim_{(N,\varepsilon)\to(\infty,0)} \mu^{-1}\int_{\mathbb{R}^3} U_{\mu,\widetilde{\beta}}(z) f_{\widetilde{\beta}}(z)\,\mathrm{d}z \int_{\mathbb{R}^2} |\chi(y)|^4\,\mathrm{d}y.$$

The proof of this proposition is given in Section 6.5. Note that the contributions to the derivative $\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}(t)$ fall into two categories:

- The terms (42)–(43) in $\gamma^<$ equal $\gamma_{a,<}$ from Proposition 3.5, and (44) is exactly $\gamma_{b,<}$ with interaction potential $U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$. Hence, estimating $\gamma^<$ is equivalent to estimating the functional $\alpha^<_{\xi,U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}}(t)$, which arises from $\alpha^<_{\xi,w_\mu}(t)$ by replacing the interaction w_μ by $U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$. Since $U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$ for any $\eta \in (0, 1-\widetilde{\beta})$ (Lemma 6.4), this is an interaction in the NLS scaling regime, which was covered in the previous section.
- γ_a to γ_f can be understood as remainders from this substitution. γ_a collects the contributions coming from the fact that the N-body wave function interacts with a three-dimensional external trap $V^\parallel$, while only $V^\parallel$ evaluated on the plane $y = 0$ enters in the effective equation (7). Since this is an effect of the strong confinement, it has no equivalent in the three-dimensional problem [30], but the same contribution occurs in the situation of a cigar-shaped confinement [5]. The terms γ_b to γ_f are analogous to the corresponding expressions in [30] and [5].

The physical idea behind the replacement is that low-energy scattering at any potential is to leading order described by the scattering length. Note that $f_{\widetilde{\beta}} \approx 1$ on $\operatorname{supp} U_{\mu,\widetilde{\beta}}$, hence $U_{\mu,\widetilde{\beta}} \approx U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$ and consequently the scattering length of $w_{\mu,\beta} - U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$ is approximately zero by construction (40). This implies that a sufficiently distant test particle with very low energy cannot resolve the difference between the two potentials.

Proposition 3.11. *Assume* A1 – A4, *let* $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$ *and let*

$$\max\left\{\tfrac{\gamma+1}{2\gamma}, \tfrac{5}{6}\right\} < d < \widetilde{\beta} < 1, \qquad 0 < \xi < \min\left\{\tfrac{1-\widetilde{\beta}}{2}, \tfrac{3-\vartheta\widetilde{\beta}}{2(\vartheta-1)}\right\}.$$

Then, for sufficiently small μ,

$$\left|\gamma^<(t)\right| \;\lesssim\; \mathfrak{e}^3_1(t)\alpha^<_{\xi,w_\mu} + \mathfrak{e}^4_1(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}} + \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}} + \varepsilon^{\frac{1-\widetilde{\beta}}{2}} + N^{-d+\frac{5}{6}}\right),$$

$$\left|\gamma_a(t)\right| \;\lesssim\; \mathfrak{e}^4_1(t)\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{1+\xi-\frac{\widetilde{\beta}}{2}},$$

$$\begin{aligned}
\left|\gamma_b(t)\right| &\lesssim \mathfrak{e}_1^3(t)\,\varepsilon^{\frac{1+\widetilde{\beta}}{2}},\\
\left|\gamma_c(t)\right| &\lesssim \mathfrak{e}_1^3(t)\left(\varepsilon^{\frac{1+\widetilde{\beta}}{2}}+\left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{\widetilde{\beta}}{2}-\xi}\right),\\
\left|\gamma_d(t)\right| &\lesssim \mathfrak{e}_1^3(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{1+\xi-\widetilde{\beta}}+\varepsilon^{\frac{1+\widetilde{\beta}}{2}}\right),\\
\left|\gamma_e(t)\right| &\lesssim \mathfrak{e}_1^3(t)\,\varepsilon^{\frac{1+\widetilde{\beta}}{2}},\\
\left|\gamma_f(t)\right| &\lesssim \mathfrak{e}_1^3(t)\,\varepsilon^{\frac{1+\widetilde{\beta}}{2}}.
\end{aligned}$$

Remark 5. (a) To estimate $\gamma^<$, observe first that we have chosen $\widetilde{\beta}$ such that $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$ for some η, and such that assumption *A4* with parameters $(\Theta,\Gamma)_1 = (\vartheta,\gamma)$ makes the sequence (N,ε) at the same time $(\Theta,\Gamma)_{\widetilde{\beta}}$ -admissible/moderately confining. Consequently, Proposition 3.6 yields

$$|\gamma^<(t)| \lesssim \alpha^<_{\xi,U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)+\mathcal{o}(1) = \left\langle\!\left\langle\psi^{N,\varepsilon},\widehat{m}\psi^{N,\varepsilon}\right\rangle\!\right\rangle+\left|E^{\psi^{N,\varepsilon}(t)}_{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)-\mathcal{E}^{\Phi(t)}_{b_{\widetilde{\beta}}}(t)\right|+\mathcal{o}(1)\,. \tag{54}$$

However, this does not yet complete the estimate for $\gamma^<$ since we need to bound all expressions in Proposition 3.10 in terms of $\alpha^<_{\xi,w_\mu} = \left\langle\!\left\langle\psi^{N,\varepsilon},\widehat{m}\psi^{N,\varepsilon}\right\rangle\!\right\rangle+\left|E^{\psi^{N,\varepsilon}(t)}_{w_\mu}(t)-\mathcal{E}^{\Phi(t)}_{b_1}(t)\right|$, up to contributions $\mathcal{o}(1)$. By construction of $f_{\widetilde{\beta}}$, it follows that $b_{\widetilde{\beta}} = b_1$ (see (86) in Lemma 6.4), hence $\mathcal{E}^{\Phi(t)}_{b_{\widetilde{\beta}}}(t) = \mathcal{E}^{\Phi(t)}_{b_1}(t)$. On the other hand, heuristic arguments[3] indicate that $E^{\psi^{N,\varepsilon}(t)}_{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)$ and $E^{\psi^{N,\varepsilon}(t)}_{w_\mu}(t)$ differ by an error of order $\mathcal{O}(1)$, which implies that the right hand side of (54) is different from $\alpha^<_{\xi,w_\mu}(t)$ by $\mathcal{O}(1)$.

By Remark 4c, this energy difference enters only in the estimate of (31) in $\gamma^{(4)}_{b,<}$ via $\|\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2 \lesssim \alpha^<_{\xi,U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)+\mathcal{o}(1)$. For the Gross–Pitaevskii scaling of the interaction, $\|\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2$ is not asymptotically zero because the microscopic structure described by $f_{\widetilde{\beta}}$ lives on the same length scale as the interaction and thus contributes a kinetic energy of $\mathcal{O}(1)$. However, as this kinetic energy is concentrated around the scattering centres, one can show a similar bound for the kinetic energy on a subset $\mathcal{A}_1$ of $\mathbb{R}^{3N}$, where appropriate holes around these centres are cut out (Definition 6.5). This is done in Section 6.3, where we show in Lemma 6.7 that

$$\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2 \lesssim \alpha^<_{\xi,w_\mu}(t)+\mathcal{o}(1)\,.$$

The proof of this lemma is similar to the corresponding proof in [5, Lemma 4.12], which, in turn, adjusts ideas from [30] to the problem with dimensional reduction. However, since one key tool for the estimate is the Gagliardo–Nirenberg–Sobolev inequality in the x-coordinates, the estimates depend in a non-trivial way on the

[3]See [5, pp. 1019–1020]. Essentially, when evaluated on the trial function ψ_{cor} from (41), the energy difference is to leading order given by $N\left\langle\!\left\langle\psi_{\mathrm{cor}}(t),(w_\mu^{(12)}-(U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)})\psi_{\mathrm{cor}}(t)\right\rangle\!\right\rangle \sim N\int\mathrm{d}z_1|\varphi^\varepsilon(t,z_1)|^2\int\mathrm{d}z|f_{\widetilde{\beta}}(z)|^2(w_\mu(z) - U_{\mu,\widetilde{\beta}}(z)) \sim \mu^{-1}\int\mathrm{d}z g_{\widetilde{\beta}}(z)w_\mu(z)f_{\widetilde{\beta}}(z) \geq \mu^{-1}g_{\widetilde{\beta}}(\mu)\int\mathrm{d}z w_\mu(z)f_{\widetilde{\beta}}(z) \sim 8\pi a^2$, where we have dropped all sub-leading contributions.

dimension of x. As one consequence, our estimate requires the moderate confinement condition with parameter $\gamma > 1$, where no such restriction was needed in [5].

Finally, we adapt the estimate of (31). In distinction to the corresponding proof in [5, Section 4.5.1], we need to integrate by parts in two steps to be able to control the logarithmic divergences that are due to the two-dimensional Green's function. Inspired by an idea in [30], we introduce two auxiliary potentials $\overline{\overline{v}}_{\mu^{\beta_2}}$ and $\overline{\overline{\nu}}_1$ such that $\|\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}}\|_{L^1(\mathbb{R}^2)} = \|\overline{\overline{v}}_{\mu^{\beta_2}}\|_{L^1(\mathbb{R}^2)} = \|\overline{\overline{\nu}}_1\|_{L^1(\mathbb{R}^2)}$, define $\overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}$ and $\overline{\overline{h}}_{\mu^{\beta_2},1}$ as the solutions of $\Delta_x\overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}} = \overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}} - \overline{\overline{v}}_{\mu^{\beta_2}}$ and $\Delta_x\overline{\overline{h}}_{\mu^{\beta_2},1} = \overline{\overline{v}}_{\mu^{\beta_2}} - \overline{\overline{\nu}}_1$, and write $\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}} = \Delta_x\overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}} + \Delta_x\overline{\overline{h}}_{\mu^{\beta_2},1} + \overline{\overline{\nu}}_1$. The expressions depending on $\overline{\overline{\nu}}_1$ can be controlled immediately, while we integrate the remainders by parts in x, making use of different properties of $\overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}$ and $\overline{\overline{h}}_{\mu^{\beta_2},1}$ (Lemma 5.6b). Subsequently, we insert identities $\mathbb{1} = \mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}_1}}$, where $\overline{\mathcal{A}_1}$ denotes the complement of $\mathcal{A}_1$. On the one hand, this yields $\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^{\Phi}\psi^{N,\varepsilon}(t)\|$, which can be controlled by the new energy lemma (Lemma 6.7). On the other hand, we obtain terms containing $\mathbb{1}_{\overline{\mathcal{A}_1}}$, which we estimate by exploiting the smallness of $\overline{\mathcal{A}_1}$. The full argument is given in Section 6.6.1.

(b) The remainders γ_a to γ_f are estimated in Sections 6.6.2, and work, for the most part, analogously to the corresponding proofs in [5, Sections 4.5.2 – 4.5.7]. The only exception is γ_c, where the strategy from [5] produces too many factors ε^{-1}. Instead, we estimate the x- and y-contributions to the scalar product $(\nabla g_{\widetilde{\beta}})\cdot\nabla\widehat{r} = (\nabla_x g_{\widetilde{\beta}})\cdot\nabla_x\widehat{r} + (\partial_y g_{\widetilde{\beta}})\partial_y\widehat{r}$ separately. To control the y-part, we integrate by parts in y and use the moderate confinement condition with $\gamma > 1$. Again, this is different from the situation in [5], where the corresponding term γ_c could be estimated without any restriction on the sequence (N,ε).

3.3 Proof of Theorem 1

Let $0 \leq T < T^{\text{ex}}_{V^{\parallel}}$. For $\beta \in (0,1)$, Proposition 3.6 implies that

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha^<_{\xi,\,w_{\mu,\beta}}(t)\right| \lesssim \mathfrak{e}^3_\beta(t)\alpha^<_{\xi,\,w_\mu}(t) + \mathfrak{e}^3_\beta(t)R_{\eta,\beta,\delta,\sigma,\xi}(N,\varepsilon)$$

for almost every $t \in [0,T]$ and sufficiently small μ, where

$$R^<_{\eta,\beta,\delta,\xi}(N,\varepsilon) := \left(\tfrac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}} + \left(\tfrac{\varepsilon^3}{\mu^\beta}\right)^{\frac{1}{2}} + \tfrac{\mu^\beta}{\varepsilon} + \mu^\eta + \varepsilon^{\frac{1-\beta}{2}} + N^{-\sigma} + N^{-\frac{\beta}{2}}$$

with $0 < \sigma < \min\{\frac{1-3\xi}{4}, \beta - \xi\}$. Since $t \mapsto \alpha^<_{\xi,\,w_{\mu,\beta}}(t)$ is non-negative and absolutely continuous on $[0,T]$, the differential version of Grönwall's inequality (see e.g. [13, Appendix B.2.j]) yields

$$\alpha^<_{\xi,\,w_{\mu,\beta}}(t) \lesssim \mathrm{e}^{\int_0^t \mathfrak{e}^3_\beta(s)\,\mathrm{d}s}\left(\alpha^<_{\xi,\,w_{\mu,\beta}}(0) + \int_0^t \mathfrak{e}^3_\beta(s)\,\mathrm{d}s\right)$$

for all $t \in [0,T]$. Since $\mathfrak{e}_\beta(t)$ is bounded uniformly in N and ε by (14) and with $R^<_{\eta,\beta,\delta,\xi}(N,\varepsilon) \to 0$ as $(N,\varepsilon) \to (\infty,0)$, this implies (16) and (17) by Lemma 3.4.

For $\beta = 1$, observe first that Proposition 3.9 implies that the correction term in $\alpha_{\xi,w_\mu}(t)$ is bounded by ε uniformly in $t \in [0,T]$, provided μ is sufficiently small. Hence, $t \mapsto \alpha_{\xi,w_\mu}(t) + \varepsilon$ is non-negative and absolutely continuous and

$$\alpha^<_{\xi,\,w_\mu}(t) \lesssim \alpha_{\xi,w_\mu}(t) + \varepsilon < \alpha_{\xi,w_\mu}(t) + R_{\gamma,\vartheta,\xi}(N,\varepsilon)$$

for

$$R_{\gamma,\vartheta,\xi}(N,\varepsilon) = \left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}} + \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\gamma^2}} + \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{\widetilde{\beta}}{2}-\xi} + \varepsilon^{\frac{1-\widetilde{\beta}}{2}} + N^{-d+\frac{5}{6}}$$

with $\max\{\frac{\gamma+1}{2\gamma}, \frac{5}{6}\} < d < \widetilde{\beta} < \frac{3}{\vartheta}$. Consequently, Proposition 3.11 yields

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}(\alpha_{\xi,w_\mu}(t) + \varepsilon)\right| \lesssim \mathfrak{e}_1^4(t)\left(\alpha_{\xi,w_\mu}(t) + R_{\gamma,\vartheta,\xi}(N,\varepsilon)\right)$$

for almost every $t \in [0,T]$ and sufficiently small μ, which, as before, implies the statement of the theorem because both ε and $R_{\gamma,\vartheta,\xi}(N,\varepsilon)$ converge to zero as $(N,\varepsilon) \to \infty$.

4 Preliminaries

We will from now on always assume that assumptions *A1* – *A4* are satisfied.

Definition 4.1. Let $\mathcal{M} \subseteq \{1,\dots,N\}$. Define $\mathcal{H}_\mathcal{M} \subseteq L^2(\mathbb{R}^{3N})$ as the subspace of functions which are symmetric in all variables in $\mathcal{M}$, i.e. for $\psi \in \mathcal{H}_\mathcal{M}$,

$$\psi(z_1,...,z_j,...,z_k,...,z_N) = \psi(z_1,...,z_k,...,z_j,...,z_N) \qquad \forall\, j,k \in \mathcal{M}.$$

Lemma 4.2. *Let $f : \mathbb{N}_0 \to \mathbb{R}_0^+$, $d \in \mathbb{Z}$, $\rho \in \{a,b\}$ and $\nu \in \{c,d,e,f\}$. Further, let $\mathcal{M}_1, \mathcal{M}_{1,2} \subseteq \{1,2,...,N\}$ with $1 \in \mathcal{M}_1$ and $1,2 \in \mathcal{M}_{1,2}$. Then*

(a) $\|\widehat{f}\|_{\mathrm{op}} = \|\widehat{f}_d\|_{\mathrm{op}} = \|\widehat{f}^{\frac{1}{2}}\|^2_{\mathrm{op}} = \sup\limits_{0\leq k\leq N} f(k)$,

(b) $\|\widehat{m}^\rho\|_{\mathrm{op}} \leq N^{-1+\xi}$, $\|\widehat{m}^\nu\|_{\mathrm{op}} \lesssim N^{-2+3\xi}$ *and* $\|\widehat{r}\|_{\mathrm{op}} \lesssim N^{-1+\xi}$,

(c) $\widehat{n}^2 = \frac{1}{N}\sum\limits_{j=1}^N q_j$,

(d) $\|\widehat{f}q_1\psi\|^2 \leq \frac{N}{|\mathcal{M}_1|}\|\widehat{f}\widehat{n}\psi\|^2$ *for* $\psi \in \mathcal{H}_{\mathcal{M}_1}$,

$\|\widehat{f}q_1q_2\psi\|^2 \leq \frac{N^2}{|\mathcal{M}_{1,2}|(|\mathcal{M}_{1,2}|-1)}\|\widehat{f}\,\widehat{n}^2\psi\|^2$ *for* $\psi \in \mathcal{H}_{\mathcal{M}_{1,2}}$,

$\|\widehat{m}^\rho_d\, q_1\psi^{N,\varepsilon}(t)\| \lesssim N^{-1}$,

(e) $\|\nabla_1\widehat{f}q_1\psi\| \lesssim \|\widehat{f}\|_{\mathrm{op}}\|\nabla_1 q_1\psi\|$ *for* $\psi \in L^2(\mathbb{R}^{3N})$,

$\|\nabla_{x_1}\widehat{f}q_1^\Phi\psi\| \lesssim \|\widehat{f}\|_{\mathrm{op}}\|\nabla_{x_1}q_1^\Phi\psi\|$ *for* $\psi \in L^2(\mathbb{R}^{3N})$,

(f) $\|\nabla_2\widehat{f}q_1q_2\psi\| \leq \frac{N}{|\mathcal{M}_1|-1}\|\widehat{f}\,\widehat{n}\|_{\mathrm{op}}\|\nabla_2 q_2\psi\|$ *for* $\psi \in \mathcal{H}_{\mathcal{M}_1}$,

$\|\nabla_{x_2}\widehat{f}q_1^\Phi q_2^\Phi\psi\| \leq \frac{N}{|\mathcal{M}_1|-1}\|\widehat{f}\,\widehat{n}\|_{\mathrm{op}}\|\nabla_{x_2}q_2^\Phi\psi\|$ *for* $\psi \in \mathcal{H}_{\mathcal{M}_1}$.

Proof. [4], Lemmas 4.1 and 4.5 and Corollary 4.6 and [5], Lemma 4.1. □

Lemma 4.3. *Let $f, g : \mathbb{N}_0 \to \mathbb{R}_0^+$ be any weights and $i, j \in \{1, \ldots, N\}$.*

(a) For $k \in \{0, \ldots, N\}$,

$$\widehat{f}\,\widehat{g} = \widehat{fg} = \widehat{g}\widehat{f}, \qquad \widehat{f}p_j = p_j\widehat{f}, \qquad \widehat{f}q_j = q_j\widehat{f}, \qquad \widehat{f}P_k = P_k\widehat{f}.$$

(b) Define $Q_0 := p_j$, $Q_1 := q_j$, $\widetilde{Q}_0 := p_ip_j$, $\widetilde{Q}_1 \in \{p_iq_j, q_ip_j\}$ and $\widetilde{Q}_2 := q_iq_j$. Let S_j be an operator acting non-trivially only on coordinate j and T_{ij} only on coordinates i and j. Then for $\mu, \nu \in \{0, 1, 2\}$

$$Q_\mu \widehat{f} S_j Q_\nu = Q_\mu S_j \widehat{f}_{\mu-\nu} Q_\nu \quad \text{and} \quad \widetilde{Q}_\mu \widehat{f} T_{ij} \widetilde{Q}_\nu = \widetilde{Q}_\mu T_{ij} \widehat{f}_{\mu-\nu} \widetilde{Q}_\nu.$$

(c)

$$[T_{ij}, \widehat{f}] = [T_{ij}, p_ip_j(\widehat{f} - \widehat{f}_2) + (p_iq_j + q_ip_j)(\widehat{f} - \widehat{f}_1)].$$

Proof. [4], Lemma 4.2. □

Lemma 4.4. *Let $f : \mathbb{N}_0 \to \mathbb{R}_0^+$.*

(a) The operators P_k and $\widehat{f}$ are continuously differentiable as functions of time, i.e.,

$$P_k,\ \widehat{f} \in \mathcal{C}^1\big(\mathbb{R}, \mathcal{L}\left(L^2(\mathbb{R}^{3N})\right)\big)$$

for $0 \leq k \leq N$. Moreover,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{f} = \mathrm{i}\Big[\widehat{f}, \sum_{j=1}^{N} h_\beta^{(j)}(t)\Big],$$

where $h_\beta^{(j)}(t)$ denotes the one-particle operator corresponding to $h_\beta(t)$ from (7) acting on the j^{th} coordinate.

(b) $\Big[-\partial_{y_j}^2 + \frac{1}{\varepsilon^2}V^\perp(\frac{y_j}{\varepsilon}), \widehat{f}\Big] = 0$ for $1 \leq j \leq N$.

Proof. [4], Lemma 4.3. □

Lemma 4.5. *Let $\psi \in L_+^2(\mathbb{R}^{3N})$ be normalised and $f \in L^\infty(\mathbb{R}^2)$. Then*

$$\Big|\langle\!\langle \psi, f(x_1)\psi \rangle\!\rangle - \langle \Phi(t), f\Phi(t)\rangle_{L^2(\mathbb{R}^2)}\Big| \lesssim \|f\|_{L^\infty(\mathbb{R}^2)} \langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle\,.$$

Proof. [4], Lemma 4.7. □

Lemma 4.6. *Let $\Gamma, \Lambda \in L^2(\mathbb{R}^{3N}) \in \mathcal{H}_\mathcal{M}$ such that $j \notin \mathcal{M}$ and $k, l \in \mathcal{M}$ with $j \neq k \neq l \neq j$. Let $O_{j,k}$ be an operator acting non-trivially only on coordinates j and k, denote by r_k and s_k operators acting only on k^{th} coordinate, and let $F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^d$ for $d \in \mathbb{N}$. Then*

(a) $|\langle\!\langle \Gamma, O_{j,k}\Lambda \rangle\!\rangle| \leq \|\Gamma\| \Big(|\langle\!\langle O_{j,k}\Lambda, O_{j,l}\Lambda \rangle\!\rangle| + |\mathcal{M}|^{-1}\|O_{j,k}\Lambda\|^2\Big)^{\frac{1}{2}}$.

(b) $|\langle\!\langle r_kF(z_j, z_k)s_k\Gamma, r_lF(z_j, z_l)s_l\Gamma \rangle\!\rangle| \leq \|s_kF(z_j, z_k)r_k\Gamma\|^2$.

(c) $|\langle\!\langle \Gamma, r_k F(z_j,z_k) s_k \Lambda \rangle\!\rangle| \leq \|\Gamma\| \left(\|s_k F(z_j,z_k) r_k \Lambda\|^2 + |\mathcal{M}|^{-1} \|r_k F(z_j,z_k) s_k \Lambda\|^2 \right)^{\frac{1}{2}}$.

Proof. [4], Lemma 4.8 and [5], Lemma 4.4. □

Lemma 4.7. *Let $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$. Then for sufficiently small ε,*

(a) $\|\Phi(t)\|_{L^2(\mathbb{R}^2)} = 1,$

$\|\Phi(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\Phi(t)\|_{H^2(\mathbb{R}^2)} \leq \mathfrak{e}_\beta(t),$

$\|\nabla_x \Phi(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\Phi(t)\|_{H^3(\mathbb{R}^2)} \leq \mathfrak{e}_\beta(t),$

$\|\Delta_x \Phi(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\Phi(t)\|_{H^4(\mathbb{R}^2)} \leq \mathfrak{e}_\beta(t),$

(b) $\|\chi^\varepsilon\|_{L^2(\mathbb{R})} = 1, \quad \|\tfrac{\mathrm{d}}{\mathrm{d}y}\chi^\varepsilon\|_{L^2(\mathbb{R})} \lesssim \varepsilon^{-1},$

$\int_{\mathbb{R}} |\chi^\varepsilon(y)|^4 \,\mathrm{d}y = \varepsilon^{-1} \int_{\mathbb{R}} |\chi(y)|^4 \,\mathrm{d}y,$

$\|\chi^\varepsilon\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon^{-\frac{1}{2}}, \quad \|\tfrac{\mathrm{d}}{\mathrm{d}y}\chi^\varepsilon\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon^{-\frac{3}{2}},$

(c) $\|\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mathfrak{e}_\beta(t)\varepsilon^{-\frac{1}{2}},$

$\|\nabla\varphi^\varepsilon(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \mathfrak{e}_\beta(t)\varepsilon^{-\frac{3}{2}},$

$\|\nabla|\varphi^\varepsilon(t)|^2\|_{L^2(\mathbb{R}^3)} \lesssim \mathfrak{e}_\beta(t)\varepsilon^{-\frac{3}{2}}.$

Proof. Part (a) follows from the Sobolev embedding theorem [1, Theorem 4.12, Part I A] and by definition of $\mathfrak{e}_\beta$. Part (b) is an immediate consequence of (6), and part (c) is implied by (a) and (b). □

Lemma 4.8. *Fix $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$ and let $j,k \in \{1,\dots,N\}$. Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be measurable functions such that $|g(z_j,z_k)| \leq G(z_k - z_j)$ and $|h(x_j,x_k)| \leq H(x_k - x_j)$ almost everywhere for some $G : \mathbb{R}^3 \to \mathbb{R}$, $H : \mathbb{R}^2 \to \mathbb{R}$. Let $t_j \in \{p_j, \nabla_{x_j} p_j\}$ and $t^\Phi_j \in \{p^\Phi_j, \nabla_{x_j} p^\Phi_j\}$. Then*

(a) $\|(t_j)^\dagger\, g(z_j,z_k) t_j\|_{\mathrm{op}} \lesssim \mathfrak{e}^2_\beta(t)\varepsilon^{-1}\|G\|_{L^1(\mathbb{R}^3)}$ *for* $G \in L^1(\mathbb{R}^3)$,

(b) $\|g(z_j,z_k) t_j\|_{\mathrm{op}} = \|t^\dagger_j\, g(z_j,z_k)\|_{\mathrm{op}} \lesssim \mathfrak{e}_\beta(t)\varepsilon^{-\frac{1}{2}}\|G\|_{L^2(\mathbb{R}^3)}$ *for* $G \in L^2 \cap L^\infty(\mathbb{R}^3)$,

(c) $\|g(z_j,z_k)\nabla_j p_j\|_{\mathrm{op}} \lesssim \mathfrak{e}_\beta(t)\varepsilon^{-\frac{3}{2}}\|G\|_{L^2(\mathbb{R}^3)}$ *for* $G \in L^2(\mathbb{R}^3)$,

(d) $\|h(x_j,x_k) t^\Phi_j\|_{\mathrm{op}} = \|(t^\Phi_j)^\dagger h(x_j,x_k)\|_{\mathrm{op}} \leq \mathfrak{e}_\beta(t)\|H\|_{L^2(\mathbb{R}^2)}$ *for* $H \in L^2 \cap L^\infty(\mathbb{R}^2)$.

Proof. Analogously to [4], Lemma 4.10. □

Lemma 4.9. *Let ε be sufficiently small and fix $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$. Then for $\beta \in (0,1]$*

(a) $\|\nabla_{x_1} p^\Phi_1\|_{\mathrm{op}} \leq \mathfrak{e}_\beta(t), \qquad \|\Delta_{x_1} p^\Phi_1\|_{\mathrm{op}} \leq \mathfrak{e}_\beta(t),$

$\|\partial_{y_1} p^{\chi^\varepsilon}_1\|_{\mathrm{op}} \lesssim \varepsilon^{-1}, \qquad \|\partial^2_{y_1} p^{\chi^\varepsilon}_1\|_{\mathrm{op}} \lesssim \varepsilon^{-2},$

$\|q^{\chi^\varepsilon}_1 \psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}_\beta(t)\varepsilon, \qquad \|\nabla_{x_1} q^\Phi_1 \psi\| \lesssim \mathfrak{e}_\beta(t), \qquad \|\partial_{y_1} q^{\chi^\varepsilon}_1 \psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_\beta(t),$

$\|\nabla_{x_1}\psi^{N,\varepsilon}(t)\| \leq \mathfrak{e}_\beta(t), \qquad \|\partial_{y_1}\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1}, \qquad \|\nabla_1\psi^{N,\varepsilon}(t)\| \lesssim \varepsilon^{-1},$

(b) $\left\| \sqrt{w_{\mu,\beta}^{(12)}} \psi^{N,\varepsilon}(t) \right\| \lesssim \mathfrak{e}_\beta(t) N^{-\frac{1}{2}}$,

(c) $\|w_{\mu,\beta}^{(12)} \psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_\beta(t) N^{-\frac{1}{2}} \mu^{\frac{1}{2}-\frac{3\beta}{2}}$,

(d) $\|p_1 \mathbb{1}_{\operatorname{supp} w_{\mu,\beta}}(z_1 - z_2)\|_{\mathrm{op}} = \|\mathbb{1}_{\operatorname{supp} w_{\mu,\beta}}(z_1 - z_2) p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}_\beta(t) \mu^{\frac{3\beta}{2}} \varepsilon^{-\frac{1}{2}}$,

(e) $\|p_1 w_{\mu,\beta}^{(12)} \psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_\beta^2(t) N^{-1}$.

Proof. Analogously to [4], Lemma 4.11 and [5], Lemma 4.7. For parts (c) and (e), note that for $\beta \in (0,1)$,

$$\|w_{\mu,\beta}\|_{L^1(\mathbb{R}^3)} \sim \mu\, b_{\beta,N,\varepsilon} \leq \mu |b_{\beta,N,\varepsilon} - b_\beta| + \mu\, b_\beta \lesssim \mu \tag{55}$$

since $w_{\mu,\beta} \in \mathcal{W}_{\beta,\eta}$ for some $\eta > 0$. For $\beta = 1$, $\|w_\mu\|_{L^1(\mathbb{R}^3)} = \mu \|w\|_{L^1(\mathbb{R}^3)} \lesssim \mu$ by scaling. □

Lemma 4.10. *Let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that $f(t,\cdot) \in \mathcal{C}^1(\mathbb{R}^3)$ and $\partial_y f(t,\cdot) \in L^\infty(\mathbb{R}^3)$ for any $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$. Then*

(a) $\|(f(t,z_1) - f(t,(x_1,0)) p_1^{\chi^\varepsilon} \psi^{N,\varepsilon}(t)\| \leq \varepsilon \|\partial_y f(t)\|_{L^\infty(\mathbb{R}^3)}$,

(b) $\|(f(t,z_1) - f(t,(x_1,0))\psi^{N,\varepsilon}(t)\| \leq \varepsilon \left(\mathfrak{e}_\beta(t) \|f(t)\|_{L^\infty(\mathbb{R}^3)} + \|\partial_y f(t)\|_{L^\infty(\mathbb{R}^3)} \right)$.

Proof. Analogously to [4], Lemma 4.12. □

Lemma 4.11. *Let $c \in \mathbb{R}$. Then*

(a) $N^{-c} \ln N < N^{-c^-}$, $\qquad \varepsilon^c \ln \varepsilon^{-1} < \varepsilon^{c^-}$, $\qquad \mu^c \ln \mu^{-1} < \mu^{c^-}$,

(b) $\varepsilon^c \ln N < (\Theta - 1)\varepsilon^{c^-} \lesssim \begin{cases} \frac{\delta-\beta}{\beta} \varepsilon^{c^-} & \beta \in (0,1)\,, \\ \varepsilon^{c^-} & \beta = 1\,, \end{cases}$

$$N^{-c} \ln \varepsilon^{-1} < \tfrac{1}{\Gamma - 1} N^{-c^-} = \begin{cases} \frac{\beta}{1-\beta} N^{-c^-} & \beta \in (0,1)\,, \\ \frac{1}{\gamma-1} N^{-c^-} & \beta = 1\,, \end{cases}$$

(c) $N^{-c} \ln \mu^{-1} < \frac{\Gamma}{\Gamma-1} N^{-c^-} = \begin{cases} \frac{1}{1-\beta} N^{-c^-} & \beta \in (0,1)\,, \\ \frac{\gamma}{\gamma-1} N^{-c^-} & \beta = 1\,, \end{cases}$

$$\varepsilon^c \ln \mu^{-1} < \Theta\, \varepsilon^{c^-} \lesssim \begin{cases} \frac{\delta}{\beta}\, \varepsilon^{c^-} & \beta \in (0,1)\,, \\ \varepsilon^{c^-} & \beta = 1\,. \end{cases}$$

Proof. Observe that $N < \varepsilon^{-\Theta+1}$ and $\varepsilon^{-1} < N^{\frac{1}{\Gamma-1}}$ due to admissibility and moderate confinement, hence $\ln N < (\Theta - 1) \ln \varepsilon^{-1}$ and $\ln \varepsilon^{-1} < \frac{1}{\Gamma-1} \ln N$. □

5 Proofs for $\beta \in (0,1)$

5.1 Proof of Proposition 3.5

The proof works analogously to the proof of Proposition 3.7 in [4] and we provide only the main steps for convenience of the reader. From now on, we will drop the time dependence of Φ, φ^ε and $\psi^{N,\varepsilon}$ in the notation and abbreviate $\psi^{N,\varepsilon} \equiv \psi$. The time derivative of $\alpha^<_{\xi,w_{\mu,\beta}}(t)$ is bounded by

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha^<_{\xi,w_{\mu,\beta}}(t)\right| \leq \left|\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\langle\psi,\widehat{m}\psi\rangle\rangle\right| + \left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left|E^\psi_{w_{\mu,\beta}}(t) - \mathcal{E}^\Phi_{b_\beta}(t)\right|\right|. \tag{56}$$

For the second term in (56), note that

$$\begin{aligned}\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left|E^\psi_{w_{\mu,\beta}}(t) - \mathcal{E}^\Phi_{b_\beta}(t)\right|\right| &= \left|\tfrac{\mathrm{d}}{\mathrm{d}t}\left(E^\psi_{w_{\mu,\beta}}(t) - \mathcal{E}^\Phi_{b_\beta}(t)\right)\right| \\ &= \left|\left\langle\!\!\left\langle\psi,\dot{V}^{\parallel}(t,z_1)\psi\right\rangle\!\!\right\rangle - \left\langle\Phi,\dot{V}^{\parallel}\left(t,(x,0)\right)\Phi\right\rangle\right|\end{aligned}$$

for almost every $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$ by [25, Theorem 6.17] because $t \mapsto \tfrac{\mathrm{d}}{\mathrm{d}t}\left(E^\psi_{w_{\mu,\beta}}(t) - \mathcal{E}^\Phi_{b_\beta}(t)\right)$ is continuous due to assumption *A3*. The first term in (56) yields

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\langle\psi,\widehat{m}\psi\rangle\rangle &= -2N\Im\left\langle\!\!\left\langle\psi, q_1\widehat{m}^a_{-1}\left(V^\parallel(t,z_1) - V^\parallel\left(t,(x_1,0)\right)\right)p_1\psi\right\rangle\!\!\right\rangle && (57)\\ &\quad -2N(N-1)\Im\left\langle\!\!\left\langle\psi, q_1p_2\widehat{m}^a_{-1}Z^{(12)}_\beta p_1p_2\psi\right\rangle\!\!\right\rangle && (58)\\ &\quad -N(N-1)\Im\left\langle\!\!\left\langle\psi, q_1q_2\widehat{m}^b_{-2}w^{(12)}_{\mu,\beta}p_1p_2\psi\right\rangle\!\!\right\rangle && (59)\\ &\quad -2N(N-1)\Im\left\langle\!\!\left\langle\psi, q_1q_2\widehat{m}^a_{-1}Z^{(12)}_\beta p_1q_2\psi\right\rangle\!\!\right\rangle, && (60)\end{aligned}$$

which follows from Lemmas 4.3 and 4.4. Expanding $q = q^{\chi^\varepsilon} + p^{\chi^\varepsilon}q^\Phi$ in (58) to (60) and subsequently estimating $N\widehat{m}^a_{-1} \leq \widehat{l}$ and $N\widehat{m}^b_{-2} \leq \widehat{l}$ for $\widehat{l} \in \mathcal{L}$ from (19) concludes the proof. □

5.2 Proof of Proposition 3.6

In this section, we will again drop the time dependence of $\psi^{N,\varepsilon}(t)$, $\varphi^\varepsilon(t)$ and $\Phi(t)$ and abbreviate $\psi^{N,\varepsilon} \equiv \psi$. Besides, we will always take $\widehat{l} \in \mathcal{L}$ from (19), hence Lemma 4.2 implies the bounds

$$\|\widehat{l}\|_{\mathrm{op}} \lesssim N^\xi\,, \qquad \|\widehat{l}_d\, q_1\psi\| \lesssim 1$$

for $d \in \mathbb{Z}$.

5.2.1 Estimate of $\gamma_{a,<}(t)$ and $\gamma^{(1)}_{b,<}(t)$

The bounds of $\gamma_{a,<}(t)$ and $\gamma^{(1)}_{b,<}(t)$ are established analogously to [4], Sections 4.4.1 and 4.4.2, and we summarise the main steps of the argument for convenience of the reader. With Lemmas 4.5, 4.10 and 4.2d, we obtain

$$|\gamma_{a,<}(t)| \lesssim \mathfrak{e}^3_\beta(t)\varepsilon + \mathfrak{e}_\beta(t)\langle\langle\psi,\widehat{n}\psi\rangle\rangle\,.$$

By Lemmas 4.7 and 4.2d and since $w_{\mu,\beta} \in \mathcal{W}_{\beta,\eta}$, $\gamma^{(1)}_{b,<}(t)$ can be estimated as

$$\begin{aligned}
|(23)| &\leq \left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi\psi, p_1^{\chi^\varepsilon} p_2(Nw^{(12)}_{\mu,\beta} - b_{\beta,N,\varepsilon}|\Phi(x_1)|^2)p_1p_2\psi \right\rangle\!\!\right\rangle\right| \\
&\quad + \left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi\psi, p_1^{\chi^\varepsilon} p_2 \left(b_{\beta,N,\varepsilon} - \tfrac{N}{N-1}b_\beta\right)|\Phi(x_1)|^2 p_1p_2\psi \right\rangle\!\!\right\rangle\right| \\
&\lesssim \left|\left\langle\!\!\left\langle \widehat{l}q_1^\Phi\psi, p_1^{\chi^\varepsilon} p_2\mathcal{G}(x_1)p_1^\Phi\psi \right\rangle\!\!\right\rangle\right| + \mathfrak{e}^2_\beta(t)\left(N^{-1} + \mu^\eta\right),
\end{aligned}$$

where

$$\mathcal{G}(x_1) := N\int\limits_{\mathbb{R}} |\chi^\varepsilon(y_1)|^2\,\mathrm{d}y_1 \left(\int\limits_{\mathbb{R}^3} |\varphi^\varepsilon(z_1-z)|^2 w_{\mu,\beta}(z)\,\mathrm{d}z - |\varphi^\varepsilon(z_1)|^2\|w_{\mu,\beta}\|_{L^1(\mathbb{R}^3)}\right). \tag{61}$$

Note that for any $g \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} g(z_1-z)w_{\mu,\beta}(z)\,\mathrm{d}z = g(z_1)\|w_{\mu,\beta}\|_{L^1(\mathbb{R}^3)} + R(z_1)$ with

$$|R(z_1)| := \left|\int\limits_{\mathbb{R}^3}\mathrm{d}z\int\limits_0^1 \nabla g(z_1 - sz)\cdot z w_{\mu,\beta}(z)\,\mathrm{d}s\right| \leq \sup_{\substack{s\in[0,1]\\ z\in\mathbb{R}^3}} |\nabla g(z_1-sz)| \int\limits_{\mathbb{R}^3}\mathrm{d}z|z|w_{\mu,\beta}(z).$$

Since $|z| \lesssim \mu^\beta$ for $z \in \operatorname{supp} w_{\mu,\beta}$ and by (55), this implies $\|R\|^2_{L^2(\mathbb{R}^3)} \lesssim \mu^{2\beta+2}\|\nabla g\|^2_{L^2(\mathbb{R}^3)}$, which, by density, extends to $g = |\varphi^\varepsilon|^2 \in H^1(\mathbb{R}^3)$. Hence,

$$\|\mathcal{G}\|_{L^2(\mathbb{R}^2)} \lesssim N\||\chi^\varepsilon|^2\|_{L^2(\mathbb{R})}\mu^{\beta+1}\|\nabla|\varphi^\varepsilon|^2\| \lesssim \tfrac{\mu^\beta}{\varepsilon}\mathfrak{e}_\beta(t)$$

by Hölder's inequality and Lemma 4.7. Using Lemmas 4.8d and 4.2d, we obtain

$$|(23)| \lesssim \mathfrak{e}^2_\beta(t)\left(\tfrac{\mu^\beta}{\varepsilon} + N^{-1} + \mu^\eta\right).$$

5.2.2 Estimate of $\gamma^{(2)}_{b,<}(t)$

The key idea for the estimate $\gamma^{(2)}_{b,<}(t)$ is to integrate by parts on a ball with radius ε, using a smooth cut-off function to prevent contributions from the boundary.

Definition 5.1. Define $h_\varepsilon : \mathbb{R}^3 \to \mathbb{R}$, $z \mapsto h_\varepsilon(z)$, by

$$h_\varepsilon(z) := \begin{cases} \dfrac{1}{4\pi}\left(\displaystyle\int\limits_{\mathbb{R}^3}\frac{w_{\mu,\beta}(\zeta)}{|z-\zeta|}\,\mathrm{d}\zeta - \int\limits_{\mathbb{R}^3}\frac{\varepsilon}{|\zeta|}\frac{w_{\mu,\beta}(\zeta)}{|\zeta^*-z|}\,\mathrm{d}\zeta\right) & \text{for } |z|<\varepsilon, \\ 0 & \text{else,}\end{cases}$$

where $\zeta^* := \frac{\varepsilon^2}{|\zeta|^2}\zeta$. Further, define $H_\varepsilon : \mathbb{R}^3 \to [0,1]$, $z \mapsto H_\varepsilon(z)$, by

$$H_\varepsilon(z) := \begin{cases} 1 & \text{for } |z| \leq \varrho_\beta, \\ \mathfrak{h}_\varepsilon(|z|) & \text{for } \varrho_\beta < |z| < \varepsilon, \\ 0 & \text{for } |z| \geq \varepsilon,\end{cases}$$

where $\mathfrak{h}_\varepsilon : (\varrho_\beta, \varepsilon) \to (0,1)$, $r \mapsto \mathfrak{h}_\varepsilon(r)$, is a smooth, decreasing function as in [4, Definition 4.15] with $\lim_{r\to\varrho_\beta}\mathfrak{h}_\varepsilon(r) = 1$ and $\lim_{r\to\varepsilon}\mathfrak{h}_\varepsilon(r) = 0$. We will abbreviate

$$h^{(ij)}_\varepsilon := h_\varepsilon(z_i - z_j), \qquad H^{(ij)}_\varepsilon := H_\varepsilon(z_i - z_j).$$

Lemma 5.2. *Let $\mu \ll \varepsilon$. Then*

(a) h_ε solves the problem $\Delta h_\varepsilon = w_{\mu,\beta}$ with boundary condition $h_\varepsilon\big|_{|z|=\varepsilon} = 0$ in the sense of distributions,

(b) $\|\nabla h_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \mu^{1-\frac{\beta}{2}}$,

(c) $\|H_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim 1, \quad \|H_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{3}{2}}, \quad \|\nabla H_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \lesssim \varepsilon^{-1}, \quad \|\nabla H_\varepsilon\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^{\frac{1}{2}}$.

Proof. The proof of Lemma 5.2 works analogously to Lemmas 4.12 and 4.13 in [4] and we briefly recall the argument for part (b) for convenience of the reader. First, we define $h_\varepsilon^{(1)}(z) := \int_{\mathbb{R}^3} \frac{w_{\mu,\beta}(\zeta)}{|z-\zeta|}\,\mathrm{d}\zeta$ and $h_\varepsilon^{(2)}(z) := \int_{\mathbb{R}^3} \frac{\varepsilon}{|\zeta|}\frac{w_{\mu,\beta}(\zeta)}{|\zeta^*-z|}\,\mathrm{d}\zeta$. To estimate $|\nabla h_\varepsilon^{(1)}|$, note that $|\zeta| \leq \varrho_\beta \lesssim \mu^\beta$ for $\zeta \in \operatorname{supp} w_{\mu,\beta}$. For $|z| \leq 2\varrho_\beta$, this implies $|z-\zeta| \leq 3\varrho_\beta \lesssim \mu^\beta$, hence $|\nabla h_\varepsilon^{(1)}(z)| \lesssim \mu^{1-2\beta}$. For $2\varrho_\beta \leq |z| \leq \varepsilon$, we find $|z-\zeta| \geq \frac{1}{2}|z|$, hence $|\nabla h_\varepsilon^{(1)}(z)| \lesssim \mu|z|^{-2}$.

For $|h_\varepsilon^{(2)}|$, observe that $\zeta \in \operatorname{supp} w_{\mu,\beta}$ implies $|\zeta^*| \geq \varepsilon^2\varrho_\beta^{-1}$, hence, for μ small enough that $\varepsilon\varrho_\beta^{-1} > 2$, we obtain $|z| \leq \varepsilon < \frac{1}{2}\varepsilon^2\varrho_\beta^{-1} \leq \frac{1}{2}|\zeta^*|$. Consequently, $|\zeta^* - z| \geq |\frac{1}{2}\varepsilon^2|\zeta|^{-1}$, which yields $|\nabla h_\varepsilon^{(2)}| \lesssim \varepsilon^{-3}\|w_{\mu,\beta}\|_{L^\infty(\mathbb{R}^3)} \int_{\operatorname{supp} w_{\mu,\beta}} |\zeta|^3\,\mathrm{d}|\zeta| \lesssim \varepsilon^{-3}\mu^{1+\beta}$. Part (b) follows from this by integration over the finite range of $\operatorname{supp} h_\varepsilon$. Part (c) is obvious. □

We now use this lemma to estimate $\gamma_{b,<}^{(2)}$. Let $t_2 \in \{p_2, q_2, q_2^\Phi p_2^{\chi^\varepsilon}\}$. As $H_\varepsilon(z_1 - z_2) = 1$ for $z_1 - z_2 \in \operatorname{supp} w_{\mu,\beta}$ and besides $\operatorname{supp} H_\varepsilon = \overline{B_\varepsilon(0)}$, Lemma 5.2a implies

$$
\begin{aligned}
|(24)| &= N\left|\left\langle\!\!\left\langle \widehat{l}t_2 q_1^{\chi^\varepsilon}\psi, H_\varepsilon^{(12)}\Delta_1 h_\varepsilon^{(12)} p_1 p_2\psi\right\rangle\!\!\right\rangle\right| \\
&\leq N\left|\left\langle\!\!\left\langle \widehat{l}q_1^{\chi^\varepsilon}\psi, t_2 H_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})\cdot p_2\nabla_1 p_1\psi\right\rangle\!\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\!\left\langle \widehat{l}q_1^{\chi^\varepsilon}\psi, t_2(\nabla_1 H_\varepsilon^{(12)})\cdot(\nabla_1 h_\varepsilon^{(12)})p_2 p_1\psi\right\rangle\!\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\!\left\langle \nabla_1\widehat{l}q_1^{\chi^\varepsilon}\psi, t_2 H_\varepsilon^{(12)}(\nabla_1 h_\varepsilon^{(12)})p_2 p_1\psi\right\rangle\!\!\right\rangle\right| \\
&\lesssim N\|\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2 H_\varepsilon^{(12)}\|_{\mathrm{op}}^2\|(\nabla_1 h_\varepsilon^{(12)})\cdot\nabla_1 p_1\|_{\mathrm{op}}^2 + N^{-1}\|(\nabla_1 h_\varepsilon^{(12)})\nabla_1 p_1\|_{\mathrm{op}}^2\Big)^{\frac{1}{2}} \\
&\quad + N\|\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2(\nabla_1 h_\varepsilon^{(12)})\|_{\mathrm{op}}^2\|(\nabla_1 H_\varepsilon^{(12)})p_1\|_{\mathrm{op}}^2 \\
&\qquad\qquad + N^{-1}\|\nabla H_\varepsilon\|_{L^\infty(\mathbb{R}^3)}^2\|(\nabla_1 h_\varepsilon^{(12)})p_2\|_{\mathrm{op}}^2\Big)^{\frac{1}{2}} \\
&\quad + N\|\nabla_1\widehat{l}q_1^{\chi^\varepsilon}\psi\|\Big(\|p_2 H_\varepsilon^{(12)}\|_{\mathrm{op}}^2\|(\nabla_1 h_\varepsilon^{(12)})p_1\|_{\mathrm{op}}^2 + N^{-1}\|(\nabla_1 h_\varepsilon^{(12)})p_2\|_{\mathrm{op}}^2\Big)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}_\beta^3(t)\left(N^{\xi+\frac{\beta}{2}}\varepsilon^{\frac{3-\beta}{2}} + N^\xi\mu^{\frac{1-\beta}{2}}\right),
\end{aligned}
$$

where the boundary terms upon integration by parts vanish because $H_\varepsilon(|z|) = 0$ for $|z| = \varepsilon$, and where we have used Lemmas 4.6, 4.2, 4.8, 4.9a and 5.2. Similarly, one computes

$$
\begin{aligned}
|(25)| &\lesssim \mathfrak{e}_\beta^3(t)N^{\xi+\frac{\beta}{2}}\varepsilon^{\frac{3-\beta}{2}}, \\
|(26)| &\lesssim \mathfrak{e}_\beta^3(t)N^{\xi+\frac{\beta}{2}}\varepsilon^{\frac{3-\beta}{2}}, \\
|(27)| &\lesssim \mathfrak{e}_\beta^3(t)\left(N^{\xi+\frac{\beta}{2}}\varepsilon^{\frac{3-\beta}{2}} + \mu^{\frac{1-\beta}{2}}\right).
\end{aligned}
$$

The bound for $\gamma_{b,<}^{(2)}$ follows from this because $N^\xi \mu^{\frac{1-\beta}{2}} = N^{\frac{-1+\beta+2\xi}{2}} \varepsilon^{\frac{1-\beta}{2}} \leq \varepsilon^{\frac{1-\beta}{2}}$ for $\xi \leq \frac{1-\beta}{2}$ and since the admissibility condition implies for $\xi \leq \frac{3-\delta}{2} \cdot \frac{\beta}{\delta-\beta}$ that

$$N^{\xi+\frac{\beta}{2}} \varepsilon^{\frac{3-\beta}{2}} = \left(\frac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}+\frac{1}{2}} \varepsilon^{\frac{3-\delta}{2}-\frac{\delta-\beta}{\beta}\xi} \leq \left(\frac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}+\frac{1}{2}} .$$

5.2.3 Preliminary estimates for the integration by parts

To control $\gamma_{b,<}^{(3)}(t)$ and $\gamma_{b,<}^{(4)}(t)$, we define the quasi two-dimensional interaction potentials $\overline{w_{\mu,\beta}}(x_1 - x_2, y_1)$ and $\overline{\overline{w_{\mu,\beta}}}(x_1 - x_2)$, which result from integrating out one or both transverse variables of the three-dimensional pair interaction $w_{\mu,\beta}(z_1 - z_2)$, and integrate by parts in x. In this section, we provide the required lemmas and definitions in a somewhat generalised form, which allows us to directly apply the results in Sections 5.2.4, 5.2.5, 5.3 and 6.6.1.

Definition 5.3. Let $\sigma \in (0, 1]$ and define $\overline{\mathcal{V}}_\sigma$ as the set containing all functions

$$\overline{\omega}_\sigma : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, \qquad (x, y) \mapsto \overline{\omega}_\sigma(x, y)$$

such that

$$\begin{cases} (a) & \operatorname{supp} \overline{\omega}_\sigma(\cdot, y) \subseteq \{x \in \mathbb{R}^2 : |x| \leq \sigma\} \text{ for all } y \in \mathbb{R}, \\ (b) & \|\overline{\omega}_\sigma\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \lesssim N^{-1}\sigma^{-2}, \\ (c) & \sup\limits_{y\in\mathbb{R}} \|\overline{\omega}_\sigma(\cdot, y)\|_{L^1(\mathbb{R}^2)} \lesssim N^{-1}, \\ (d) & \sup\limits_{y\in\mathbb{R}} \|\overline{\omega}_\sigma(\cdot, y)\|_{L^2(\mathbb{R}^2)} \lesssim N^{-1}\sigma^{-1}. \end{cases}$$

Further, define the set

$$\overline{\overline{\mathcal{V}}}_\sigma := \left\{ \overline{\overline{\omega}}_\sigma : \mathbb{R}^2 \to \mathbb{R}^2 : \exists\, \overline{\omega}_\sigma \in \overline{\mathcal{V}}_\sigma \text{ s.t. } \overline{\overline{\omega}}_\sigma(x) = \int_{\mathbb{R}} \mathrm{d}y\, |\chi^\varepsilon(y)|^2 \overline{\omega}_\sigma(x, y) \right\} .$$

Note that $\operatorname{supp} \overline{\overline{\omega}}_\sigma \subseteq \{x \in \mathbb{R}^2 : |x| \leq \sigma\}$ and, since χ^ε is normalised, the estimates for the norms of $\overline{\overline{\omega}}_\sigma$ coincide with the respective estimates for $\overline{\omega}_\sigma$. Next, we define the quasi two-dimensional interaction potentials $\overline{w_{\mu,\beta}}$ and $\overline{\overline{w_{\mu,\beta}}}$ as well as the auxiliary potentials needed for the integration by parts, and show that they are contained in the sets $\overline{\mathcal{V}}_\sigma$ and $\overline{\overline{\mathcal{V}}}_\sigma$, respectively, for suitable choices of σ.

Definition 5.4. Let $w_{\mu,\beta} \in \mathcal{W}_{\beta,\eta}$ for some $\eta > 0$ and define

$$\overline{w_{\mu,\beta}} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, \qquad (x, y) \mapsto \overline{w_{\mu,\beta}}(x, y) := \int_{\mathbb{R}} \mathrm{d}\widetilde{y}\, |\chi^\varepsilon(\widetilde{y})|^2 w_{\mu,\beta}(x, y - \widetilde{y}), \tag{62}$$

$$\overline{\overline{w_{\mu,\beta}}} : \mathbb{R}^2 \to \mathbb{R}, \qquad x \mapsto \overline{\overline{w_{\mu,\beta}}}(x) := \int_{\mathbb{R}} \mathrm{d}y\, |\chi^\varepsilon(y)|^2\, \overline{w_{\mu,\beta}}(x, y). \tag{63}$$

For $\rho \in (\varrho_\beta, 1]$, define

$$\overline{v}_\rho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto \overline{v}_\rho(x, y) := \begin{cases} \frac{1}{\pi}\rho^{-2} \|\overline{w_{\mu,\beta}}(\cdot, y)\|_{L^1(\mathbb{R}^2)} & \text{for } |x| < \rho, \\ 0 & \text{else,} \end{cases} \tag{64}$$

$$\overline{\overline{v}}_\rho : \mathbb{R}^2 \to \mathbb{R}\,, \qquad x \mapsto \overline{\overline{v}}_\rho(x) := \int\limits_{\mathbb{R}} \mathrm{d}y\, |\chi^\varepsilon(y)|^2 \overline{v}_\rho(x,y)\,. \tag{65}$$

It can easily be verified that $\overline{\overline{w_{\mu,\beta}}}$ and $\overline{\overline{v}}_\rho$ can equivalently be written as

$$\begin{aligned} \overline{\overline{w_{\mu,\beta}}}(x) &= \int\limits_{\mathbb{R}} \mathrm{d}y_1 |\chi^\varepsilon(y_1)|^2 \int_{\mathbb{R}} \mathrm{d}y_2 |\chi^\varepsilon(y_2)|^2 w_{\mu,\beta}(x, y_1 - y_1)\,, \\ \overline{\overline{v}}_\rho(x) &= \begin{cases} \frac{1}{\pi}\rho^{-2} \|\overline{\overline{w_{\mu,\beta}}}\|_{L^1(\mathbb{R}^2)} & \text{for } |x| < \rho, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Besides, note that

$$\begin{aligned} p_2^{\chi^\varepsilon} w_{\mu,\beta}^{(12)} p_2^{\chi^\varepsilon} &= \overline{w_{\mu,\beta}}(x_1 - x_2, y_1) p_2^{\chi^\varepsilon}\,, \\ p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} w_{\mu,\beta}^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} &= \overline{\overline{w_{\mu,\beta}}}(x_1 - x_2) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon}\,. \end{aligned}$$

Lemma 5.5. *For $\overline{w_{\mu,\beta}}$, $\overline{\overline{w_{\mu,\beta}}}$, $\overline{v}_\rho$ and $\overline{\overline{v}}_\rho$ from Definition 5.4, it holds that*

(a) $\overline{w_{\mu,\beta}} \in \overline{\mathcal{V}}_{\varrho_\beta}$, $\qquad \overline{\overline{w_{\mu,\beta}}} \in \overline{\overline{\mathcal{V}}}_{\varrho_\beta}$, $\qquad \overline{v}_\rho \in \overline{\mathcal{V}}_\rho$, $\qquad \overline{\overline{v}}_\rho \in \overline{\overline{\mathcal{V}}}_\rho$,

(b) $\|\overline{w_{\mu,\beta}}(\cdot,y)\|_{L^1(\mathbb{R}^2)} = \|\overline{v}_\rho(\cdot,y)\|_{L^1(\mathbb{R}^2)}$ *for any* $y \in \mathbb{R}$,

$\|\overline{\overline{w_{\mu,\beta}}}\|_{L^1(\mathbb{R}^2)} = \|\overline{\overline{v}}_\rho\|_{L^1(\mathbb{R}^2)}$.

Proof. It suffices to derive the respective estimates for $\overline{w_{\mu,\beta}}(\cdot,y)$ and $\overline{v}_\rho(\cdot,y)$ uniformly in $y \in \mathbb{R}$. For instance, Lemma 4.7 and (55) yield

$$\begin{aligned} |\overline{w_{\mu,\beta}}(x,y)| &\leq \|\chi^\varepsilon\|^2_{L^\infty(\mathbb{R})} \int\limits_{y-\varrho_\beta}^{y+\varrho_\beta} \mathrm{d}y_1\, \mathbb{1}_{|y-y_1|\leq \varrho_\beta} w_{\mu,\beta}(x, y - y_1) \\ &\lesssim \varepsilon^{-1}\mu^{1-2\beta} \sim N^{-1}\varrho_\beta^{-2}\,, \\ \|\overline{v}_\rho(\cdot,y)\|_{L^1(\mathbb{R}^2)} &= \frac{1}{\rho^2\pi}\|\overline{w_{\mu,\beta}}(\cdot,y)\|_{L^1(\mathbb{R}^2)} \int\limits_{\mathbb{R}^2} \mathbb{1}_{|x|\leq\rho}\,\mathrm{d}x = \|\overline{w_{\mu,\beta}}(\cdot,y)\|_{L^1(\mathbb{R}^2)} \lesssim N^{-1}\,, \end{aligned}$$

and the remaining parts are verified analogously. □

In analogy to electrostatics, let us now define the "potentials" $\overline{h}_{\sigma_1,\sigma_2}$ and $\overline{\overline{h}}_{\sigma_1,\sigma_2}$ corresponding to the "charge distributions" $\overline{\omega}_{\sigma_1} - \overline{\omega}_{\sigma_2}$ and $\overline{\overline{\omega}}_{\sigma_1} - \overline{\overline{\omega}}_{\sigma_2}$, respectively.

Lemma 5.6. *Let $0 < \sigma_1 < \sigma_2 \leq 1$, $\overline{\omega}_{\sigma_1} \in \overline{\mathcal{V}}_{\sigma_1}$ and $\overline{\omega}_{\sigma_2} \in \overline{\mathcal{V}}_{\sigma_2}$ such that for any $y \in \mathbb{R}$*

$$\|\overline{\omega}_{\sigma_1}(\cdot,y)\|_{L^1(\mathbb{R}^2)} = \|\overline{\omega}_{\sigma_2}(\cdot,y)\|_{L^1(\mathbb{R}^2)}\,.$$

Define

$$\begin{aligned} \overline{h}_{\sigma_1,\sigma_2} : \mathbb{R}^2 \times \mathbb{R} &\to \mathbb{R} \\ (x,y) &\mapsto \overline{h}_{\sigma_1,\sigma_2}(x,y) := \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \mathrm{d}\xi \ln|x-\xi| \Big(\overline{\omega}_{\sigma_1}(\xi,y) - \overline{\omega}_{\sigma_2}(\xi,y)\Big) \end{aligned} \tag{66}$$

and

$$\begin{aligned}\overline{\overline{h}}_{\sigma_1,\sigma_2} : \mathbb{R}^2 &\to \mathbb{R}\\ x &\mapsto \overline{\overline{h}}_{\sigma_1,\sigma_2}(x) := \int_{\mathbb{R}} \mathrm{d}y\, |\chi^\varepsilon(y)|^2 \overline{h}_{\sigma_1,\sigma_2}(x,y)\,. \end{aligned} \tag{67}$$

Let $y \in \mathbb{R}$ *and* $\big(h_{\sigma_1,\sigma_2}, \omega_{\sigma_1}, \omega_{\sigma_2}\big) \in \Big\{\Big(\overline{h}_{\sigma_1,\sigma_2}(\cdot,y), \overline{\omega}_{\sigma_1}(\cdot,y), \overline{\omega}_{\sigma_2}(\cdot,y)\Big),\ \Big(\overline{\overline{h}}_{\sigma_1,\sigma_2}, \overline{\overline{\omega}}_{\sigma_1}, \overline{\overline{\omega}}_{\sigma_2}\Big)\Big\}$.

(a) h_{σ_1,σ_2} *satisfies*

$$\Delta_x h_{\sigma_1,\sigma_2} = \omega_{\sigma_1} - \omega_{\sigma_2}$$

in the sense of distributions, and

$$\operatorname{supp} h_{\sigma_1,\sigma_2} \subseteq \big\{x \in \mathbb{R}^2 : |x| \le \sigma_2\big\}\ ,$$

(b) $\|h_{\sigma_1,\sigma_2}\|_{L^2(\mathbb{R}^2)} \lesssim N^{-1}\sigma_2\left(1 + \ln \sigma_2^{-1}\right)$,

$\|\nabla_x h_{\sigma_1,\sigma_2}\|_{L^2(\mathbb{R}^2)} \lesssim N^{-1}\left(\ln \sigma_1^{-1}\right)^{\frac{1}{2}}$.

Proof. The first part of (a) follows immediately from [25, Theorem 6.21]. For the second part, Newton's theorem [25, Theorem 9.7] states that for $|x| \ge \sigma_2$,

$$\overline{h}_{\sigma_1,\sigma_2}(x,y) = \frac{1}{2\pi}\ln|x| \int_{\mathbb{R}^2} (\overline{\omega}_{\sigma_1}(\xi,y) - \overline{\omega}_{\sigma_2}(\xi,y))\,\mathrm{d}\xi = 0$$

as $\|\overline{\omega}_{\sigma_1}(\cdot,y)\|_{L^1(\mathbb{R}^2)} = \|\overline{\omega}_{\sigma_2}(\cdot,y)\|_{L^1(\mathbb{R}^2)}$. Besides, [25, Theorem 9.7] yields the estimate

$$\big|\overline{h}_{\sigma_1,\sigma_2}(x,y)\big| \le \frac{1}{2\pi}\big|\ln|x|\big| \int_{\mathbb{R}^2} (\overline{\omega}_{\sigma_1}(\xi,y) + \overline{\omega}_{\sigma_2}(\xi,y))\,\mathrm{d}\xi \lesssim N^{-1}\big|\ln|x|\big|$$

by definition of $\overline{\omega}$. Hence,

$$\|\overline{h}_{\sigma_1,\sigma_2}(\cdot,y)\|^2_{L^2(\mathbb{R}^2)} \ \lesssim\ N^{-2}\int_0^{\sigma_2} r(\ln r)^2\,\mathrm{d}r \ \lesssim\ N^{-2}\sigma_2^2(1+\ln\sigma_2^{-1})^2\,.$$

To derive the second part of (b), let us define the abbreviations

$$\overline{h}^{(1)}_{\sigma_1,\sigma_2}(x,y) := \int_{\mathbb{R}^2} \mathrm{d}\xi \ln|x-\xi|\,\overline{\omega}_{\sigma_1}(\xi,y), \qquad \overline{h}^{(2)}_{\sigma_1,\sigma_2}(x,y) := \int_{\mathbb{R}^2} \mathrm{d}\xi \ln|x-\xi|\,\overline{\omega}_{\sigma_2}(\xi,y)\,.$$

To estimate $\nabla_x\overline{h}^{(1)}_{\sigma_1,\sigma_2}$, let $y \in \mathbb{R}$ and consider $\xi \in \operatorname{supp}\overline{\omega}_{\sigma_1}(\cdot,y)$, hence $|\xi| \le \sigma_1$. If $|x| \le 2\sigma_1$, we have $|x-\xi| \le |x| + |\xi| \le 3\sigma_1$, hence

$$|\nabla_x\overline{h}^{(1)}_{\sigma_1,\sigma_2}(x,y)| \lesssim \|\overline{\omega}_{\sigma_1}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})}\int_0^{3\sigma_1}\mathrm{d}r \lesssim N^{-1}\sigma_1^{-1}\,.$$

If $2\sigma_1 < |x| \le \sigma_2$, this implies $|x-\xi| \ge |x| - |\xi| \ge |x| - \sigma_1 \ge \frac{1}{2}|x|$, and one concludes

$$|\nabla_x\overline{h}^{(1)}_{\sigma_1,\sigma_2}(x,y)| \le \tfrac{2}{|x|}\int_{\mathbb{R}^2}\overline{\omega}_{\sigma_1}(\xi,y)\,\mathrm{d}\xi \lesssim N^{-1}\tfrac{1}{|x|}\,.$$

To estimate $\nabla_x \overline{h}^{(2)}_{\sigma_1,\sigma_2}$, note that $|x-\xi| \leq |x| + |\xi| \leq 2\sigma_2$ for $x \in \operatorname{supp} \overline{h}_{\sigma_1,\sigma_2}(\cdot, y)$ and $\xi \in \operatorname{supp} \overline{\omega}_{\sigma_2}$, hence

$$|\nabla_x \overline{h}^{(2)}_{\sigma_1,\sigma_2}(x,y)| \leq \sup \|\overline{\omega}_{\sigma_2}\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})} \int\limits_{|\xi'\leq 2\sigma_2} \mathrm{d}|\xi'| \lesssim N^{-1}\sigma_2^{-1}.$$

Part (b) follows from integrating over $|x| \leq \sigma_2$. □

5.2.4 Estimate of $\gamma^{(3)}_{b,<}(t)$

To derive a bound for $\gamma^{(3)}_{b,<}$, observe first that both terms (28) and (29) contain the interaction $\overline{w_{\mu,\beta}}$. We add and subtract $\overline{v}_\rho$ from Definition 5.4 for suitable choices of ρ, i.e.,

$$\begin{aligned}\overline{w_{\mu,\beta}}(x_1-x_2,y_1) &= \overline{w_{\mu,\beta}}(x_1-x_2,y_1) - \overline{v}_\rho(x_1-x_2,y_1) + \overline{v}_\rho(x_1-x_2,y_1)\\ &= \Delta_{x_1}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1) + \overline{v}_\rho(x_1-x_2,y_1)\end{aligned}$$

by Lemma 5.6, which is applicable by Lemma 5.5.

Estimate of (28). Due to the symmetry of ψ, (28) can be written as

$$(28) = N\left|\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2^\Phi \widehat{l} p_2^{\chi^\varepsilon} w^{(12)}_{\mu,\beta} p_2^{\chi^\varepsilon} p_1^{\chi^\varepsilon} p_2^\Phi q_1^\Phi \psi\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2^\Phi \widehat{l} p_2^{\chi^\varepsilon} w^{(12)}_{\mu,\beta} p_2^{\chi^\varepsilon} p_1^{\chi^\varepsilon} p_1^\Phi q_2^\Phi \psi\right\rangle\!\!\right\rangle\right|,$$

hence with $(s_1^\Phi, t_2^\Phi) \in \{(p_1^\Phi, q_2^\Phi), (q_1^\Phi, p_2^\Phi)\}$ and for some $\rho \in (\varrho_\beta, 1]$,

$$\begin{aligned}|(28)| \;\leq\;& N\left|\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2^\Phi p_2^{\chi^\varepsilon}\left(\Delta_{x_2}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1)\right) p_1^{\chi^\varepsilon}\widehat{l}_1 s_1^\Phi t_2^\Phi\psi\right\rangle\!\!\right\rangle\right| &(68)\\ &+N\left|\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2^\Phi p_2^{\chi^\varepsilon}\overline{v}_\rho(x_1-x_2,y_1) p_1^{\chi^\varepsilon}\widehat{l}_1 s_1^\Phi t_2^\Phi\psi\right\rangle\!\!\right\rangle\right|. &(69)\end{aligned}$$

Since $s_1^\Phi t_2^\Phi$ contains in both cases a projector p^Φ and a projector q^Φ, the second term is easily estimated as

$$(69) \leq N\|q_1^{\chi^\varepsilon}\psi\|\|\widehat{l}_1 q_1^\Phi\psi\|\|p_1^\Phi \overline{v}_\rho(x_1-x_2,y_1)\|_{\mathrm{op}} \lesssim \mathfrak{e}^2_\beta(t)\varepsilon\rho^{-1}$$

by Lemmas 4.8d and 4.2d. For (68), note first that for $(s_1^\Phi, t_2^\Phi) = (q_1^\Phi, p_2^\Phi)$,

$$\begin{aligned}\|(\nabla_{x_2}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1))\nabla_{x_2} p_2^\Phi q_1^\Phi p_1^{\chi^\varepsilon}\widehat{l}_1\psi\| &\leq \|(\nabla_{x_2}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1))\nabla_{x_2}p_2^\Phi\|^2_{\mathrm{op}}\|\widehat{l}_1 q_1^\Phi\psi\|\\ &\lesssim \mathfrak{e}_\beta(t)N^{-1}(\ln\mu^{-1})^{\frac{1}{2}}\end{aligned}$$

and for $(s_1^\Phi, t_2^\Phi) = (p_1^\Phi, q_2^\Phi)$,

$$\begin{aligned}\|(\nabla_{x_2}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1))p_1^\Phi\nabla_{x_2} q_2^\Phi p_1^{\chi^\varepsilon}\widehat{l}_1\psi\| &\leq \|(\nabla_{x_2}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1))p_1^\Phi\|^2_{\mathrm{op}}\|\nabla_{x_2} q_2^\Phi\widehat{l}_1\psi\|\\ &\lesssim \mathfrak{e}^2_\beta(t)N^{-1+\xi}(\ln\mu^{-1})^{\frac{1}{2}},\end{aligned}$$

where we have used that $\varrho_\beta \sim \mu^\beta$. Hence, integration by parts in x_2 yields with Lemma 4.6

$$|(68)| \;\leq\; N\left|\left\langle\!\!\left\langle q_1^{\chi^\varepsilon}\psi, q_2^\Phi p_2^{\chi^\varepsilon}(\nabla_{x_2}\overline{h}_{\varrho_\beta,\rho}(x_1-x_2,y_1))p_1^{\chi^\varepsilon}\nabla_{x_2} t_2^\Phi \widehat{l}_1 s_1^\Phi\psi\right\rangle\!\!\right\rangle\right|$$

$$
\begin{aligned}
&+N\left|\left\langle\!\!\left\langle\nabla_{x_2}q_2^\Phi p_2^{\chi^\varepsilon}\psi, q_1^{\chi^\varepsilon}(\nabla_{x_2}\overline{h}_{\varrho\beta,\rho}(x_1-x_2,y_1))t_2^\Phi p_1^{\chi^\varepsilon}\widehat{l}_1 s_1^\Phi\psi\right\rangle\!\!\right\rangle\right| \\
\lesssim\;& N\|q_1^{\chi^\varepsilon}\psi\|\|(\nabla_{x_2}\overline{h}_{\varrho\beta,\rho}(x_1-x_2,y_1))\nabla_{x_2}t_2^\Phi s_1^\Phi p_1^{\chi^\varepsilon}\widehat{l}_1\psi\| \\
&+N\|\nabla_{x_2}q_2^\Phi p_2^{\chi^\varepsilon}\psi\|\Big(\|p_1^{\chi^\varepsilon}s_1^\Phi(\nabla_{x_2}\overline{h}_{\varrho\beta,\rho}(x_1-x_2,y_1))t_2^\Phi\widehat{l}_1 q_1^{\chi^\varepsilon}\psi\|^2 \\
&+N^{-1}\|(\nabla_{x_2}\overline{h}_{\varrho\beta,\rho}(x_1-x_2,y_1))t_2^\Phi s_1^\Phi p_1^{\chi^\varepsilon}\widehat{l}_1\psi\|^2\Big)^{\frac12} \\
\lesssim\;& \mathfrak{e}_\beta^3(t)(N^\xi\varepsilon+N^{-\frac12})(\ln\mu^{-1})^{\frac12}\,.
\end{aligned}
$$

Estimate of (29). For this term, we choose $\rho=1$ and integrate by parts in x_2. This yields

$$
\begin{aligned}
|(29)| \;\leq\;& N\left|\left\langle\!\!\left\langle\widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\overline{v}_1(x_1-x_2,y_1)p_2^\Phi q_1^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&+N\left|\left\langle\!\!\left\langle\widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\left(\nabla_{x_2}\overline{h}_{\varrho\beta,1}(x_1-x_2,y_1)\right)\cdot\nabla_{x_2}p_2^\Phi q_1^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
&+N\left|\left\langle\!\!\left\langle\nabla_{x_2}\widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\left(\nabla_{x_2}\overline{h}_{\varrho\beta,1}(x_1-x_2,y_1)\right)p_2^\Phi q_1^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| \\
\leq\;& N\|q_1^{\chi^\varepsilon}\psi\|\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|\Big(\|p_2^\Phi\overline{v}_1(x_1-x_2,y_1)\|_{\mathrm{op}} \\
&\qquad+\|\left(\nabla_{x_2}\overline{h}_{\varrho\beta,1}(x_1-x_2,y_1)\right)\nabla_{x_2}p_2^\Phi\|_{\mathrm{op}}\Big) \\
&+N\|q_1^{\chi^\varepsilon}\psi\|\|p_2^\Phi\left(\nabla_{x_2}\overline{h}_{\varrho\beta,1}(x_1-x_2,y_1)\right)\|_{\mathrm{op}}\|\nabla_{x_2}\widehat{l}q_1^\Phi q_2^\Phi\psi\| \\
\lesssim\;& \mathfrak{e}_\beta^3(t)\varepsilon(\ln\mu^{-1})^{\frac12}
\end{aligned}
$$

by Lemmas 4.2, 4.9a, 4.8d and 5.6. Together, the estimates for (28) and (29) yield

$$
|\gamma_{b,<}^{(3)}(t)|\lesssim\mathfrak{e}_\beta^3(t)\left(N^\xi\varepsilon+N^{-\frac12}\right)(\ln\mu^{-1})^{\frac12}\lesssim\mathfrak{e}_\beta^3(t)\left(\tfrac{1}{1-\beta}N^{-1^-}+\tfrac{\delta}{\beta}N^{2\xi}\varepsilon^{2^-}\right)^{\frac12}
$$

by Lemma 4.11. Since $\beta\in(0,1)$ and $3-\delta\in(0,2)$ as $\delta\in(1,3)$, this implies

$$
|\gamma_{b,<}^{(3)}(t)|\lesssim\mathfrak{e}_\beta^3(t)\left(\tfrac{1}{1-\beta}N^{-\beta}+\tfrac{\delta}{\beta}N^{2\xi}\varepsilon^{3-\delta}\right)^{\frac12}\,,
$$

which yields the final bound for $\gamma_{b,<}^{(3)}$ because, by admissibility and since $\xi\leq\frac{3-\delta}{2}\frac{\beta}{\delta-\beta}$,

$$
N^\xi\varepsilon^{\frac{3-\delta}{2}}=\left(\tfrac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}}\varepsilon^{\frac{3-\delta}{2}-\frac{\delta-\beta}{\beta}\xi}\leq\left(\tfrac{\varepsilon^\delta}{\mu^\beta}\right)^{\frac{\xi}{\beta}}\,.
$$

5.2.5 Estimate of $\gamma_{b,<}^{(4)}(t)$

First, observe that

$$
|(32)|\;\lesssim\;\|\widehat{l}q_1q_2\psi\|\|q_2\psi\|\|\Phi\|^2_{L^\infty(\mathbb{R}^2)}\;\lesssim\;\mathfrak{e}_\beta^2(t)\left\langle\!\!\left\langle\psi,\widehat{n}\psi\right\rangle\!\!\right\rangle.
$$

Since both terms (30) and (31) contain the quasi two-dimensional interaction $\overline{\overline{w_{\mu,\beta}}}$, we integrate by parts in x as before, using that

$$
\overline{\overline{w_{\mu,\beta}}}(x_1-x_1)=\Delta_{x_1}\overline{\overline{h}}_{\varrho\beta,\rho}(x_1-x_2)+\overline{\overline{v}}_\rho(x_1-x_2)
$$

and choose $\rho = N^{-\beta_1}$ for $\beta_1 = \min\left\{\frac{1+\xi}{4}, \beta\right\}$ in (30) and $\rho = 1$ in (31). In the sequel, we abbreviate

$$\overline{\overline{w_{\mu,\beta}}}^{(12)} := \overline{\overline{w_{\mu,\beta}}}(x_1 - x_2)\,, \qquad \overline{\overline{v}}_\rho^{(12)} := \overline{\overline{v}}_\rho(x_1 - x_2)\,, \qquad \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} := \overline{\overline{h}}_{\varrho_\beta,\rho}(x_1 - x_2).$$

Estimate of (30). Integration by parts in x_1 yields with Lemma 4.3b

$$\begin{aligned} |(30)| \;\leq\; & N\left|\left\langle\!\!\left\langle \widehat{l}^{\frac{1}{2}} q_1^\Phi q_2^\Phi \psi, \overline{\overline{v}}_\rho^{(12)} p_1 p_2 \widehat{l}_2^{\frac{1}{2}} \psi \right\rangle\!\!\right\rangle\right| && (70)\\ & + N\left|\left\langle\!\!\left\langle \nabla_{x_1} \widehat{l} q_1^\Phi q_2^\Phi \psi, (\nabla_{x_1} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)}) p_1 p_2 \psi \right\rangle\!\!\right\rangle\right| && (71)\\ & + N\left|\left\langle\!\!\left\langle \widehat{l} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)}) \cdot \nabla_{x_1} p_1 p_2 \psi \right\rangle\!\!\right\rangle\right|. && (72) \end{aligned}$$

For the first term, we obtain with Lemmas 4.6c, 4.8d and for $\rho = N^{-\beta_1}$

$$\begin{aligned} |(70)| \;&\lesssim\; N \|\widehat{l}^{\frac{1}{2}} q_1^\Phi \psi\| \left(\|p_2 \overline{\overline{v}}_\rho^{(12)} p_1 \widehat{l}_2^{\frac{1}{2}} q_1^\Phi \psi\|^2 + N^{-1} \|\overline{\overline{v}}_\rho^{(12)} p_2^\Phi\|_{\mathrm{op}}^2 \|\widehat{l}_2^{\frac{1}{2}} \psi\|^2 \right)^{\frac{1}{2}} \\ &\lesssim\; \mathfrak{e}_\beta^2(t) \left(\langle\!\langle \psi, \widehat{n}\psi \rangle\!\rangle + N^{-\frac{1}{2}+\frac{\xi}{2}+\beta_1} \right), \end{aligned}$$

where we used that $\overline{\overline{v}}_\rho = \sqrt{\overline{\overline{v}}_\rho}\sqrt{\overline{\overline{v}}_\rho}$ since $\overline{\overline{v}}_\rho \geq 0$ and consequently

$$\|p_2 \overline{\overline{v}}_\rho^{(12)} p_1\|_{\mathrm{op}}^2 \;\leq \|p_2^\Phi \sqrt{\overline{\overline{v}}_\rho^{(12)}}\|_{\mathrm{op}}^2 \|\sqrt{\overline{\overline{v}}_\rho^{(12)}} p_1^\Phi\|_{\mathrm{op}}^2 \lesssim \mathfrak{e}_\beta^4(t) \|\overline{\overline{v}}_\rho\|_{L^1(\mathbb{R}^2)}^2 \lesssim \mathfrak{e}_\beta^4(t) N^{-2}. \qquad (73)$$

To estimate (71) and (72), observe first that for any operator s_1 acting only on the first coordinate,

$$\begin{aligned} &\left\langle\!\!\left\langle q_2^\Phi (\nabla_{x_2} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)}) s_1 p_2 \widetilde{\psi}, q_3^\Phi (\nabla_{x_3} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)}) s_1 p_3 \widetilde{\psi} \right\rangle\!\!\right\rangle \\ &= \; - \left\langle\!\!\left\langle \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} s_1 \nabla_{x_2} p_2 q_3^\Phi \widetilde{\psi}, (\nabla_{x_3} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)}) s_1 p_3 q_2^\Phi \widetilde{\psi} \right\rangle\!\!\right\rangle \\ &\quad - \left\langle\!\!\left\langle \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} s_1 p_2 q_3^\Phi \widetilde{\psi}, (\nabla_{x_3} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)}) s_1 p_3 \nabla_{x_2} q_2^\Phi \widetilde{\psi} \right\rangle\!\!\right\rangle \\ &= \; \left\langle\!\!\left\langle \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} s_1 \nabla_{x_2} p_2 \nabla_{x_3} q_3^\Phi \widetilde{\psi}, \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)} s_1 p_3 q_2^\Phi \widetilde{\psi} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} s_1 \nabla_{x_2} p_2 q_3^\Phi \widetilde{\psi}, \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)} s_1 \nabla_{x_3} p_3 q_2^\Phi \widetilde{\psi} \right\rangle\!\!\right\rangle \\ &\quad + \left\langle\!\!\left\langle \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} s_1 p_2 \nabla_{x_3} q_3^\Phi \widetilde{\psi}, \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)} s_1 p_3 \nabla_{x_2} q_2^\Phi \widetilde{\psi} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)} s_1 p_2 q_3^\Phi \widetilde{\psi}, \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)} s_1 \nabla_{x_3} p_3 \nabla_{x_2} q_2^\Phi \widetilde{\psi} \right\rangle\!\!\right\rangle \\ &\lesssim \; \mathfrak{e}_\beta^2(t) \|\overline{\overline{h}}_{\varrho_\beta,\rho}\|_{L^2(\mathbb{R}^2)}^2 \left(\|s_1 q_2^\Phi \widetilde{\psi}\|^2 + \|s_1 \nabla_{x_2} q_2^\Phi \widetilde{\psi}\|^2 \right) \qquad (74) \end{aligned}$$

by Lemmas 4.2e and 4.9a. With Lemmas 4.6, 4.2 and 5.6b, we thus obtain for $\rho = N^{-\beta_1}$

$$\begin{aligned} |(71)| \;\lesssim\; N \|\nabla_{x_1} \widehat{l} q_1^\Phi \psi\| \Bigg(& \left\langle\!\!\left\langle q_2^\Phi (\nabla_{x_2} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)}) p_1 p_2 \psi, q_3^\Phi (\nabla_{x_3} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(13)}) p_1 p_3 \psi \right\rangle\!\!\right\rangle \\ & + N^{-1} \|(\nabla_{x_1} \overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)}) p_1^\Phi\|_{\mathrm{op}}^2 \Bigg)^{\frac{1}{2}} \end{aligned}$$

$$\lesssim \mathfrak{e}^3_\beta(t)\left(N^{-\beta_1+\xi}\ln N + N^{-\frac{1}{2}+\xi}(\ln\mu^{-1})^{\frac{1}{2}}\right),$$

$$|(72)| \lesssim N\|\widehat{l}q_1^\Phi\psi\|\left(\left\langle\!\!\left\langle q_2^\Phi(\nabla_{x_2}\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho})\nabla_{x_1}p_1p_2\psi, q_3^\Phi(\nabla_{x_3}\overline{\overline{h}}^{(13)}_{\varrho_\beta,\rho})\nabla_{x_1}p_1p_3\psi\right\rangle\!\!\right\rangle\right.$$
$$\left.+N^{-1}\|(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho})p_2^\Phi\|^2_{\mathrm{op}}\|\nabla_{x_1}p_1^\Phi\|^2_{\mathrm{op}}\right)^{\frac{1}{2}}$$
$$\lesssim \mathfrak{e}^2_\beta(t)\left(N^{-\beta_1}\ln N + N^{-\frac{1}{2}}(\ln\mu^{-1})^{\frac{1}{2}}\right).$$

Together, this yields with Lemma 4.11

$$|(30)| \lesssim \mathfrak{e}^2_\beta(t)\left\langle\!\!\left\langle\psi,\widehat{n}\psi\right\rangle\!\!\right\rangle + \mathfrak{e}^3_\beta(t)\left(N^{-\frac{1}{2}+\frac{\xi}{2}+\beta_1} + (\tfrac{1}{1-\beta})^{\frac{1}{2}}N^{-(\frac{1}{2}-\xi)^-} + N^{-(\beta_1-\xi)^-}\right)$$

Note that for $\beta_1 = \min\{\frac{1+\xi}{4},\beta\}$ and since $\xi < \frac{1}{3}$, it holds that $N^{-\beta_1+\xi} > N^{-\frac{1}{2}+\xi}$ and that $-\frac{1}{2}+\frac{\xi}{2}+\beta_1 < -\beta_1+\xi$. Hence,

$$|(30)| \lesssim \mathfrak{e}^2_\beta(t)\left\langle\!\!\left\langle\psi,\widehat{n}\psi\right\rangle\!\!\right\rangle + \mathfrak{e}^3_\beta(t)N^{-(\beta_1-\xi)^-}.$$

Estimate of (31). Observe first that for $j\in\{0,1\}$,

$$\|p_1^\Phi(\nabla_{x_2}\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho})\widehat{l}_jq_1^\Phi q_2^\Phi\psi\|^2 \tag{75}$$
$$= \||\Phi(x_1)\rangle\langle\nabla_{x_1}\Phi(x_1)|\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho}\widehat{l}_jq_1^\Phi q_2^\Phi\psi\|^2 + \|p_1^\Phi\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho}\nabla_{x_1}\widehat{l}_jq_1^\Phi q_2^\Phi\psi\|^2$$
$$+\left(\left\langle\!\!\left\langle|\Phi(x_1)\rangle\langle\nabla_{x_1}\Phi(x_1)|\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho}\widehat{l}_jq_1^\Phi q_2^\Phi\psi, p_1^\Phi\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho}\nabla_{x_1}\widehat{l}_jq_1^\Phi q_2^\Phi\psi\right\rangle\!\!\right\rangle + \mathrm{h.c.}\right)$$
$$\lesssim \|\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho}\nabla_{x_1}p_1^\Phi\|^2_{\mathrm{op}}\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|^2 + \|\overline{\overline{h}}^{(12)}_{\varrho_\beta,\rho}p_1^\Phi\|^2_{\mathrm{op}}\|\nabla_{x_1}\widehat{l}q_1^\Phi q_2^\Phi\psi\|^2$$
$$\lesssim \mathfrak{e}^2_\beta(t)\|\overline{\overline{h}}_{\varrho_\beta,\rho}\|^2_{L^2(\mathbb{R}^2)}\left(\left\langle\!\!\left\langle\psi,\widehat{n}\psi\right\rangle\!\!\right\rangle + \|\nabla_{x_1}q_1^\Phi\psi\|^2\right). \tag{76}$$

Integration by parts in x_2 with $\rho = 1$ yields with Lemmas 4.3b, 5.6, 4.9a and 4.11

$$|(31)| \leq N\left|\left\langle\!\!\left\langle\widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\overline{\overline{v}}^{(12)}_1q_2^\Phi p_1^\Phi\psi\right\rangle\!\!\right\rangle\right|$$
$$+N\left|\left\langle\!\!\left\langle\widehat{l}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}(\nabla_{x_2}\overline{\overline{h}}^{(12)}_{\varrho_\beta,1})p_1^\Phi\nabla_{x_2}q_2^\Phi\psi\right\rangle\!\!\right\rangle\right|$$
$$+N\left|\left\langle\!\!\left\langle\nabla_{x_2}q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}(\nabla_{x_2}\overline{\overline{h}}^{(12)}_{\varrho_\beta,1})p_1^\Phi\widehat{l}_1q_2^\Phi\psi\right\rangle\!\!\right\rangle\right|$$
$$\lesssim N\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|\|\overline{\overline{v}}^{(12)}_1p_1^\Phi\|_{\mathrm{op}}\|q_2^\Phi\psi\| + N\|\nabla_{x_2}q_2^\Phi\psi\|\|p_1^\Phi(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_\beta,1})\widehat{l}q_1^\Phi q_2^\Phi\psi\|$$
$$+N\|\nabla_{x_2}q_2^\Phi\psi\|\left(\|p_1^\Phi(\nabla_{x_2}\overline{\overline{h}}^{(12)}_{\varrho_\beta,1})\widehat{l}_1q_1^\Phi q_2^\Phi\psi\|^2\right.$$
$$\left.+N^{-1}\|(\nabla_{x_2}\overline{\overline{h}}^{(12)}_{\varrho_\beta,1})p_1^\Phi\|^2_{\mathrm{op}}\|\widehat{l}_1q_2^\Phi\psi\|^2\right)^{\frac{1}{2}}$$
$$\lesssim \mathfrak{e}_\beta(t)\left(\left\langle\!\!\left\langle\psi,\widehat{n}\psi\right\rangle\!\!\right\rangle + \|\nabla_{x_1}q_1^\Phi\psi\|^2 + \tfrac{1}{1-\beta}N^{-1^-}\right).$$

With Lemma 5.7 below, we obtain

$$|(31)| \lesssim \mathfrak{e}^3_\beta(t)\alpha^<_{\xi,w_{\mu,\beta}}(t) + \mathfrak{e}^4_\beta(t)\left(\frac{\mu^\beta}{\varepsilon} + \left(\frac{\varepsilon^3}{\mu^\beta}\right)^{\frac{1}{2}} + N^{-\beta_2^-} + \mu^\eta + \mu^{\frac{1-\beta}{2}}\right)$$

for $\beta_2 = \min\left\{\beta, \frac{1}{4}\right\}$. Together, the estimates of (30) and (31) yield

$$|\gamma^{(4)}_{b,<}(t)| \lesssim \mathfrak{e}^3_\beta(t)\,\alpha^<_{\xi,w_{\mu,\beta}}(t) + \mathfrak{e}^4_\beta(t)\left(\frac{\mu^\beta}{\varepsilon} + \left(\frac{\varepsilon^3}{\mu^\beta}\right)^{\frac{1}{2}} + N^{-(\beta_1-\xi)^-} + \mu^\eta + \mu^{\frac{1-\beta}{2}}\right).$$

5.3 Estimate of the kinetic energy for $\beta \in (0,1)$

Lemma 5.7. *For $\beta_2 = \min\left\{\frac{1}{4}, \beta\right\}$ and sufficiently small μ,*

$$\|\nabla_{x_1} q_1^\Phi \psi\|^2 \lesssim \mathfrak{e}^2_\beta(t)\alpha^<_{\xi,w_{\mu,\beta}}(t) + \mathfrak{e}^3_\beta(t)\left(\frac{\mu^\beta}{\varepsilon} + \left(\frac{\varepsilon^3}{\mu^\beta}\right)^{\frac{1}{2}} + N^{-\beta_2^-} + \mu^\eta + \mu^{\frac{1-\beta}{2}}\right).$$

Proof. Analogously to the proof of Lemma 4.21 in [4], we expand

$$\begin{aligned}
&E_{w_{\mu,\beta}}(\Psi) - \mathcal{E}_{b_\beta}(\Phi)\\
&\gtrsim \|\nabla_{x_1} q_1^\Phi \psi\|^2 + N\|\sqrt{w^{(12)}_{\mu,\beta}}(1-p_1p_2)\psi\|^2 && (77)\\
&\quad - \left|\|\nabla_{x_1} p_1^\Phi\psi\|^2 - \|\nabla_x\Phi\|^2_{L^2(\mathbb{R}^2)}\right| - \left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}} q_1^\Phi\psi, \Delta_{x_1} p_1^\Phi\left(q_1^{\chi^\varepsilon}\widehat{n}^{\frac{1}{2}} + p_1^{\chi^\varepsilon}\widehat{n}_1^{\frac{1}{2}}\right)\psi\right\rangle\!\!\right\rangle\right| && (78)\\
&\quad - \left|\left\langle\!\!\left\langle \psi, p_1p_2\left(Nw^{(12)}_{\mu,\beta} - b_\beta|\Phi(x_1)|^2\right)p_1p_2\psi\right\rangle\!\!\right\rangle\right| - \|\sqrt{w^{(12)}_{\mu,\beta}}p_1p_2\psi\|^2 && (79)\\
&\quad -N\left|\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}}q_1\psi, p_2 w^{(12)}_{\mu,\beta}p_1p_2\widehat{n}_2^{\frac{1}{2}}\psi\right\rangle\!\!\right\rangle\right| && (80)\\
&\quad -N\left|\left\langle\!\!\left\langle \psi, q_1q_2 w^{(12)}_{\mu,\beta}p_1p_2\psi\right\rangle\!\!\right\rangle\right| && (81)\\
&\quad - \left|\left\langle\!\!\left\langle \psi, (1-p_1p_2)|\Phi(x_1)|^2 p_1p_2\psi\right\rangle\!\!\right\rangle\right| - \left|\left\langle\!\!\left\langle \psi, (1-p_1p_2)|\Phi(x_1)|^2(1-p_1p_2)\psi\right\rangle\!\!\right\rangle\right| && (82)\\
&\quad - \left|\left\langle\!\!\left\langle \psi, |\Phi(x_1)|^2\psi\right\rangle\!\!\right\rangle - \left\langle \Phi, |\Phi(x_1)|^2\Phi\right\rangle\right| && (83)\\
&\quad - \left|\left\langle\!\!\left\langle \psi, V^\parallel(t,z_1)\psi\right\rangle\!\!\right\rangle - \left\langle \Phi, V^\parallel(t,(x_1,0))\Phi\right\rangle\right|. && (84)
\end{aligned}$$

Note that the second term in (77) is non-negative. For (78), we observe that

$$\|\nabla_{x_1} p_1^\Phi\psi\|^2 - \|\nabla_x\Phi\|^2_{L^2(\mathbb{R}^2)} = -\|\nabla_x\Phi\|^2_{L^2(\mathbb{R}^2)}\|q_1^\Phi\psi\|^2 \lesssim \mathfrak{e}^2_\beta(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle$$

and $\left\langle\!\!\left\langle \widehat{n}^{-\frac{1}{2}} q_1^\Phi\psi, \Delta_{x_1} p_1^\Phi \widehat{n}^{\frac{1}{2}}\psi\right\rangle\!\!\right\rangle \lesssim \mathfrak{e}^2_\beta(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle$. Making use of $\mathcal{G}(x)$ from (61) and Lemma 4.8, we find $|(79)| \lesssim \mathfrak{e}^2_\beta(t)\left(\frac{\mu^\beta}{\varepsilon} + N^{-1} + \mu^\eta\right)$ and $|(80)| \lesssim \mathfrak{e}_\beta(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle$. Insertion of $\widehat{n}^{\frac{1}{2}}\widehat{n}^{-\frac{1}{2}}$ yields $|(82)| \lesssim \mathfrak{e}^2_\beta(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle$. As a consequence of Lemmas 4.5 and 4.10, $|(83)| + |(84)| \lesssim \mathfrak{e}^2_\beta(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle + \mathfrak{e}^3_\beta(t)\varepsilon$. Finally, we decompose $|(81)|$ as

$$\begin{aligned}
|(81)| \lesssim\; & N\left|\left\langle\!\!\left\langle \psi, q_1^{\chi^\varepsilon} q_2 w^{(12)}_{\mu,\beta}p_1p_2\psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle q_2^{\chi^\varepsilon}\psi, q_1^\Phi p_1^{\chi^\varepsilon} w^{(12)}_{\mu,\beta}p_1p_2\psi\right\rangle\!\!\right\rangle\right|\\
&+N\left|\left\langle\!\!\left\langle q_1^\Phi q_2^\Phi\psi, p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon} w^{(12)}_{\mu,\beta}p_1p_2\psi\right\rangle\!\!\right\rangle\right|.
\end{aligned}$$

Analogously to the bound of (24) (Section 5.2.2), the first line is bounded by

$$\mathfrak{e}^3_\beta(t)(\varepsilon^{\frac{3}{2}}\mu^{-\frac{\beta}{2}} + \mu^{\frac{1-\beta}{2}}),$$

and the second line yields

$$\mathfrak{e}^2_\beta(t)\left\langle\!\left\langle \psi, \widehat{n}\psi\right\rangle\!\right\rangle + \mathfrak{e}^3_\beta(t)N^{-\beta_2^-}$$

for $\beta_2 = \min\left\{\beta, \frac{1}{4}\right\}$ as in the estimate of (30) (Section 5.2.5). □

6 Proofs for $\beta = 1$

6.1 Microscopic structure

This section collects properties of the scattering solution $f_{\widetilde{\beta}}$ and its complement $g_{\widetilde{\beta}}$.

Lemma 6.1. *Let $f_{\widetilde{\beta}}$ and $\varrho_{\widetilde{\beta}}$ as in Definition 3.7 and j_μ as in* (35)*. Then*

(a) $f_{\widetilde{\beta}}$ is a non-negative, non-decreasing function of $|z|$,

(b) $f_{\widetilde{\beta}}(z) \geq j_\mu(z)$ for all $z \in \mathbb{R}^3$ and there exists $\kappa_{\widetilde{\beta}} \in \big(1, \frac{\mu^{\widetilde{\beta}}}{\mu^{\widetilde{\beta}} - \mu a}\big)$ such that

$$f_{\widetilde{\beta}}(z) = \kappa_{\widetilde{\beta}} j_\mu(z)$$

for $|z| \leq \mu^{\widetilde{\beta}}$,

(c) $\varrho_{\widetilde{\beta}} \sim \mu^{\widetilde{\beta}}$,

(d) $\|\mathbb{1}_{|z_1 - z_2| < \varrho_{\widetilde{\beta}}} \nabla_1 \psi\|^2 + \frac{1}{2} \left\langle\!\!\left\langle \psi, (w_\mu^{(12)} - U_{\mu,\widetilde{\beta}}^{(12)})\psi \right\rangle\!\!\right\rangle \geq 0$ for any $\psi \in \mathcal{D}(\nabla_1)$.

Proof. Parts (a) to (c) are proven in [5, Lemma 4.9]. For part (d), see [30, Lemma 5.1(3)]. □

Lemma 6.2. *For $g_{\widetilde{\beta}}$ as in Definition 3.7 and sufficiently small ε,*

(a) $|g_{\widetilde{\beta}}(z)| \lesssim \frac{\mu}{|z|}$,

(b) $\|g_{\widetilde{\beta}}\|_{L^2(\mathbb{R}^3)} \lesssim \mu^{1+\frac{\widetilde{\beta}}{2}}$,

(c) $\|\nabla g_{\widetilde{\beta}}\|_{L^2(\mathbb{R}^3)} \lesssim \mu^{\frac{1}{2}}$,

(d) $\|g_{\widetilde{\beta}}^{(12)} \psi^{N,\varepsilon}(t)\| \lesssim N^{-1}$,

(e) $\|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_1(t)\mu^{\widetilde{\beta}}\varepsilon^{-\frac{1}{3}} = \mathfrak{e}_1(t) N^{-\widetilde{\beta}} \varepsilon^{\widetilde{\beta}-\frac{1}{3}}$,

(f) $\|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}(\,\cdot\,, y_1 - y_2)}(x_1 - x_2)\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_1(t)\mu^{\frac{p-1}{p}\widetilde{\beta}}$ for any fixed $p \in [1, \infty)$.

Proof. Parts (a) to (c) are proven in [5, Lemmas 4.10 and 4.11]. Assertion (d) works analogously as [5, Lemma 4.10c]. For (e), we obtain similarly to [5, Lemma 4.10e]

$$\|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}}(z_1 - z_2)\psi\|^2 \lesssim \mu^{2\widetilde{\beta}} \int \mathrm{d}z_N \cdots \mathrm{d}z_2 \left(\int \mathrm{d}z_1 |\psi(z_1, ..., z_N)|^6 \right)^{\frac{2}{6}} ,$$

where we have used Hölder's inequality in the $\mathrm{d}z_1$ integration. Now we substitute $z_1 \mapsto \widetilde{z}_1 = (x_1, \frac{y_1}{\varepsilon})$ and use Sobolev's inequality in the $\mathrm{d}\widetilde{z}_1$-integration, noting that $\nabla_{\widetilde{z}_1} = (\nabla_{x_1}, \varepsilon\partial_{y_1})$ and $\mathrm{d}\widetilde{z}_1 = \varepsilon\, \mathrm{d}z_1$. This yields

$$\left(\int \mathrm{d}z_1 |\psi(z_1, ..., z_N)|^6 \right)^{\frac{2}{6}} = \left(\varepsilon \int \mathrm{d}\widetilde{z}_1 |\psi((x_1, \varepsilon\widetilde{y}_1), z_2, ..., z_N)|^6 \right)^{\frac{2}{6}}$$

$$\lesssim \varepsilon^{\frac{1}{3}} \int \mathrm{d}\widetilde{z}_1 |\nabla_{\widetilde{z}_1} \psi((x_1, \varepsilon\widetilde{y}_1), z_2, ..., z_N)|^2$$
$$= \varepsilon^{-\frac{2}{3}} \int \mathrm{d}z_1 \left(|\nabla_{x_1} \psi(z_1, ..., z_N)|^2 + \varepsilon^2 |\partial_{y_1} \psi(z_1, ..., z_N)|^2 \right) .$$

The statement then follows with Lemma 4.9a. For part (f), recall the two-dimensional Gagliardo–Nirenberg–Sobolev inequality: for $2 < q < \infty$ and $f \in H^1(\mathbb{R}^2)$,

$$\|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{q-2}{q}} \|f\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \geq S_q \|f\|_{L^q(\mathbb{R}^2)} , \tag{85}$$

where S_q is a positive constant which is finite for $2 < q < \infty$ (e.g. [25, Theorem 8.5(ii)] and [26, Equation (2.2.5)]). Consequently, $\|f\|_{L^q(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \|\nabla f\|_{L^2(\mathbb{R}^2)}^{\frac{q-2}{q}}$ for each fixed $q \in (2, \infty)$. Hence, for any fixed $p \in (1, \infty)$ and $\psi \in L^2(\mathbb{R}^{3N}) \cap \mathcal{D}(\nabla_{x_1})$,

$$\|\mathbb{1}_{\mathrm{supp}\, g_{\widetilde{\beta}}}(x_1 - x_2)\psi\|^2$$
$$\leq \int \mathrm{d}z_N \cdots \mathrm{d}y_1 \left(\int_{\mathbb{R}^2} \mathbb{1}_{|x| \leq \varrho_{\widetilde{\beta}}} \mathrm{d}x \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^2} \mathrm{d}x_1 |\psi(z_1, ..., z_N)|^{2p} \right)^{\frac{2}{2p}}$$
$$\lesssim \mu^{\frac{2\widetilde{\beta}(p-1)}{p}} \int \mathrm{d}z_N \cdots \mathrm{d}y_1 \left(\int_{\mathbb{R}^2} \mathrm{d}x_1 |\psi(z_1, ..., z_N)|^2 \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} \mathrm{d}x_1 |\nabla_{x_1} \psi(z_1, ..., z_N)|^2 \right)^{\frac{p-1}{p}}$$
$$\leq \mu^{\frac{2\widetilde{\beta}(p-1)}{p}} \|\psi\|^{\frac{2}{p}} \|\nabla_{x_1}\psi\|^{\frac{2(p-1)}{p}} ,$$

where we have used Hölder's inequality in the $\mathrm{d}x_1$ integration, applied (85), and finally used again Hölder in the $\mathrm{d}z_N \cdots \mathrm{d}y_1$ integration. □

6.2 Characterisation of the auxiliary potential $U_{\mu,\widetilde{\beta}}$

In this section, we show that both $U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$ and $U_{\mu,\widetilde{\beta}}$ from Definition 3.7 are contained in the set $\mathcal{W}_{\widetilde{\beta},\eta}$ from Definition 2.2, which admits the transfer of results obtained in Section 5 to these interaction potentials.

Lemma 6.3. *The family $U_{\mu,\widetilde{\beta}}$ is contained in $\mathcal{W}_{\widetilde{\beta},\eta}$ for any $\eta > 0$.*

Proof. Note that $\mu^{-1} \int_{\mathbb{R}^3} U_{\mu,\widetilde{\beta}}(z)\, \mathrm{d}z = \frac{4\pi}{3} a(\varrho_{\widetilde{\beta}}^3 \mu^{-3\widetilde{\beta}} - 1) = \frac{4\pi}{3} ac$ for some $c > 0$ by Lemma 6.1c, hence $b_{\widetilde{\beta},N,\varepsilon}(U_{\mu,\widetilde{\beta}}) = \lim_{(N,\varepsilon)\to(\infty,0)} b_{\widetilde{\beta},N,\varepsilon}(U_{\mu,\widetilde{\beta}})$. The remaining requirements are easily verified. □

Lemma 6.4. *Let $0 < \eta < 1 - \widetilde{\beta}$. Then the family $U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$ is contained in $\mathcal{W}_{\widetilde{\beta},\eta}$.*

Proof. As before, it only remains to show that $U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}$ satisfies part (d) of Definition 2.2. To see this, observe that

$$\mu^{-1} \int_{\mathbb{R}^3} U_{\mu,\widetilde{\beta}}(z) f_{\widetilde{\beta}}(z)\, \mathrm{d}z \overset{(40)}{=} \mu^{-1} \int_{B_\mu(0)} w_\mu(z) f_{\widetilde{\beta}}(z) \overset{6.1\mathrm{b}}{=} \mu^{-1} \kappa_{\widetilde{\beta}} \int_{B_\mu(0)} w_\mu(z) j_\mu(z) \overset{(37)}{=} \kappa_{\widetilde{\beta}} 8\pi a ,$$

hence $b_{\widetilde{\beta},N,\varepsilon}(U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}) = \kappa_{\widetilde{\beta}} 8\pi a \int_{\mathbb{R}} |\chi(y)|^4 \,\mathrm{d}y$. By Lemma 6.1b, this implies

$$\lim_{(N,\varepsilon)\to(\infty,0)} b_{\widetilde{\beta},N,\varepsilon}(U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}) = 8\pi a \int_{\mathbb{R}} |\chi(y)|^4 \,\mathrm{d}y = b_1 \tag{86}$$

and

$$|b_{\widetilde{\beta},N,\varepsilon}(U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}) - b_1| = 8\pi a (\kappa_{\widetilde{\beta}} - 1) \int_{\mathbb{R}} |\chi(y)|^4 \,\mathrm{d}y \overset{6.1\mathrm{b}}{\lesssim} \frac{\mu a}{\mu^{\widetilde{\beta}} - \mu a} \lesssim \mu^{1-\widetilde{\beta}}.$$

□

6.3 Estimate of the kinetic energy for $\beta = 1$

The main goal of this section is to provide a bound for the kinetic energy of the part of $\psi^{N,\varepsilon}(t)$ with at least one particle orthogonal to $\Phi(t)$. Since the predominant part of the kinetic energy is caused by the microscopic structure and thus concentrated in neighbourhoods of the scattering centres, we will consider the part of the kinetic energy originating from the complement of these neighbourhoods and prove that it is subleading. The first step is to define the appropriate neighbourhoods $\overline{\mathcal{C}}_j$ as well as sufficiently large balls $\overline{\mathcal{A}}_j \supset \overline{\mathcal{C}}_j$ around them.

Definition 6.5. Let $\max\left\{\frac{\gamma+1}{2\gamma}, \frac{5}{6}\right\} < d < \widetilde{\beta}$, $j,k \in \{1,\dots,N\}$, and define

$$\begin{aligned} a_{j,k} &:= \left\{(z_1,\dots,z_N) : |z_j - z_k| < \mu^d\right\}, \\ c_{j,k} &:= \left\{(z_1,\dots,z_N) : |z_j - z_k| < \varrho_{\widetilde{\beta}}\right\}, \\ a^x_{j,k} &:= \left\{(z_1,\dots,z_N) : |x_j - x_k| < \mu^d\right\}. \end{aligned}$$

Then the subsets $\overline{\mathcal{A}}_j$, $\overline{\mathcal{B}}_j$, $\overline{\mathcal{C}}_j$ and $\overline{\mathcal{A}}^x_j$ of $\mathbb{R}^{3N}$ are defined as

$$\overline{\mathcal{A}}_j := \bigcup_{k\neq j} a_{j,k}, \qquad \overline{\mathcal{B}}_j := \bigcup_{k,l\neq j} a_{k,l}, \qquad \overline{\mathcal{C}}_j := \bigcup_{k\neq j} c_{j,k}, \qquad \overline{\mathcal{A}}^x_j := \bigcup_{k\neq j} a^x_{j,k}$$

and their complements are denoted by $\mathcal{A}_j$, $\mathcal{B}_j$, $\mathcal{C}_j$ and $\mathcal{A}^x_j$, e.g., $\mathcal{A}_j := \mathbb{R}^{3N} \setminus \overline{\mathcal{A}}_j$.

The sets $\overline{\mathcal{A}}_j$ and $\overline{\mathcal{A}}^x_j$ contain all N-particle configurations where at least one other particle is sufficiently close to particle j or where the projections in the x-direction are close, respectively. The sets $\mathcal{B}_j$ consist of all N-particle configurations where particles can interact with particle j but are mutually too distant to interact among each other.

Note that the characteristic functions $\mathbb{1}_{\mathcal{A}^x_1}$ and $\mathbb{1}_{\overline{\mathcal{A}}^x_1}$ do not depend on any y-coordinate, and $\mathbb{1}_{\mathcal{B}_1}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$ are independent of z_1. Hence, the multiplication operators corresponding to these functions commute with all operators that act non-trivially only on the y-coordinates or on z_1, respectively. Some useful properties of these cut-off functions are collected in the following lemma.

Lemma 6.6. *Let $\overline{\mathcal{A}}_1$, $\overline{\mathcal{A}}^x_1$ and $\overline{\mathcal{B}}_1$ as in Definition 6.5. Then*

(a) $\|\mathbb{1}_{\overline{\mathcal{A}}_1} p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}_1(t)\mu^{\frac{3d-1}{2}}$, $\qquad \|\mathbb{1}_{\overline{\mathcal{A}}_1} \nabla_{x_1} p_1\|_{\mathrm{op}} \lesssim \mathfrak{e}_1(t)\,\mu^{\frac{3d-1}{2}}$,

(b) $\|\mathbb{1}_{\overline{\mathcal{A}}_1}\psi\| \lesssim \mu^{d-\frac{1}{3}}\left(\|\nabla_{x_1}\psi\| + \varepsilon\|\partial_{y_1}\psi\|\right)$ *for any* $\psi \in L^2(\mathbb{R}^{3N}) \cap \mathcal{D}(\nabla_1)$,

(c) $\|\mathbb{1}_{\overline{\mathcal{A}}_1}\partial_{y_1}p_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_1(t)\varepsilon^{-1}\mu^{d-\frac{1}{3}}$,

(d) $\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\| \lesssim \mu^{d-\frac{1}{3}}\left(\sum\limits_{k=2}^{N}(\|\nabla_{x_k}\psi\|^2 + \varepsilon^2\|\partial_{y_k}\psi\|^2)\right)^{\frac{1}{2}}$ *for any* $\psi \in L^2(\mathbb{R}^{3N}) \cap \mathcal{D}(\nabla_1)$,

(e) $\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_1(t)N^{\frac{1}{2}}\mu^{d-\frac{1}{3}} = \mathfrak{e}_1(t)N^{-d+\frac{5}{6}}\varepsilon^{d-\frac{1}{3}}$,

(f) $\|\mathbb{1}_{\overline{\mathcal{A}}_1^x}\psi\| \lesssim (N\mu^{2d})^{\frac{p-1}{2p}}\|\psi\|^{\frac{1}{p}}\|\nabla_{x_1}\psi\|^{\frac{p-1}{p}}$ *for any fixed* $p \in (1,\infty)$, $\psi \in L^2(\mathbb{R}^{3N}) \cap \mathcal{D}(\nabla_{x_1})$,

(g) $\|\mathbb{1}_{\overline{\mathcal{A}}_1^x}q_1^{\chi^\varepsilon}\psi^{N,\varepsilon}(t)\| \lesssim \mathfrak{e}_1(t)\varepsilon^{\frac{1}{p}}(N\mu^{2d})^{\frac{p-1}{2p}}$ *for any fixed* $p \in (1,\infty)$.

Proof. The proof of parts (a) to (e) works analogously to the proof of [5, Lemma 4.13]: one first observes that in the sense of operators, $\mathbb{1}_{\overline{\mathcal{A}}_1} \le \sum_{k=2}^N \mathbb{1}_{a_{1,k}}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1} \le \sum_{k=2}^N \mathbb{1}_{\overline{\mathcal{A}}_k}$, concludes that $\int_{\mathbb{R}^3}\mathbb{1}_{a_{1,k}}(z_1,z_k)\,\mathrm{d}z_1 \lesssim \mu^{3d}$, and proceeds as in the proof of Lemma 6.2e. The proofs of (f) and (g) work analogously to the proof of Lemma 6.2f, where one uses the estimate $\int_{\mathbb{R}^2}\mathbb{1}_{\overline{\mathcal{A}}_1^x}(x_1,...,x_N)\,\mathrm{d}x_1 \lesssim N\mu^{2d}$. □

Lemma 6.7. *Let* $1 > \widetilde{\beta} > d > \max\left\{\frac{\gamma+1}{2\gamma}, \frac{5}{6}\right\}$. *Then, for sufficiently small* μ,

$$\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi^{N,\varepsilon}(t)\|^2 \quad\lesssim\quad \mathfrak{e}_1^2(t)\alpha^<_{\xi,w_\mu}(t) + \mathfrak{e}_1^3(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}} + \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\beta\gamma^2}} + \mu^{\frac{1-\widetilde{\beta}}{2}} + N^{-d+\frac{5}{6}}\right).$$

Proof. We will in the following abbreviate $\psi^{N,\varepsilon}(t) \equiv \psi$ and $\Phi(t) \equiv \Phi$. Analogously to [5, Lemma 4.12], we decompose the energy difference as

$$\begin{aligned}
&E^\psi_{w_\mu}(t) - \mathcal{E}^\Phi_{b_1}(t)\\
&\ge \|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2 - \left|\left\langle\!\!\left\langle\nabla_{x_1}q_1^\Phi\psi, \mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}p_1^\Phi q_1^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle\right| &(87)\\
&\quad+\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_{x_1}\psi\|^2 + \left\langle\!\!\left\langle\psi, (-\partial_{y_1}^2 + \tfrac{1}{\varepsilon^2}V^\perp(\tfrac{y_1}{\varepsilon}) - \tfrac{E_0}{\varepsilon^2})\psi\right\rangle\!\!\right\rangle\\
&\qquad\qquad+\tfrac{N-1}{2}\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1}\left(w_\mu^{(12)} - U^{(12)}_{\mu,\widetilde{\beta}}\right)\psi\right\rangle\!\!\right\rangle &(88)\\
&\quad+2\Re\left\langle\!\!\left\langle\nabla_{x_1}p_1\psi, \mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1\psi\right\rangle\!\!\right\rangle &(89)\\
&\quad+\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}p_1\psi\|^2 - \|\nabla_x\Phi\|^2_{L^2(\mathbb{R})} &(90)\\
&\quad+\tfrac{b_1}{2}\left(\left\langle\!\!\left\langle\psi, |\Phi(x_1)|^2\psi\right\rangle\!\!\right\rangle - \left\langle\Phi, |\Phi|^2\Phi\right\rangle\right) + \left\langle\!\!\left\langle\psi, V^\parallel(t,z_1)\psi\right\rangle\!\!\right\rangle - \left\langle\Phi, V^\parallel(t,(x,0))\Phi\right\rangle &(91)\\
&\quad+\tfrac{N-1}{2}\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1}p_1p_2U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle - \tfrac{b_1}{2}\left\langle\!\!\left\langle\psi, |\Phi(x_1)|^2\psi\right\rangle\!\!\right\rangle &(92)\\
&\quad+(N-1)\Re\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1}(p_1q_2 + q_1p_2)U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle &(93)\\
&\quad+(N-1)\Re\left\langle\!\!\left\langle\psi, \mathbb{1}_{\mathcal{B}_1}q_1q_2U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle. &(94)
\end{aligned}$$

The first line is easily controlled as

$$(87) \gtrsim \|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2 - \mathfrak{e}_1^3(t)\varepsilon\,.$$

To estimate (88), note that $(c_{1,k} \cap \mathcal{B}_1) \cap (c_{1,l} \cap \mathcal{B}_1) = \emptyset$ by Definition 6.5 and since $d < \widetilde{\beta}$ implies $\varrho_{\widetilde{\beta}} < 2\varrho_{\widetilde{\beta}} < \mu^d$. Consequently, $\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1} \geq \mathbb{1}_{\overline{\mathcal{C}}_1}\mathbb{1}_{\mathcal{B}_1} = \mathbb{1}_{\mathcal{B}_1}\sum\limits_{k=2}^N \mathbb{1}_{c_{1,k}} = \mathbb{1}_{\mathcal{B}_1}\sum\limits_{k=2}^N \mathbb{1}_{|z_1-z_k|\leq\varrho_{\widetilde{\beta}}}$, which yields with Lemma 6.1d

$$\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\nabla_1\psi\|^2 + \tfrac{N-1}{2}\left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\psi, \left(w_\mu^{(12)} - U_{\mu,\widetilde{\beta}}^{(12)}\right)\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle \geq 0\,.$$

To use this for (88), we must extract a contribution $\|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\partial_{y_1}\psi\|^2$ from the remaining expression $\langle\!\langle\psi, (-\partial_{y_1}^2 + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2})\psi\rangle\!\rangle$. To this end, recall that χ^ε is the ground state of $\partial_{y_1}^2 + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon})$ corresponding to the eigenvalue $\frac{E_0}{\varepsilon^2}$, hence $O_{y_1} := -\partial_{y_1}^2 + \frac{1}{\varepsilon^2}V^\perp(\frac{y_1}{\varepsilon}) - \frac{E_0}{\varepsilon^2}$ is a positive operator and $O_{y_1}\psi = O_{y_1}q_1^{\chi^\varepsilon}\psi$. Since $\mathbb{1}_{\overline{\mathcal{A}}_1^x}$ and $\mathbb{1}_{\overline{\mathcal{B}}_1}$ and their complements commute with any operator that acts non-trivially only on y_1 and since $\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\overline{\mathcal{B}}_1}\psi$ and $\mathbb{1}_{\mathcal{A}_1^x}\psi$ are contained in the domain of O_{y_1} if this holds for ψ, we find

$$\begin{aligned}
\langle\!\langle\psi, O_{y_1}\psi\rangle\!\rangle &= \left\langle\!\!\left\langle \mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1}q_1^{\chi^\varepsilon}\psi, O_{y_1}\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1}q_1^{\chi^\varepsilon}\psi\right\rangle\!\!\right\rangle \\
&\quad + \left\langle\!\!\left\langle (\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\overline{\mathcal{B}}_1} + \mathbb{1}_{\mathcal{A}_1^x})\psi, O_{y_1}(\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\overline{\mathcal{B}}_1} + \mathbb{1}_{\mathcal{A}_1^x})\psi\right\rangle\!\!\right\rangle \\
&\geq \|\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1}\partial_{y_1}q_1^{\chi^\varepsilon}\psi\|^2 - \varepsilon^{-2}\|(V^\perp - E_0)_-\|_{L^\infty(\mathbb{R})}\|\mathbb{1}_{\overline{\mathcal{A}}_1^x}q_1^{\chi^\varepsilon}\psi\|^2 \\
&\gtrsim \|\mathbb{1}_{\overline{\mathcal{A}}_1^x}\mathbb{1}_{\mathcal{B}_1}\partial_{y_1}\psi\|^2 - 2\left|\left\langle\!\!\left\langle \mathbb{1}_{\mathcal{B}_1}\partial_{y_1}q_1^{\chi^\varepsilon}\psi, \partial_{y_1}p_1^{\chi^\varepsilon}\mathbb{1}_{\overline{\mathcal{A}}_1^x}\psi\right\rangle\!\!\right\rangle\right| - \varepsilon^{-2}\|\mathbb{1}_{\overline{\mathcal{A}}_1^x}q_1^{\chi^\varepsilon}\psi\|^2 \\
&\quad - \|\mathbb{1}_{\overline{\mathcal{A}}_1^x}\partial_{y_1}p_1^{\chi^\varepsilon}\psi\|^2 \\
&\gtrsim \|\mathbb{1}_{\overline{\mathcal{A}}_1}\mathbb{1}_{\mathcal{B}_1}\partial_{y_1}\psi\|^2 - \mathfrak{e}_1^2(t)\left(\varepsilon^{-1}(N\mu^{2d})^{\frac{p-1}{2p}} - \varepsilon^{-2+\frac{2}{p}}(N\mu^{2d})^{\frac{p-1}{p}}\right)
\end{aligned}$$

for any fixed $p \in (1,\infty)$ by Lemma 6.6. Note that we have used in the last line the fact that $\mathbb{1}_{\overline{\mathcal{A}}_1^x} \geq \mathbb{1}_{\overline{\mathcal{A}}_1}$ in the sense of operators as $\overline{\mathcal{A}}_1 \subseteq \overline{\mathcal{A}}_1^x$. Now choose $p = 1 + \frac{2}{\gamma(2d-1)-1}$, which is contained in $(1,\infty)$ as $2d-1 > \frac{1}{\gamma}$ because $d > \frac{1}{2} + \frac{1}{2\gamma}$. This yields

$$\varepsilon^{-1}(N\mu^{2d})^{\frac{p-1}{2p}} = \left(N^{-1}\varepsilon^{1-\gamma}\right)^{\frac{p-1}{2p}(2d-1)}\varepsilon^{\frac{1}{2p}(\gamma(2d-1)(p-1)-p-1)} = \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{p-1}{p}(2d-1)} < \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}}$$

because, since $\gamma > 1$ and $d < \widetilde{\beta}$,

$$\tfrac{p-1}{p}(2d-1) = \tfrac{2(2d-1)}{\gamma(2d-1)+1} > \tfrac{2d-1}{d\gamma} > \tfrac{1}{\widetilde{\beta}\gamma^2}\,.$$

For the second expression in the brackets, recall that $d > \frac{1}{2\gamma} + \frac{1}{2}$ by Definition 6.5, hence

$$\begin{aligned}
\varepsilon^{-2+\frac{2}{p}}(N\mu^{2d})^{\frac{p-1}{p}} &= \left(N^{-1}\varepsilon^{1-\gamma}\right)^{\frac{p-1}{p}(2d-1)}\varepsilon^{\frac{p-1}{p}((\gamma-1)(2d-1)-2+2d)} \\
&< \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{p-1}{p}(2d-1)} < \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}}\,.
\end{aligned}$$

Consequently,

$$(88) \gtrsim -\mathfrak{e}_1^2(t)\left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}}\,.$$

Analogously to the estimates of (48) to (50) in [5, Lemma 4.12], we obtain

$$|(89)| \lesssim \mathfrak{e}_1^2(t)\left(\langle\!\langle\psi, \widehat{n}\psi\rangle\!\rangle + \mu^{\frac{3d-1}{2}}\right),$$

$$|(90)| \lesssim \mathfrak{e}_1^2(t)\left(\mu^{3d-1}+\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\right),$$
$$|(91)| \lesssim \mathfrak{e}_1^2(t)\,\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle+\mathfrak{e}_1^3(t)\varepsilon\,,$$

where we decomposed $\mathbb{1}_{\mathcal{A}_1}=\mathbb{1}-\mathbb{1}_{\overline{\mathcal{A}}_1}$ and used that $\|\nabla_{x_1}p_1\psi\|^2=\|\nabla_x\Phi\|^2_{L^2(\mathbb{R}^2)}\|p_1\psi\|^2$ as well as Lemmas 4.3b, 4.5, 4.7a, 4.9a, 4.10 and 6.6a. Analogously to the corresponding terms (51) and (52) in [5, Lemma 4.12], we write (92) as

$$\tfrac{N-1}{2}\left\langle\!\!\left\langle(\mathbb{1}-\mathbb{1}_{\overline{\mathcal{B}}_1})\psi,p_1p_2\left((U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}+(U_{\mu,\widetilde{\beta}}g_{\widetilde{\beta}})^{(12)}\right)p_1p_2(\mathbb{1}-\mathbb{1}_{\overline{\mathcal{B}}_1})\psi\right\rangle\!\!\right\rangle-\langle\!\langle\psi,b_1|\Phi(x_1)|^2\psi\rangle\!\rangle$$

and control the contribution with $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}$ and without $\mathbb{1}_{\overline{\mathcal{B}}_1}$ by means of $\mathcal{G}(x)$ as in (61), using the respective estimates from Section 5.2.1 since $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}\in\mathcal{W}_{\widetilde{\beta},\eta}$ for $\eta\in(0,1-\widetilde{\beta})$. For the remainders of (92), note that $\|U_{\mu,\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)}\lesssim\mu$ and that

$$\|U_{\mu,\widetilde{\beta}}g_{\widetilde{\beta}}\|_{L^1(\mathbb{R}^3)}=a\mu^{1-3\widetilde{\beta}}\int_{\operatorname{supp}U_{\mu,\widetilde{\beta}}}\mathrm{d}z|g_{\widetilde{\beta}}(z)|\le a\mu^{1-3\widetilde{\beta}}g_{\widetilde{\beta}}(\mu^{\widetilde{\beta}})\int_{\operatorname{supp}U_{\mu,\widetilde{\beta}}}\mathrm{d}z\lesssim\mu^{2-\widetilde{\beta}}\,.$$

For (93), we decompose $\mathbb{1}_{\mathcal{B}_1}=\mathbb{1}-\mathbb{1}_{\overline{\mathcal{B}}_1}$ and insert $\widehat{n}^{\frac12}\widehat{n}^{-\frac12}$ into the term with identities on both sides. This leads to the bounds

$$|(92)| \lesssim \mathfrak{e}_1^3(t)\left(\tfrac{\mu^{\widetilde{\beta}}}{\varepsilon}+\mu^{1-\widetilde{\beta}}+N^{-1}+N^{-d+\frac56}\varepsilon^{d-\frac13}\right),$$
$$|(93)| \lesssim \mathfrak{e}_1^3(t)\left(N^{-d+\frac56}\varepsilon^{d-\frac13}+\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\right).$$

Finally, for the last term of the energy difference, we decompose $q=q^{\chi^\varepsilon}+p^{\chi^\varepsilon}q^{\Phi}$, which yields

$$|(94)| \lesssim N\left|\left\langle\!\!\left\langle\mathbb{1}_{\mathcal{B}_1}\psi,q_1^{\chi^\varepsilon}q_2U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\mathbb{1}_{\mathcal{B}_1}\psi\right\rangle\!\!\right\rangle\right|+N\left|\left\langle\!\!\left\langle\psi,q_1^{\chi^\varepsilon}q_2^{\Phi}p_2^{\chi^\varepsilon}U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\psi\right\rangle\!\!\right\rangle\right| \tag{95}$$
$$+N\left|\left\langle\!\!\left\langle\mathbb{1}_{\overline{\mathcal{B}}_1}\psi,q_2^{\chi^\varepsilon}q_1^{\Phi}p_1^{\chi^\varepsilon}U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\psi\right\rangle\!\!\right\rangle\right| \tag{96}$$
$$+N\left|\left\langle\!\!\left\langle\mathbb{1}_{\mathcal{B}_1}\psi,q_2^{\chi^\varepsilon}q_1^{\Phi}p_1^{\chi^\varepsilon}U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\right\rangle\!\!\right\rangle\right| \tag{97}$$
$$+N\left|\left\langle\!\!\left\langle\psi,q_1^{\Phi}q_2^{\Phi}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\psi\right\rangle\!\!\right\rangle\right| \tag{98}$$
$$+N\left|\left\langle\!\!\left\langle\mathbb{1}_{\overline{\mathcal{B}}_1}\psi,q_1^{\Phi}q_2^{\Phi}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\psi\right\rangle\!\!\right\rangle\right| \tag{99}$$
$$+N\left|\left\langle\!\!\left\langle\mathbb{1}_{\mathcal{B}_1}\psi,q_1^{\Phi}q_2^{\Phi}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}U^{(12)}_{\mu,\widetilde{\beta}}p_1p_2\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\right\rangle\!\!\right\rangle\right|, \tag{100}$$

where we used the symmetry under the exchange $1\leftrightarrow2$ of the second term in the first line. For (95), note that $\mathbb{1}_{\mathcal{B}_1}\psi$ is symmetric in $\{2,\dots,N\}$ and commutes with ∇_1 and $q_1^{\chi^\varepsilon}$, hence we obtain, analogously to the estimate of (24) (Section 5.2.2), the bound

$$|(95)|\lesssim\mathfrak{e}_1^3(t)\left(N^{\frac{\widetilde{\beta}}{2}}\varepsilon^{\frac{3-\widetilde{\beta}}{2}}+\mu^{\frac{1-\widetilde{\beta}}{2}}\right)<\mathfrak{e}_1^3(t)\left(\left(\tfrac{\varepsilon^{\vartheta}}{\mu}\right)^{\frac{\widetilde{\beta}}{2}}+\mu^{\frac{1-\widetilde{\beta}}{2}}\right)$$

since

$$N^{\frac{\widetilde{\beta}}{2}}\varepsilon^{\frac{3-\widetilde{\beta}}{2}} = \left(N\varepsilon^{\vartheta-1}\right)^{\frac{\widetilde{\beta}}{2}}\varepsilon^{\frac{3-\vartheta\widetilde{\beta}}{2}} \leq \left(N\varepsilon^{\vartheta-1}\right)^{\frac{\widetilde{\beta}}{2}}$$

for $\widetilde{\beta} \leq \frac{3}{\vartheta}$. For the second line and third line, note that $p_1^{\chi^\varepsilon} U^{(12)}_{\mu,\widetilde{\beta}} p_1^{\chi^\varepsilon} = p_1^{\chi^\varepsilon}\overline{U_{\mu,\widetilde{\beta}}}(x_1 - x_2, y_2)$, with $\overline{U_{\mu,\widetilde{\beta}}}$ as in Definition 5.4, which is sensible since $U_{\mu,\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$ for any $\eta > 0$. Hence, with $\overline{v}_\rho$ and $\overline{h}_{\varrho_{\widetilde{\beta}},\rho}$ as in Definition 5.4 and Lemma 5.6, we obtain with the choice $\rho = 1$

$$\begin{aligned}
|(96)| &\lesssim N\left|\left\langle\!\left\langle q_1^\Phi \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, q_2^{\chi^\varepsilon}\overline{v}_\rho(x_1-x_2,y_2)p_1p_2\psi\right\rangle\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\nabla_{x_1}q_1^\Phi\psi, q_2^{\chi^\varepsilon}(\nabla_{x_1}\overline{h}_{\varrho_{\widetilde{\beta}},1}(x_1-x_2,y_2))p_1p_2\psi\right\rangle\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\left\langle q_1^\Phi \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, q_2^{\chi^\varepsilon}(\nabla_{x_1}\overline{h}^{(12)}_{\varrho_{\widetilde{\beta}},1})\nabla_{x_1}p_1p_2\psi\right\rangle\!\right\rangle\right| \\
&\lesssim N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\|\overline{v}_\rho(x_1-x_2,y_2)p_1^\Phi\|_{\mathrm{op}}\left(\|q_2^{\chi^\varepsilon}\psi\|^2 + N^{-1}\right)^{\frac{1}{2}} \\
&\quad + N\|\nabla_{x_1}q_1^\Phi\psi\|\|(\nabla_{x_1}\overline{h}_{\varrho_{\widetilde{\beta}},1}(x_1-x_2,y_2))p_1^\Phi\|_{\mathrm{op}}\left(\|q_2^{\chi^\varepsilon}\psi\|^2 + N^{-1}\right)^{\frac{1}{2}} \\
&\quad + N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\|(\nabla_{x_1}\overline{h}_{\varrho_{\widetilde{\beta}},1}(x_1-x_2,y_2))\cdot\nabla_{x_1}p_1^\Phi\|_{\mathrm{op}}\left(\|q_2^{\chi^\varepsilon}\psi\|^2 + N^{-1}\right)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}_1^3(t)(\ln\mu^{-1})^{\frac{1}{2}}(\varepsilon + N^{-\frac{1}{2}})
\end{aligned}$$

by Lemmas 4.6c, 4.9a, 5.6 and 6.6e. Similarly, but without the need for Lemma 4.6c, we obtain with $\rho = 1$

$$|(97)| \lesssim \mathfrak{e}_1^3(t)N^{-d+\frac{5}{6}}\varepsilon^{d-\frac{1}{3}}(\ln\mu^{-1})^{\frac{1}{2}}\,.$$

Analogously to the bound of (30) in Section 5.2.5, using $\overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\rho}$ with the choice $\rho = N^{-\frac{1}{4}}$ and suitably inserting $\widehat{n}^{\frac{1}{2}}\widehat{n}^{-\frac{1}{2}}$, we obtain

$$|(98)| \lesssim \mathfrak{e}_1^2(t)\left\langle\!\left\langle\psi,\widehat{n}\psi\right\rangle\!\right\rangle + \mathfrak{e}_1^3(t)N^{-\frac{1}{4}^-}\,.$$

Finally, with the choice $\rho = N^{-\frac{1}{2}}$, the last two lines can be bounded as

$$\begin{aligned}
|(99)| &\lesssim N\left|\left\langle\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\psi, q_1^\Phi q_2^\Phi \overline{\overline{v}}^{(12)}_\rho p_1p_2\psi\right\rangle\!\right\rangle\right| + N\left|\left\langle\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}q_1^\Phi\psi, q_2^\Phi(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\rho})\cdot\nabla_{x_1}p_1p_2\psi\right\rangle\!\right\rangle\right| \\
&\quad + N\left|\left\langle\!\left\langle \mathbb{1}_{\overline{\mathcal{B}}_1}\nabla_{x_1}q_1^\Phi\psi, q_2^\Phi(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\rho})p_1p_2\psi\right\rangle\!\right\rangle\right| \\
&\leq N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\left(\|p_1^\Phi\overline{\overline{v}}^{(12)}_\rho p_2^\Phi\|^2_{\mathrm{op}} + N^{-1}\|\overline{\overline{v}}^{(12)}_\rho p_1^\Phi\|^2_{\mathrm{op}}\right)^{\frac{1}{2}} \\
&\quad + N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\|(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\rho})\cdot\nabla_{x_1}p_1^\Phi\|_{\mathrm{op}} \\
&\quad + N\|\nabla_{x_1}q_1^\Phi\psi\|\Bigg(\left\langle\!\left\langle q_2^\Phi(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\rho})p_1p_2\psi, q_3^\Phi(\nabla_{x_1}\overline{\overline{h}}^{(13)}_{\varrho_{\widetilde{\beta}},\rho})p_1p_3\psi\right\rangle\!\right\rangle \\
&\qquad\qquad + N^{-1}\|(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\rho})p_1^\Phi\|^2_{\mathrm{op}}\Bigg)^{\frac{1}{2}} \\
&\lesssim \mathfrak{e}_1^3(t)\left(N^{-d+\frac{5}{6}}\varepsilon^{d-\frac{1}{3}} + N^{-\frac{1}{2}}\right)\left(\ln\mu^{-1}\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
|(100)| &\lesssim N\|\mathbb{1}_{\overline{\mathcal{B}}_1}\psi\|\left(\|p_2^\Phi\overline{\overline{v}}_\rho^{(12)}p_1^\Phi\|_{\mathrm{op}} + N^{-\frac12}\|p_2^\Phi\overline{\overline{v}}_\rho^{(12)}\|_{\mathrm{op}} + \|(\nabla_{x_1}\overline{\overline{h}}_{\varrho_\beta,\rho}^{(12)})p_1^\Phi\|_{\mathrm{op}}\mathfrak{e}_1(t)\right)\\
&\lesssim \mathfrak{e}_1^2(t)N^{-d+\frac56}\varepsilon^{d-\frac13}(\ln\mu^{-1})^{\frac12}\,,
\end{aligned}
$$

where we used (74) with $s_1 = p_1$ as well as (73) and Lemmas 5.6, 6.6e, 4.11 and 4.9a. Hence, we obtain with Lemma 4.11

$$
\begin{aligned}
|(94)| &\lesssim \mathfrak{e}_1^3(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}} + \mu^{\frac{1-\widetilde{\beta}}{2}} + \tfrac{\gamma}{\gamma-1}N^{-\frac12^-} + \varepsilon^{1^-} + N^{-d+\frac56}\varepsilon^{(d-\frac13)^-}\right) + \mathfrak{e}_1^2(t)\,\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\\
&\lesssim \mathfrak{e}_1^3(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}} + \mu^{\frac{1-\widetilde{\beta}}{2}} + N^{-d+\frac56}\right) + \mathfrak{e}_1^2(t)\,\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\,,
\end{aligned}
$$

where we have used that $-\frac14 < -d+\frac56$ and that $\varepsilon \ll N^{-d+\frac56}\varepsilon^{d-\frac13}$, which follows because

$$
\varepsilon N^{d-\frac56}\varepsilon^{\frac13-d} = \left(N\varepsilon^{\vartheta-1}\right)^{d-\frac56}\varepsilon^{\frac12-\vartheta(d-\frac56)} \le \left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{d-\frac56} \ll 1
$$

since $\vartheta \le 3$. All estimates together imply

$$
\begin{aligned}
|E_{w_\mu}^\psi(t) - \mathcal{E}_{b_1}^\Phi(t)| &\ge \|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2 - \mathfrak{e}_1^2(t)\,\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle\\
&\quad -\mathfrak{e}_1^3(t)\left(\mu^{\frac{1-\widetilde{\beta}}{2}} + \left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}} + N^{-d+\frac56} + \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}}\right)\,,
\end{aligned}
$$

where we have used that $3d-1 > 1-\widetilde{\beta}$ as $\widetilde{\beta} > d > \frac56$ and that $\frac{\mu^{\widetilde{\beta}}}{\varepsilon} < \left(\frac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}}$ because, since $\widetilde{\beta} > \frac12 + \frac{1}{2\gamma} > \frac1\gamma$,

$$
\tfrac{\mu^{\widetilde{\beta}}}{\varepsilon} = \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\widetilde{\beta}}\varepsilon^{\gamma\widetilde{\beta}-1} < \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\widetilde{\beta}} < \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\widetilde{\beta}\gamma^2}}
$$

□

6.4 Proof of Proposition 3.9

Recalling that $\widehat{r} = p_1p_2\widehat{m}^b + (p_1q_2+q_1p_2)\widehat{m}^a$, we conclude immediately

$$
\begin{aligned}
N^2&\left|\left\langle\!\!\left\langle \mathbb{1}_{\mathrm{supp}\,g_{\widetilde{\beta}}}(z_1-z_2)\psi, g_{\widetilde{\beta}}^{(12)}(p_1p_2\widehat{m}^b + (p_1q_2+q_1p_2)\widehat{m}^a)\psi\right\rangle\!\!\right\rangle\right|\\
&\lesssim \mathfrak{e}_1^2(t)N^{-\frac{3\widetilde{\beta}}{2}+\xi}\varepsilon^{\frac16+\frac{3\widetilde{\beta}}{2}} \;<\; \mathfrak{e}_1^2(t)\varepsilon^{\frac{17}{12}}
\end{aligned}
$$

by Lemmas 6.2 and 4.2a and because $\widetilde{\beta} > \frac56$. For fixed $t \in [0, T_{V^\parallel}^{\mathrm{ex}})$ and sufficiently small ε, $\mathfrak{e}_1^2(t)\varepsilon^{\frac{5}{12}} \lesssim 1$, hence this is bounded by ε.

6.5 Proof of Proposition 3.10

This proof is analogous to the proof of [5, Proposition 3.2], and we sketch the main steps for convenience of the reader. In the sequel, we abbreviate $\psi^{N,\varepsilon} \equiv \psi$ and $\Phi(t) \equiv \Phi$. Since

$$
\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}^<(t) - N(N-1)\Re\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)}\widehat{r}\psi\right\rangle\!\!\right\rangle\right)\,,
$$

Proposition 3.5 implies that for almost every $t \in [0, T^{\mathrm{ex}}_{V^\parallel})$,

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\xi,w_\mu}\right| \leq |\gamma_{a,<}(t)| + \left|\gamma_{b,<}(t) - N(N-1)\Re\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle \psi, g^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle\right)\right| . \tag{101}$$

The second term in (101) gives

$$\begin{aligned}
&-N(N-1)\Re\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\left\langle\!\!\left\langle \psi, g^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle\right)\\
&\quad= N(N-1)\Im\left\langle\!\!\left\langle \psi, g^{(12)}_{\widetilde{\beta}}\left[H_\mu(t) - \sum_{j=1}^{N} h_j(t), \widehat{r}\right]\psi\right\rangle\!\!\right\rangle \qquad (102)\\
&\qquad+N(N-1)\Im\left\langle\!\!\left\langle \psi, \left[H_\mu(t), g^{(12)}_{\widetilde{\beta}}\right]\widehat{r}\psi\right\rangle\!\!\right\rangle . \qquad (103)
\end{aligned}$$

In (102), we write $\sum_{i<j} w^{(ij)}_\mu = w^{(12)}_\mu + \sum_{j=3}^{N}\left(w^{(1j)}_\mu + w^{(2j)}_\mu\right) + \sum_{3\leq i<j\leq N} w^{(ij)}_\mu$ and use the identity $w^{(12)}_\mu - b_1(|\Phi(x_1)|^2 + |\Phi(x_2)|^2) = Z^{(12)} - \frac{N-2}{N-1}b_1(|\Phi(x_1)|^2 + |\Phi(x_2)|^2)$. This yields

$$(102) = \gamma_a(t) + \gamma_d(t) + \gamma_e(t) + \gamma_f(t) + N(N-1)\Im\left\langle\!\!\left\langle \psi, g^{(12)}_{\widetilde{\beta}}\left[Z^{(12)}, \widehat{r}\right]\psi\right\rangle\!\!\right\rangle .$$

For (103), note that

$$\left[H_\mu(t), g^{(12)}_{\widetilde{\beta}}\right]\widehat{r}\psi = \left(w^{(12)}_\mu - U^{(12)}_{\beta_1}\right) f^{(12)}_{\widetilde{\beta}}\widehat{r}\psi - 2(\nabla_1 g^{(12)}_{\widetilde{\beta}})\cdot\nabla_1\widehat{r}\psi - 2(\nabla_2 g^{(12)}_{\widetilde{\beta}})\cdot\nabla_2\widehat{r}\psi,$$

hence

$$(103) = \gamma_c(t) + N(N-1)\Im\left\langle\!\!\left\langle \psi, \left(w^{(12)}_\mu - U^{(12)}_{\mu,\widetilde{\beta}}\right) f^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle .$$

The expressions $\gamma_{a,<}(t)$, $\gamma_{b,<}t)$ together with the remaining terms from (102) and (103) yield

$$\begin{aligned}
&\gamma_{a,<}(t) + N(N-1)\Im\Bigg(- \left\langle\!\!\left\langle \psi, Z^{(12)}\widehat{r}\psi\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \psi, (1 - f^{(12)}_{\widetilde{\beta}})\left[Z^{(12)}, \widehat{r}\right]\psi\right\rangle\!\!\right\rangle\\
&\qquad\qquad + \left\langle\!\!\left\langle \psi, (w^{(12)}_\mu - U^{(12)}_{\mu,\widetilde{\beta}}) f^{(12)}_{\widetilde{\beta}}\widehat{r}\psi\right\rangle\!\!\right\rangle\Bigg)\\
&= \gamma_{a,<}(t) - N(N-1)\Im\left\langle\!\!\left\langle \psi, g^{(12)}_{\widetilde{\beta}}\widehat{r}Z^{(12)}\psi\right\rangle\!\!\right\rangle\\
&\quad -N(N-1)\Im\left\langle\!\!\left\langle \psi, \left(U^{(12)}_{\mu,\widetilde{\beta}} - \tfrac{b_1}{N-1}\left(|\Phi(x_1)|^2 + |\Phi(x_2)|^2\right)\right)(1 - g^{(12)}_{\widetilde{\beta}})\widehat{r}\psi\right\rangle\!\!\right\rangle\\
&= \gamma^<(t) + \gamma_b(t) ,
\end{aligned}$$

where we used that $\Im\left\langle\!\!\left\langle \psi, \widetilde{Z}^{(12)}\widehat{r}\psi\right\rangle\!\!\right\rangle = \Im\left\langle\!\!\left\langle \psi, \widetilde{Z}^{(12)}\widehat{m}\psi\right\rangle\!\!\right\rangle$ and that

$$Z^{(12)} f^{(12)}_{\widetilde{\beta}} = \left(w^{(12)}_\mu - U^{(12)}_{\mu,\widetilde{\beta}}\right) f^{(12)}_{\widetilde{\beta}} + U^{(12)}_{\mu,\widetilde{\beta}} f^{(12)}_{\widetilde{\beta}} - \tfrac{b_1}{N-1}\left(|\Phi(x_1)|^2 + |\Phi(x_2)|^2\right) f^{(12)}_{\widetilde{\beta}}. \quad \square$$

6.6 Proof of Proposition 3.11

6.6.1 Estimate of $\gamma^<(t)$

To estimate $\gamma^<(t)$, we apply Proposition 3.6 to the interaction potential $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}$, which makes sense since $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$ for $\eta \in (0, 1-\widetilde{\beta})$ by Lemma 6.4. Besides, we need to verify that the sequence (N,ε), which satisfies *A4* with $(\Theta,\Gamma)_1 = (\vartheta,\gamma)$, is also admissible and moderately confining with parameters $(\Theta,\Gamma)_{\widetilde{\beta}} = (\delta/\widetilde{\beta}, 1/\widetilde{\beta})$ for some $\delta \in (1,3)$. We make the choice $\delta = \vartheta\widetilde{\beta}$.

By assumption, $1 > \widetilde{\beta} > \frac{\gamma+1}{2\gamma} > \frac{1}{\gamma} > \frac{1}{\vartheta}$. Hence, $\delta = \vartheta\widetilde{\beta} \in (1,3)$ and we find

$$\frac{\varepsilon^{\delta/\widetilde{\beta}}}{\mu} = \frac{\varepsilon^{\vartheta}}{\mu}\,, \qquad \frac{\mu}{\varepsilon^{1/\widetilde{\beta}}} = \frac{\mu}{\varepsilon^{\gamma}}\,\varepsilon^{\gamma-1/\widetilde{\beta}} \leq \frac{\mu}{\varepsilon^{\gamma}}\,.$$

Since Proposition 3.6 requires the parameter $0 < \xi < \min\left\{\frac{1}{3}, \frac{1-\widetilde{\beta}}{2}, \widetilde{\beta}, \frac{3-\delta}{2}\cdot\frac{\widetilde{\beta}}{\delta-\widetilde{\beta}}\right\}$, we choose $0 < \xi < \min\left\{\frac{1-\widetilde{\beta}}{2}, \frac{3-\vartheta\widetilde{\beta}}{2(\vartheta-1)}\right\}$.

Proposition 3.6 provides a bound for $\gamma^<(t)$, which, however, depends on $\alpha^<_{\xi,U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)$ and consequently on the energy difference $|E^{\psi}_{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t) - \mathcal{E}^{\Phi}_{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)|$. Note that $\alpha^<_{\xi,U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}(t)$ enters only in the estimate of

$$|(31)| \leq N\left|\left\langle\!\!\left\langle \widehat{l}q_1^{\Phi}\psi, q_2^{\Phi}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}(U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}p_2^{\Phi}q_1^{\Phi}\psi\right\rangle\!\!\right\rangle\right|$$

in $\gamma^{(4)}_{b,<}(t)$. Hence, we need a new estimate of (31) by means of Lemma 6.7 to obtain a bound in terms of $|E^{\psi}_{w_{\mu,\beta}}(t) - \mathcal{E}^{\Phi}_{b_\beta}(t)|$. Since $U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}} \in \mathcal{W}_{\widetilde{\beta},\eta}$, we can define $\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}} \in \overline{\overline{\mathcal{V}}}_{\varrho_{\widetilde{\beta}}}$ as in Definition 5.4,

$$p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}(U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}})^{(12)}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon} = \overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}}^{(12)}p_1^{\chi^\varepsilon}p_2^{\chi^\varepsilon}\,.$$

and perform an integration by parts in two steps: first, we replace $\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}}$ by the potential $\overline{\overline{v}}_{\mu^{\beta_2}} \in \overline{\overline{\mathcal{V}}}_{\mu^{\beta_2}}$ from Definition 5.4, namely

$$\overline{\overline{v}}_{\mu^{\beta_2}}(x) = \begin{cases} \frac{1}{\pi}\mu^{-2\beta_2}\|\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}}\|_{L^1(\mathbb{R}^2)} & \text{for } |x| < \mu^{\beta_2}\,, \\ 0 & \text{else}\,, \end{cases}$$

where we have chosen $\rho = \mu^{\beta_2}$ for some $\beta_2 \in (0,\widetilde{\beta})$. Subsequently, we replace this potential by $\overline{\overline{\nu}}_1 \in \overline{\overline{\mathcal{V}}}_1$ with $\rho = 1$, where $\overline{\overline{v}}_{\mu^{\beta_2}}$ plays the role of $\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}}$, i.e.,

$$\overline{\overline{\nu}}_1(x) := \begin{cases} \frac{1}{\pi}\|\overline{\overline{v}}_{\mu^{\beta_2}}\|_{L^1(\mathbb{R}^2)} & \text{for } |x| < 1\,, \\ 0 & \text{else.} \end{cases}$$

By construction,

$$\|\overline{\overline{U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}}}}\|_{L^1(\mathbb{R}^2)} = \|\overline{\overline{v}}_{\mu^{\beta_2}}\|_{L^1(\mathbb{R}^2)} = \|\overline{\overline{\nu}}_1\|_{L^1(\mathbb{R}^2)},$$

hence, by Lemma 5.6a, the functions $\overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}$ and $\overline{\overline{h}}_{\mu^{\beta_2},1}$ as defined in (67) satisfy the equations

$$\Delta_x \overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}} = \overline{\overline{U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}}}} - \overline{\overline{v}}_{\mu^{\beta_2}} \, , \qquad \Delta_x \overline{\overline{h}}_{\mu^{\beta_2},1} = \overline{\overline{v}}_{\mu^{\beta_2}} - \overline{\overline{\nu}}_1 \, .$$

Hence,

$$p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} (U_{\mu,\widetilde{\beta}} f_{\widetilde{\beta}})^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} = \left(\Delta_x \overline{\overline{h}}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}} + \Delta_x \overline{\overline{h}}_{\mu^{\beta_2},1} + \overline{\overline{\nu}}_1 \right) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} \, ,$$

and consequently

$$\begin{aligned} |(31)| \;\leq\; & N \left| \left\langle\!\!\left\langle p_1^{\chi^\varepsilon} \nabla_{x_1} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}) p_2 q_1^\Phi \widehat{l}_1 \psi \right\rangle\!\!\right\rangle \right| & (104) \\ & + N \left| \left\langle\!\!\left\langle p_1^{\chi^\varepsilon} \widehat{l} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}) p_2 \nabla_{x_1} q_1^\Phi \psi \right\rangle\!\!\right\rangle \right| & (105) \\ & + N \left| \left\langle\!\!\left\langle \nabla_{x_1} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} p_2^\Phi \widehat{l}_1 q_1^\Phi \psi \right\rangle\!\!\right\rangle \right| & (106) \\ & + N \left| \left\langle\!\!\left\langle \widehat{l} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} p_2^\Phi \nabla_{x_1} q_1^\Phi \psi \right\rangle\!\!\right\rangle \right| & (107) \\ & + N \left| \left\langle\!\!\left\langle \widehat{l} q_1^\Phi q_2^\Phi \psi, \overline{\overline{\nu}}_1^{(12)} p_1^{\chi^\varepsilon} p_2^{\chi^\varepsilon} p_2^\Phi q_1^\Phi \psi \right\rangle\!\!\right\rangle \right| \, . & (108) \end{aligned}$$

With Lemma 4.6a, the first two lines can be bounded as

$$\begin{aligned} (104) \;\lesssim\; & N \mathfrak{e}_1(t) \Big(\left\langle\!\!\left\langle q_2^\Phi (\nabla_{x_2} \overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}) p_2 q_1^\Phi \widehat{l}_1 \psi, q_3^\Phi (\nabla_{x_3} \overline{\overline{h}}^{(13)}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}) p_3 q_1^\Phi \widehat{l}_1 \psi \right\rangle\!\!\right\rangle \\ & \qquad + N^{-1} \| \nabla_{x_1} \overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}} p_2^\Phi \|^2_{\mathrm{op}} \Big)^{\frac{1}{2}} \\ \lesssim\; & \mathfrak{e}_1^3(t) \left(\mu^{\beta_2} + N^{-\frac{1}{2}} \right) \left(\ln \mu^{-1} \right)^{\frac{1}{2}} \, , \\ (105) \;\lesssim\; & N \| \nabla_{x_1} q_1^\Phi \psi \| \| p_2^\Phi (\nabla_{x_2} \overline{\overline{h}}^{(12)}_{\varrho_{\widetilde{\beta}},\mu^{\beta_2}}) \widehat{l} q_1^\Phi q_2^\Phi \psi \| \;\lesssim\; \mathfrak{e}_1^2(t) \mu^{\beta_2} \ln \mu^{-1} \, , \end{aligned}$$

where we used for (104) the estimate (74) with $s_1 = q_1^\Phi$ and $\widetilde{\psi} = \widehat{l}_1 \psi$ and for (105) the estimate (76) and applied Lemma 5.6b. To estimate (106) and (107), we insert identities $\mathbb{1} = \mathbb{1}_{\mathcal{A}_1} + \mathbb{1}_{\overline{\mathcal{A}}_1}$ to be able to use Lemma 6.7:

$$\begin{aligned} (106) + (107) \;\leq\; & N \left| \left\langle\!\!\left\langle \nabla_{x_1} q_1^\Phi \psi, \mathbb{1}_{\overline{\mathcal{A}}_1} q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_2 p_1^{\chi^\varepsilon} q_1^\Phi \widehat{l}_1 \psi \right\rangle\!\!\right\rangle \right| & (109) \\ & + N \left| \left\langle\!\!\left\langle \nabla_{x_1} q_1^\Phi \psi, \mathbb{1}_{\overline{\mathcal{A}}_1} p_2 p_1^{\chi^\varepsilon} (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) \widehat{l} q_1^\Phi q_2^\Phi \psi \right\rangle\!\!\right\rangle \right| & (110) \\ & + N \left| \left\langle\!\!\left\langle \mathbb{1}_{\mathcal{A}_1} \nabla_{x_1} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_2 p_1^{\chi^\varepsilon} q_1^\Phi \widehat{l}_1 \psi \right\rangle\!\!\right\rangle \right| & (111) \\ & + N \left| \left\langle\!\!\left\langle \widehat{l} q_1^\Phi \psi, q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_2 p_1^{\chi^\varepsilon} \mathbb{1}_{\mathcal{A}_1} \nabla_{x_1} q_1^\Phi \psi \right\rangle\!\!\right\rangle \right| \, . & (112) \end{aligned}$$

By Lemma 6.6b, we find for $\widetilde{\psi} \in L^2(\mathbb{R}^{3N})$ and with $x = (x^{(1)}, x^{(2)})$

$$\begin{aligned} & \| \mathbb{1}_{\overline{\mathcal{A}}_1} q_2^\Phi (\nabla_{x_1} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_2^\Phi \widetilde{\psi} \|^2 \\ & \quad = \| \mathbb{1}_{\overline{\mathcal{A}}_1} q_2^\Phi (\partial_{x_1^{(1)}} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_2^\Phi \widetilde{\psi} \|^2 + \| \mathbb{1}_{\overline{\mathcal{A}}_1} q_2^\Phi (\partial_{x_1^{(2)}} \overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}) p_2^\Phi \widetilde{\psi} \|^2 \end{aligned}$$

$$\lesssim \mu^{d-\frac{1}{3}}\Big(\|(\partial^2_{x_1^{(1)}}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1})p_2^\Phi\widetilde{\psi}\|^2+\|(\partial^2_{x_1^{(2)}}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1})p_2^\Phi\widetilde{\psi}\|^2+\|(\partial_{x_1^{(1)}}\partial_{x_1^{(2)}}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1})p_2^\Phi\widetilde{\psi}\|^2$$
$$+\|(\partial_{x_1^{(1)}}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1})p_2^\Phi\nabla_{x_1}\widetilde{\psi}\|^2+\|(\partial_{x_1^{(2)}}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1})p_2^\Phi\nabla_{x_1}\widetilde{\psi}\|^2$$
$$+\varepsilon^2\|(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}p_2^\Phi)\partial_{y_1}\widetilde{\psi}\|^2\Big),$$

and analogously for the respective expression in (110). Note that for $i,j\in\{1,2\}$ and $F\in L^2(\mathbb{R}^2)$ with Fourier transform $\widehat{F}(k)$, it holds that $\|\partial_{x^{(j)}}F\|^2_{L^2(\mathbb{R}^2)}\leq\|\nabla_x F\|^2_{L^2(\mathbb{R}^2)}$ and that

$$\|\partial_{x^{(i)}}\partial_{x^{(j)}}F\|^2_{L^2(\mathbb{R}^2)}=\|k^{(i)}k^{(j)}\widehat{F}\|^2_{L^2(\mathbb{R}^2)}\lesssim\|((k^{(1)})^2+(k^{(2)})^2)\widehat{F}\|^2_{L^2(\mathbb{R}^2)}=\|\Delta_x F\|^2_{L^2(\mathbb{R}^2)}.$$

Hence, we conclude with Lemma 4.8d that

$$\begin{aligned}(109)+(110)\;&\lesssim\; N\|\nabla_{x_1}q_1^\Phi\psi\|\mu^{d-\frac{1}{3}}\mathfrak{e}_1(t)\Big(\|\Delta_x\overline{\overline{h}}_{\mu^{\beta_2},1}\|_{L^2(\mathbb{R}^2)}\|\widehat{l}q_1^\Phi\psi\|\\&\quad+\|\nabla_x\overline{\overline{h}}_{\mu^{\beta_2},1}\|_{L^2(\mathbb{R}^2)}\|\nabla_{x_1}\widehat{l}q_1^\Phi\psi\|+\varepsilon\|\nabla_x\overline{\overline{h}}_{\mu^{\beta_2},1}\|_{L^2(\mathbb{R}^2)}\|\partial_{y_1}p_1^{\chi^\varepsilon}\|_{\mathrm{op}}\|\widehat{l}q_1^\Phi\psi\|\Big)\\&\lesssim\;\mathfrak{e}_1^3(t)\left(\mu^{d-\beta_2-\frac{1}{3}}+N^\xi\mu^{d-\frac{1}{3}}(\ln\mu^{-1})^{\frac{1}{2}}\right),\end{aligned}$$

which follows because $\Delta_x\overline{\overline{h}}_{\mu^{\beta_2},1}=\overline{\overline{v}}_{\mu^{\beta_2}}-\overline{\overline{\nu}}_1$. For the next two lines, note that $\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi$ is symmetric in$\{2,...,N\}$, hence we can apply Lemma 4.3a. Similarly to the estimate that led to (74), integrating by parts twice yields

$$\begin{aligned}(111)\;\lesssim\; N\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|\Big(&\|\widehat{l}_1q_1^\Phi q_2^\Phi\psi\|^2\|\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\nabla_{x_2}p_2^\Phi\|^2_{\mathrm{op}}+\|p_2\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\nabla_{x_2}\widehat{l}_1q_1^\Phi q_2^\Phi\psi\|^2\\&+N^{-1}\|(\nabla_{x_1}\overline{\overline{h}}_{\mu^{\beta_2},1})p_2\|^2_{\mathrm{op}}\Big)^{\frac{1}{2}}.\end{aligned}$$

Further, proceeding as in (76), we find

$$(112)\;\lesssim\; N\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|\left(\|\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\nabla_{x_1}p_1^\Phi\|_{\mathrm{op}}\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|+\|p_1\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\nabla_{x_1}\widehat{l}q_1^\Phi q_2^\Phi\psi\|\right).$$

By Lemmas 4.2d, 5.6b and 6.6b, we obtain for $j\in\{0,1\}$

$$\begin{aligned}&\|p_1\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\widehat{l}_jq_2^\Phi\nabla_{x_1}q_1^\Phi\psi\|^2\\&\lesssim\;\|p_1\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\widehat{l}_jq_2^\Phi\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2+\left|\left\langle\!\left\langle\nabla_{x_1}q_1^\Phi\psi,\mathbb{1}_{\overline{\mathcal{A}}_1}\widehat{l}_jq_2^\Phi\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}p_1\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\widehat{l}_jq_2^\Phi\nabla_{x_1}q_1^\Phi\psi\right\rangle\!\right\rangle\right|\\&\quad+\left|\left\langle\!\left\langle\nabla_{x_1}q_1^\Phi\psi,\mathbb{1}_{\overline{\mathcal{A}}_1}\widehat{l}_jq_2^\Phi\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}p_1\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\widehat{l}_jq_2^\Phi\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\right\rangle\!\right\rangle\right|\\&\lesssim\;\|p_1\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\|^2_{\mathrm{op}}\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2\\&\quad+\|\nabla_{x_1}q_1^\Phi\psi\|^2\mu^{d-\frac{1}{3}}\|\widehat{l}\|_{\mathrm{op}}\Big(\|(\nabla_{x_1}\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1})p_1^\Phi\|_{\mathrm{op}}+\|\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}\nabla_{x_1}p_1^\Phi\|_{\mathrm{op}}\\&\qquad+\varepsilon\|\partial_{y_1}p_1^{\chi^\varepsilon}\|_{\mathrm{op}}\|\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}p_1^\Phi\|_{\mathrm{op}}\Big)\|\overline{\overline{h}}^{(12)}_{\mu^{\beta_2},1}p_1^\Phi\|_{\mathrm{op}}\\&\lesssim\;\mathfrak{e}_1^2(t)N^{-2}\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2+\mathfrak{e}_1^4(t)N^{-2+\xi}\mu^{d-\frac{1}{3}}(\ln\mu^{-1})^{\frac{1}{2}}.\end{aligned}$$

Combining these estimates, we conclude with Lemma 6.7

$$\begin{aligned}
&(111)+(112)\\
&\quad\lesssim \mathfrak{e}_1(t)\left(\|\mathbb{1}_{\mathcal{A}_1}\nabla_{x_1}q_1^\Phi\psi\|^2+\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle+N^{-1}\ln\mu^{-1}\right)+\mathfrak{e}_1^3(t)N^\xi\mu^{d-\frac13}(\ln\mu^{-1})^{\frac12}\\
&\quad\lesssim \mathfrak{e}_1^3(t)\alpha^<_{\xi,w_\mu}(t)+\mathfrak{e}_1^4(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}}+\left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\beta\gamma^2}}+\mu^{\frac{1-\widetilde{\beta}}{2}}+N^{-d+\frac56}\right)
\end{aligned}$$

Finally,

$$(108)\;\lesssim\; N\|\widehat{l}q_1^\Phi q_2^\Phi\psi\|\|q_1^\Phi\psi\|\|\overline{\overline{\nu}}_1^{(12)}p_2^\Phi\|_{\mathrm{op}}\lesssim\mathfrak{e}_1(t)\,\langle\!\langle\psi,\widehat{n}\psi\rangle\!\rangle$$

by Lemmas 4.2, 4.8d and by Definition 5.3 of $\overline{\overline{\mathcal{V}}}_1$. With the choice $\beta_2=\frac{3d-1}{6}>\frac{1-\widetilde{\beta}}{2}$, all estimates together yield

$$|(31)|\lesssim\mathfrak{e}_1^3(t)\alpha^<_{\xi,w_\mu}+\mathfrak{e}_1^4(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}}+\left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\beta\gamma^2}}+\mu^{\frac{1-\widetilde{\beta}}{2}}+N^{-d+\frac56}\right).$$

In combination with the remaining bounds from Proposition 3.6, evaluated for $\widetilde{\beta}$, $\eta=(1-\widetilde{\beta})^-$ and $\delta=\vartheta\widetilde{\beta}$, we obtain

$$|\gamma^<(t)|\;\lesssim\;\mathfrak{e}_1^3(t)\alpha^<_{\xi,w_\mu}+\mathfrak{e}_1^4(t)\left(\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{\frac{\widetilde{\beta}}{2}}+\left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{1}{\beta\gamma^2}}+\varepsilon^{\frac{1-\widetilde{\beta}}{2}}+N^{-d+\frac56}\right).$$

6.6.2 Estimate of the remainders $\gamma_a(t)$ to $\gamma_f(t)$

The estimates of $\gamma_a(t)$, $\gamma_b(t)$ as well as the bounds for $\gamma_d(t)$ to $\gamma_f(t)$ work mostly analogously to the respective estimates in [5, Section 4.5], hence we merely sketch the main steps for completeness.

Recalling that $\widehat{r}:=\widehat{m}^bp_1p_2+\widehat{m}^a(p_1q_2+q_1p_2)$, one concludes with Lemmas 4.10, 6.2b and 4.2b that

$$\begin{aligned}
|\gamma_a(t)|&\lesssim N^3\|(V^\|(t,z_1)-V^\|(t,(x_1,0)))\psi\|\|g^{(12)}_{\widetilde{\beta}}p_1\|_{\mathrm{op}}\left(\|\widehat{m}^a\|_{\mathrm{op}}+\|\widehat{m}^b\|_{\mathrm{op}}\right)\\
&\lesssim \mathfrak{e}_1^4(t)N^{1+\xi-\frac{\widetilde{\beta}}{2}}\varepsilon^{\frac{3+\widetilde{\beta}}{2}}<\mathfrak{e}_1^4(t)\left(\tfrac{\varepsilon^\vartheta}{\mu}\right)^{1+\xi-\frac{\widetilde{\beta}}{2}}
\end{aligned}$$

since $\widetilde{\beta}>\frac56$, $\xi<\frac{1}{12}$ and $\vartheta\le3$. To estimate $\gamma_b(t)$, note first that $b_{\widetilde{\beta}}=b(U_{\mu,\widetilde{\beta}}f_{\widetilde{\beta}})=b_1$ by (86), hence $(47)=0$. The two remaining terms can be controlled as

$$\begin{aligned}
|(46)|&\lesssim N\|\Phi\|^2_{L^\infty(\mathbb{R})}\|g^{(12)}_{\widetilde{\beta}}p_1\|_{\mathrm{op}}\left(\|\widehat{m}^a\|_{\mathrm{op}}+\|\widehat{m}^b\|_{\mathrm{op}}\right)\\
&\lesssim \mathfrak{e}_1^3(t)N^{-1-\frac{\widetilde{\beta}}{2}+\xi}\varepsilon^{\frac{1+\widetilde{\beta}}{2}}\;<\;\mathfrak{e}_1^3(t)\varepsilon^{\frac{1+\widetilde{\beta}}{2}},\\
|(48)|&\lesssim N^2\|p_1g^{(12)}_{\widetilde{\beta}}\|_{\mathrm{op}}\left(\|\widehat{m}^a\|_{\mathrm{op}}+\|\widehat{m}^b\|_{\mathrm{op}}\right)\|p_1\left(w^{(12)}_\mu-\tfrac{b_1}{N-1}(|\Phi(x_1)|^2+|\Phi(x_2)|^2)\right)\psi\|\\
&\lesssim \mathfrak{e}_1^3(t)N^{-1-\frac{\widetilde{\beta}}{2}+\xi}\varepsilon^{\frac{1+\widetilde{\beta}}{2}}\;<\;\mathfrak{e}_1^3(t)\varepsilon^{\frac{1+\widetilde{\beta}}{2}}
\end{aligned}$$

as a consequence of Lemmas 4.2b, 4.7a, 4.9e and 6.2b. The first term of $\gamma_d(t)$ yields

$$|(50)|\;\lesssim\;N^3\|\mathbb{1}_{\mathrm{supp}\,g_{\widetilde{\beta}}}(z_1-z_2)\psi\|\|g^{(12)}_{\widetilde{\beta}}p_1\|_{\mathrm{op}}\|\Phi\|^2_{L^\infty(\mathbb{R})}\left(\|\widehat{m}^a\|_{\mathrm{op}}+\|\widehat{m}^b\|_{\mathrm{op}}\right)$$

$$\lesssim \quad \mathfrak{e}_1^4(t) N^{1+\xi-\frac{3\widetilde{\beta}}{2}} \varepsilon^{\frac{3\widetilde{\beta}}{2}+\frac{1}{6}} < \mathfrak{e}_1^4(t)\varepsilon$$

since $\widetilde{\beta} > \frac{5}{6}$ and $\xi < \frac{1}{12}$. For the second term of $\gamma_d(t)$, we write $\widehat{r} = \widehat{m}^a(p_1+p_2) + (\widehat{m}^b - 2\widehat{m}^a)p_1p_2$, apply Lemma 4.3c with $\widehat{m}^c$ and $\widehat{m}^d$ from Definition 3.2, and observe that $g_{\widetilde{\beta}}^{(12)} w_\mu^{(13)} \neq 0$ implies $|z_2 - z_3| \leq 2\varrho_{\widetilde{\beta}}$ because $|z_1 - z_2| \leq \varrho_{\widetilde{\beta}}$ for $z_1 - z_2 \in \operatorname{supp} g_{\widetilde{\beta}}$ and $|z_1 - z_3| \leq \mu$ for $z_1 - z_3 \in \operatorname{supp} w_\mu$. This leads to

$$\begin{aligned}
|(51)| \quad &\lesssim \quad N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_2 \left[\mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3) w_\mu^{(13)}, p_1p_3\widehat{m}^d + (p_1q_3 + q_1p_3)\widehat{m}^c \right] \psi \right\rangle\!\!\right\rangle \right| \\
&\quad + N^3 \left| \left\langle\!\!\left\langle p_1 \mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3) g_{\widetilde{\beta}}^{(12)} w_\mu^{(13)} \psi, \mathbb{1}_{B_{2\varrho_{\widetilde{\beta}}}(0)}(z_2 - z_3)\widehat{m}^a \psi \right\rangle\!\!\right\rangle \right| \\
&\quad + N^3 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_1(\widehat{m}^a + p_2(\widehat{m}^b - 2\widehat{m}^a))p_1 w_\mu^{(13)} \psi \right\rangle\!\!\right\rangle \right| \\
&\quad + N^3 \left| \left\langle\!\!\left\langle w_\mu^{(13)} \psi, g_{\widetilde{\beta}}^{(12)} p_2 \mathbb{1}_{\operatorname{supp} w_\mu}(z_1 - z_3) p_1(\widehat{m}^b - 2\widehat{m}^a)\psi \right\rangle\!\!\right\rangle \right| \\
&\lesssim \quad \mathfrak{e}_1^3(t) \left(N^{-1-\frac{\widetilde{\beta}}{2}+3\xi} \varepsilon^{\frac{1+\widetilde{\beta}}{2}} + N^{1+\xi-\widetilde{\beta}} \varepsilon^{\widetilde{\beta}-\frac{1}{3}} + N^{-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{\frac{1+\widetilde{\beta}}{2}} \right) \\
&< \quad \mathfrak{e}_1^3(t) \left(\left(\frac{\varepsilon^\vartheta}{\mu} \right)^{1+\xi-\widetilde{\beta}} + \varepsilon^{\frac{1+\widetilde{\beta}}{2}} \right)
\end{aligned}$$

since $\widetilde{\beta} > \frac{5}{6}$ and $\xi < \frac{1}{12}$ and where we have estimated $\|\mathbb{1}_{B_{2\varrho_{\widetilde{\beta}}}(0)}(z_2 - z_3)\widehat{m}^a\psi\|^2$ analogously to Lemma 6.2e. Using Lemma 4.3c, the relation

$$\begin{aligned}
&p_3p_4(\widehat{r} - \widehat{r}_2) + (p_3q_4 + q_3p_4)(\widehat{r} - \widehat{r}_1) \\
&\quad = \quad (p_1q_2 + q_1p_2)(p_3q_4 + q_3p_4)\widehat{m}^c + (p_1q_2 + q_1p_2)p_3p_4\widehat{m}^d \\
&\qquad\quad + p_1p_2(p_3q_4 + q_3p_4)\widehat{m}^e + p_1p_2p_3p_4\widehat{m}^f \,,
\end{aligned}$$

and the symmetry of ψ, we obtain

$$\begin{aligned}
|\gamma_e(t)| \quad &\lesssim \quad N^4 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_1q_2 \left[w_\mu^{(34)}, p_3q_4\widehat{m}^c + p_3p_4\widehat{m}^d \right] \psi \right\rangle\!\!\right\rangle \right| \\
&\quad + N^4 \left| \left\langle\!\!\left\langle \psi, g_{\widetilde{\beta}}^{(12)} p_1p_2 \left[w_\mu^{(34)}, p_3q_4\widehat{m}^e + p_3p_4\widehat{m}^f \right] \psi \right\rangle\!\!\right\rangle \right| \\
&\lesssim \quad N^4 \|p_3 w_\mu^{(34)}\psi\| \|g_{\widetilde{\beta}}^{(12)} p_1\|_{\mathrm{op}} \left(\|\widehat{m}^c\|_{\mathrm{op}} + \|\widehat{m}^d\|_{\mathrm{op}} \right) \\
&\lesssim \quad \mathfrak{e}_1^3(t) N^{-\frac{\widetilde{\beta}}{2}+3\xi} \varepsilon^{\frac{1+\widetilde{\beta}}{2}} \; < \; \mathfrak{e}_1^3(t) \varepsilon^{\frac{1+\widetilde{\beta}}{2}}
\end{aligned}$$

by Lemmas 4.9e, 6.2b and Lemma 4.2b. Finally,

$$|\gamma_f(t)| \quad \lesssim \quad N^2 \mathfrak{e}_1^2(t) \|p_2 g_{\widetilde{\beta}}^{(12)}\|_{\mathrm{op}} \left(\|\widehat{m}^a\|_{\mathrm{op}} + \|\widehat{m}^b\|_{\mathrm{op}} \right) \lesssim \mathfrak{e}_1^3(t) N^{-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{\frac{1+\widetilde{\beta}}{2}} \; < \; \mathfrak{e}_1^3(t) \varepsilon^{\frac{1+\widetilde{\beta}}{2}} \,.$$

The last remaining term left to estimate is $\gamma_c(t)$, where we follow a different path than in [5]: we decompose the scalar product of the gradients into its x- and y-component and subsequently integrate by parts, making use of the fact that $\nabla_{x_1} g_{\widetilde{\beta}}^{(12)} = -\nabla_{x_2} g_{\widetilde{\beta}}^{(12)}$

and analogously for y. Taking the maximum over $s_2 \in \{p_2, q_2\}$ and $\widehat{l} \in \mathcal{L}$ from (19), this results in

$$|\gamma_c(t)| \lesssim N\left|\left\langle\!\!\left\langle\psi, (\nabla_{x_1} g_{\widetilde{\beta}}^{(12)}) \cdot \nabla_{x_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle\psi, (\nabla_{x_2} g_{\widetilde{\beta}}^{(12)}) p_2 \cdot \nabla_{x_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| \quad (113)$$

$$+N\left|\left\langle\!\!\left\langle p_2^{\chi^\varepsilon}\psi, (\partial_{y_2} g_{\widetilde{\beta}}^{(12)}) \partial_{y_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle p_2^{\chi^\varepsilon}\psi, (\partial_{y_2} g_{\widetilde{\beta}}^{(12)}) p_2 \partial_{y_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| \quad (114)$$

$$+N\left|\left\langle\!\!\left\langle q_2^{\chi^\varepsilon}\psi, (\partial_{y_2} g_{\widetilde{\beta}}^{(12)}) \partial_{y_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle q_2^{\chi^\varepsilon}\psi, (\partial_{y_2} g_{\widetilde{\beta}}^{(12)}) p_2 \partial_{y_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| . \quad (115)$$

With Lemmas 4.2b, 4.8, 4.9a and 6.2, the first line is easily estimated as

$$\begin{aligned}
(113) \lesssim\ & N\left|\left\langle\!\!\left\langle\nabla_{x_1}\psi, g_{\widetilde{\beta}}^{(12)} \nabla_{x_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle\nabla_{x_2}\psi, g_{\widetilde{\beta}}^{(12)} \nabla_{x_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| \\
& +N\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)} \Delta_{x_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle\psi, g_{\widetilde{\beta}}^{(12)} \nabla_{x_2} p_2 \nabla_{x_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| \\
\lesssim\ & \mathfrak{e}_1^3(t) N^{-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{\frac{1+\widetilde{\beta}}{2}} < \mathfrak{e}_1^3(t)\varepsilon^{\frac{1+\widetilde{\beta}}{2}} .
\end{aligned}$$

For the second line, we conclude with Lemma 6.2f that for any fixed $p \in (1, \infty)$,

$$\begin{aligned}
(114) \lesssim\ & N\left|\left\langle\!\!\left\langle\partial_{y_2} p_2^{\chi^\varepsilon} \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}(\cdot, y_1-y_2)}(x_1-x_2)\psi, g_{\widetilde{\beta}}^{(12)} \partial_{y_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| \\
& +N\left|\left\langle\!\!\left\langle\partial_{y_2} p_2^{\chi^\varepsilon} \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}(\cdot, y_1-y_2)}(x_1-x_2)\psi, g_{\widetilde{\beta}}^{(12)} p_2 \partial_{y_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| \\
& +N\left|\left\langle\!\!\left\langle p_2^{\chi^\varepsilon} \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}(\cdot, y_1-y_2)}(x_1-x_2)\psi, g_{\widetilde{\beta}}^{(12)} \partial_{y_1} p_1 \partial_{y_2} \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| \\
& +N\left|\left\langle\!\!\left\langle p_2^{\chi^\varepsilon} \mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}(\cdot, y_1-y_2)}(x_1-x_2)\psi, g_{\widetilde{\beta}}^{(12)} \partial_{y_2} p_2 \partial_{y_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| \\
& \lesssim N^{1+\xi}\varepsilon^{-1} \|\mathbb{1}_{\operatorname{supp} g_{\widetilde{\beta}}(\cdot, y_1-y_2)}(x_1-x_2)\psi\| \left(\|g_{\widetilde{\beta}}^{(12)} \partial_{y_1} p_1\|_{\mathrm{op}} + \|g_{\widetilde{\beta}}^{(12)} p_1\|_{\mathrm{op}} \varepsilon^{-1}\right) \\
& \lesssim \mathfrak{e}_1^2(t) N^{\xi - \frac{3\widetilde{\beta}}{2} + \frac{\widetilde{\beta}}{p}} \varepsilon^{-\frac{3}{2} + \frac{3\widetilde{\beta}}{2} - \frac{\widetilde{\beta}}{p}} .
\end{aligned}$$

With the choice $p = \frac{\gamma+1}{\gamma-1}$, we obtain

$$\begin{aligned}
N^{\xi - \frac{3\widetilde{\beta}}{2} + \frac{\widetilde{\beta}}{p}} \varepsilon^{-\frac{3}{2} + \frac{3\widetilde{\beta}}{2} - \frac{\widetilde{\beta}}{p}} &= \left(N^{-1}\varepsilon^{1-\gamma}\right)^{\frac{3\widetilde{\beta}}{2} - \xi - \frac{\widetilde{\beta}}{p}} \varepsilon^{\gamma\widetilde{\beta}(\frac{3}{2} - \frac{\gamma-1}{\gamma+1}) - \frac{3}{2} - \xi(\gamma-1)} \\
&\leq \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{\widetilde{\beta}}{2} - \xi} \varepsilon^{(\gamma-1)(\frac{1}{4} - \xi)} < \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{\widetilde{\beta}}{2} - \xi}
\end{aligned}$$

since $\widetilde{\beta} > \frac{\gamma+1}{2\gamma}$ and $\xi < \frac{1}{4}$. Finally, the last line yields

$$\begin{aligned}
(115) \lesssim\ & N\left|\left\langle\!\!\left\langle\partial_{y_2} q_2^{\chi^\varepsilon}\psi, g_{\widetilde{\beta}}^{(12)} \partial_{y_1} p_1 \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle q_2^{\chi^\varepsilon}\psi, g_{\widetilde{\beta}}^{(12)} \partial_{y_1} p_1 \partial_{y_2} \widehat{l} s_2 \psi\right\rangle\!\!\right\rangle\right| \\
& +N\left|\left\langle\!\!\left\langle\partial_{y_2} q_2^{\chi^\varepsilon}\psi, g_{\widetilde{\beta}}^{(12)} p_2 \partial_{y_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| + N\left|\left\langle\!\!\left\langle q_2^{\chi^\varepsilon}\psi, g_{\widetilde{\beta}}^{(12)} \partial_{y_2} p_2 \partial_{y_1} \widehat{l} q_1 \psi\right\rangle\!\!\right\rangle\right| \\
\lesssim\ & \mathfrak{e}_1^2(t) N^{-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{-\frac{1-\widetilde{\beta}}{2}} < \left(\tfrac{\mu}{\varepsilon^\gamma}\right)^{\frac{\widetilde{\beta}}{2} - \xi} ,
\end{aligned}$$

where the last inequality follows because

$$N^{-\frac{\widetilde{\beta}}{2}+\xi} \varepsilon^{-\frac{1-\widetilde{\beta}}{2}} = \left(N^{-1}\varepsilon^{1-\gamma}\right)^{\frac{\widetilde{\beta}}{2} - \xi} \varepsilon^{\frac{\gamma\widetilde{\beta}}{2} - \frac{1}{2} - \xi(\gamma-1)} < \left(N^{-1}\varepsilon^{1-\gamma}\right)^{\frac{\widetilde{\beta}}{2} - \xi}$$

as $\widetilde{\beta} > \frac{\gamma+1}{2\gamma}$ and $\xi < \frac{1}{4}$.

Acknowledgments

I thank Stefan Teufel for helpful remarks and for his involvement in the closely related joint project [5]. Helpful discussions with Serena Cenatiempo and Nikolai Leopold are gratefully acknowledged. This work was supported by the German Research Foundation within the Research Training Group 1838 "Spectral Theory and Dynamics of Quantum Systems".

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B.2. Higher order corrections to the mean-field description of the dynamics of interacting bosons

Higher order corrections to the mean-field description of the dynamics of interacting bosons

Lea Boßmann[*], Nataša Pavlović[†], Peter Pickl[‡], and Avy Soffer[§]

Abstract

In this paper, we introduce a novel method for deriving higher order corrections to the mean-field description of the dynamics of interacting bosons. More precisely, we consider the dynamics of N d-dimensional bosons for large N. The bosons initially form a Bose–Einstein condensate and interact with each other via a pair potential of the form $(N-1)^{-1}N^{d\beta}v(N^\beta\cdot)$ for $\beta\in[0,\frac{1}{4d})$. We derive a sequence of N-body functions which approximate the true many-body dynamics in $L^2(\mathbb{R}^{dN})$-norm to arbitrary precision in powers of N^{-1}. The approximating functions are constructed as Duhamel expansions of finite order in terms of the first quantised analogue of a Bogoliubov time evolution.

1 Introduction

We consider a system of N bosons in $\mathbb{R}^d$, $d\geq 1$, interacting with each other via pair interactions in the mean field scaling regime. The Hamiltonian of the system is given by

$$H^\beta(t) := \sum_{j=1}^N \left(-\Delta_j + V^{\mathrm{ext}}(t,x_j)\right) + \frac{1}{N-1}\sum_{i<j} v^\beta(x_i-x_j)\,. \tag{1}$$

Here, V^{ext} denotes some possibly time-dependent external potential, and the interaction potential v^β is defined as

$$v^\beta(x) := N^{d\beta}v(N^\beta x)\,, \qquad \beta\in[0,\tfrac{1}{d})\,, \tag{2}$$

for some bounded, spherically symmetric and compactly supported function $v:\mathbb{R}^d\to\mathbb{R}$. In the following, we will make use of the abbreviation

$$v_{ij}^\beta := v^\beta(x_i-x_j)\,.$$

[*]Fachbereich Mathematik, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. lea.bossmann@uni-tuebingen.de

[†]Department of Mathematics, The University of Texas at Austin, 2515 Speedway Stop C1200, Austin, TX 78712, USA. natasa@math.utexas.edu

[‡]Mathematisches Institut, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany; and Duke Kunshan University, No. 8 Duke Avenue, Kunshan, Jiangsu Province, 215316, China. pickl@math.lmu.de

[§]Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA. soffer@math.rugers.edu

Note that the prefactor $(N-1)^{-1}$ in front of v^β is chosen such that the interaction energy and the kinetic energy per particle are of the same order. The mean inter-particle distance is of order $N^{-\frac{1}{d}}$ and therefore much smaller than the range of the interaction, which scales as $N^{-\beta}$. Hence, on average, every particle interacts with many other particles, and the interactions are weak since $(N-1)^{-1}N^{d\beta} \to 0$ as $N \to \infty$. This implies that we consider a mean-field regime. In particular, the case $\beta = 0$ is known as the Hartree scaling regime.

In this paper, we study the time evolution of the N-body system for large N when the bosons initially exhibit Bose–Einstein condensation. We impose suitable conditions on the external potential $V^{\text{ext}}(t)$ such that $H^\beta(t)$ is self-adjoint on $\mathcal{D}(H^\beta(t)) = H^2(\mathbb{R}^{dN})$ for each $t \in \mathbb{R}$. Consequently, $H^\beta(t)$ generates a unique family of unitary time evolution operators $\{U(t,s)\}_{t,s\in\mathbb{R}}$ via the Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}U(t,s) = H^\beta(t)U(t,s)\,, \qquad U(s,s) = \mathbb{1}\,. \tag{3}$$

The N-body wave function at time $t \in \mathbb{R}$ is determined by

$$\psi(t) = U(t,0)\psi(0) \tag{4}$$

for some initial datum $\psi(0) = \psi_0 \in L^2_{\text{sym}}(\mathbb{R}^{dN})$. Due to the interactions, the characterisation of the time evolution $U(t,s)$ is a difficult problem. Even if the system was initially in a factorised state, where all particles are independent of each other, the interactions instantaneously correlate the particles such that an explicit formula for $U(t,s)$ is quite inaccessible.

To describe $U(t,s)$ approximatively, one observes that the dynamics of the many-body system can be decomposed into

- the dynamics of the condensate wave function $\varphi(t) \in L^2(\mathbb{R}^d)$, and
- the dynamics of the fluctuations around the (time-evolved) condensate.

More precisely, the N-body wave function $\psi(t)$ can be written as

$$\psi(t) = \sum_{k=0}^{N} \varphi(t)^{\otimes(N-k)} \otimes_s \xi^{(k)}_{\varphi(t)} \tag{5}$$

for some $\xi_{\varphi(t)} = \big(\xi^{(k)}_{\varphi(t)}\big)_{k=0}^N \in \mathcal{F}^{\leq N}\left(\{\varphi(t)\}^\perp\right)$, where

$$\mathcal{F}^{\leq N}\left(\{\varphi\}^\perp\right) := \bigoplus_{k=0}^{N} \bigotimes_{\text{sym}}^{k} \{\varphi\}^\perp \tag{6}$$

is the truncated bosonic Fock space over the orthogonal complement in $L^2(\mathbb{R}^d)$ of the span of $\varphi \in L^2(\mathbb{R}^d)$. A definition of $\xi^{(k)}_{\varphi(t)}$ will be given in (20). Further, $\otimes_s$ denotes the symmetric tensor product, which is for $\psi_a \in L^2(\mathbb{R}^{da})$, $\psi_b \in L^2(\mathbb{R}^{db})$ defined as

$$\begin{aligned}&(\psi_a \otimes_s \psi_b)(x_1, \dots, x_{a+b})\\ &\qquad := \frac{1}{\sqrt{a!\,b!\,(a+b)!}} \sum_{\sigma\in\mathfrak{S}_{a+b}} \psi_a(x_{\sigma(1)}, \dots, x_{\sigma(a)})\,\psi_b(x_{\sigma(a+1)}, \dots, x_{\sigma(a+b)})\,,\end{aligned}$$

where $\mathfrak{S}_{a+b}$ denotes the set of all permutations of $a+b$ elements. The addend $k=0$ in (5) describes the condensate, while the terms $k \in \{1, \dots, N\}$ correspond to the fluctuations. In the following, we will refer to $\xi^{(k)}_\varphi(t)$ as *k-particle fluctuation.*

1.1 A first order approximation to the N-body dynamics

A first approximation to the N-body dynamics is provided by the time evolution of the condensate wave function. Its dynamics yield a macroscopic description of the Bose gas, which, in the limit $N \to \infty$, coincides with the true dynamics in the sense of reduced density matrices. In order to formulate this mathematically, one assumes that the system is initially in a Bose–Einstein condensate with condensate wave function φ_0, i.e.,

$$\lim_{N\to\infty} \mathrm{Tr}\left|\gamma^{(1)}(0) - |\varphi_0\rangle\langle\varphi_0|\right| = 0\,,$$

where

$$\gamma^{(1)}(t) := \mathrm{Tr}_{2,\dots,N}|\psi(t)\rangle\langle\psi(t)|$$

is the one-particle reduced density matrix of $\psi(t)$ at time t. Then it has been shown, see e.g. [1, 2, 11, 13, 18, 19, 32, 53], that

$$\lim_{N\to\infty} \mathrm{Tr}\left|\gamma^{(1)}(t) - |\varphi(t)\rangle\langle\varphi(t)|\right| = 0 \tag{7}$$

for any $t \in \mathbb{R}$, where $\varphi(t)$ is the solution of the Hartree equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi(t) = \left(-\Delta + V^{\mathrm{ext}}(t) + \overline{v}^{\varphi(t)} - \mu^{\varphi(t)}\right)\varphi(t) =: h^{\varphi(t)}(t)\varphi(t) \tag{8}$$

with initial datum $\varphi(0) = \varphi_0$ and with

$$\overline{v}^{\varphi(t)}(x) := \left(v^\beta * |\varphi(t)|^2\right)(x) := \int_{\mathbb{R}^d} v^\beta(x-y)|\varphi(t,y)|^2\,\mathrm{d}y\,. \tag{9}$$

Note that for $\beta = 0$, the equation (8) is the N-independent Hartree (NLH) equation. For $\beta > 0$, the evolution is N-dependent and converges to the non-linear Schrödinger (NLS) dynamics with N-independent coupling parameter $\int v$ in the limit $N \to \infty$. The parameter $\mu^{\varphi(t)}$ is a real-valued phase factor, which we choose as

$$\mu^{\varphi(t)} := \tfrac{1}{2}\int_{\mathbb{R}^d} \mathrm{d}x|\varphi(t,x)|^2\,\overline{v}^{\varphi(t)}(x) \;=\; \tfrac{1}{2}\int_{\mathbb{R}^d}\mathrm{d}x\int_{\mathbb{R}^d}\mathrm{d}y\,|\varphi(t,x)|^2|\varphi(t,y)|^2 v^\beta(x-y) \tag{10}$$

for later convenience. For the convergence with respect to reduced densities, this phase is irrelevant since it cancels in the projection $|\varphi(t)\rangle\langle\varphi(t)|$.

One way to prove the convergence (7), and consequently to derive the NLH/ NLS equation from a system of N bosons, is via the so-called BBGKY[1] hierarchy, which was prominently used in the works of Lanford for the study of classical mechanical systems in the infinite particle limit [36, 37]. The first derivation of the NLH equation via the BBGKY hierarchy was given by Spohn in [54], and this was further pursued, e.g., in [1, 2, 20, 21]. About a decade ago, Erdős, Schlein and Yau fully developed the BBGKY hierarchy approach to the derivation of the NLH/NLS equation in their seminal works including [18, 19]. Subsequently, a crucial step of this method was revisited by Klainerman and Machedon in [33], based on reformulating combinatorial argument in [18, 19] and a viewpoint inspired by methods of non-linear PDEs. This, in turn, motivated many recent works on the derivation of dispersive PDEs, including [11, 12,

[1](Bogoliubov-Born-Green-Kirkwood-Yvon)

13, 14, 15, 32, 53]. In [52], Rodnianski and Schlein introduced yet another method for proving (7), which uses coherent states on Fock space and was inspired by techniques of quantum field theory and the pioneering work of Hepp [29].

In the context of the current paper, the most relevant works on the derivation of the NLH/NLS equation are due to Pickl [50, 51], who introduced an efficient method for deriving effective equations from the many-body dynamics, transforming the physical idea behind the mean-field description of an N-body system into a mathematical algorithm. Instead of describing the condensate as the vacuum of a Fock space of fluctuations, this approach remains in the N-body setting and uses projection operators to factor out the condensate. This strategy was successfully applied to prove effective dynamics for N-boson systems in various situations, e.g., [4, 8, 17, 30, 31, 34, 40, 41].

A much stronger notion of distance than the one expressed in (7) is provided by the $L^2(\mathbb{R}^{dN})$-norm. Whereas closeness in the sense of reduced densities implies that the majority of the particles (up to a relative number that vanishes as $N \to \infty$) is in the state $\varphi(t)$, the norm approximation requires the control of all N particles. In particular, this implies that the fluctuations around the condensate can no longer be omitted from the description. In this sense, the norm approximation of $\psi(t)$ can be understood as next-to-leading order correction to the mean-field description.

For the dynamics $U(t,s)$, a norm approximation in d dimensions was proven in [38] for $\beta = 0$ and $V^{\text{ext}} = 0$ under quite general assumptions on the interaction potential v. In [44], this result was extended to the range $\beta \in [0, \frac{1}{3})$ for the three-dimensional defocusing case, and in [45], the focusing case in dimensions one and two was treated for $\beta > 0$ and $\beta \in (0,1)$, respectively. In these works, the authors consider initial data of the form

$$\psi_0 = \sum_{k=0}^{N} {\varphi_0}^{\otimes(N-k)} \otimes_s \chi^{(k)}(0) \tag{11}$$

for some appropriate initial fluctuation vector $\chi(0) := (\chi^{(k)}(0))_{k=0}^{\infty} \in \mathcal{F}(\{\varphi_0\}^{\perp})$. It is then shown that there exist constants $C, C' > 0$ such that

$$\left\| \psi(t) - \sum_{k=0}^{N} \varphi(t)^{\otimes(N-k)} \otimes_s \chi^{(k)}(t) \right\|^2_{L^2(\mathbb{R}^{dN})} \le C \mathrm{e}^{C't} N^{-\delta}\,, \tag{12}$$

where $\delta = 1$ for $\beta = 0$, $\delta = 1 - 3\beta$ for the three-dimensional defocusing case with $\beta \in [0, \frac{1}{3})$, and $\delta = \frac{1}{2}$ and $\delta < \frac{1}{3}(1-\beta)$ for the one- and two-dimensional focusing case, respectively. The fluctuations $\chi(t) = (\chi^{(k)}(t))_{k=0}^{\infty} \in \mathcal{F}(\{\varphi(t)\}^{\perp})$ at time $t > 0$ are determined by the Bogoliubov evolution,

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\chi(t) = \mathbb{H}_{\text{Bog}}(t)\chi(t)\,. \tag{13}$$

Here, $\mathbb{H}_{\text{Bog}}(t)$ denotes the Bogoliubov Hamiltonian[2], an effective Hamiltonian in Fock space which is quadratic in the number of creation and annihilation operators.

[2] Written in second quantized form, $\mathbb{H}_{\text{Bog}}(t)$ is defined as

$$\mathbb{H}_{\text{Bog}}(t) := \int_{\mathbb{R}^d} a_x^* \left(h^{\varphi(t)}(t,x) + K_1(t) \right) a_x \, \mathrm{d}x + \int_{\mathbb{R}^d} \mathrm{d}x \int_{\mathbb{R}^d} \mathrm{d}y \left(K_2(t,x,y) a_x^* a_y^* + \overline{K_2(t,x,y)} a_x a_y \right),$$

where a_x^* and a_x denote the operator-valued distributions corresponding to the usual creation and annihilation operators on $\mathcal{F}(L^2(\mathbb{R}^d))$. Besides, $K_1(t) := Q(t)\widetilde{K}_1(t)Q(t)$ with $Q(t) := 1 - |\varphi(t)\rangle\langle\varphi(t)|$, where $\widetilde{K}_1$ is the Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$ with kernel $\widetilde{K}_1(t,x,y) := \varphi(t,x)v^\beta(x-y)\overline{\varphi(t,y)}$.

For three dimensions and scaling parameter $\beta = 0$, a similar result was obtained in [42, 43] via a first quantised approach. More precisely, denote

$$p_j^{\varphi(t)} := |\varphi(t, x_j)\rangle\langle\varphi(t, x_j)|$$

and

$$q_j^{\varphi(t)} := \mathbb{1} - p_j^{\varphi(t)}.$$

The auxiliary N-particle Hamiltonian $\widetilde{H}^{\varphi(t)}(t)$ is defined by subtracting from $H^\beta(t)$ in each coordinate the mean-field Hamiltonian $h^{\varphi(t)}(t)$ from (8), inserting identities

$$(p_i^{\varphi(t)} + q_i^{\varphi(t)})(p_j^{\varphi(t)} + q_j^{\varphi(t)})$$

on both sides of the difference, and discarding all terms which are cubic, $\mathcal{C}^{\varphi(t)}$, or quartic, $\mathcal{Q}^{\varphi(t)}$, in the number of projections $q^{\varphi(t)}$ (see Lemma 2.2). This yields

$$\widetilde{H}^{\varphi(t)}(t) := \sum_{j=1}^N h_j^{\varphi(t)}(t) + \tfrac{1}{N-1} \sum_{i<j} \left(p_i^{\varphi(t)} q_j^{\varphi(t)} v_{ij}^\beta q_i^{\varphi(t)} p_j^{\varphi(t)} + p_i^{\varphi(t)} p_j^{\varphi(t)} v_{ij}^\beta q_i^{\varphi(t)} q_j^{\varphi(t)} + \text{h.c.} \right), \tag{14}$$

which has a quadratic structure comparable to that of the Bogoliubov-Hamiltonian $\mathbb{H}_{\text{Bog}}(t)$: all terms in $H^\beta(t) - \sum_j h_j^{\varphi(t)}(t)$, which form an effective two-body potential, contain exactly two projectors $q^{\varphi(t)}$ onto the complement of the condensate wave function, while $\mathbb{H}_{\text{Bog}}(t)$ is quadratic in the creation and annihilation operators of the fluctuations. However, $\widetilde{H}^{\varphi(t)}(t)$ is particle number conserving and acts on the N-body Hilbert space $L^2(\mathbb{R}^{dN})$, i.e., it determines the evolution of both condensate wave function and fluctuations. In contrast, $\mathbb{H}_{\text{Bog}}(t)$ operates on the fluctuation Fock space $\mathcal{F}\left(\{\varphi(t)\}^\perp\right)$, does not conserve the particle number, and exclusively concerns the dynamics of the fluctuations with respect to the condensate wave function evolving according to (8).

Under appropriate assumptions on the initial datum ψ_0, the time evolution $\widetilde{U}_\varphi(t,s)$ generated by $\widetilde{H}^{\varphi(t)}(t)$ approximates the actual time evolution $U(t,s)$. More precisely, there exist constants $C, C' > 0$ such that

$$\left\| \left(U(t,0) - \widetilde{U}_\varphi(t,0) \right) \psi_0 \right\|^2_{L^2(\mathbb{R}^{dN})} \leq C \mathrm{e}^{C' t^2} N^{-1} \tag{15}$$

[42, Theorem 2.6]. Further, in the limit $N \to \infty$, the fluctuations in $\widetilde{U}_\varphi(t,0)\psi_0$ coincide with the solutions of the Bogoliubov evolution equation: let $\xi_{\varphi_0} = \left(\xi_{\varphi_0}^{(k)}\right)_{k=0}^N$ denote the fluctuations around $\varphi_0{}^{\otimes N}$ in the initial state ψ_0 under the decomposition (5), let $\widetilde{\xi}_{\varphi(t)} = \left(\widetilde{\xi}_{\varphi(t)}^{(k)}\right)_{k=0}^N$ denote the fluctuations around $\varphi(t)^{\otimes N}$ in $\widetilde{U}_\varphi(t,0)\psi_0$, and let $\chi(t) = \left(\chi^{(k)}(t)\right)_{k\geq 0}$ denote the solutions of (13) with initial datum ξ_{φ_0} for $0 \leq k \leq N$ and $\xi_{\varphi_0}^{(k)} = 0$ for $k > N$. Then

$$\sum_{k=0}^N \left\| \widetilde{\xi}_{\varphi(t)}^{(k)} - \chi^{(k)}(t) \right\|^2_{L^2(\mathbb{R}^{dk})} \leq C \mathrm{e}^{C' t^2} N^{-1} \tag{16}$$

Further, $K_2(t) := (Q(t) \otimes Q(t))\, \widetilde{K}_2(t)$, where the kernel of the two-body function $\widetilde{K}_2(t)$ is given by $\widetilde{K}_2(t,x,y) := \varphi(t,x) v^\beta(x-y) \varphi(t,y)$ (e.g. [44, Equation (31)]).

[42, Lemma 2.8]. Hence, the combination of (15) and (16) yields (12), with a different time-dependent constant but the same N-dependence.

Beyond the mean field regime, a statement similar to (12) was shown in [46] for the range $\beta \in [0, \frac{1}{2})$. For larger values of the scaling parameter, the evolutions of $\varphi(t)$ and $\xi_{\varphi(t)}$ do not (approximately) decouple any more as a consequence of the short-scale structure related to the two-body scattering process. For $\beta \in (0,1)$, an accordingly adjusted variant of (12) for appropriately modified initial data was obtained in [9] in the three-dimensional defocusing case. Similar estimates for the many-body evolution of appropriate classes of Fock space initial data have been obtained in [6, 16, 22, 23, 25, 26, 27, 28, 35, 52] for various ranges of the scaling parameter. A related result for Bose gases with large volume and large density was proved in [49].

1.2 Higher order approximations to the N-body dynamics

In this paper, we introduce a novel method for deriving a more precise characterisation of the dynamics, which approximates the N-body wave function to arbitrary order in powers of N^{-1}. This is achieved by constructing a sequence of N-body wave functions, which are defined via an iteration of Duhamel's formula with the time evolution $\widetilde{U}_\varphi(t,s)$ generated by $\widetilde{H}^{\varphi(t)}(t)$. We work in the first quantized framework as was the case, e.g., in [42].

- In our first result, we estimate the growth of the first A moments of the number of fluctuations when the system evolves under the dynamics $U(t,s)$ or $\widetilde{U}_\varphi(t,s)$. Estimates of this kind are often needed to derive effective descriptions of the dynamics of interacting bosons, e.g., in [5, 6, 10, 42, 49, 52]. Our proof extends comparable statements for $\beta = 0$ and $d = 3$ obtained in [42, Lemma 2.1] and [52, Proposition 3.3], and for Bose gases with large volume and large density in [49, Corollary 4.2]. The estimate is given in Proposition 2.4 and holds for $\beta \in [0, \frac{1}{2d})$ in case of the dynamics $U(t,s)$, and for the full mean-field range $\beta \in [0, \frac{1}{d})$ in case of the dynamics $\widetilde{U}_\varphi(t,s)$.

In the remainder of the paper, we assume that for some $A \in \{1, ..., N\}$, the first A moments of the number of fluctuations in the initial state are sub-leading (see Assumption A3). More precisely, let $\gamma \in (0,1]$. We assume that for all $a \in \{0, ..., A\}$, there exists some constant $C(a)$ depending only on a such that

$$\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \xi_{\varphi_0} \right\rangle \leq C(a) N^{(1-\gamma)a} . \tag{17}$$

Here, ξ_{φ_0} denotes the fluctuation vector corresponding to the initial state ψ_0 as in (5), and $\mathcal{N}_{\varphi_0}$ is the number operator on the Fock space $\mathcal{F}^{\leq N}\left(\{\varphi_0\}^\perp\right)$ of fluctuations around ${\varphi_0}^{\otimes N}$. Note that $\gamma = 0$ corresponds to the trivial bound $\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \xi_{\varphi_0} \right\rangle \leq N^a$. In this sense, our assumption states that the expected number of fluctuations in ψ_0 is sub-leading. Clearly, the larger we choose γ, the stronger is the assumption.

- Under these conditions, we show in Corollary 2.5 that at any time t and for sufficiently large N, the first A moments of the number of fluctuations remain sub-leading, and the N-dependence $N^{(1-\gamma)a}$ in (17) is replaced by $N^{c(\beta,\gamma)a}$ for some $(1-\gamma) \leq c(\beta,\gamma) < 1$.

- In our second and main result (Theorem 1), we prove higher order corrections to the norm approximation (12) for the scaling regime $\beta \in [0, \frac{1}{4d})$. This is to be understood in the following sense: we construct a sequence of N-body wave functions $\{\psi_\varphi^{(a)}(t)\}_{a\in\mathbb{N}} \subset L^2(\mathbb{R}^{dN})$ such that, for sufficiently large N,

$$\|\psi(t) - \psi_\varphi^{(a)}(t)\|^2_{L^2(\mathbb{R}^{dN})} \leq C(t) N^{-a\delta(\beta,\gamma)} \tag{18}$$

  for some time-dependent constant $C(t)$. The exponent $\delta(\beta,\gamma)$ is positive, depends on β and γ and is determined in Theorem 1.

  The first element of the approximating sequence $\{\psi_\varphi^{(a)}(t)\}$ is given by

$$\psi_\varphi^{(1)}(t) = \widetilde{U}_\varphi(t,0)\psi_0 \,.$$

  For $a = 1$, the estimate (18) is thus well known since it coincides with the norm approximation (15) and consequently with (12). To obtain the next higher correction with respect to N, we add an appropriate correction term to $\psi_\varphi^{(1)}(t)$. We expand the difference $\big(U(t,s) - \widetilde{U}_\varphi(t,s)\big)\psi_0$ using Duhamel's formula, identify the leading order contribution, and approximate it by replacing $U(t,s)$ with $\widetilde{U}_\varphi(t,s)$. This leads to the second element

$$\psi_\varphi^{(2)}(t) = \widetilde{U}_\varphi(t,0)\psi_0 + \mathrm{i}\int_0^t \widetilde{U}_\varphi(t,s)\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0 \,\mathrm{d}s \,.$$

  For the third element, we expand the difference $\big(U(t,s) - \widetilde{U}_\varphi(t,s)\big)\psi_0$ to the next order, using Duhamel's formula twice, and subsequently follow the same strategy as before. In this way, we construct all higher elements of the sequence as Duhamel expansions with finitely many terms, each of which exclusively contains ψ_0, the auxiliary time evolution $\widetilde{U}_\varphi(t,s)$, and the cubic and quartic interaction terms $\mathcal{C}^{\varphi(t)}$ and $\mathcal{Q}^{\varphi(t)}$. The precise definition of $\psi_\varphi^{(a)}(t)$ for any a, as well as a more detailed explanation of the construction, is provided in Definition 2.2 and the preceding discussion.

We note that higher order approximations of the reduced density matrices were obtained by Paul and Pulvirenti in [47] for $\beta = 0$ and factorized initial data, based on the method of kinetic errors from the paper by Paul, Pulvirenti and Simonella [48]. For $j \in \{1, \ldots, N\}$, the authors of [47] construct a sequence $\{F_j^{N,n}(t)\}_{n\in\mathbb{N}}$ of trace class operators on $L^2(\mathbb{R}^{jd})$, which approximate the j-particle reduced density matrix $\gamma^{(j)}(t)$ of the system with increasing accuracy up to arbitrary precision. The approximating operators $F_j^{N,n}(t)$ can be determined by a number of operations scaling with n. They depend on the initial data as well as the knowledge of the solution of the Hartree equation and its linearization around this solution.

Due to different methods used, it is not straightforward to compare the results of [47] with the results of this paper. However, we list some features of our paper that differ from the operator-based method of kinetic errors [47, 48]. In contrast to the approach in [47], we derive approximations directly for the time-evolved N-body wave function. Our construction is in terms of the Bogoliubov time evolution $\widetilde{U}_\varphi$ instead of the linearized Hartree flow, and it is implemented as a robust algorithm that requires an

a-dependent, N-independent number of explicit calculations to compute the a'th order approximation. Moreover, the results obtained in this paper cover more generic initial data satisfying (17) and include positive values of β.

Notation. In the following, any expression C that is independent of both N and t will be referred to as a constant. Note that constants may depend on all fixed parameters of the model such as φ_0, ψ_0, v and $V^{\text{ext}}(0)$. Further, we denote $A \lesssim B$ and $A \gtrsim B$ to indicate that there exists a constant $C > 0$ such that $A \le CB$, resp. $A \ge CB$, and abbreviate

$$\langle\,\cdot\,,\,\cdot\,\rangle_{L^2(\mathbb{R}^{dN})} =: \langle\,\cdot\,,\,\cdot\,\rangle\,, \qquad \|\cdot\|_{L^2(\mathbb{R}^{dN})} =: \|\cdot\|, \qquad \|\cdot\|_{\mathcal{L}(L^2(\mathbb{R}^{dN}))} =: \|\cdot\|_{\text{op}}.$$

Finally, we use the notation

$$\lfloor r \rfloor := \max\{z \in \mathbb{Z} :\, z \le r\}\ , \qquad \lceil r \rceil := \min\{z \in \mathbb{Z} :\, z \ge r\}$$

for $r \in \mathbb{R}$.

2 Main results

2.1 Framework and assumptions

Let us first recall from [39, 42, 43] the explicit decomposition of an N-body wave function ψ in terms of a condensate $\varphi^{\otimes N}$ and k-particle fluctuations around this condensate. To this end, recall the following projections introduced in [50]:

Definition 2.1. Let $\varphi \in L^2(\mathbb{R}^d)$. Define the orthogonal projections on $L^2(\mathbb{R}^d)$

$$p^\varphi := |\varphi\rangle\langle\varphi|, \qquad q^\varphi := \mathbb{1} - p^\varphi$$

and the corresponding projection operators on $L^2(\mathbb{R}^{dN})$

$$p_j^\varphi := \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{j-1} \otimes p^\varphi \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N-j} \quad \text{and} \quad q_j^\varphi := \mathbb{1} - p_j^\varphi\,.$$

For $0 \le k \le N$, define the many-body projections

$$P_k^\varphi \;:=\; \sum_{\substack{J \subseteq \{1,\dots,N\} \\ |J| = k}} \prod_{j \in J} q_j^\varphi \prod_{l \notin J} p_l^\varphi \;=\; \frac{1}{(N-k)!k!} \sum_{\sigma \in \mathfrak{S}_N} q^\varphi_{\sigma(1)} \cdots q^\varphi_{\sigma(k)} p^\varphi_{\sigma(k+1)} \cdots p^\varphi_{\sigma(N)}$$

and $P_k^\varphi = 0$ for $k < 0$ and $k > N$. Further, for any function $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ and any $j \in \mathbb{Z}$, define the operators $\widehat{f^\varphi}, \widehat{f_j^\varphi} \in \mathcal{L}\left(L^2(\mathbb{R}^{dN})\right)$ by

$$\widehat{f^\varphi} := \sum_{k=0}^{N} f(k) P_k^\varphi, \qquad \widehat{f_j^\varphi} := \sum_{n=-j}^{N-j} f(n+j) P_n^\varphi\,.$$

We will in particular need the operators $\widehat{n^\varphi}$ and $\widehat{m^\varphi}$ corresponding to the weights

$$n(k) := \sqrt{\tfrac{k}{N}}\,, \qquad m(k) := \sqrt{\tfrac{k+1}{N}}\,.$$

The part of ψ in the condensate is given by $P_0^\varphi\psi$, and the part of ψ corresponding to k particles fluctuating around the condensate is precisely $P_k^\varphi\psi$ for $k \geq 1$. By construction, $P_k^\varphi P_{k'}^\varphi = \delta_{k,k'} P_k^\varphi$. Besides, the identity $\sum_{k=0}^N P_k^\varphi = \mathbb{1}$ implies

$$\psi \;=\; \sum_{k=0}^{N} P_k^\varphi \psi \;=:\; \sum_{k=0}^{N} \varphi^{\otimes(N-k)} \otimes_s \xi_\varphi^{(k)} \tag{19}$$

for some $\xi_\varphi^{(k)} \in L^2(\mathbb{R}^{dk})$. To determine the explicit form of $\xi_\varphi^{(k)}$, observe that by Definition 2.1,

$$\begin{aligned} &P_k^\varphi \psi(x_1, ..., x_N) \\ &= \frac{1}{(N-k)!k!} \sum_{\sigma \in \mathfrak{S}_N} \varphi(x_{\sigma(k+1)}) \cdots \varphi(x_{\sigma(N)})\, q^\varphi_{\sigma(1)} \cdots q^\varphi_{\sigma(k)} \times \\ &\quad \times \int\limits_{\mathbb{R}^d} \mathrm{d}y_1 \cdots \int\limits_{\mathbb{R}^d} \mathrm{d}y_{N-k}\, \overline{\varphi(y_1)} \cdots \overline{\varphi(y_{N-k})}\, \psi(x_{\sigma(1)}, ..., x_{\sigma(k)}, y_1, ..., y_{N-k}) \\ &=: \left(\varphi^{\otimes(N-k)} \otimes_s \xi_\varphi^{(k)}\right)(x_1, ..., x_N)\,, \end{aligned}$$

where, by definition of the symmetric tensor product,

$$\begin{aligned} &\xi_\varphi^{(k)}(x_1, ..., x_k) := \\ &= \sqrt{\tbinom{N}{k}}\, q_1^\varphi \cdots q_k^\varphi \int\limits_{\mathbb{R}^d} \mathrm{d}\widetilde{x}_{k+1} \cdots \int \mathrm{d}\widetilde{x}_N\, \overline{\varphi(\widetilde{x}_{k+1})} \cdots \overline{\varphi(\widetilde{x}_N)}\, \psi(x_1, ..., x_k, \widetilde{x}_{k+1}, ..., \widetilde{x}_N)\,. \end{aligned} \tag{20}$$

Obviously, $\xi_\varphi^{(k)}$ is symmetric under permutations of all of its coordinates, and $\xi_\varphi^{(k)}$ is orthogonal to φ in every coordinate, i.e.,

$$\int_{\mathbb{R}^d} \overline{\varphi(x_j)}\, \xi_\varphi^{(k)}(x_1, ..., x_j, ..., x_N)\, \mathrm{d}x_j = 0\,, \qquad p_j^\varphi \xi_\varphi^{(k)} = 0\,, \qquad q_j^\varphi \xi_\varphi^{(k)} = \xi_\varphi^{(k)} \tag{21}$$

for every $j \in \{1, ..., k\}$. Hence, $\xi_\varphi^{(k)} \in \bigotimes_{\mathrm{sym}}^k \{\varphi\}^\perp$. The fluctuations $\xi_\varphi^{(k)}$, $k \in \{0, ..., N\}$, define a vector $\xi_\varphi := \left(\xi_\varphi^{(0)}, \xi_\varphi^{(1)}, ..., \xi_\varphi^{(N)}\right)$ in the truncated Fock space $\mathcal{F}^{\leq N}\left(\{\varphi\}^\perp\right)$ defined in (6). The relation between the N-body state ψ and the corresponding fluctuation vector ξ_φ is given by the unitary map

$$\mathfrak{U}_N^\varphi : L^2(\mathbb{R}^{dN}) \to \mathcal{F}^{\leq N}\left(\{\varphi\}^\perp\right), \quad \psi \mapsto \mathfrak{U}_N^\varphi \psi := \xi_\varphi\,, \tag{22}$$

where ξ_φ is defined by (20). The vacuum $(1, 0, ..., 0)$ of $\mathcal{F}^{\leq N}\left(\{\varphi\}^\perp\right)$ corresponds to the condensate $\varphi^{\otimes N}$, and the probability of k particles being outside the condensate equals

$$\|\xi_\varphi^{(k)}\|^2_{L^2(\mathbb{R}^{dk})} = \tbinom{N}{k} \|q_1^\varphi \cdots q_k^\varphi p_{k+1}^\varphi \cdots p_N^\varphi \psi\|^2 = \|P_k^\varphi \psi\|^2 \tag{23}$$

by (20). The number operator $\mathcal{N}_\varphi$ on $\mathcal{F}^{\leq N}\left(\{\varphi\}^\perp\right)$, counting the number of fluctuations, is defined by its action

$$(\mathcal{N}_\varphi\, \xi_\varphi)^{(k)} := k\, \xi_\varphi^{(k)}\,.$$

The expected number of fluctuations around the condensate $\varphi^{\otimes N}$ in the state ψ is thus given by

$$\begin{aligned}\langle \xi_\varphi, \mathcal{N}_\varphi\, \xi_\varphi \rangle_{\mathcal{F}^{\leq N}(\{\varphi\}^\perp)} &= \sum_{k=0}^{N} k \|\xi_\varphi^{(k)}\|^2_{L^2(\mathbb{R}^{dk})} = \sum_{k=0}^{N} k \|P_k^\varphi \psi\|^2 = N \left\langle \psi, \sum_{k=0}^{N} \tfrac{k}{N} P_k^\varphi \psi \right\rangle \\ &= N \|\widehat{n^\varphi}\psi\|^2\end{aligned} \tag{24}$$

with $\widehat{n^\varphi}$ from Definition 2.1.

Let us now state our assumptions on the model (1) and on the initial data.

A1 *Interaction potential.* Let $v : \mathbb{R}^d \to \mathbb{R}$ be spherically symmetric and bounded uniformly in N, i.e., $\|v\|_{L^\infty(\mathbb{R}^d)} \lesssim 1$. Further, assume that $\operatorname{supp} v \subseteq \{x \in \mathbb{R}^d : |x| \lesssim 1\}$.

A2 *External potential.* Let $V^{\text{ext}} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $V^{\text{ext}}(\cdot, x) \in \mathcal{C}(\mathbb{R})$ for each $x \in \mathbb{R}^d$ and $V^{\text{ext}}(t, \cdot) \in L^\infty(\mathbb{R}^d)$ for each $t \in \mathbb{R}$.

A3 *Initial data.* Let $\psi_0 \in \mathcal{D}(H^\beta(0)) \cap L^2_{\text{sym}}(\mathbb{R}^{dN})$ and $\varphi_0 \in H^k(\mathbb{R}^d)$, $k = \lceil \frac{d}{2} \rceil$, both be normalised. Let $\gamma \in (0, 1]$ and $A \in \mathbb{N}$. Assume that for any $a \in \{0, \dots, A\}$, there exists a set of non-negative, a-dependent constants $\{\mathfrak{C}_a\}_{0 \leq a \leq A}$ with $\mathfrak{C}_0 = 1$ such that, for sufficiently large N,

$$\left\| \left(\widehat{m^{\varphi_0}} \right)^a \psi_0 \right\|^2 \leq \mathfrak{C}_a\, N^{-\gamma a}\,.$$

Our analysis is valid as long as the solution $\varphi(t)$ of the non-linear equation (8) exists in $H^k(\mathbb{R}^d)$-sense for $k = \lceil \frac{d}{2} \rceil$. The maximal time of $H^k(\mathbb{R}^d)$-existence, $T^{\text{ex}}_{d,v,V^{\text{ext}}}$, is defined as

$$T^{\text{ex}}_{d,v,V^{\text{ext}}} := \sup \left\{ t \in \mathbb{R}_0^+ : \|\varphi(t)\|_{H^k(\mathbb{R}^d)} < \infty \text{ for } k = \lceil \tfrac{d}{2} \rceil \right\} \tag{25}$$

and depends on the dimension d, the sign of $\overline{v}^{\varphi(t)}$, and the regularity of the external trap $V^{\text{ext}}(t)$.

Assumptions A1 and A2 are rather standard in the rigorous treatment of interacting many-boson systems. Note that we make no assumption on the sign of the potential or its scattering length and thus cover both repulsive and attractive interactions. Besides, we admit a large class of time-dependent external traps V^{ext}, with basically the only restriction that $V^{\text{ext}}(t)$ must not obstruct the self-adjointness of $H^\beta(t)$ on $H^2(\mathbb{R}^{dN})$.

The third assumption provides a bound on the expected number of fluctuations around the condensate ${\varphi_0}^{\otimes N}$ in the initial state ψ_0. Note that while $\gamma = 0$ is the trivial bound, the condition becomes more restrictive as γ increases. We have chosen this particular formulation of A3 for later convenience[3]. However, its physical meaning is better understood from one of the following two equivalent versions of A3:

[3]Note that the operators $\widehat{n^\varphi}$ and $\widehat{m^\varphi}$ are equivalent in the sense that they are related via (36), namely $(\widehat{n^\varphi})^{2a} \leq (\widehat{m^\varphi})^{2a} \leq 2^a (\widehat{n^\varphi})^{2a} + N^{-a}$, hence all results in terms of $\widehat{n^\varphi}$ can be translated to the corresponding statements in terms of $\widehat{n^\varphi}$. We chose to work with $\widehat{m^\varphi}$ instead of $\widehat{n}^\varphi$ because this makes in particular Proposition 2.4 easier to write. For example, in terms of $\widehat{n^\varphi}$, Proposition 2.4b reads

$$\|(\widehat{n^\varphi})^j \widetilde{U}_\varphi(t,s)\psi\|^2 \lesssim C_j^{t,s} \sum_{n=0}^{j} N^{n(-1+d\beta)} \left(2^{j-n} \|(\widehat{n^\varphi})^{j-n}\psi\|^2 + N^{-j+n} \right),$$

which contains an additional term N^{-j+n}. Since the proof of our main result requires an iteration of this proposition, the version with $\widehat{n^\varphi}$ is more practicable.

A3′ Let $\psi_0 \in \mathcal{D}(H^\beta(0)) \cap L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$ and $\varphi_0 \in H^k(\mathbb{R}^d)$, $k = \lceil \frac{d}{2} \rceil$, both be normalised. Let $\gamma \in (0,1]$ and $A \in \mathbb{N}$. Assume that for any $a \in \{0, \ldots, A\}$, there exists a set of non-negative, a-dependent constants $\{\mathfrak{C}'_a\}_{0 \le a \le A}$ with $\mathfrak{C}'_0 = 1$ such that, for sufficiently large N,

$$\|q_1^{\varphi_0} \cdots q_a^{\varphi_0} \psi_0\|^2 \le \mathfrak{C}'_a \, N^{-\gamma a}.$$

A3″ Let $\psi_0 \in \mathcal{D}(H^\beta(0)) \cap L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$ and $\varphi_0 \in H^k(\mathbb{R}^d)$, $k = \lceil \frac{d}{2} \rceil$, both be normalised. Let $\gamma \in (0,1]$, $A \in \mathbb{N}$ and $\xi_{\varphi_0} = \mathfrak{U}_N^{\varphi_0} \psi_0$. Assume that for any $a \in \{0, \ldots, A\}$, there exists a set of non-negative, a-dependent constants $\{\mathfrak{C}''_a\}_{0 \le a \le A}$ with $\mathfrak{C}''_0 = 1$ such that, for sufficiently large N,

$$\left\langle \xi_{\varphi_0}, \mathcal{N}^a_{\varphi_0} \, \xi_{\varphi_0} \right\rangle_{\mathcal{F}^{\le N}\left(\{\varphi_0\}^\perp\right)} = \sum_{k=0}^{N} k^a \|\xi^{(k)}_{\varphi_0}\|^2_{L^2(\mathbb{R}^{dk})} \le \mathfrak{C}''_a \, N^{(1-\gamma)a} \, .$$

The equivalence A3 ⇔ A3′ ⇔ A3″ follows immediately from Lemma 2.1, whose proof is postponed to Section 3.1.

Lemma 2.1. *Let $a \in \{1, \ldots, N\}$ and $\varphi \in L^2(\mathbb{R}^d)$. Let $\psi \in L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$ and $\xi_\varphi = U_N^\varphi \psi$. Then*

(a) $$\left\|q_1^\varphi \cdots q_a^\varphi \psi\right\|^2 \le \left\|\left(\widehat{m^\varphi}\right)^a \psi\right\|^2 \le 4^a a! \sum_{j=1}^{a} N^{-a+j} \left\|q_1^\varphi \cdots q_j^\varphi \psi\right\|^2 + N^{-a} \, ,$$

(b) $$\left\langle \xi_\varphi, \mathcal{N}^a_\varphi \, \xi_\varphi \right\rangle_{\mathcal{F}^{\le N}\left(\{\varphi\}^\perp\right)} \le N^a \left\|\left(\widehat{m^\varphi}\right)^a \psi\right\|^2 \le 1 + 2^a \left\langle \xi_\varphi, \mathcal{N}^a_\varphi \, \xi_\varphi \right\rangle_{\mathcal{F}^{\le N}\left(\{\varphi\}^\perp\right)} \, .$$

Hence, A3 can be understood as follows: Let $A \in \mathbb{N}$ and consider sufficiently large N such that $A = \mathcal{O}(1)$ with respect to N, i.e. $A \lesssim 1$. Then we assume that for any $a \le A$, the part of the wave function with any a particles outside the condensate is at most of order $N^{-\gamma a}$.

Equivalently, A3 states that the first $A \lesssim 1$ moments of the number of fluctuations must be sub-leading with respect to the particle number; for $\gamma = 1$, they must even be bounded uniformly in N. Here, "sub-leading" means that the moments of the relative number of fluctuations, i.e., the expectation values of $(\mathcal{N}_{\varphi(t)}/N)^A$, vanish as $N \to \infty$. This, in turn, provides a bound on the high components of the fluctuation vector: for example, $\sum_{k=0}^N k^A \|\xi^{(k)}_{\varphi_0}\|^2_{L^2(\mathbb{R}^{dk})} \lesssim N^{(1-\gamma)A}$ implies $\|\xi^{(N)}_{\varphi_0}\|^2_{L^2(\mathbb{R}^{dk})} \lesssim N^{-\gamma A}$. In other words, it must be very unlikely that significantly many particles are outside the condensate, whereas we impose no restriction on fluctuations involving only few particles (with respect to N).

As soon as a becomes comparable to N, i.e., $a \gtrsim N$, the constants $\mathfrak{C}_a^{(\prime,\prime\prime)}$ are N-dependent and the assumption is trivially satisfied. However, note that we demand that N be large enough that $A \lesssim 1$.

The simplest example of an N-body state satisfying A3 is the product state $\psi = {\varphi_0}^{\otimes N}$. Whereas the ground state of non-interacting bosons ($v = 0$) is of this form, the ground state as well as the lower excited states of interacting systems are not close to an exact product with respect to the $L^2(\mathbb{R}^{dN})$-norm due to the correlation structure related to the interactions.

Besides, it seems reasonable to expect that states exhibiting Bose–Einstein condensation satisfy A3 for some (possibly very small) γ, as it is well known that

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma^{(1)} - |\varphi\rangle\langle\varphi|\right| = 0 \quad \Leftrightarrow \quad \lim_{N\to\infty} \|(\widehat{m^\varphi})^j\psi\|^2 = 0 \text{ for any } j > 0$$

(e.g. [50, Lemma 2.3]). Note, however, that we require a certain minimal size of γ, which is strictly greater than $\frac{2}{3}$ (Theorem 1).

To the best of our knowledge, there exists only one rigorous result in [43, Chapter 3] that identifies situations where a Bose gas satisfies assumption A3. This work concerns a homogeneous Bose gas on the d-dimensional torus and is restricted to the scaling $\beta = 0$. For this case, it is shown that the ground state as well as the lower excited states fulfil assumption A3 with $\gamma = 1$ (and consequently for all $\gamma \in (0, 1]$). More precisely, let φ_0 be the minimizer of the Hartree functional on the torus with ground state energy E_0, and let ψ_n denote the n'th excited state with energy E_n. Then the author proves that there exist constants $C, D > 0$ such that $\|P_a^{\varphi_0}\psi_n\|^2 \leq C\mathrm{e}^{-Da}$ for all $(E_n - E_0) \leq a \leq N$. As a corollary of this statement, it is shown that there exists $C_a > 0$ such that

$$\langle\psi_n, q_1^{\varphi_0}\cdots q_a^{\varphi_0}\psi_n\rangle \leq N^{-a}C_a\left(1 + (E_n - E_0)^a\right),$$

which implies that assumption A3$'$ is satisfied.

Let us conclude the discussion of our assumptions with a remark on the relation between A3 and the so-called Wick property of quasi-free states[4]. In [39, Theorem A.1], it was shown that the ground state of $\mathbb{H}_{\mathrm{Bog}}$ is a quasi-free state, which, via the map $\mathfrak{U}_N^\varphi$, defines an N-body state ψ_{Bog} that converges to the actual ground state ψ_0 in norm as $N \to \infty$ [39, Theorem 2.2]. For a quasi-free state χ on a Fock space $\mathcal{F}$, it is known (e.g. [44, Lemma 5]) that for every $a \geq 1$, there exists a constant $C_a > 0$ such that

$$\langle\chi, \mathcal{N}^a\chi\rangle_{\mathcal{F}} \leq C_a\left(1 + \langle\chi, \mathcal{N}\chi\rangle_{\mathcal{F}}\right)^a.$$

Hence, A3$^{(\prime,\prime\prime)}$ holds with $\gamma = 1$ for quasi-free states. Since it is somewhat similar to the Wick property, it is referred to as *quasi-free type property* in [43].

Finally, let us recall from (14) the Hamiltonian $\widetilde{H}^{\varphi(t)}(t)$ introduced in [42, 43],

$$\begin{aligned}\widetilde{H}^{\varphi(t)}(t) &= \sum_{j=1}^N h_j^{\varphi(t)}(t)\\ &\quad + \frac{1}{N-1}\sum_{i<j}\left(p_i^{\varphi(t)}q_j^{\varphi(t)}v_{ij}^\beta q_i^{\varphi(t)}p_j^{\varphi(t)} + p_i^{\varphi(t)}p_j^{\varphi(t)}v_{ij}^\beta q_i^{\varphi(t)}q_j^{\varphi(t)} + \text{h.c.}\right),\end{aligned}$$

which can be understood as first-quantised analogue of a Bogoliubov Hamiltonian. As pointed out in the introduction, $\widetilde{H}^{\varphi(t)}(t)$ differs from $H^\beta(t)$ precisely by terms with three or four projectors $q^{\varphi(t)}$, denoted by $\mathcal{C}^{\varphi(t)}$ and $\mathcal{Q}^{\varphi(t)}$. In this sense, it is a quadratic Hamiltonian comparable to $\mathbb{H}_{\mathrm{Bog}}(t)$.

[4] A state χ in a Fock space $\mathcal{F}(\mathfrak{H})$ over a Hilbert space $\mathfrak{H}$ is called *quasi-free* if it has a finite particle number expectation and satisfies Wick's Theorem: For all n and for all $f_1, ..., f_n \in \mathfrak{H}$ and for $a^\sharp$ either the creation or the annihilation operator, $\left\langle\chi, a^\sharp(f_1)a^\sharp(f_2)\dots a^\sharp(f_{2n-1})\chi\right\rangle = 0$ and $\left\langle\chi, a^\sharp(f_1)a^\sharp(f_2)\dots a^\sharp(f_{2n})\chi\right\rangle = \sum_{\sigma\in P_{2n}}\prod_{j=1}^n\left\langle\chi, a^\sharp(f_{\sigma(2j-1)})a^\sharp(f_{\sigma(2j)})\chi\right\rangle$, where $P_{2n} = \{\sigma \in \mathfrak{S}_{2n} : \sigma(2j-1) < \min\{\sigma(2j), \sigma(2j+1)\}\forall j\}$ is the set of pairings.

Lemma 2.2.
$$H^\beta(t) = \widetilde{H}^{\varphi(t)}(t) + \mathcal{C}^{\varphi(t)} + \mathcal{Q}^{\varphi(t)}\,,$$
where
$$\begin{aligned}\mathcal{C}^{\varphi(t)} &:= \frac{1}{N-1}\sum_{i<j}\Big(q_i^{\varphi(t)}q_j^{\varphi(t)}\Big(v_{ij}^\beta - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j)\Big)\Big(q_i^{\varphi(t)}p_j^{\varphi(t)} + p_i^{\varphi(t)}q_j^{\varphi(t)}\Big) \\ &\qquad\qquad + \text{h.c.}\Big)\,, \\ \mathcal{Q}^{\varphi(t)} &:= \frac{1}{N-1}\sum_{i<j} q_i^{\varphi(t)}q_j^{\varphi(t)}\Big(v_{ij}^\beta - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)}\Big) q_i^{\varphi(t)}q_j^{\varphi(t)}\,.\end{aligned}$$

Proof.
$$\begin{aligned}H^\beta(t) &= \sum_{j=1}^N h_j^{\varphi(t)}(t) + \frac{1}{N-1}\sum_{i<j} v_{ij}^\beta - \sum_{j=1}^N \overline{v}^{\varphi(t)}(x_j) + N\mu^{\varphi(t)} \\ &= \sum_{j=1}^N h_j^{\varphi(t)}(t) + \frac{1}{N-1}\sum_{i<j}\Big(v_{ij}^\beta - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)}\Big)\end{aligned}$$

Now one inserts identities $\mathbb{1} = (p_i^{\varphi(t)} + q_i^{\varphi(t)})(p_j^{\varphi(t)} + q_j^{\varphi(t)})$ before and after the expression in the brackets and uses the relations
$$p_i^{\varphi(t)}v_{ij}^\beta p_i^{\varphi(t)} = \overline{v}^{\varphi(t)}(x_j)p_i^{\varphi(t)}, \qquad p_i^{\varphi(t)}\overline{v}^{\varphi(t)}(x_i)p_i^{\varphi(t)} = 2\mu^{\varphi(t)}p_i^{\varphi(t)}\,,$$
which concludes the proof. □

The time evolution generated by $\widetilde{H}^{\varphi(t)}(t)$ is denoted by $\widetilde{U}_\varphi(t,s)$, and its well-posedness is recalled in the following lemma.

Lemma 2.3. *Let $t \in \big[0, T^{\text{ex}}_{d,v,V^{\text{ext}}}\big)$. Then $\widetilde{H}^{\varphi(t)}(t)$ is self-adjoint on $\mathcal{D}(\widetilde{H}^{\varphi(t)}(t)) = H^2(\mathbb{R}^{dN})$ and generates a unique family of unitary time evolution operators $\widetilde{U}_\varphi(t,s)$. $\widetilde{U}_\varphi(t,s)$ is strongly continuous jointly in s,t and leaves $H^2(\mathbb{R}^{dN})$ invariant. For an initial datum $\psi_0 \in L^2_{\text{sym}}(\mathbb{R}^{dN})$, the corresponding N-body wave function at time $t \in \mathbb{R}$ will be denoted by*
$$\widetilde{\psi}_\varphi(t) = \widetilde{U}_\varphi(t,0)\psi_0\,. \tag{26}$$

Proof. As a consequence of the Sobolev embedding theorem (e.g. [3, Theorem 4.12, Part IA]), $\|\varphi(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \|\varphi(t)\|_{H^k(\mathbb{R}^d)}$ for $k = \lceil\frac{d}{2}\rceil$. Hence, by definition (25) of $T^{\text{ex}}_{d,v,V^{\text{ext}}}$, $\mu^{\varphi(t)}$ and $(N-1)\overline{v}^{\varphi(t)}$ are bounded uniformly in N for $t \in \big[0, T^{\text{ex}}_{d,v,V^{\text{ext}}}\big)$. Further, $t \mapsto \widetilde{H}^{\varphi(t)}(t)\psi$ is Lipschitz for all $\psi \in H^2(\mathbb{R}^{dN})$ because of (8), since $t \mapsto V^{\text{ext}}(t) \in \mathcal{L}(L^2(\mathbb{R}^d))$ is continuous and as $\frac{\mathrm{d}}{\mathrm{d}t}p^{\varphi(t)} = \mathrm{i}[p^{\varphi(t)}, h^{\varphi(t)}(t)]$. Hence, the statement of the lemma follows from [24]. □

2.2 Control of higher moments of the number of fluctuations

In our first result, we prove bounds on the growth of the expected number of fluctuations under the time evolution. We consider both the actual N-body dynamics $U(t,s)$ and the dynamics $\widetilde{U}_\varphi(t,s)$ generated by the Hamiltonian $\widetilde{H}^{\varphi(t)}(t)$. The estimates are stated for $\|(\widehat{m^\varphi})^a\psi\|^2$ as these expressions are required for the proof of our main theorem. By Lemma 2.1, they easily translate to bounds on the corresponding quantities $\|q_1\cdots q_a\psi\|^2$ and $\left\langle\xi_\varphi,\mathcal{N}_\varphi^a\xi_\varphi\right\rangle$. The proofs of Proposition 2.4 and Corollary 2.5 are postponed to Section 3.2.

Proposition 2.4. *Let* $\beta\in[0,\frac{1}{d})$*, assume* A1 *and* A2 *and let* $\psi\in L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$*. Let* $s\in\mathbb{R}$*,* $\varphi(s)\in H^k(\mathbb{R}^d)$ *for* $k=\lceil\frac{d}{2}\rceil$*, and let* $\varphi(t)$ *be the solution of* (8) *with initial datum* $\varphi(s)$*. Then it holds for* $t\in\left[s,s+T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\right)$ *and* $j\in\{1,\dots,N\}$ *that*

(a) for any $b\in\mathbb{N}_0$*,*

$$\left\|\left(\widehat{m^{\varphi(t)}}\right)^j U(t,s)\psi\right\|^2 \lesssim C_j^{t,s}\sum_{n=0}^{j}N^{n(-1+d\beta)}\left\|\left(\widehat{m^{\varphi(s)}}\right)^{j-n}\psi\right\|^2 + 2^b C_b^{t,s}\sum_{n=0}^{b}N^{n(-1+d\beta)+d\beta b}\left\|\left(\widehat{m^{\varphi(s)}}\right)^{b-n}\psi\right\|^2,$$

(b)

$$\left\|\left(\widehat{m^{\varphi(t)}}\right)^j \widetilde{U}_\varphi(t,s)\psi\right\|^2 \lesssim C_j^{t,s}\sum_{n=0}^{j}N^{n(-1+d\beta)}\left\|\left(\widehat{m^{\varphi(s)}}\right)^{j-n}\psi\right\|^2,$$

where $C_j^{t,s}:=j!\,3^{j(j+1)}\mathrm{e}^{9^j\int_s^t\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\mathrm{d}s_1}$.

Proposition 2.4 provides an extension to positive β of [42, Lemma 2.1], where a comparable statement was shown for $\beta=0$, $\gamma=1$ and $d=3$ with a similar method. Under the additional assumption A3 on the initial data, this implies the following estimates:

Corollary 2.5. *Assume* A1 – A2 *and* A3 *with* $\gamma\in(0,1]$ *and* $A\in\{1,\dots,N\}$*. Let* $\psi(t)$*,* $\widetilde{\psi}_\varphi(t)$ *and* $\varphi(t)$ *denote the solutions of* (4), (26) *and* (8) *with initial data* ψ_0 *and* φ_0 *from* A3*. Then for* $t\in\left[0,T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\right)$*, sufficiently large* N *and* $a\in\{0,\dots,A\}$*, it holds that*

(a) for $\beta\in[0,\frac{1}{2d})$*,*

$$\left\|\left(\widehat{m^{\varphi(t)}}\right)^a\psi(t)\right\|^2 \lesssim a\,C_a^t\,N^{-a(1-d\beta)}\qquad\text{for } 1-d\beta\leq\gamma\leq 1$$

and for $\beta\in[0,\frac{1}{d})$*,*

$$\left\|\left(\widehat{m^{\varphi(t)}}\right)^a\psi(t)\right\|^2 \lesssim a\,\mathfrak{C}_a\,C_a^t\,N^{-\gamma a}\qquad\text{for } d\beta<\gamma\leq 1-d\beta\,,$$

(b) for $\beta\in[0,\frac{1}{d})$*,*

$$\left\|\left(\widehat{m^{\varphi(t)}}\right)^a\widetilde{\psi}_\varphi(t)\right\|^2 \lesssim \begin{cases} a\,C_a^t\,N^{-a(1-d\beta)} & \text{for } 1-d\beta\leq\gamma\leq 1\,,\\ a\,\mathfrak{C}_a\,C_a^t\,N^{-\gamma a} & \text{for } 0<\gamma\leq 1-d\beta\end{cases}$$

with $C_a^t:=C_a^{t,0}$.

At the threshold $\gamma = 1 - d\beta$, the leading order terms in the sums in Proposition 2.4 change, hence we obtain two different estimates. The additional restrictions on β and γ in part (a) stem from the second sum in Proposition 2.4a. Only if either $\beta < \frac{1}{2d}$ or $\gamma > d\beta$, it is possible to choose b sufficiently large that the first sum dominates for large N.

By Lemma 2.1, Corollary 2.5 yields estimates on the growth of the first A moments of the fluctuation number, given A3 with parameters A and γ. Let $\xi_{\varphi_0} = \mathfrak{U}_N^{\varphi_0}\psi_0$, $\xi_{\varphi(t)} = \mathfrak{U}_N^{\varphi(t)}\psi(t) = \mathfrak{U}_N^{\varphi(t)}U(t,0)\psi_0$ and $\widetilde{\xi}_{\varphi(t)} = \mathfrak{U}_N^{\varphi(t)}\widetilde{\psi}_\varphi(t) = \mathfrak{U}_N^{\varphi(t)}\widetilde{U}_\varphi(t,0)\psi_0$. Then, for sufficiently large N and for all $a \in \{0, ..., A\}$, we obtain for $\beta \in [0, \frac{1}{2d})$

$$\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \, \xi_{\varphi_0} \right\rangle \lesssim N^{(1-\gamma)a} \quad \Rightarrow \quad \left\langle \xi_{\varphi(t)}, \mathcal{N}_{\varphi(t)}^a \xi_{\varphi(t)} \right\rangle \lesssim C_a^t N^{d\beta a} \qquad 1 - d\beta \le \gamma \le 1 \,,$$

and for $\beta \in [0, \frac{1}{d})$

$$\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \, \xi_{\varphi_0} \right\rangle \lesssim N^{(1-\gamma)a} \quad \Rightarrow \quad \left\langle \xi_{\varphi(t)}, \mathcal{N}_{\varphi(t)}^a \xi_{\varphi(t)} \right\rangle \lesssim C_a^t N^{(1-\gamma)a} \qquad d\beta < \gamma \le 1 - d\beta \,,$$

$$\left\langle \xi_{\varphi_0}, \mathcal{N}_{\varphi_0}^a \, \xi_{\varphi_0} \right\rangle \lesssim N^{(1-\gamma)a} \quad \Rightarrow \quad \begin{cases} \left\langle \widetilde{\xi}_{\varphi(t)}, \mathcal{N}_{\varphi(t)}^a \widetilde{\xi}_{\varphi(t)} \right\rangle \lesssim C_a^t N^{d\beta a} & 1 - d\beta \le \gamma \le 1 \,, \\ \left\langle \widetilde{\xi}_{\varphi(t)}, \mathcal{N}_{\varphi(t)}^a \widetilde{\xi}_{\varphi(t)} \right\rangle \lesssim C_a^t N^{(1-\gamma)a} & 0 < \gamma < 1 - d\beta \,, \end{cases}$$

where we estimated $a, \mathfrak{C}_a, \mathfrak{C}''_a \lesssim 1$ for the sake of readability. For $\beta = 0$, both time evolutions preserve the property A3$''$ exactly, i.e., with the same power γ of N, up to a constant growing rapidly in t and a. For $\beta > 0$, the conservation is exact only for small γ, whereas one looses some power of N for larger γ. Further, note that for the range $\gamma \in (0, d\beta)$, we do not obtain a non-trivial estimate for the fluctuations $\xi_{\varphi(t)}$ in $U(t,0)\psi_0$.

2.3 Higher order corrections to the norm approximation

Based on the estimates obtained in Proposition 2.4, our main result establishes corrections of any order to the norm approximations (12) and (15): under assumption A3 on the initial data, we construct a sequence $\{\psi_\varphi^{(a)}\}_{a\in\mathbb{N}} \subset L^2(\mathbb{R}^{dN})$ such that

$$\|\psi(t) - \psi_\varphi^{(a)}(t)\|^2 \le C(t) N^{-a\delta(\beta,\gamma)}$$

for some $\delta(\beta,\gamma) > 0$, which may depend on β as well as on the parameter γ from assumption A3. For reasons given below, our analysis is restricted to the scaling regime $\beta \in [0, \frac{1}{4d})$.

As explained in the introduction, it is well known that the actual time evolution $\psi(t)$ is close to the evolution $\widetilde{\psi}_\varphi(t)$ from (26) in norm. Hence, the first element of the approximating sequence $\{\psi_\varphi^{(a)}\}_{a\in\mathbb{N}}$ is determined by

$$\psi_\varphi^{(1)}(t) := \widetilde{U}_\varphi(t,0)\psi_0 \,.$$

Using Duhamel's formula, the difference between $U(t,s)\psi$ and $\widetilde{U}_\varphi(t,s)\psi$ can be expressed as

$$U(t,s)\psi = \widetilde{U}_\varphi(t,s)\psi - \mathrm{i}\int_s^t U(t,r)\left(\mathcal{C}^{\varphi(r)} + \mathcal{Q}^{\varphi(r)}\right)\widetilde{U}_\varphi(r,s)\psi \,\mathrm{d}r \qquad (27)$$

for any $\psi \in L^2(\mathbb{R}^{dN})$. Consequently,

$$\begin{aligned}\|\psi(t) - \psi^{(1)}_\varphi(t)\| &= \left\| -\mathrm{i}\int_0^t U(t,s)\left(\mathcal{C}^{\varphi(s)} + \mathcal{Q}^{\varphi(s)}\right)\widetilde{U}_\varphi(s,0)\psi_0\,\mathrm{d}s\right\| \\ &\le \int_0^t \|\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|\,\mathrm{d}s + \int_0^t \|\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|\,\mathrm{d}s \qquad (28)\end{aligned}$$

by the triangle inequality and as a consequence of the unitarity of $U(t,s)$. The leading order contribution in (28) is the term containing $\mathcal{C}^{\varphi(s)}$ because the cubic interaction terms are larger than the quartic ones in the following sense:

Lemma 2.6. *Let $\psi \in L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$ and denote by $\varphi(t)$ the solution of (8) with initial datum $\varphi_0 \in H^k(\mathbb{R}^d)$, $k = \lceil \frac{d}{2} \rceil$. Then for any $j \in \mathbb{N}_0$ and $t \in \left[0, T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\right)$,*

(a) $\|\left(\widehat{m^{\varphi(t)}}\right)^j \mathcal{Q}^{\varphi(t)}\psi\|^2 \lesssim N^{2+2d\beta}\|\left(\widehat{m^{\varphi(t)}}\right)^{4+j}\psi\|^2$,

(b) $\|\left(\widehat{m^{\varphi(t)}}\right)^j \mathcal{C}^{\varphi(t)}\psi\|^2 \lesssim 4^j\|\varphi(t)\|^2_{H^k(\mathbb{R}^d)} N^{2+d\beta}\|\left(\widehat{m^{\varphi(t)}}\right)^{3+j}\psi\|^2$.

The proof of this lemma is postponed to Section 3.3. For $j = 0$, it gives a bound on the cubic and quartic terms; the more general statement $j \ge 0$ is included for later convenience.

When applying Lemma 2.6 to (28), we obtain expressions like $\|(\widehat{m^{\varphi(s)}})^j\widetilde{U}_\varphi(s,0)\psi_0\|^2$. To be able to use assumption A3 on the initial data, we need to interchange, in a sense, the order of $\widetilde{U}_\varphi(s,0)$ and $(\widehat{m^{\varphi(s)}})^j$. This is where Proposition 2.4 comes into play: from part 2.4b, it follows for sufficiently large N that

$$\begin{aligned}\|\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|^2 &\overset{2.6}{\lesssim} N^{2+d\beta}\|(\widehat{m^{\varphi(s)}})^3\widetilde{U}_\varphi(s,0)\psi_0\|^2 \\ &\overset{2.4\mathrm{b}}{\lesssim} C_3^s\, N^{2+d\beta}\sum_{n=0}^{3} N^{n(-1+d\beta)}\|(\widehat{m^{\varphi_0}})^{3-n}\psi_0\|^2 \\ &\overset{\mathrm{A3}}{\lesssim} C_3^s\, N^{2+d\beta}\sum_{n=0}^{3}\mathfrak{C}_{3-n}\, N^{n(-1+d\beta+\gamma)-3\gamma}\,.\end{aligned}$$

As in Corollary 2.5, the size of γ determines the leading order term in the sum: for $\gamma \ge 1 - d\beta$, the dominant contribution issues from $n = 3$, whereas otherwise the addend corresponding to $n = 0$ is of leading order. Consequently,

$$\|\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|^2 \lesssim \begin{cases} C_3^s\, N^{-1+4d\beta} & \text{for } 1 - d\beta \le \gamma \le 1\,, \\ \mathfrak{C}_3\, C_3^s\, N^{2+d\beta-3\gamma} & \text{for } \frac{2+d\beta}{3} < \gamma \le 1 - d\beta\,. \end{cases} \qquad (29)$$

To ensure that (29) converges to zero as $N \to \infty$, we have restricted the range of parameters γ admitted by assumption A3 to $\gamma \in (\frac{2+d\beta}{3}, 1]$. Besides, in the first case, the bound is only small for $\beta < \frac{1}{4d}$, and the second case is anyway only possible for $\beta < \frac{1}{4d}$. Hence, we can only cover the parameter regime $\beta \in [0, \frac{1}{4d})$. Analogously to (29), we also obtain

$$\|\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|^2 \lesssim \begin{cases} C_4^s\, N^{-2+6d\beta} & \text{for } 1 - d\beta \le \gamma \le 1\,, \\ C_4^s\, \mathfrak{C}_4\, N^{2+2d\beta-4\gamma} & \text{for } \frac{2+d\beta}{3} < \gamma \le 1 - d\beta\,. \end{cases} \qquad (30)$$

Note that $\beta < \frac{1}{4d}$ implies that $-2 + 6d\beta < -1 + 4d\beta$, and besides, it follows from $\gamma > \frac{2+3d}{3}$ and $\beta < \frac{1}{4d}$ that $2 + 2d\beta - 4\gamma < 2 + d\beta - 3\gamma$. Consequently, the contribution with $\mathcal{C}^{\varphi(s)}$ dominates in (28) for sufficiently large N, which leads to the estimate

$$\|\psi(t) - \psi_\varphi^{(1)}(t)\|^2 \lesssim N^{-\delta(\beta,\gamma)} \left(\int_0^t \sqrt{C_3^s}\, \mathrm{d}s \right)^2 \lesssim \mathrm{e}^{c(1) \int_0^t \|\varphi(s)\|^2_{H^k(\mathbb{R}^d)} \mathrm{d}s} N^{-\delta(\beta,\gamma)} \tag{31}$$

for some constant $c(1) > 0$ and with

$$\delta(\beta,\gamma) := \begin{cases} 1 - 4d\beta & \text{for } \; 1 - d\beta \le \gamma \le 1\,, \\ -2 - d\beta + 3\gamma & \text{for } \; \frac{2+d\beta}{3} < \gamma \le 1 - d\beta\,. \end{cases} \tag{32}$$

This yields (18) for $n = 1$.

To construct the second element $\psi_\varphi^{(2)}(t)$ of the approximating sequence, we need to extract from (27) the relevant contributions such that $\|\psi(t) - \psi_\varphi^{(2)}(t)\|^2 \le C(t)N^{-2\delta(\beta,\gamma)}$. As a consequence of Lemma 2.6, we define

$$\psi_\varphi^{(2)}(t) \;\; := \;\; \widetilde{U}_\varphi(t,0)\psi_0 - \mathrm{i} \int_0^t \mathrm{d}s\, \widetilde{U}_\varphi(t,s)\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\,,$$

which equals the leading order contribution in (27) but with the true time evolution $U(t,s)$ replaced by $\widetilde{U}_\varphi(t,s)$. Put differently, the leading order contribution is cancelled but for the difference between $U(t,s)$ and $\widetilde{U}_\varphi(t,s)$. Since this difference is evaluated on $\mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0$, which is small in norm, this is an improvement compared to the first order approximation $\psi_\varphi^{(1)}(t)$. To verify this, let us compute the difference between $\psi(t)$ and $\psi_\varphi^{(2)}(t)$. Using twice Duhamel's formula, we obtain

$$\begin{aligned} &\psi(t) - \psi_\varphi^{(2)}(t) \\ &\quad = \; -\mathrm{i} \int_0^t \left(U(t,s) - \widetilde{U}_\varphi(t,s) \right) \mathcal{C}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\, \mathrm{d}s \\ &\qquad\quad -\mathrm{i} \int_0^t U(t,s)\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\, \mathrm{d}s \\ &\quad = \; - \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2\, U(t,s_2) \left(\mathcal{C}^{\varphi(s_2)} + \mathcal{Q}^{\varphi(s_2)} \right) \widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0 \\ &\qquad\quad -\mathrm{i} \int_0^t U(t,s)\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\, \mathrm{d}s. \end{aligned}$$

Due to the unitarity of $U(t,s)$, we obtain with the triangle inequality

$$\begin{aligned} \|\psi(t) - \psi_\varphi^{(2)}(t)\| \;\; \le \;\; & \int_0^t \mathrm{d}s_1 \int_{s_2}^t \mathrm{d}s_2 \|\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\| \\ & + \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \|\mathcal{Q}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\| \\ & + \int_0^t \mathrm{d}s \|\mathcal{Q}^{\varphi(s)}\widetilde{U}_\varphi(s,0)\psi_0\|\,. \end{aligned} \tag{33}$$

The leading order term in (33) can be estimated as

$$
\begin{aligned}
&\|\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2\\
&\overset{2.6,2.4b}{\lesssim} \quad N^{2+d\beta}C_3^{s_2-s_1}\|\varphi(s_2)\|^2_{H^k(\mathbb{R}^d)}\sum_{n=0}^{3}N^{n(-1+d\beta)}\|(\widehat{m^{\varphi(s_1)}})^{3-n}\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2\\
&\overset{2.6,2.4b}{\lesssim} \quad N^{4+2d\beta}C_3^{s_2-s_1}\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\|\varphi(s_2)\|^2_{H^k(\mathbb{R}^d)}\times\\
&\qquad\qquad\times\sum_{n=0}^{3}\sum_{l=0}^{6-n}4^{3-n}C_{6-n}^{s_1}N^{(n+l)(-1+d\beta)}\|(\widehat{m^{\varphi_0}})^{6-n-l}\psi_0\|^2\\
&\overset{A3}{\lesssim} \quad N^{-2+2d\beta}C_3^{s_2-s_1}\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\|\varphi(s_2)\|^2_{H^k(\mathbb{R}^d)}\times\\
&\qquad\qquad\times\sum_{n=0}^{3}\sum_{l=0}^{6-n}4^{3-n}C_{6-n}^{s}\,\mathfrak{C}_{6-n-l}\,N^{(n+l)(-1+d\beta+\gamma)-6\gamma}\,.
\end{aligned}
$$

As before, considering the two ranges of γ separately yields for sufficiently large N

$$
\begin{aligned}
&\|\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2\\
&\qquad\lesssim C_3^{s_2-s_1}\,C_6^{s_1}\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\|\varphi(s_2)\|^2_{H^k(\mathbb{R}^d)}\,N^{-2\delta(\beta,\gamma)}
\end{aligned}
$$

with $\delta(\beta,\gamma)$ from (32), where we have used that C_a^t is increasing in a. Analogously, the second term can be estimated as

$$
\begin{aligned}
&\|\mathcal{Q}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\|^2\\
&\qquad\lesssim\begin{cases} C_4^{s_2-s_1}\,C_7^{s_1}\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\,N^{-3+10d\beta} & \text{for } 1-d\beta\leq\gamma\leq 1\,,\\ C_4^{s_2-s_1}\,C_7^{s_1}\,\mathfrak{C}_7\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\,N^{4+3d\beta-7\gamma} & \text{for } \frac{2+d\beta}{3}<\gamma\leq 1-d\beta\,,\end{cases}
\end{aligned}
$$

and the third term was already treated in (30). Combining all bounds, we obtain

$$
\begin{aligned}
\|\psi(t)-\psi_\varphi^{(2)}(t)\|^2 &\lesssim \left(\int_0^t\sqrt{C_6^{s_1}}\,\mathrm{d}s_1\int_{s_1}^t\sqrt{C_3^{s_2-s_1}}\,\mathrm{d}s_2\right)^2 N^{-2\delta(\beta,\gamma)}\\
&\lesssim \mathrm{e}^{c(2)\int_0^t\|\varphi(s)\|^2_{H^k(\mathbb{R}^d)}\mathrm{d}s}\,N^{-2\delta(\beta,\gamma)}
\end{aligned}
$$

for some $c(2)>0$, which yields (18) for $n=2$.

Iterating Duhamel's formula (27) $(a-1)$ times, we construct $\psi_\varphi^{(a)}(t)$ as an expansion with $a-1$ terms, where the last term contains the true time evolution $U(t,s)$ and all others exclusively contain $\widetilde{U}_\varphi(t,s)$. Consequently, to construct $\psi_\varphi^{(2)}(t)$, we iterate (27) once more, which yields

$$
\begin{aligned}
&\Big(U(t,0)-\widetilde{U}_\varphi(t,0)\Big)\,\psi\\
&\qquad=-\mathrm{i}\int_0^t\mathrm{d}s\,\widetilde{U}_\varphi(t,s)\left(\mathcal{C}^{\varphi(s)}+\mathcal{Q}^{\varphi(s)}\right)\widetilde{U}_\varphi(s,0)\psi\\
&\qquad\quad-\int_0^t\mathrm{d}s_1\int_{s_1}^t\mathrm{d}s_2\,U(t,s_2)\left(\mathcal{C}^{\varphi(s_2)}+\mathcal{Q}^{\varphi(s_2)}\right)\widetilde{U}_\varphi(s_2,s_1)\times
\end{aligned}
$$

$$\times \left(\mathcal{C}^{\varphi(s_1)} + \mathcal{Q}^{\varphi(s_1)}\right) \widetilde{U}_\varphi(s_1,0)\psi \,.$$

The leading order contributions issue from the first integral and from the expression with two cubic interaction terms. Analogously to above, they determine the next element $\psi_\varphi^{(3)}$ of the sequence $\{\psi_\varphi^{(a)}\}_{a\in\mathbb{N}}$ as

$$\begin{aligned}\psi_\varphi^{(3)}(t) \;&:=\; \widetilde{U}_\varphi(t,0)\psi - \mathrm{i}\int_0^t \mathrm{d}s\, \widetilde{U}_\varphi(t,s)\left(\mathcal{C}^{\varphi(s)} + \mathcal{Q}^{\varphi(s)}\right)\widetilde{U}_\varphi(s,0)\psi_0\\ &\quad -\int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2\, \widetilde{U}_\varphi(t,s_2)\,\mathcal{C}^{\varphi(s_2)}\widetilde{U}_\varphi(s_2,s_1)\,\mathcal{C}^{\varphi(s_1)}\widetilde{U}_\varphi(s_1,0)\psi_0\,,\end{aligned}$$

and similar calculations as before yield $\|\psi(t)-\psi_\varphi^{(3)}(t)\|^2 \lesssim C(t)N^{-3\delta(\beta,\gamma)}$. Continuing the iteration of (27), we obtain for any $a\geq 1$ and $s_0=0$ the expansion

$$\begin{aligned}\psi(t) \;&=\; \sum_{n=0}^{a-1}(-\mathrm{i})^n \int\limits_0^t \mathrm{d}s_1 \int\limits_{s_1}^t \mathrm{d}s_2 \cdots \int\limits_{s_{n-1}}^t \mathrm{d}s_n\, \widetilde{U}_\varphi(t,s_n)\left(\mathcal{C}^{\varphi(s_n)}+\mathcal{Q}^{\varphi(s_n)}\right)\widetilde{U}_\varphi(s_n,s_{n-1})\times\\ &\qquad\qquad \times\cdots\widetilde{U}_\varphi(s_2,s_1)\left(\mathcal{C}^{\varphi(s_1)}+\mathcal{Q}^{\varphi(s_1)}\right)\widetilde{U}_\varphi(s_1,0)\psi_0\\ &\quad +(-\mathrm{i})^a \int\limits_0^t \mathrm{d}s_1 \int\limits_{s_1}^t \mathrm{d}s_2 \cdots \int\limits_{s_{a-1}}^t \mathrm{d}s_a\, U(t,s_a)\left(\mathcal{C}^{\varphi(s_a)}+\mathcal{Q}^{\varphi(s_a)}\right)\widetilde{U}_\varphi(s_a,s_{a-1})\times\\ &\qquad\qquad \times\cdots\widetilde{U}_\varphi(s_2,s_1)\left(\mathcal{C}^{\varphi(s_1)}+\mathcal{Q}^{\varphi(s_1)}\right)\widetilde{U}_\varphi(s_1,0)\psi_0\\ &=\; \sum_{n=0}^{a-1}\prod_{\nu=1}^{n}\left(-\mathrm{i}\int_{s_{\nu-1}}^t \mathrm{d}s_\nu\right)\widetilde{U}_\varphi(t,s_n)\prod_{\ell=0}^{n-1}\Bigg(\left(\mathcal{C}^{\varphi(s_{n-\ell})}+\mathcal{Q}^{\varphi(s_{n-\ell})}\right)\times\\ &\qquad\qquad \times\widetilde{U}_\varphi(s_{n-\ell},s_{n-\ell-1})\Bigg)\psi_0\\ &\quad +\prod_{\nu=1}^{a}\left(-\mathrm{i}\int_{s_{\nu-1}}^t \mathrm{d}s_\nu\right)U(t,s_a)\prod_{\ell=0}^{a-1}\Bigg(\left(\mathcal{C}^{\varphi(s_{a-\ell})}+\mathcal{Q}^{\varphi(s_{a-\ell})}\right)\times\\ &\qquad\qquad \times\widetilde{U}_\varphi(s_{a-\ell},s_{a-\ell-1})\Bigg)\psi_0\,. \qquad (34)\end{aligned}$$

All products are to be understood as ordered, i.e. $\prod_{\ell=0}^{L} P_\ell := P_0P_1\cdots P_L$ for $L\in\mathbb{N}$ and any expressions P_ℓ. Extracting the leading contributions in each order, we construct the sequence $\{\psi_\varphi^{(a)}(t)\}_{a\in\mathbb{N}}$ as follows:

Definition 2.2. Let $I_1^{\varphi(t)} := \mathcal{C}^{\varphi(t)}$ and $I_2^{\varphi(t)} := \mathcal{Q}^{\varphi(t)}$. Define the set

$$\mathcal{S}_n^{(k)} := \left\{(j_1,\ldots,j_n):\ j_\ell\in\{1,2\} \text{ for } \ell=1,\ldots,n \text{ and } \sum_{\ell=1}^{n} j_\ell = k\right\},$$

i.e., the set of n-tuples with elements in $\{1,2\}$ such that the elements of each tuple add to k. Define for $n\in\mathbb{N}$ and $n\leq k\leq 2n$

$$T_n^{(k)} := \sum_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}}(-\mathrm{i})^n\prod_{\nu=1}^{n}\left(\int\limits_{s_{\nu-1}}^{t}\mathrm{d}s_\nu\right)\widetilde{U}_\varphi(t,s_n)\prod_{\ell=0}^{n-1}\left(I_{j_{n-\ell}}^{\varphi(s_{n-\ell})}\widetilde{U}_\varphi(s_{n-\ell},s_{n-\ell-1})\right)\psi_0$$

$$= (-\mathrm{i})^n \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \, \widetilde{U}_\varphi(t,s_n) \times$$
$$\times \sum_{(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}} \left(I_{j_n}^{\varphi(s_n)} \widetilde{U}_\varphi(s_n,s_{n-1}) I_{j_{n-1}}^{\varphi(s_{n-1})} \cdots \widetilde{U}_\varphi(s_2,s_1) I_{j_1}^{\varphi(s_1)} \right) \widetilde{U}_\varphi(s_1,0)\psi_0 \,,$$

where $s_0 := 0$. As above, the products are ordered. For $n = k = 0$, let $T_0^{(0)} := \widetilde{U}_\varphi(t,0)\psi_0$, and $T_n^{(k)} := 0$ for $k < n$ and $k > 2n$. Hence, $T_n^{(k)}$ is an n-dimensional integral where the integrand contains all possible combinations of $I_{j_l}^{\varphi(s_l)}$ such that $\sum_{l=1}^n j_l = k$.
Finally, the elements of the sequence $\{\psi_\varphi^{(a)}\}_{a\in\mathbb{N}}$ are defined as

$$\psi_\varphi^{(a)}(t) := \sum_{k=0}^{a-1} \sum_{n=\lceil \frac{k}{2} \rceil}^{k} T_n^{(k)} = \sum_{n=0}^{a-1} \sum_{k=n}^{\min\{2n,a-1\}} T_n^{(k)}.$$

Theorem 1. *Let $\beta \in [0, \frac{1}{4d})$ and assume* A1 – A3 *with $A \in \{1,\dots,N\}$ and $\gamma \in (\frac{2+d\beta}{3}, 1]$. Let $\psi(t)$ and $\varphi(t)$ denote the solutions of* (4) *and* (8) *with initial data ψ_0 and φ_0 from* A3*, respectively, and let $\psi_\varphi^{(a)}(t)$ be defined as in Definition 2.2. Then for sufficiently large N, $t \in \big[0, T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\big)$ and $a \in \{1,\dots,\lfloor \frac{A}{6} \rfloor\}$, there exists a constant $c(a)$ such that*

$$\|\psi(t) - \psi_\varphi^{(a)}(t)\|^2 \lesssim \mathrm{e}^{c(a)\int_0^t \|\varphi(s)\|^2_{H^k(\mathbb{R}^d)}\mathrm{d}s} N^{-a\delta(\beta,\gamma)},$$

where

$$\delta(\beta,\gamma) = \begin{cases} 1 - 4d\beta & \text{for } 1 - d\beta \le \gamma \le 1\,, \\ 3\gamma - 2 - d\beta & \text{for } \frac{2+d\beta}{3} < \gamma \le 1 - d\beta\,. \end{cases}$$

Hence, given any desired precision of the approximation, there exists some $a \in \mathbb{N}$ such that the corresponding function $\psi_\varphi^{(a)}(t)$ approximates the actual N-body dynamics $\psi(t)$ to this order for large N. To compute $\psi_\varphi^{(a)}(t)$, an a-dependent number of steps is required, as well as the knowledge of the first quantised Bogoliubov time evolution. Put differently, all higher order corrections to the norm approximation follow from the (first order) norm approximation $\widetilde{U}_\varphi(t,0)\psi_0$ after an N-independent number of operations. We cover initial states where the first A moments of the number of fluctuations are sub-leading, where A depends on a but is independent of N.

3 Proofs

3.1 Preliminaries

Lemma 3.1. *Let $\varphi_0 \in H^k(\mathbb{R}^d)$ for $k = \lceil \frac{d}{2} \rceil$, $t \in \big[0, T^{\mathrm{ex}}_{d,v,V^{\mathrm{ext}}}\big)$ and $\varphi(t)$ the solution of* (8) *with inital datum φ_0.*

(a) Let $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $|f(z_j,z_k)| \le F(z_k - z_j)$ almost everywhere for some $F : \mathbb{R}^d \to \mathbb{R}$. Then

$$\|p_1^{\varphi(t)} f(x_1,x_2)\|_{\mathrm{op}} \lesssim \|\varphi(t)\|_{H^k(\mathbb{R}^d)} \|F\|_{L^2(\mathbb{R}^d)}.$$

(b) Let $f:\mathbb{N}_0\to\mathbb{R}_0^+$. Then $P_k^{\varphi(t)}$, $\widehat{f^{\varphi(t)}}\in\mathcal{C}^1\big(\mathbb{R},\mathcal{L}\left(L^2(\mathbb{R}^{dN})\right)\big)$ for $0\le k\le N$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{f^{\varphi(t)}}=\mathrm{i}\Big[\widehat{f^{\varphi(t)}},\sum_{j=1}^N h_j^{\varphi(t)}(t)\Big],$$

where $h_j^{\varphi(t)}(t)$ denotes the one-particle operator $h^{\varphi(t)}(t)$ from (8) acting on the j^{th} coordinate.

Proof. For part (a), see, e.g., [51, Lemma 4.1] and note that $\|\varphi(t)\|_{L^\infty(\mathbb{R}^d)}\lesssim\|\varphi(t)\|_{H^k(\mathbb{R}^d)}$ by the Sobolev embedding theorem. Part (b) can be shown as in the proof of [51, Lemma 6.2]. □

Lemma 3.2. *Let $\psi\in L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$, $\varphi\in L^2(\mathbb{R}^d)$ and $f:\mathbb{N}_0\to\mathbb{R}_0^+$.*

(a) $\left(\widehat{n^\varphi}\right)^2=\frac{1}{N}\sum\limits_{j=1}^N q_j^\varphi$.

(b) Let $a\in\{1,\dots,N\}$. Then for $j\in\{0,\dots,a\}$,

$$\|q_1^\varphi\cdots q_a^\varphi\psi\|^2\le\|q_1^\varphi\cdots q_j^\varphi\left(\widehat{n^\varphi}\right)^{a-j}\psi\|^2.$$

(c) In particular, this implies

$$\left\|\widehat{f^\varphi}q_1^\varphi\psi\right\|^2\le\left\|\widehat{f^\varphi}\widehat{n^\varphi}\psi\right\|^2,\qquad\left\|\widehat{f^\varphi}q_1^\varphi q_2^\varphi\psi\right\|^2\le\left\|\widehat{f^\varphi}\left(\widehat{n^\varphi}\right)^2\psi\right\|^2.$$

Proof. For simplicity, let us drop all superscripts φ. Part (a) is shown e.g. in [51, Lemma 4.1]. For part (b), observe that for any $1\le j\le N$,

$$\begin{aligned}\|q_1\cdots q_j\psi\|^2&=\tfrac{j-1}{N}\langle\psi,q_1\cdots q_j\psi\rangle+\tfrac{N-j+1}{N}\langle\psi,q_1\cdots q_j\psi\rangle\\&\le\tfrac{1}{N}\langle\psi,q_1\cdots q_{j-1}\left(j-1+(N-j+1)q_j\right)\psi\rangle\\&=\left\langle\psi,q_1\cdots q_{j-1}\left(\tfrac{1}{N}\sum_{l=1}^N q_l\right)\psi\right\rangle=\|q_1\cdots q_{j-1}\widehat{n}\psi\|^2\end{aligned}$$

by part (a). Since $\widehat{n}\psi$ is again symmetric, the statement follows by iteration. □

Lemma 3.3. *Denote by T_{ij} an operator acting non-trivially only on coordinates i and j.*

(a) Let $\varphi\in L^2(\mathbb{R}^d)$, let $f,g:\mathbb{N}_0\to\mathbb{R}_0^+$ be any weights and $i,j\in\{1,\dots,N\}$. Let $Q_0^\varphi:=p_i^\varphi p_j^\varphi$, $Q_1^\varphi\in\{p_i^\varphi q_j^\varphi,q_i^\varphi p_j^\varphi\}$ and $Q_2^\varphi:=q_i^\varphi q_j^\varphi$. Then, for $\mu,\nu\in\{0,1,2\}$,

$$Q_\mu^\varphi\,\widehat{f^\varphi}\,T_{ij}\,Q_\nu^\varphi=Q_\mu^\varphi\,T_{ij}\widehat{f^\varphi}_{\mu-\nu}\,Q_\nu^\varphi.$$

(b) Let $\Gamma,\Lambda\in L^2(\mathbb{R}^{dN})$ be symmetric under the exchange of coordinates in a subset $\mathcal{M}\subseteq\{1,\dots,N\}$ such that $j\notin\mathcal{M}$ and $k,l\in\mathcal{M}$. Then

$$|\langle\Gamma,T_{j,k}\Lambda\rangle|\le\|\Gamma\|\Big(|\langle T_{j,k}\Lambda,T_{j,l}\Lambda\rangle|+|\mathcal{M}|^{-1}\|T_{j,k}\Lambda\|^2\Big)^{\frac{1}{2}}.$$

Proof. [51, Lemma 4.1] and [7, Lemma 4.7]. □

Proof of Lemma 2.1. Let us for simplicity drop all superscripts φ. First, observe that

$$\begin{cases} n(k)^{2a} = \left(\frac{k}{N}\right)^a \leq \left(\frac{k+1}{N}\right)^a = m(k)^{2a} & \text{for } k \geq 0, \\ m(k)^{2a} \leq \left(\frac{2k}{N}\right)^a = 2^a n(k)^{2a} & \text{for } k \geq 1\,, \end{cases} \tag{35}$$

hence

$$\widehat{n}^{2a} \leq \widehat{m}^{2a} \leq 2^a \widehat{n}^{2a} + N^{-a} \tag{36}$$

in the sense of operators. The first part of (a) follows from Lemma 3.2b and the first line in (35). For the second part, Lemma 3.2a implies

$$\|\widehat{n}^a\psi\|^2 \;=\; \left\langle \psi, \left(\frac{1}{N}\sum_{j=1}^N q_j\right)^a \psi \right\rangle = N^{-a}\left\langle \psi, \sum_{a_1+\cdots+a_N=a} \binom{a}{a_1,\ldots,a_N} q_1^{a_1}\cdots q_N^{a_N}\psi \right\rangle$$

for $a_1,\ldots,a_N \in \{0,\ldots,a\}$. Due to the symmetry of ψ, since there are $\binom{a-1}{j-1}$ possibilities to write a as the sum of j positive integers and with $\binom{a}{a_1,\ldots,a_N} \leq a!$, this yields

$$\|\widehat{n}^a\psi\|^2 \;=\; \frac{a!}{N^a}\sum_{j=1}^a \binom{N}{j}\binom{a-1}{j-1}\|q_1\cdots q_j\psi\|^2\,.$$

Further, note that

$$\max_{j=\{1,\ldots,a-1\}} \binom{a-1}{j-1} = \binom{a-1}{\lceil\frac{a-1}{2}\rceil} = \frac{(a-1)!}{\lceil\frac{a-1}{2}\rceil!\lfloor\frac{a-1}{2}\rfloor!} \leq 2^{a-1}\,, \tag{37}$$

and $\binom{N}{j} \leq N^j$, hence

$$\|\widehat{m}^a\psi\|^2 \leq N^{-a}\left(1 + 2^{2a-1}a!\sum_{j=1}^a \tbinom{N}{j}\|q_1\cdots q_j\psi\|^2\right)\,.$$

Part (b) follows from (24) and (36). □

3.2 Proof of Proposition 2.4

Proof of Proposition 2.4. The proof of this proposition is essentially an adaptation of the proof of [49, Corollary 4.2]. We begin with part (a). Let $\psi \in L^2(\mathbb{R}^{dN})$ symmetric, $s \in \mathbb{R}$ and $f : \mathbb{N}_0 \to \mathbb{R}_0^+$ some weight function. Define

$$\alpha_{\psi,\varphi,s}(f;t) := \left\langle U(t,s)\psi, \widehat{f^{\varphi(t)}}\, U(t,s)\psi \right\rangle. \tag{38}$$

and

$$Z_{ij}^\beta := \left(v_{ij}^\beta - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)}\right). \tag{39}$$

Let us for the moment abbreviate $U(t,s)\psi =: \psi_t$. By Lemma 3.1b,

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\psi,\varphi,s}(f;t) && (40)\\
&= \mathrm{i}\left\langle \psi_t, \left[H^\beta(t) - \sum_{j=1}^N h_j^{\varphi(t)}(t),\, \widehat{f^{\varphi(t)}} \right] \psi_t \right\rangle\\
&= \mathrm{i}\tfrac{N}{2}\left\langle \psi_t, \left[Z_{12}^\beta,\, \widehat{f^{\varphi(t)}} \right] \psi_t \right\rangle\\
&= 2N\Im \left\langle \psi_t, \left(\widehat{f^{\varphi(t)}} - \widehat{f^{\varphi(t)}_{-1}} \right) q_1^{\varphi(t)} p_2^{\varphi(t)} Z_{12}^\beta p_1^{\varphi(t)} p_2^{\varphi(t)} \psi_t \right\rangle && (41)\\
&\quad + N\Im \left\langle \psi_t, \left(\widehat{f^{\varphi(t)}} - \widehat{f^{\varphi(t)}_{-2}} \right)^{\frac12} q_1^{\varphi(t)} q_2^{\varphi(t)} v_{12}^\beta p_1^{\varphi(t)} p_2^{\varphi(t)} \left(\widehat{f^{\varphi(t)}_2} - \widehat{f^{\varphi(t)}} \right)^{\frac12} \psi_t \right\rangle && (42)\\
&\quad + 2N\Im \left\langle \psi_t, \left(\widehat{f^{\varphi(t)}} - \widehat{f^{\varphi(t)}_{-1}} \right)^{\frac12} q_1^{\varphi(t)} q_2^{\varphi(t)} Z_{12}^\beta p_1^{\varphi(t)} q_2^{\varphi(t)} \left(\widehat{f^{\varphi(t)}_1} - \widehat{f^{\varphi(t)}} \right)^{\frac12} \psi_t \right\rangle, && (43)
\end{aligned}
$$

where we have inserted $\mathbb{1} = (p_1^{\varphi(t)} + q_1^{\varphi(t)})(p_2^{\varphi(t)} + q_2^{\varphi(t)})$ on both sides of the commutator and used Lemma 3.3a. Since $q_1^{\varphi(t)} p_2^{\varphi(t)} Z_{12}^\beta p_1^{\varphi(t)} p_2^{\varphi(t)} = 0$, we conclude that (41) equals zero. From now on, we will for simplicity drop the superscripts $\varphi(t)$. Let

$$
\begin{aligned}
L_f := \Bigg\{ &\sum_{k=2}^N (f(k) - f(k-2)) P_k^{\varphi(t)},\ \sum_{k=1}^N (f(k) - f(k-1))\, P_k^{\varphi(t)},\\
&\sum_{k=0}^{N-2} (f(k+2) - f(k))\, P_k^{\varphi(t)},\ \sum_{k=0}^{N-1} (f(k+1) - f(k))\, P_k^{\varphi(t)} \Bigg\}. \qquad (44)
\end{aligned}
$$

Since, for example, $\left(\widehat{f} - \widehat{f_{-2}}\right)^{\frac12} q_1 q_2 = \left(\sum_{k=2}^N (f(k) - f(k-2)) P_k^{\varphi(t)} \right)^{\frac12} q_1 q_2$, this yields

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\psi,\varphi,s}(f;t)\\
&\lesssim \max_{\widehat{l}\in L_f} \left\{ N \left| \left\langle \psi_t, \widehat{l}^{\frac12} q_1 q_2 v_{12}^\beta p_1 p_2 \widehat{l}^{\frac12} \psi_t \right\rangle \right| + N \left| \left\langle \psi_t, \widehat{l}^{\frac12} q_1 q_2 Z_{12}^\beta p_1 q_2 \widehat{l}^{\frac12} \psi_t \right\rangle \right| \right\}. \qquad (45)
\end{aligned}
$$

By Lemmas 3.1 and 3.2 and since $\|v^\beta\|^2_{L^2(\mathbb{R}^d)} \lesssim N^{d\beta}$, the first term in (45) leads to

$$
\begin{aligned}
&N \left| \left\langle \psi_t, \widehat{l}^{\frac12} q_1 q_2 v_{12}^\beta p_1 p_2 \widehat{l}^{\frac12} \psi_t \right\rangle \right|\\
&\lesssim N \|\widehat{l}^{\frac12} q_1 \psi_t\| \left(\left\langle q_2 v_{12}^\beta p_2\, \widehat{l}^{\frac12} p_1 \psi_t, q_3 v_{13}^\beta p_3\, \widehat{l}^{\frac12} p_1 \psi_t \right\rangle + N^{-1} \|q_2 v_{12}^\beta p_2 p_1\, \widehat{l}^{\frac12} \psi_t\|^2 \right)^{\frac12}\\
&\lesssim N \|\widehat{l}^{\frac12}\, q_1 \psi_t\| \left(\|\widehat{l}^{\frac12} q_3 \psi_t\| \|p_1 p_2 v_{12}^\beta v_{13}^\beta p_3 p_1\|_{\mathrm{op}} \|\widehat{l}^{\frac12} q_2 \psi_t\| + N^{-1} \|v_{12}^\beta p_2\|^2_{\mathrm{op}} \|\widehat{l}^{\frac12} \psi_t\|^2 \right)^{\frac12}\\
&\lesssim N \left\langle \psi_t, \widehat{l}\,\widehat{n}^2 \psi_t \right\rangle^{\frac12} \left(\left\langle \psi_t, \widehat{l}\,\widehat{n}^2 \psi_t \right\rangle + N^{-1+d\beta} \left\langle \psi_t, \widehat{l} \psi_t \right\rangle \right)^{\frac12} \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}, \qquad (46)
\end{aligned}
$$

To obtain the estimate in the last line, note first that

$$
\|p_1 p_2 v_{13}^\beta v_{12}^\beta p_1 p_3\|_{\mathrm{op}} = \|p_1 v_{13}^\beta p_2 p_3 v_{12}^\beta p_1\|_{\mathrm{op}} = \|p_1 v_{12}^\beta p_2\|^2_{\mathrm{op}}.
$$

Now we decompose $v^\beta = v^\beta_+ - v^\beta_-$ into its positive and negative part such that $v^\beta_\pm \geq 0$, hence $v^\beta_\pm(x) = \sqrt{v^\beta_\pm(x)}\sqrt{v^\beta_\pm(x)}$, which leads to

$$\begin{aligned}
\|p_1 v^\beta_{12} p_2\|_{\mathrm{op}} &= \|p_1 (v^\beta_+ - v^\beta_-)_{12} p_2\|_{\mathrm{op}} \\
&\leq \|p_1 \sqrt{(v^\beta_+)_{12}}\sqrt{(v^\beta_+)_{12}} p_2\|_{\mathrm{op}} + \|p_1 \sqrt{(v^\beta_-)_{12}}\sqrt{(v^\beta_-)_{12}} p_2\|_{\mathrm{op}} \\
&\lesssim \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)} \left(\|v^\beta_+\|_{L^1(\mathbb{R}^d)} + \|v^\beta_-\|_{L^1(\mathbb{R}^d)}\right) \\
&= \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)} \|v^\beta\|_{L^1(\mathbb{R}^d)} \lesssim \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}
\end{aligned}$$

by Lemma 3.1. The second term in (45) can be estimated as

$$\begin{aligned}
N\left|\left\langle \psi_t, \widehat{l}^{\frac{1}{2}} q_1 q_2 Z^\beta_{12} p_1 q_2 \widehat{l}^{\frac{1}{2}} \psi_t \right\rangle\right| &\lesssim N \|\widehat{l}^{\frac{1}{2}} q_1 q_2 \psi_t\| \|\widehat{l}^{\frac{1}{2}} \widehat{n} \psi_t\| \|Z^\beta_{12} p_1\|_{\mathrm{op}} \\
&\lesssim N^{1+\frac{d\beta}{2}} \left\langle \psi_t, \widehat{l}\, \widehat{n}^4 \psi_t \right\rangle^{\frac{1}{2}} \left\langle \psi_t, \widehat{l}\, \widehat{n}^2 \psi_t \right\rangle^{\frac{1}{2}} \|\varphi(t)\|_{H^k(\mathbb{R}^d)} \quad (47)
\end{aligned}$$

Now we choose for f the family of weight functions $w^j_\lambda : k \mapsto (w_\lambda(k))^j$ given by

$$w_\lambda(k) := \begin{cases} \dfrac{k+1}{N^\lambda} & 0 \leq k \leq N^\lambda - 1, \\ 1 & \text{else} \end{cases} \tag{48}$$

for some $0 < \lambda \leq 1 - d\beta$ and $j \in \{0, \ldots, N\}$. The set corresponding to L_f from (44) is called $L_{w^j_\lambda}$. To bound the operators in $L_{w^j_\lambda}$, note that for any $a, b \in \mathbb{N}_0$, $a > b$,

$$\begin{aligned}
(k+a)^j - (k+b)^j &= \tbinom{j}{j-1} k^{j-1}(a-b) + \tbinom{j}{j-2} k^{j-2}(a^2 - b^2) + \ldots + (a^j - b^j) \\
&\leq j a^j \left(\tbinom{j-1}{j-1} k^{j-1} + \tbinom{j-1}{j-2} k^{j-2} + \ldots + \tbinom{j-1}{1} k + 1\right) \\
&= j a^j (k+1)^{j-1},
\end{aligned}$$

where we have used in the second line that for every $1 \leq m \leq j-1$,

$$\tbinom{j}{m} = \tfrac{j(j-1)!}{(j-m)((j-1)-m)!m!} = \tfrac{j}{j-m}\tbinom{j-1}{m} \leq j\tbinom{j-1}{m},$$

and that $a^j \geq a^\ell - b^\ell$ for any $1 \leq \ell \leq j$ and $j \geq 1$ (the statement is trivial for $j = 0$). Since $w_\lambda(k) \leq \frac{k+1}{N^\lambda}$ for all k, especially also if $k > N^\lambda - 1$, we conclude that

$$\begin{aligned}
(w_\lambda(k))^j - (w_\lambda(k-1))^j &\leq \tfrac{(k+1)^j - k^j}{N^{\lambda j}} \leq j\tfrac{(k+1)^{j-1}}{N^{\lambda j}} = j\tfrac{w_\lambda(k)^{j-1}}{N^\lambda} && 1 \leq k \leq N^\lambda - 1, \\
(w_\lambda(k+1))^j - (w_\lambda(k))^j &\leq \tfrac{(k+2)^j - (k+1)^j}{N^{\lambda j}} \leq j2^j\tfrac{(k+1)^{j-1}}{N^{\lambda j}} = j2^j\tfrac{w_\lambda(k)^{j-1}}{N^\lambda} && 0 \leq k \leq N^\lambda - 1, \\
(w_\lambda(k+2))^j - (w_\lambda(k))^j &\leq \tfrac{(k+3)^j - (k+1)^j}{N^{\lambda j}} \leq j3^j\tfrac{(k+1)^{j-1}}{N^{\lambda j}} = j3^j\tfrac{w_\lambda(k)^{j-1}}{N^\lambda} && 0 \leq k \leq N^\lambda - 1.
\end{aligned}$$

Besides, one computes analogously to above that $(k+1)^j - (k-1)^j \leq 2j(k+1)^{j-1}$, hence

$$(w_\lambda(k))^j - (w_\lambda(k-2))^j \leq \tfrac{(k+1)^j - k^j}{N^{\lambda j}} \leq 2j\tfrac{(k+1)^{j-1}}{N^{\lambda j}} = 2j\tfrac{w_\lambda(k)^{j-1}}{N^\lambda} \qquad 2 \leq k \leq N^\lambda - 1\,.$$

Finally, $w_\lambda(k)=1$ for $k>N^\lambda-1$, hence the above estimates imply

$$\begin{aligned}(w_\lambda(k))^j-(w_\lambda(k-1))^j &\le j\tfrac{(k+1)^{j-1}}{N^{\lambda j}}\le j2^{j-1}N^{-\lambda}=j2^{j-1}\tfrac{w_\lambda(k)^{j-1}}{N^\lambda} && N^\lambda-1<k\le N^\lambda,\\ (w_\lambda(k))^j-(w_\lambda(k-2))^j &\le 2j\tfrac{(k+1)^{j-1}}{N^{\lambda j}}\le j2^{j}N^{-\lambda}=j2^{j}\tfrac{w_\lambda(k)^{j-1}}{N^\lambda} && N^\lambda-1\le k\le N^\lambda.\end{aligned}$$

For all other values of k, the differences yield zero. Thus, every element of $L_{w_\lambda^j}$ can be bounded, in the sense of operators, by the operator corresponding to the weight function

$$l_\lambda^j(k)=\begin{cases} j3^j\,\dfrac{w_\lambda(k)^{j-1}}{N^\lambda} & 0\le k\le N^\lambda,\\ 0 & \text{else.}\end{cases} \tag{49}$$

Besides, since $l_\lambda^j(k)=0$ for $k>N^\lambda+1$, one obtains

$$\begin{aligned} l_\lambda^j(k)n^2(k) &\le j3^jN^{-1}w_\lambda^j(k), && (50)\\ l_\lambda^j(k)n^4(k) &\le j3^jw_\lambda^j(k)\tfrac{k}{N^2}\le j3^jw_\lambda^j(k)\tfrac{N^\lambda+1}{N^2}\lesssim j3^jN^{-2+\lambda}w_\lambda^j(k). && (51)\end{aligned}$$

Inserting (49) to (51) into (46) and (47) and using that $\lambda\le 1-d\beta$ implies $N^{\frac{d\beta+\lambda-1}{2}}\le 1$, we conclude that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\alpha_{\psi,\varphi,s}(w_\lambda^j;t)\lesssim j3^j\|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}\left(\alpha_{\psi,\varphi,s}(w_\lambda^j;t)+N^{d\beta-\lambda}\alpha_{\psi,\varphi,s}(w_\lambda^{j-1};t)\right). \tag{52}$$

Now we apply Grönwall's inequality, for now on using the abbreviations $\alpha_{\psi,\varphi,s}(w_\lambda^j;t)=:\alpha_j(t)$ and $I_t:=\int_s^t\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\,\mathrm{d}s_1$. This yields

$$\begin{aligned}\alpha_j(t) &\lesssim \mathrm{e}^{j3^jI_t}\left(\alpha_j(s)+j3^jN^{d\beta-\lambda}\int_s^t\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\alpha_{j-1}(s_1)\,\mathrm{d}s_1\right)\\ &\le \mathrm{e}^{j3^jI_t}\alpha_j(s)+j3^j\mathrm{e}^{j(3^j+3^{j-1})I_t}I_tN^{d\beta-\lambda}\alpha_{j-1}(s)\\ &\quad +j(j-1)3^{j+(j-1)}\mathrm{e}^{j(3^j+3^{j-1})I_t}I_t^2N^{2(d\beta-\lambda)}\int_s^t\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\alpha_{j-2}(s_1)\,\mathrm{d}s_1\\ &\lesssim \mathrm{e}^{j3^jI_t}\alpha_j(s)\\ &\quad +j3^j\mathrm{e}^{j(3^j+3^{j-1})I_t}I_tN^{d\beta-\lambda}\alpha_{j-1}(s)\\ &\quad +j(j-1)3^{j+(j-1)}\mathrm{e}^{j(3^j+3^{j-1}+3^{j-2})I_t}I_t^2N^{2(d\beta-\lambda)}\alpha_{j-2}(s)\\ &\quad +j(j-1)(j-2)3^{j+(j-1)+(j-2)}\mathrm{e}^{j(3^j+3^{j-1}+3^{j-2})I_t}I_t^2N^{3(d\beta-\lambda)}\times\\ &\qquad\qquad\times\int_s^t\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\alpha_{j-3}(s_1)\,\mathrm{d}s_1\\ &\lesssim \dots\\ &\lesssim \sum_{n=0}^j\tfrac{j!}{(j-n)!}3^{\frac{n(2j+1-n)}{2}}\mathrm{e}^{2j3^jI_t}I_t^nN^{n(d\beta-\lambda)}\alpha_{j-n}(s)\,,\end{aligned}$$

where we have used that all integrands are non-negative and thus the upper boundary of all integrals could be replaced by t. Written explicitly, this gives

$$\alpha_{\psi,\varphi,s}(w_\lambda^j;t)\lesssim C_j^{t,s}\sum_{n=0}^jN^{n(d\beta-\lambda)}\alpha_{\psi,\varphi,s}(w_\lambda^{j-n};s)=C_j^{t,s}\sum_{n=0}^jN^{n(d\beta-\lambda)}\left\langle\psi,\widehat{w_\lambda}^{j-n}\psi\right\rangle, \tag{53}$$

with

$$C_j^{t,s} := j!\, 3^{j(j+1)} \mathrm{e}^{9^j \int_s^t \|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)} \mathrm{d}s_1},$$

where we have estimated $I_t^j \mathrm{e}^{2j3^j I_t} < \mathrm{e}^{9^j I_t}$. To relate this estimate to $\|\widehat{m}^j\psi\|^2$, observe that for $0 \le k \le N$,

$$w_\lambda^j(k) \le \left(\tfrac{k+1}{N^\lambda}\right)^j = \left(\tfrac{k+1}{N}\right)^j N^{j(1-\lambda)} = m^{2j}(k) N^{j(1-\lambda)},$$

and

$$m^{2j}(k) \;=\; \left(\tfrac{k+1}{N}\right)^j \le \begin{cases} \left(\tfrac{k+1}{N^\lambda}\right)^j N^{-j(1-\lambda)} \;=\; w_\lambda^j(k) N^{-j(1-\lambda)} & \text{for } 0 \le k \le N^\lambda - 1, \\ 2^j \;=\; 2^j w_\lambda^b(k) \;\text{ for any } b \in \mathbb{N} & \text{for } N^\lambda - 1 \le k \le N. \end{cases}$$

Consequently, $m^{2j}(k) \le N^{-j(1-\lambda)} w_\lambda^j(k) + w_\lambda^b(k)$, and we conclude

$$\begin{aligned} \alpha_{\psi,\varphi,s}(w_\lambda^j;t) &= \left\langle \psi_t, \widehat{w_\lambda}^j \psi_t \right\rangle \le N^{j(1-\lambda)} \left\langle \psi_t, \widehat{m}^{2j}\psi_t \right\rangle = N^{j(1-\lambda)} \|\widehat{m}^j \psi_t\|^2, \\ \|\widehat{m}^j\psi_t\|^2 &= \left\langle \psi_t, \widehat{m}^{2j}\psi_t \right\rangle \le N^{-j(1-\lambda)} \alpha_{\psi,\varphi,s}(w_\lambda^j;t) + 2^j \alpha_{\psi,\varphi,s}(w_\lambda^b;t) \end{aligned}$$

for any $b \in \mathbb{N}$. Inserting these estimates into (53) yields

$$\begin{aligned} &\|\widehat{m}^j U(t,s)\psi\|^2 \\ &\quad\lesssim\; C_j^{t,s} \sum_{n=0}^{j} N^{n(-1+d\beta)} \|\widehat{m}^{j-n}\psi\|^2 + 2^j C_b^{t,s} \sum_{n=0}^{b} N^{n(-1+d\beta)+b(1-\lambda)} \|\widehat{m}^{b-n}\psi\|^2. \end{aligned}$$

To minimise the second term, we choose the maximal $\lambda = 1 - d\beta$, which concludes the proof of part (a).

The proof of part (b) is much simpler since we now consider the time evolution $\widetilde{U}_\varphi(t,s)$. The term corresponding to (43) vanishes, which implies that we may directly consider the weights $m^{2j}(k)$ instead of taking the detour via $w_\lambda^j(k)$. Analogously to (38), we define

$$\widetilde{\alpha}_{\psi,\varphi,s}(f;t) := \left\langle \widetilde{U}_\varphi(t,s)\psi, \widehat{f^{\varphi(t)}}\, \widetilde{U}_\varphi(t,s)\psi \right\rangle.$$

We will now abbreviate $\widetilde{U}_\varphi(t,s)\psi =: \widetilde{\psi}_t$. In this notation,

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{\alpha}_{\psi,\varphi,s}(f;t) \\ &\quad= \mathrm{i}\left\langle \widetilde{\psi}_t, \left[\widetilde{H}^{\varphi(t)}(t) - \sum_{j=1}^{N} h_j^{\varphi(t)}(t), \widehat{f^{\varphi(t)}} \right] \widetilde{\psi}_t \right\rangle \\ &\quad= \mathrm{i}\tfrac{N}{2} \left\langle \widetilde{\psi}_t, \left[p_1^{\varphi(t)} q_2^{\varphi(t)} v_{12}^\beta q_1^{\varphi(t)} p_2^{\varphi(t)} + \text{h.c.}\,, \widehat{f^{\varphi(t)}} \right] \widetilde{\psi}_t \right\rangle \\ &\qquad + \mathrm{i}\tfrac{N}{2} \left\langle \widetilde{\psi}_t, \left[p_1^{\varphi(t)} p_2^{\varphi(t)} v_{12}^\beta q_1^{\varphi(t)} q_2^{\varphi(t)} + \text{h.c.}\,, \widehat{f^{\varphi(t)}} \right] \widetilde{\psi}_t \right\rangle \\ &\quad= -N\Im \left\langle \widetilde{\psi}_t, q_1^{\varphi(t)} q_2^{\varphi(t)} \left(\widehat{f^{\varphi(t)}} - \widehat{f^{\varphi(t)}_{-2}} \right)^{\frac{1}{2}} v_{12}^\beta p_1^{\varphi(t)} p_2^{\varphi(t)} \left(\widehat{f_2^{\varphi(t)}} - \widehat{f^{\varphi(t)}} \right)^{\frac{1}{2}} \widetilde{\psi}_t \right\rangle. \end{aligned}$$

We now evaluate this expression for the weight $m^{2j}(k)$, i.e.

$$\widetilde{\alpha}_{\psi,\varphi,s}(m^{2j};t) = \left\| \left(\widehat{m^{\varphi(t)}}\right)^j \widetilde{\psi}_t \right\|^2 .$$

This corresponds to $w^j_\lambda(k)$ with the choice $\lambda = 1$ in (48). Consequently, we define $l^j(k) := j3^j N^{-1} m^{2(j-1)}(k)$ analogously to (49) and conclude that $m^{2j}(k) - m^{2j}(k-2) \leq l^j(k)$ and $m^{2j}(k+2) - m^{2j}(k) \leq l^j(k)$. Analogously to the estimate of the first term in (45) and using the relation (50) for $\lambda = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \| \left(\widehat{m^{\varphi(t)}}\right)^j \widetilde{\psi} \|^2 \lesssim j3^j \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)} \left(\| \left(\widehat{m^{\varphi(t)}}\right)^j \widetilde{\psi} \|^2 + N^{-1+d\beta} \| \left(\widehat{m^{\varphi(t)}}\right)^{j-1} \widetilde{\psi} \|^2 \right).$$

The same Grönwall argument which led to (53) concludes the proof. □

Proof of Corollary 2.5. From Proposition 2.4a and the assumptions on the initial data, we conclude that for every $b \in \mathbb{N}$ and sufficiently large N,

$$\begin{aligned} \left\| \left(\widehat{m^{\varphi(t)}}\right)^a \psi(t) \right\|^2 \quad &\lesssim \quad C^t_a \sum_{n=0}^{a} \mathfrak{C}_{a-n}\, N^{n(-1+d\beta+\gamma)-\gamma a} \\ &\quad + 2^b C^t_b \sum_{n=0}^{b} \mathfrak{C}_{b-n}\, N^{n(-1+d\beta+\gamma)-b(\gamma-d\beta)} . \end{aligned}$$

If $\gamma \geq 1 - d\beta$, the leading order terms in both sums are the ones with maximal n, hence

$$\left\| \left(\widehat{m^{\varphi(t)}}\right)^a \psi(t) \right\|^2 \lesssim (a+1) C^t_a\, N^{a(-1+d\beta)} + (b+1) C^t_b\, N^{b(-1+2d\beta)}.$$

If one chooses $b > a\frac{1-d\beta}{1-2d\beta}$ for fixed $\beta < \frac{1}{2d}$, the second term is for sufficiently large N dominated by the first one. For $\gamma < 1 - d\beta$, the leading order terms are those with $n = 0$, hence

$$\left\| \left(\widehat{m^{\varphi(t)}}\right)^a \psi(t) \right\|^2 \lesssim (a+1) C^t_a\, \mathfrak{C}_a\, N^{-\gamma a} + (b+1) 2^b C^t_b\, \mathfrak{C}_b\, N^{-b(\gamma-d\beta)} ,$$

which yields a non-trivial bound only for $\gamma > d\beta$. Part (b) follows analogously from part (b) of Proposition 2.4 without the restrictions on β and γ that are due to the second sum. □

3.3 Proof of Theorem 1

Proof of Lemma 2.6. We use the abbreviation $Z^\beta_{ij} = v^\beta_{ij} - \overline{v}^{\varphi(t)}(x_i) - \overline{v}^{\varphi(t)}(x_j) + 2\mu^{\varphi(t)}$ as in (39), and drop all superscripts $\varphi(t)$ in $p^{\varphi(t)}$, $q^{\varphi(t)}$ and $\widehat{m^{\varphi(t)}}$ for simplicity. By Lemma 3.3a, $\mathcal{Q}^{\varphi(t)} \widehat{m}^a = \widehat{m}^a \mathcal{Q}^{\varphi(t)}$, hence

$$\begin{aligned} \|\widehat{m}^a \mathcal{Q}^{\varphi(t)} \psi\|^2 \quad &= \quad \tfrac{1}{(N-1)^2} \sum_{i<j} \sum_{k<l} \left\langle \widehat{m}^a \psi, q_i q_j Z^\beta_{ij} q_i q_j q_k q_l Z^\beta_{kl} q_k q_l \widehat{m}^a \psi \right\rangle \\ &= \quad \tfrac{N}{2(N-1)} \left\langle \widehat{m}^a \psi, q_1 q_2 Z^\beta_{12} q_1 q_2 Z^\beta_{12} q_1 q_2 \widehat{m}^a \psi \right\rangle \end{aligned}$$

$$
\begin{aligned}
&+\tfrac{N(N-2)}{N-1}\left\langle \widehat{m}^a\psi, q_1q_2Z_{12}^{\beta}q_1q_2q_3Z_{13}^{\beta}q_1q_3\widehat{m}^a\psi\right\rangle \\
&+\tfrac{N(N-2)(N-3)}{4(N-1)}\left\langle \widehat{m}^a\psi, q_1q_2Z_{12}^{\beta}q_1q_2q_3q_4Z_{34}^{\beta}q_3q_4\widehat{m}^a\psi\right\rangle \\
\lesssim\ & N^{2d\beta}\left(\|q_1q_2\widehat{m}^a\psi\|^2 + N\|q_1q_2q_3\widehat{m}^a\psi\|^2 + N^2\|q_1q_2q_3q_4\widehat{m}^a\psi\|^2\right),
\end{aligned}
$$

where we have used that $\|Z_{ij}^{\beta}\|_{L^\infty(\mathbb{R}^d)} \lesssim N^{d\beta}$ by Young's inequality. Now observe that

$$
\begin{aligned}
\tbinom{N}{2}\|q_1q_2\widehat{m}^a\psi\|^2 &= \sum_{i<j}\langle \widehat{m}^a\psi, q_iq_j\widehat{m}^a\psi\rangle < \sum_{i,j}\langle \widehat{m}^a\psi, q_iq_j\widehat{m}^a\psi\rangle \\
&< \sum_{i,j,k,l}\langle \widehat{m}^a\psi, q_iq_jq_kq_l\widehat{m}^a\psi\rangle ,
\end{aligned}
$$

hence

$$
\begin{aligned}
\|q_1q_2\widehat{m}^a\psi\|^2 &\lesssim N^{-2}\sum_{i,j,k,l}\langle \widehat{m}^a\psi, q_iq_jq_kq_l\widehat{m}^a\psi\rangle = N^2\left\langle \widehat{m}^a\psi, \left(\tfrac{1}{N}\sum_{j=1}^{N}q_j\right)^4\widehat{m}^a\psi\right\rangle \\
&= N^2\left\langle \widehat{m}^a\psi, \widehat{n}^8\widehat{m}^a\psi\right\rangle < N^2\|\widehat{m}^{4+a}\psi\|^2,
\end{aligned}
$$

by (35), and analogously

$$
\begin{aligned}
\|q_1q_2q_3\widehat{m}^a\psi\|^2 &= \tbinom{N}{3}^{-1}\sum_{i<j<k}\langle \widehat{m}^a\psi, q_iq_jq_k\widehat{m}^a\psi\rangle \\
&\lesssim N^{-3}\sum_{i,j,k,l}\langle \widehat{m}^a\psi, q_iq_jq_kq_l\widehat{m}^a\psi\rangle \lesssim N\|\widehat{m}^{4+a}\psi\|^2, \\
\|q_1q_2q_3q_4\widehat{m}^a\psi\|^2 &= \tbinom{N}{4}^{-1}\sum_{i<j<k<l}\langle \widehat{m}^a\psi, q_iq_jq_kq_l\widehat{m}^a\psi\rangle \\
&\lesssim N^{-4}\sum_{i,j,k,l}\langle \widehat{m}^a\psi, q_iq_jq_kq_l\widehat{m}^a\psi\rangle \lesssim \|\widehat{m}^{4+a}\psi\|^2.
\end{aligned}
$$

This implies part (a). For part (b), note that by Lemma 3.3a,

$$
\begin{aligned}
\widehat{m}^a\mathcal{C}^{\varphi(t)} &= \tfrac{1}{N-1}\sum_{i<j}\left(q_iq_jZ_{ij}^{\beta}(q_ip_j+p_iq_j)\right)\widehat{m}_1^a \\
&\quad+\tfrac{1}{N-1}\sum_{i<j}\left((p_iq_j+q_ip_j)Z_{ij}^{\beta}q_iq_j\right)\widehat{m}_{-1}^a.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|\widehat{m}^a\mathcal{C}^{\varphi(t)}\psi\|^2 \\
&= \tfrac{1}{(N-1)^2}\sum_{i<j}\sum_{k<l}\Big(\left\langle \widehat{m}_1^a\psi, (q_ip_j+p_iq_j)Z_{ij}^{\beta}q_iq_jq_kq_lZ_{kl}^{\beta}(p_kq_l+q_kp_l)\widehat{m}_1^a\psi\right\rangle \\
&\qquad+\left\langle \widehat{m}_1^a\psi, (q_ip_j+p_iq_j)Z_{ij}^{\beta}q_iq_j(p_kq_l+q_kp_l)Z_{kl}^{\beta}q_lq_k\widehat{m}_{-1}^a\psi\right\rangle \\
&\qquad+\left\langle \widehat{m}_{-1}^a\psi, q_iq_jZ_{ij}^{\beta}(p_iq_j+q_ip_j)q_kq_lZ_{kl}^{\beta}(p_kq_l+q_kp_l)\widehat{m}_1^a\psi\right\rangle \\
&\qquad+\left\langle \widehat{m}_{-1}^a\psi, q_iq_jZ_{ij}^{\beta}(p_iq_j+q_ip_j)(p_kq_l+q_kp_l)Z_{kl}^{\beta}q_kq_l\widehat{m}_{-1}^a\psi\right\rangle\Big)
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \; N^{d\beta}\left(\|q_1\widehat{m}_1^a\psi\|^2+\|q_1q_2\widehat{m}_{-1}^a\psi\|^2\right)\|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}\\
&\quad +N^{1+d\beta}\left(\|q_1q_2\widehat{m}_1^a\psi\|^2+\|q_1\widehat{m}_1^a\psi\|\|q_1q_2q_3\widehat{m}_{-1}^a\psi\|+\|q_1q_2\widehat{m}_{-1}^a\psi\|^2\right)\|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}\\
&\quad +N^{2+d\beta}\left(\|q_1q_2q_3\widehat{m}_1^a\psi\|^2+\|q_1q_2q_3\widehat{m}_{-1}^a\psi\|^2+\|q_1q_2\widehat{m}_1^a\psi\|\|q_1q_2q_3q_4\widehat{m}_{-1}^a\psi\|\right)\times\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}
\end{aligned}$$

similarly to the estimate of $\|\widehat{m}^a\mathcal{Q}^{\varphi(t)}\psi\|$. The last inequality follows because by Lemma 3.1a, $\|p_1Z_{12}^\beta\|^2_{\mathrm{op}}\lesssim N^{d\beta}\|\varphi(t)\|^2_{H^k(\mathbb{R}^d)}$ due to Young's inequality and since $\|v^\beta\|^2_{L^2(\mathbb{R}^d)}\lesssim N^{d\beta}$. Further, note that

$$\begin{aligned}
\widehat{m}_1^{2a} &= \left(\sum_{k=0}^{N-1}m(k+1)P_k\right)^{2a} = \left(\sum_{k=0}^{N-1}\sqrt{\tfrac{k+2}{N}}P_k\right)^{2a} \le \left(2\sum_{k=0}^{N}\sqrt{\tfrac{k+1}{N}}P_k\right)^{2a} = 4^a\widehat{m}^{2a}\,,\\
\widehat{m}_{-1}^{2a} &= \left(\sum_{k=1}^{N}m(k-1)P_k\right)^{2a} = \left(\sum_{k=1}^{N}\sqrt{\tfrac{k}{N}}P_k\right)^{2a} \le \left(\sum_{k=0}^{N}\sqrt{\tfrac{k+1}{N}}P_k\right)^{2a} = \widehat{m}^{2a}
\end{aligned}$$

in the sense of operators. As in the estimate of $\mathcal{Q}^{\varphi(t)}$, we thus obtain for $\ell\in\{-1,1\}$

$$\|q_1\widehat{m}_\ell^a\psi\|^2 \;<\; N^{-1}\sum_{i,j,k}\langle\widehat{m}_\ell^a\psi,q_iq_jq_k\widehat{m}_\ell^a\psi\rangle = N^2\left\langle\widehat{n}^3\psi,\widehat{m}_\ell^{2a}\,\widehat{n}^3\psi\right\rangle \le 2^{2a}N^2\|\widehat{m}^{a+3}\psi\|^2\,,$$

and analogously $\|q_1q_2\widehat{m}_\ell^a\psi\|<4^aN\|\widehat{m}^{a+3}\psi\|^2$ and $\|q_1q_2q_3\widehat{m}_\ell^a\psi\|<4^a\|\widehat{m}^{a+3}\psi\|^2$. Together, this implies part (b). □

Proof of Theorem 1. Let $a\in\mathbb{N}_0$ such that $6a\le A$. Recall that by Definition 2.2,

$$\psi_\varphi^{(a+1)}(t) \;=\; \sum_{n=0}^{a}\sum_{k=n}^{\min\{2n,a\}}T_n^{(k)}$$

for any $a\ge 0$, where $T_n^{(k)}$ is given by

$$T_n^{(k)}=\sum_{(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}}(-\mathrm{i})^n\prod_{\nu=1}^{n}\left(\int_{s_{\nu-1}}^{t}\mathrm{d}s_\nu\right)\widetilde{U}_\varphi(t,s_n)\,t^{(k)}_{(j_1,\dots,j_n)}\,,$$

where

$$t^{(k)}_{(j_1,\dots,j_n)}:=\begin{cases}0 & \text{for } k<n \text{ and } k>2n,\\ \psi_0 & \text{for } k=n=0,\\ \prod_{\ell=0}^{n-1}\left(I_{j_{n-\ell}}^{\varphi(s_{n-\ell})}\widetilde{U}_\varphi(s_{n-\ell},s_{n-\ell-1})\right)\psi_0 & \text{else,}\end{cases}$$

with $I_1^{\varphi(t)}=\mathcal{C}^{\varphi(t)}$ and $I_2^{\varphi(t)}=\mathcal{Q}^{\varphi(t)}$ and $(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}$. In this notation,

$$\prod_{\ell=0}^{n-1}\left(\left(\mathcal{C}^{\varphi(s_{n-\ell})}+\mathcal{Q}^{\varphi(s_{n-\ell})}\right)\widetilde{U}_\varphi(s_{n-\ell},s_{n-\ell-1}\right)=\sum_{k=n}^{2n}\sum_{(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}}t^{(k)}_{(j_1,\dots,j_n)}\,,$$

hence the Duhamel expansion (34) of $\psi(t)$ reads

$$\psi(t) = \sum_{n=0}^{a-1} \sum_{k=n}^{2n} T_n^{(k)} + \sum_{k=a}^{2a} \widetilde{T}_a^{(k)} .$$

Here, $\widetilde{T}_n^{(k)}$ is obtained from $T_n^{(k)}$ by replacing the first $\widetilde{U}_\varphi(t, s_n)$ by the full time evolution $U(t, s_n)$, i.e., for $n < k < 2n$,

$$\widetilde{T}_n^{(k)} := \sum_{(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}} (-\mathrm{i})^n \prod_{\nu=1}^{n} \left(\int\limits_{s_{\nu-1}}^{t} \mathrm{d}s_\nu \right) U(t, s_n) \prod_{l=0}^{n-1} \left(I_{j_{n-l}}^{\varphi(s_{n-l})} \widetilde{U}_\varphi(s_{n-l}, s_{n-l-1}) \right) \psi_0 .$$

Consequently,

$$\begin{aligned}
\psi(t) - \psi_\varphi^{(a+1)}(t) &= \sum_{n=0}^{a-1} \sum_{k=\min\{2n,a\}+1}^{2n} T_n^{(k)} + \sum_{k=a}^{2a} \widetilde{T}_a^{(k)} - \sum_{k=a}^{\min\{2a,a\}} T_a^{(k)} \\
&= \sum_{n=\lceil \frac{a+1}{2} \rceil}^{a-1} \sum_{k=a+1}^{2n} T_n^{(k)} + \sum_{k=a+1}^{2a} \widetilde{T}_a^{(k)} + \left(\widetilde{T}_a^{(a)} - T_a^{(a)} \right) \qquad (54)
\end{aligned}$$

since the first double sum contributes only if $2n \geq a+1$, and in this case $\min\{2n, a\} = a$. Note that for $k = n$, $j_1 = \dots = j_k = 1$, hence $T_k^{(k)}$ and $\widetilde{T}_k^{(k)}$ exclusively contain $\mathcal{C}^{\varphi(s_l)}$. Using Duhamel's formula, the last expression can thus be expanded as

$$\begin{aligned}
&\widetilde{T}_a^{(a)} - T_a^{(a)} \\
&= (-\mathrm{i})^a \int\limits_0^t \mathrm{d}s_1 \cdots \int\limits_{s_{a-1}}^t \mathrm{d}s_a \left(U(t, s_a) - \widetilde{U}_\varphi(t, s_a) \right) \mathcal{C}^{\varphi(s_a)} \widetilde{U}_\varphi(s_a, s_{a-1}) \mathcal{C}^{\varphi(s_{a-1})} \times \\
&\qquad\qquad \times \cdots \mathcal{C}^{\varphi(s_1)} \widetilde{U}_\varphi(s_1, 0) \psi_0 \\
&= (-\mathrm{i})^{a+1} \int\limits_0^t \mathrm{d}s_1 \cdots \int\limits_{s_a}^t \mathrm{d}s_{a+1} U(t, s_{a+1}) \left(\mathcal{C}^{\varphi(s_{a+1})} + \mathcal{Q}^{\varphi(s_{a+1})} \right) \widetilde{U}_\varphi(s_{a+1}, s_a) \mathcal{C}^{\varphi(s_a)} \times \\
&\qquad\qquad \times \cdots \mathcal{C}^{\varphi(s_1)} \widetilde{U}_\varphi(s_1, 0) \psi_0 \\
&= \widetilde{T}_{a+1}^{(a+1)} + (-\mathrm{i})^{a+1} \int\limits_0^t \mathrm{d}s_1 \cdots \int\limits_{s_a}^t \mathrm{d}s_{a+1} U(t, s_{a+1}) \, t_{(1,1,\dots,1,2)}^{(a+2)} . \qquad (55)
\end{aligned}$$

By unitarity of $U(t, s)$ and $\widetilde{U}_\varphi(t, s)$,

$$\begin{aligned}
\|T_n^{(k)}\| &\leq \sum_{(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}} \int\limits_0^t \mathrm{d}s_1 \cdots \int\limits_0^t \mathrm{d}s_n \|t_{(j_1,\dots,j_n)}^{(k)}\| , \\
\|\widetilde{T}_n^{(k)}\| &\leq \sum_{(j_1,\dots,j_n)\in\mathcal{S}_n^{(k)}} \int\limits_0^t \mathrm{d}s_1 \cdots \int\limits_0^t \mathrm{d}s_n \|t_{(j_1,\dots,j_n)}^{(k)}\| .
\end{aligned}$$

With this, (54) and (55) imply for $a = 0, 1$

$$\|\psi(t) - \psi_\varphi^{(1)}(t)\| = \left\| \widetilde{T}_1^{(1)} - \mathrm{i} \int_0^t \mathrm{d}s_1 U(t,s_1) t_{(2)}^{(2)} \right\| \leq 2 \max_{k\in\{1,2\}} \left\{ \int_0^t \mathrm{d}s \|t_{(k)}^{(k)}\| \right\}, \tag{56}$$

$$\begin{aligned} \|\psi(t) - \psi_\varphi^{(2)}(t)\| &= \left\| \widetilde{T}_1^{(2)} + \widetilde{T}_2^{(2)} - \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2\, U(t,s_2) t_{(1,2)}^{(3)} \right\| \\ &\leq 3 \max_{\substack{n\in\{1,2\}\\ k\in\{2,3\}}} \left\{ \sum_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}} \int_0^t \mathrm{d}s_1 \cdots \int_0^t \mathrm{d}s_n \|t_{(j_1,\ldots,j_n)}^{(k)}\| \right\} \end{aligned} \tag{57}$$

which coincides with (28) and (33). For $a \geq 2$, we find

$$\begin{aligned} &\|\psi(t) - \psi_\varphi^{(a+1)}(t)\| \\ &< a^2 \max_{\substack{n\in\{\lceil\frac{a+1}{2}\rceil,\ldots,a-1\}\\ k\in\{a+1,\ldots,2(a-1)\}}} \|T_n^{(k)}\| + a \max_{k\in\{a+1,\ldots,2a\}} \|\widetilde{T}_a^{(k)}\| + \|\widetilde{T}_{a+1}^{(a+1)}\| \\ &\quad + \int_0^t \mathrm{d}s_1 \cdots \int_{s_a}^t \mathrm{d}s_{a+1} \left\| t_{(1,1,\ldots,1,2)}^{(a+2)} \right\| \\ &\leq 2a^2 \max_{\substack{n\in\{\lceil\frac{a+1}{2}\rceil,\ldots,a+1\}\\ k\in\{a+1,\ldots,2a\}}} \left\{ \sum_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}} \int_0^t \mathrm{d}s_1 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \left\| t_{(j_1,\ldots,j_n)}^{(k)} \right\| \right\} \\ &\lesssim a^2 \max_{\substack{k\in\{a+1,\ldots,2a\}\\ n\leq k}} \left\{ \sum_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}} \int_0^t \mathrm{d}s_1 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \left\| t_{(j_1,\ldots,j_n)}^{(k)} \right\| \right\} \end{aligned} \tag{58}$$

where we used that $a + 2 \leq 2a$ for $a \geq 2$. To estimate $\|t_{(j_1,\ldots,j_n)}^{(k)}\|^2$ for $a + 1 \leq k \leq 2a$ and $n \leq k$, note first that Lemma 2.6 and Proposition 2.4b can be combined into the single statement

$$\begin{aligned} &\left\| \left(\widehat{m^{\varphi(t)}}\right)^a I_j^{\varphi(t)} \widetilde{U}_\varphi(t,s)\psi \right\|^2 \\ &\quad\lesssim 4^a \|\varphi(t)\|^2_{H^k(\mathbb{R}^d)} N^{2+d\beta j} C_{2+a+j}^{t-s} \sum_{\nu=0}^{2+j+a} N^{\nu(-1+d\beta)} \left\| \left(\widehat{m^{\varphi(s)}}\right)^{2+j+a-\nu} \psi \right\|^2 \end{aligned} \tag{59}$$

for $j \in \{1, 2\}$ and any $\psi \in L^2_{\mathrm{sym}}(\mathbb{R}^{dN})$. Hence, with $\delta_\mu := 2(n-\mu+1)+(j_n+j_{n-1}+\cdots+j_\mu)$ and $\eta_\mu := \prod_{\ell=0}^{\mu} \|\varphi(s_{n-\ell})\|^2_{H^k(\mathbb{R}^d)}$, we obtain for $n \leq k$

$$\begin{aligned} &\|t_{(j_1,\ldots,j_n)}^{(k)}\|^2 \\ &\lesssim N^{2+d\beta j_n} \sum_{\nu_1=0}^{\delta_n} C_{\delta_n}^{s_n - s_{n-1}} \eta_0\, N^{\nu_1(-1+d\beta)} \times \end{aligned}$$

$$\times \left\| \left(\widehat{m^{\varphi(s_{n-1})}}\right)^{\delta_n - \nu_1} \prod_{\ell=1}^{n-1} \left(I_{j_{n-\ell}}^{\varphi(s_{n-\ell})} \widetilde{U}_\varphi(s_{n-\ell}, s_{n-\ell-1}) \right) \psi_0 \right\|^2$$

$$\lesssim N^{2\cdot 2 + d\beta(j_n + j_{n-1})} \eta_1 \sum_{\nu_1=0}^{\delta_n} \sum_{\nu_2=0}^{\delta_{n-1}-\nu_1} 4^{\delta_n - \nu_1} C_{\delta_n}^{s_n - s_{n-1}} C_{\delta_{n-1}-\nu_1}^{s_{n-1}-s_{n-2}} N^{(\nu_1+\nu_2)(-1+d\beta)} \times$$

$$\times \left\| \left(\widehat{m^{\varphi(s_{n-2})}}\right)^{\delta_{n-1}-(\nu_1+\nu_2)} \prod_{\ell=2}^{n-1} \left(I_{j_{n-\ell}}^{\varphi(s_{n-\ell})} \widetilde{U}_\varphi(s_{n-\ell}, s_{n-\ell-1}) \right) \psi_0 \right\|^2$$

$$\lesssim \cdots$$

$$\lesssim N^{2(\mu+1)+d\beta(j_n+\cdots+j_{n-\mu})} \eta_\mu \sum_{\nu_1=0}^{\delta_n} \cdots \sum_{\nu_{\mu+1}=0}^{\delta_{n-\mu}-(\nu_1+\cdots+\nu_\mu)} C_{\delta_n}^{s_n-s_{n-1}} \cdots C_{\delta_{n-\mu}-(\nu_1+\cdots+\nu_\mu)}^{s_{n-\mu}-s_{n-\mu-1}} \times$$

$$\times 4^{\delta_n+\cdots+\delta_{n+1-\mu}-(\nu_1+\cdots+(\nu_1+\cdots+\nu_\mu)} N^{(\nu_1+\cdots+\nu_{\mu+1})(-1+d\beta)}$$

$$\times \left\| \left(\widehat{m^{\varphi(s_{n-\mu-1})}}\right)^{\delta_{n-\mu}-(\nu_1+\cdots+\nu_{\mu+1}))} \prod_{\ell=\mu+1}^{n-1} \left(I_{j_{n-\ell}}^{\varphi(s_{n-\ell})} \widetilde{U}_\varphi(s_{n-\ell}, s_{n-\ell-1}) \right) \psi_0 \right\|^2$$

$$\lesssim \cdots$$

$$\lesssim N^{2n+d\beta(j_n+\cdots+j_1)} \eta_{n-1} \sum_{\nu_1=0}^{\delta_n} \cdots \sum_{\nu_n=0}^{\delta_1-(\nu_1+\cdots+\nu_{n-1})} 4^{\delta_n+\cdots+\delta_2-(\nu_1+\cdots+(\nu_1+\cdots+\nu_{n-1}))} \times$$

$$\times C_{\delta_n}^{s_n-s_{n-1}} \cdots C_{\delta_1-(\nu_1+\cdots+\nu_{n-1})}^{s_1} N^{(\nu_1+\cdots+\nu_n)(-1+d\beta)} \left\| \left(\widehat{m^{\varphi_0}}\right)^{\delta_1-(\nu_1+\cdots+\nu_n)} \psi_0 \right\|^2 . \quad (60)$$

Since $j_1+, \ldots, +j_n = k$ and $n \le k \le 2a$, we find $\delta_1 = 2n + k \le 3k \le 6a \le A$, hence assumption A3 yields

$$\|(\widehat{m^{\varphi_0}})^{\delta_1-(\nu_1+\cdots+\nu_n)}\psi_0\|^2 \lesssim \mathfrak{C}_{\delta_1-(\nu_1+\cdots+\nu_n)} N^{-\gamma\delta_1+\gamma(\nu_1+\cdots+\nu_n)} .$$

Let us for the moment focus on the N-dependent factors in (60), thereby neglecting all other contributions to the sum. This yields

$$N^{2n+d\beta k-\gamma\delta_1} \sum_{\nu_1=0}^{\delta_n} \cdots \sum_{\nu_n=0}^{\delta_1-(\nu_1+\cdots+\nu_{n-1})} N^{(\nu_1+\cdots+\nu_n)(-1+d\beta+\gamma)} .$$

For $\gamma \ge 1 - d\beta$, the leading order term in the sum $\sum_{\nu_n}$ is the term corresponding to the choice $\nu_n = \delta_1 - (\nu_1 + \cdots + \nu_{n-1}) = 2n + k - (\nu_1 + \cdots + \nu_{n-1})$, which yields the total factor $N^{k(-1+d\beta)} N^{d\beta\delta_1} = N^{-k+2d\beta(n+k)}$. This factor is maximal for $n = k$. For $\gamma < 1 - d\beta$, the leading term corresponds to the choice $\nu_1 = \cdots = \nu_n = 0$, which yields $N^{2n(1-\gamma)+k(d\beta-\gamma)}$. Also here, the maximal contribution issues from $n = k$. In fact, the leading contributions for both ranges of γ can be summarised as $N^{-k\delta(\beta,\gamma)}$, where

$$\delta(\beta, \gamma) = \begin{cases} 1 - 4d\beta & \text{for } 1 - d\beta \le \gamma \le 1 , \\ -2 - d\beta + 3\gamma & \text{for } \frac{2+d\beta}{3} < \gamma \le 1 - d\beta \end{cases}$$

as defined in (32). Hence, for sufficiently large N, the dominating terms is the one with $n = k$, which comes from $t^{(k)}_{(j_1,\ldots,j_k)} = t^{(k)}_{(1,\ldots,1)}$.

$$\max_{(j_1,\ldots,j_n)\in\mathcal{S}_n^{(k)}} \left\| t^{(k)}_{(j_1,\ldots,j_n)} \right\| = \left\| t^{(k)}_{(1,\ldots,1)} \right\| ,$$

and (56) to (58) can be summarised as

$$\|\psi(t)-\psi_\varphi^{(a+1)}(t)\| \;\leq\; (a+1)^2 \max_{a+1\leq k\leq \max\{2a,a+2\}} \left\{ \int_0^t \mathrm{d}s_1 \cdots \int_{s_{k-1}}^t \mathrm{d}s_k \left\| t^{(k)}_{(1,\dots,1)} \right\| \right\} (61)$$

It remains to evaluate the estimate (60) for $n=k$. In this case, $j_1=\dots=j_k=1$ and $\delta_\mu=3(k-\mu+1)$. Note also that the constants C_a^t are increasing in a and t, hence $C^{s_{k-\mu}-s_{k-\mu-1}}_{\delta_{k-\mu}-(\nu_1+\dots+\nu_{\mu-1})} \leq C^{s_{k-\mu}}_{3(\mu+1)}$. Further, observe that $\delta_k+\dots+\delta_2=\frac{3}{2}k(k-1)\leq\frac{3}{2}k^2$. Consequently,

$$\|t^{(k)}_{(1,\dots,1)}\|^2 \;\lesssim\; (1+\mathfrak{C}_{3k})\, 2^{3k^2} N^{-k\delta(\beta,\gamma)} \prod_{\mu=0}^{k-1}\left((3\mu+1)C^{s_{k-\mu}}_{3(\mu+1)}\|\varphi(s_\mu)\|^2_{H^k(\mathbb{R}^d)}\right), (62)$$

where we have used that each sum $\sum_{\nu_\mu}$ in (60) contains at most $\delta_{k-\mu+1}=3\mu+1$ addends, and that the prefactor of the leading order term for $\gamma\geq 1-d\beta$ is $\mathfrak{C}_0=1$, whereas it is $\mathfrak{C}_{3k}$ for $\gamma<1-d\beta$. Consequently, for sufficiently large N, the maximum in (61) is attained for $k=a+1$. Inserting the explicit formula $C_j^{t,s}=j!\,3^{j(j+1)}\mathrm{e}^{9^jI_t}$ with $I_t=\int_s^t\|\varphi(s_1)\|^2_{H^k(\mathbb{R}^d)}\,\mathrm{d}s_1$ yields

$$\begin{aligned}\|\psi(t)-\psi_\varphi^{(a)}(t)\|^2 \;&\lesssim\; N^{-a\delta(\beta,\gamma)}\prod_{\nu=1}^{a}\left(\int_0^t \mathrm{e}^{\frac{1}{2}9^{3(\nu+1)}I_{s_\nu}}\|\varphi(s_n)\|_{H^k(\mathbb{R}^d)}\,\mathrm{d}s_\nu\right)^2\\ &\lesssim\; \mathrm{e}^{a9^{3(a+1)}I_t}I_t^{2a}\,N^{-a\delta(\beta,\gamma)} \;\lesssim\; \mathrm{e}^{9^{4a}I_t}\,N^{-a\delta(\beta,\gamma)},\end{aligned}$$

where we have bounded all a-dependent, time-independent expressions by a constant $c\lesssim 1$. □

Acknowledgements

L.B. gratefully acknowledges the support by the German Research Foundation within the Research Training Group 1838 "Spectral Theory and Dynamics of Quantum Systems", and wishes to thank Stefan Teufel, Sören Petrat and Marcello Porta for helpful discussions. N.P. gratefully acknowledges support from NSF grant DMS-1516228 and DMS-1840314. Part of this work was done when N.P. and P.P. were visiting CCNU, Wuhan. N.P. and P.P. thank A.S. for his hospitality at CCNU.

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