Surgery for extended Ricci flow systems

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Summary

This thesis contains the author's results on the extended Ricci flow system

$$\partial_t g(t) = -2\,\mathrm{Rc}_{g(t)} + 2\alpha\,\mathrm{tr}_N(\nabla u(t) \otimes \nabla u(t)),$$
$$\partial_t u(t) = \Delta_{g(t),\gamma} u(t),$$

where $\{g(t)\}_{t\in[0,T)}$ denotes a one-parameter family of smooth Riemannian metrics on a smooth manifold M, $\{u(t)\}_{t\in[0,T)}$ denotes a one-parameter family of smooth maps $u(t) : M \to (N,\gamma)$ into a target manifold (N,γ) of nonpositive sectional curvature and $\alpha > 0$ a coupling constant. The extended Ricci flow system reduces to Ricci flow, if the map u is a constant map. The motivation to study this system stems from static solutions to the Einstein vacuum equations and from Ricci flow on warped product manifolds in higher dimensions.

To understand the behaviour of solutions we need to analyze the reaction–diffusion equations satisfied by the curvature and the derivatives of the smooth maps $\{u(t)\}$. The additional difficulty in the equations for the curvature compared to Ricci flow arises from the presence of terms involving the second derivatives of the smooth maps $\{u(t)\}$, which are critical, since they have the same scaling dimension as the curvature.

The first contribution of this thesis is to show that norm of the second derivatives satisfies an improved estimate in three dimensions, which implies that effect from the coupling on the evolution of curvature is subcritical. The main application of this result is to generalize the Hamilton-Ivey estimate for Ricci flow to the extended Ricci flow system. This estimate allows a complete description of singularity models of the flow.

In our second and third contribution we focus on the special case $(N,\gamma) = (\mathbb{S}^1, g_{\mathbb{S}^1})$, which is known as List flow in the literature. While a three-manifold with positive Ricci curvature becomes more round under the evolution by Ricci flow, this is not the case for the extended system: If the function u is nonconstant, even a round sphere does not stay round under the flow. However, we are able to show that the flow converges on three-manifolds of positive Ricci curvature and large scalar curvature the to a round sphere and the function u converges to a constant.

Our third result deals with the formation of singularities in List flow. In general the curvature will blow up at some space-time points and it is not possible to continue the flow smoothly. We construct a surgery algorithm in the spirit of R. Hamilton and G. Perelman to continue the flow beyond singularities. An obstacle is to show that the energy density stays controlled along the surgically modified flow.

Zusammenfassung

In dieser Arbeit werden die Resultate des Autors für das gekoppelte Ricci-Fluss-System

$$\partial_t g(t) = -2 \operatorname{Rc}_{g(t)} + 2\alpha \operatorname{tr}_N(\nabla u(t) \otimes \nabla u(t)),$$
$$\partial_t u(t) = \Delta_{g(t),\gamma} u(t),$$

vorgestellt, wobei $\{g(t)\}_{t\in[0,T)}$ eine Einparameter-Familie von Riemannschen Metriken auf einer geschlossenen 3-Mannigfaltigkeit M bezeichnet und $\{u(t)\}_{t\in[0,T)}$ eine Einparameter-Familie von Abbildungen $u(t) : M \to (N, \gamma)$ in eine Zielmannigfaltigkeit (N, γ) nichtpositiver Schnittkrümmung bezeichnet. Falls die Abbildung u konstant ist, so reduziert sich dieses erweitere Ricci-Fluss-System auf den Ricci-Fluss. Die Motivation dieses System zu studieren rührt von der Verbindung zu statischen Lösungen der Einstein-Vakuumgleichungen her sowie von der Verbindung zu Ricci-Fluss auf gewarpten Produktmannigfaltigkeiten in höheren Dimensionen.

Um das Verhalten der Lösungen zu verstehen, müssen wir die Reaktion-Diffusionsgleichungen verstehen, die von den Krümmungen der Metrik und von den Ableitungen der glatten Abbildungen $\{u(t)\}$ erfüllt werden. Im Vergleich zum Ricci-Fluss entsteht die zusätzliche Schwierigkeit darain, dass die Gleichungen für die Krümmung der Metrik zweite Ableitungen der glatten Abbildungen $\{u(t)\}$ enthalten, welche die Skalierungsdimension wie die Krümmung haben und daher kritisch sind.

Das erste Ergebnis dieser Arbeit zeigt, dass die Norm der zweiten Ableitungen eine verbesserte Abschätzung erfüllt, die dazu führt, dass die Kopplungseffekte in den parabolischen Gleichungen für die Krümmung subkritisch sind. Die Hauptanwendung dieses Resultates liegt darin die Hamilton–Ivey Abschätzung für Ricci-Fluss auf das gekoppelte System zu generalisieren. Diese Abschätzung erlaubt eine vollständige Beschreibung der Singularitätsmodelle des Flusses.

Das zweite und dritte Resultat bezieht sich auf den Spezialfall $(N, \gamma) = (\mathbb{S}^1, g_{\mathbb{S}^1})$, welcher in der Literatur als List-Fluss bekannt ist. Während eine Drei-Mannigfaltigkeit mit positiver Ricci-Krümmung unter dem Ricci-Fluss runder wird, ist dies nicht der Fall für das gekoppelte System: Falls die Funktion u nicht konstant ist, so bleibt selbst eine runde Sphäre nicht rund unter dem Fluss. Allerdings zeigen wir, dass der Fluss für Drei-Mannigfaltigkeiten mit positiver Ricci-Krümmung und großer Skalarkrümmung zu einer runden Sphäre konvergiert und dass die Funktion u zu einer Konstanten konvergiert.

Das dritte Resultat behandelt die Entstehung von Singularitäten entlang des Flusses. Im Allgemeinen wird die Krümmung in einigen Punkten der Raumzeit explodieren und es ist nicht möglich, den Fluss glatt fortzusetzen. Wir konstruieren einen Chirurgie-Algorithmus im Geiste von R. Hamilton und G. Perelman, um den Fluss über diese Singularitäten hinweg fortzusetzen. Eine Hürde ist zu zeigen, dass die Energiedichte entlang des chirurgisch modifizierten Flusses kontrolliert bleibt.

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1. Introduction

In the second half of the twentieth century geometric flows emerged as an important tool to study the interplay of topology and curvature of smooth manifolds. Beyond the intrinsic importance of this question this lead to several applications in Mathematical Physics.
In the year 1964 J. Eells and J.H. Sampson [34] were the first to study a geometric flow. Their quest was to understand which maps between Riemannian manifolds are homotopic to a harmonic map. They evolved a given map between Riemannian manifolds by its tension field with the aim to deform this map into a harmonic map. This strategy was sucessful for target manifolds of non-negative sectional curvature: Indeed in this setting the flow subconverges to a harmonic map as time tends to infinity and this map is homotopic to the initial map.
The field essentially lay dormant until 1982, when R. Hamilton introduced in the seminal work [36] a parabolic deformation equation for Riemannian metrics, which is nowadays known as Ricci flow. In Ricci flow a one-parameter family $\{g(t)\}$ of Riemannian metrics evolves by the equation

$$\partial_t g(t) = -2\,\mathrm{Rc}_{g(t)}\,.$$

This is a weakly parabolic system due to the diffeomorphism invariance of the metric. By using the Naser–Moser implicit function theorem R. Hamilton was able to show that the above equation admits a solution for a short time (this was later simplified by D. DeTurck [30]) and that the solution can be continued unless the norm of the Riemann curvature tensor blows up. A crucial point is to show that the metric improves under the evolution by Ricci flow! The main technical insight was to generalize the scalar maximum principle for parabolic equations to a tensor maximum principle and to use this to understand the reaction-diffusion equation satisfied by the curvature.
For three-manifolds of positive Ricci curvature this turned out to be fruitful:
Positive Ricci curvature is preserved along a solution, and so is the roundness of the metric. As the curvature becomes large, the solution becomes more and more round and as the flow approaches the singular time, the metric converges after rescaling to a metric of constant positive sectional curvature. This implies a classification result for all closed three-manifolds, which admit a metric of positive Ricci curvature; indeed these manifolds are given by spherical space forms, that is quotients of the round sphere by finite groups. This result ignited the hope that one may use Ricci flow as a tool to understand the geometry of three-manifolds and to solve the Poincaré conjecture: Start Ricci flow on a homotopy three-sphere endowed with an arbitrary Riemannian metric and understand the limiting behaviour of the flow.
In the following twenty years many questions in this direction were adressed for Ricci flow: G. Huisken proved in 1985 a convergence result in higher dimensions for manifolds, whose Weyl and trace-free Ricci curvature is small compared to the scalar curvature [47], similar results were proved by C. Margerin [64] and S. Nishikawa [71]. Such a condition is necessary to understand the much more involved curvature algebra in higher dimensions. In 1986 R. Hamilton proved a classification result for four-manifolds with positive curvature operator [37]. Here the special structure of curvature in four dimensions plays a major

role. In the same work he introduced a strong tensor maximum principle, which allowed to extend the previous results from positive curvature to nonnegative curvature. In 1988 he investigated the normalized Ricci flow on surfaces [38]. Depending on the Euler characteristic of the initial surface the flow converges to a metric of positive, zero or negative constant sectional curvature. The technical tools are very different here, since one cannot compare curvatures at a point as in the higher dimensional case. Later his arguments were refined to give a new proof of the uniformization theorem for surfaces, see [27, 7, 81, 26, 3]. Another important result of R. Hamilton [39] was the discovery of a Harnack inequality for Ricci flow, which allows to compare the scalar curvature at different space time points, whenever the solution is positively curved. In a 1995 survey [40] he gave an intuitive picture of the singularity formation and showed that a solution of the flow has sectional curvature pinched towards nonnegative in the regions of high curvature (this was independently shown by T. Ivey [55]). From this result one deduces that singularity models have nonnegative sectional curvature. In 1999 R. Hamilton studied the longtime behaviour of non-singular solutions to Ricci flow on three-manifolds [42].
However, a main obstacle in applications of geometric flows to understand the topology of manifolds is the occurence of singularities in the metric, where one cannot continue the flow smoothly. Indeed, while in the above convergence results for three-manifolds with positive Ricci curvature, four-manifolds with positive curvature operator and higher dimensional manifolds with pinched curvature the metric goes singular (in the sense that $|\operatorname{Rm}|(t,p) \to \infty$ as $t \to T$ for some $p \in M$) simultaneously at all points, in general this is not the case.

In the same time span different geometric flows were used with sucess to adress classification questions for submanifolds and questions in Mathematical Relativity:
In the year 1984 G. Huisken used an extrinsic curvature flow [46], nowadays known as Mean Curvature flow, to show that every strictly convex hypersurface in $\mathbb{R}^{n+1}$, where $n \geq 2$, is diffeomorphic to a sphere. An intriguing difference to the convergence results for Ricci flow is the following: While in Ricci flow the pinching towards roundness is proven by exploiting the structure of the reaction terms by the maximum principle, in Mean Curvature flow this does not work. Instead one has to use the Michael–Simon inequality to exploit the diffusion term, which allows to obtain supremum bounds from L^p-bounds by a Stampacchia iteration argument. Shortly after G. Huisken extended this result to general ambient spaces [48] and to sufficently convenx surfaces in the sphere [49].
Later G. Huisken proved a monotonicity formula for the Gaussian volume under the evolution by Mean Curvature flow [50], which allows to classify singularity models of the flow. For harmonic map heat flow a monotonicity formula for the Gaussian energy density was established by M. Struwe [80].
At the end of the twentieth century G. Huisken and T. Ilmanen constructed weak solutions to Inverse Mean Curvature flow to prove the Riemannian Penrose inequality [52]. The Riemannian Pensore inequality was independently proved by H. Bray using a conformal flow of Riemannian metrics [11].

Despite the progress in understanding geometric flows sketched above there was scepticism whether one can understand the formation of singularities and continue the flow beyond such singularities. However, in a seminal paper in 1997 R. Hamilton introduced the technique of surgery for geometric flows [41], more precisely he studied Ricci flow on closed four-manifolds with positive isotropic curvature. He performed a quantitative analysis of the high curvature regions, so called necks — and showed how to remove these regions from the flow while preserving the a-priori assumptions on the solution. However, it is nowadays well-known to experts, see for example the remark in the abstract of [75], that these estimates do not suffice to rule out the accumulation of surgery times.

In three groundbreaking papers G. Perelman [73, 75, 74] sucessfully finished R. Hamilton's program and gave a proof of Thurston's Geometrization conjecture, which includes the elliptization conjecture and Poincaré conjecture as a special case. The proof was discussed in a series of expository works, see B. Kleiner and J. Lott [56], H.-D. Cao and X.-P. Zhu [25] and J. Morgan and G. Tian [66, 67].

In G. Perelman's first work [73] many new ideas are contained:

By introducing the $\mathcal{F}$-functional he showed that Ricci flow is almost a gradient flow, and he used the related $\mathcal{W}$-functional to show noncollapsing for the volume along the flow. Moreover, he developed a comparison geometry approach for Ricci flow. Most importantly, he developed a structure theory for the singularity models of Ricci flow, so called κ-solutions, and used it to obtain a geometric description of the high curvature regions.

In his second work [75] G. Perelman studied the structure of the solution at the first singular time and adapted the surgery algorithm of R. Hamilton. He then showed that the surgery times cannot accumulate and studied the longtime behaviour of the flow to obtain a proof of the Geometrization conjecture. In the last work [74] he sketched a short-cut to the Poincaré conjecture by showing finite time extinction for the flow. An independent argument was given by T. Colding and W. Minicozzi [28].

The field of geometric flows has expanded a lot since G. Perelman's breakthrough: In an important work C. Böhm and B. Wilking [10] introduced novel pinching sets to show that Ricci flow converges to round points for manifolds with positive curvature operator. This implies that manifolds with positive curvature operator are spherical space forms. S. Brendle and R. Schoen used Ricci flow to prove the Differentiable Sphere theorem [23] and they classified manifolds with weakly quarter pinched sectional curvatures [22]. In [12] S. Brendle weakenend the assumptions for the convergence theorem further to a condition known as PIC1.

Since the work of G. Perelman several variants of the surgery construction for solutions to geometric flows have appeared. Most notably the classification of immersed two-convex hypersurfaces in euclidean space $\mathbb{R}^{n+1}$ for $n \geq 3$ by G. Huisken and C. Sinestrari [54] and of embedded two-convex hypersurfaces in euclidean space $\mathbb{R}^3$ due to S. Brendle and G. Huisken [20] using mean curvature flow. Moreover, S. Brendle [17, 18] used Ricci flow with surgery to study manifolds with positive isotropic curvature in dimension $n \geq 12$. We will discuss others results and the technical differences between the above surgery constructions in Section 1.3 of this introduction.

In this thesis we will investigate coupled Ricci flow systems, which are known in the literature as List flow and Ricci flow coupled to harmonic map heat flow. The first system was introduced by B. List in his thesis [59], see also the article [60]. It is a special case of the second system, which was introduced by R. Buzano (formerly R. Mueller) in his thesis [69], see also the article [70].
Let M be a smooth manifold, (N, γ) a Riemannian manifold and $\alpha > 0$ a coupling constant. A one-parameter family of tuples $\{g(t), u(t)\}_{t\in[0,T)}$ consisting of a one-parameter family of smooth Riemannian metrics $g(t)$ and a one-parameter family of smooth maps $u(t) : M \to (N, \gamma)$ evolves by Ricci flow coupled to harmonic map heat flow with initial data (g_0, u_0), if for all $t \in (0, T)$ the system

$$\begin{aligned} \partial_t g(t) &= -2 \operatorname{Rc}_{g(t)} + 2\alpha \ \operatorname{tr}_N \left(\nabla u(t) \otimes \nabla u(t) \right), \\ \partial_t u(t) &= \Delta_{g(t),\gamma} u(t). \end{aligned} \tag{1}$$

is satisfied and moreover $g(0) = g_0$ and $u(0) = u_0$. The special case $(N, \gamma) = (\mathbb{S}^1, g_{\mathbb{S}^1})$ is known as List flow. The notation used in the above expression is explained in Section 2.1.
Let us explain two motivations to study these systems:
Static vacuum solutions to the Einstein equations of General Relativity correspond after a conformal change to stationary points of List flow. In particular, one might use this flow as a smoothing procedure for initial data sets, since it preserves the ADM mass. For a more detailed explanation and computations see Section 2.3.
On the other hand Ricci flow on warped product manifolds with k Ricci flat factors over an m-dimensional base M reduces to Ricci flow coupled to harmonic map heat flow on a the base manifold M^m into the target manifold $(\mathbb{R}^k, \delta)$. To understand the singularity formation in Ricci flow in higher dimensions one currently needs a strong condition on the positivity of the curvature. For the case of warped products with Ricci flat fibers over a three-dimensional base manifold we do not need such a curvature condition. For a more detailed explanation and computations see Section 2.2.

1.1. **Results.**

The first main result of this thesis is the following:

Theorem 1.
Suppose $(M, g(t), u(t))_{t\in[0,T)}$, $T < \infty$, is a solution of Ricci flow coupled to harmonic map heat flow (1) on a closed three-manifold into a target manifold (N, γ) of non-positive sectional curvature and controlled curvature. Then any singularity model has nonnegative sectional curvature and the map u is constant; hence any singularity model is given by a singularity model of three-dimensional Ricci flow.

For a more detailed statement, see Theorem 5.2.
For $(N, \gamma) = (\mathbb{S}^1, g_{\mathbb{S}^1})$ the flow reduces to List flow and List flow on three-manifolds is equivalent to Ricci flow on four-manifolds with topology $\mathbb{S}^1 \times M^3$ starting from a warped product metric. Then the above result classifies all singularity models for Ricci flow on four-manifolds $\mathbb{S}^1 \times M^3$ with warped product metrics without an assumption on the initial curvature.
By recent work of S. Brendle on ancient solutions to Ricci flow on three-manifolds [16], we actually know that at the first singular time any singularity is modeled on a round shrinking sphere $\mathbb{S}^3$, a shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$, the $\mathbb{Z}_2$-quotient of the shrinking cylinder or the Bryant soliton.

The second main result concerns the convergence of List flow on three-manifolds with positive Ricci curvature and large scalar curvature.

Theorem 2.
Let (M^3, g_0) be a Riemannian manifold and $u_0 : M \to \mathbb{R}$ a smooth function, such that

$$\mathrm{Rc} > 0, \mathrm{Rc} > \epsilon\, \mathrm{R}\, g \text{ and } \mathrm{R} \geq \frac{1}{\epsilon^2} D(\alpha c_0 + \sqrt{\alpha s_0})$$

initially, where D is a numerical constant and the constants c_0 and s_0 depend on the first and second derivatives of the initial function u_0.
Then List flow preserves these conditions, the solution extincts at some finite time $T < \infty$. Moreover, as $t \to T$ the metrics

$$\tilde{g}(t) = \frac{1}{4(T-t)} g(t)$$

converge in C^∞ to a metric of constant sectional curvature and the function u converges in C^∞ to some constant u_∞ with $\inf_{p\in M} u_0(p) \leq u_\infty \leq \sup_{p\in M} u_0(p)$.

This result is a generalization of the celebrated convergence result of R. Hamilton on three-manifolds with positive Ricci curvature [36] in the following sense: If the function u is constant along the flow, we have that $c_0 = s_0 = 0$ and the result reduces to R. Hamilton's result. We also recover R. Hamilton's result in the limit $\alpha \to 0$ of vanishing coupling.

For the third main result we use the above classification theorem for singularities to perform surgery for List flow:

Theorem 3.
Suppose (M^3, g_0) *is a closed Riemannian three-manifold and* $u_0 : M \to \mathbb{R}$ *a smooth function. Then there exists a solution to List flow with surgery with initial condition* $g(0) = g_0$ *and* $u(0) = u_0$.
Moreover, if the prime decomposition of the manifold (M, g) *does not contain any non-aspherical factors (or if the manifold* (M, g) *is simply-connected), then the flow extincts in finite time.*

These results are novel in the sense that they provide the first complete singularity classification result for extended Ricci flow systems on three-manifolds; the first convergence result for extended Ricci flow systems on three-manifolds and the first surgery result for extended Ricci flow systems.

1.2. **Related developments for extended Ricci flow systems.**

In this section we review the literature on extended Ricci flow systems.
The first extended Ricci flow system appearing in the literature is the system

$$\begin{aligned} \partial_t g(t) &= -2\operatorname{Rc}_{g(t)} + 2\alpha \nabla u(t) \otimes \nabla u(t), \\ \partial_t u(t) &= \Delta_{g(t)} u(t), \end{aligned} \tag{2}$$

where $u(t): M \to \mathbb{R}$ is a real-valued function. This system was introduced by B. List in his thesis [59], see also the article [60]. B. List showed short-time existence on manifolds of bounded curvature and bounded injectivity radius and uniqueness in the class of closed manifolds. Moreover, we constructed analogues of Perelman's $\mathcal{F}$- and $\mathcal{W}$-functional and used the latter to show noncollapsing of the flow. He also derived that singularity models of the flow are given by solutions of Ricci flow, however there is no restriction on the curvature of these solutions. By deriving interior derivative estimates for the flow B. List also showed that the class of complete asymptotically flat manifolds is preserved under this flow and the ADM mass stays constant along a smooth solution.
J. Lott and N. Sesum [62] studied List flow on closed surfaces or equivalently Ricci flow on three-manifolds with warped product structure over $\mathbb{S}^1$. They classified the behaviour of the flow depeding on the Euler characteristic $\chi(M)$ of the two-dimensional base manifold M by invoking earlier results by J. Lott [61] on the longtime behaviour of Ricci flow. Moreover, B. Guo, Z. Huang and D.H. Phong [35] showed a pseudo-locality result for List flow and showed that type-I singularities are given by gradient shrinking solitons to List flow.
The system (1), which is known as Ricci flow coupled to harmonic map heat flow or sometimes as Harmonic Ricci flow was introduced by R. Buzano (formely R. Mueller) in his thesis [69], see also the article [70].
In 2017 R. Buzano and M. Rupflin [24] showed long-time existence for Ricci flow coupled to harmonic map heat flow on surfaces of Euler characteristics $\chi(M) \leq 0$. In a recent preprint G. Di Matteo [31] studied type-I singularities of the flow.
Another extended Ricci flow system, where one couples Ricci flow to Yang-Mills flow was independently studied by J. Streets in his thesis [78] and by A. Young in her thesis [83]:

$$\begin{aligned} \partial_t g(t) &= -2\operatorname{Rc}_{g(t)} + \alpha F_A^2(t), \\ \partial_t A(t) &= -d^* F_A(t), \end{aligned}$$

where $\alpha > 0$ is a coupling constant, d^* denotes the adjoint of the exterior derivative, A denotes a connection, F_A its curvature and F_A^2 the operator square of the curvature F_A. Later J. Streets obtained a partial convergence result for surfaces and $U(1)$-bundles under an assumption on the isoperimetric constant, see [79].
Another extended Ricci flow system arises from quantum field theory: To a quantum field theory one can associate a renormalization group flow. In the simplest case one obtains at the one-loop approximation the Ricci flow. If one introduces the so called B-field and sets

$\eta = dB$, then one obtains at the one-loop approximation the evolution system

$$\partial_t g = -2\,\mathrm{Rc} + \frac{1}{2}\eta \sim \eta, \qquad (3)$$
$$\partial_t \eta = \Delta_H \eta,$$

where $\eta \in \Omega^3(M)$, Δ_H denotes the Hodge-Laplacian acting on three forms by $\Delta_H \eta = -(d\delta + \delta d)\eta$ and $\eta \sim \eta$ is given in abstract index notation by $(\eta \sim \eta)_{ab} = \eta_{acd}\eta_{bcd}$. This system was first studied by T. Oliynyk, E. Suneeta and E. Woolgar [72]. Their work actually included a scalar function u, which solves a heat equation and introduces a coupling term of the form $\nabla u \otimes \nabla u$ into the equation, similar as in List flow.
J. Lott studied Ricci flow on manifolds with abelian symmetry, see page 500ff of [61]. He derived the evolution equations and constructed analogues of the $\mathcal{F}$- and $\mathcal{W}$-functionals. T. Marxen studied Ricci flow on a certain class of noncompact warped products [65].

1.3. **Surgery constructions for geometric flows.**

In this section we explain similarities and differences between different surgery algorithms for geometric flows. In his seminal work [41] R. Hamilton made many important observations: In particular he explained how to deform the metric in the surgery algorithm (see Section 4.1 of [41, p. 47-49]) and how to preserve the a-priori estimates on the curvature through the surgery process (see Section 4.2 and 4.3 of [41, p. 49-60]). To perform the surgeries he stopped the flow, if a certain curvature three-shold R_1 is reached, then the scalar curvature is reduced by the surgery below a threeshold R_2, which is much smaller than R_1. Unfortunately, the estimates are not strong enough to rule out the accumulation of surgery times.
Several years later G. Perelman introduced new concepts, which allowed him to finish the surgery program outlined by R. Hamilton. He introduced the $\mathcal{F}$- and $\mathcal{W}$-functional and the associated λ- and μ-invariants, which are monotone under Ricci flow. Using the $\mathcal{W}$-functional or the concept of reduced volume (introduced in Section 7 of [73]) he was able to rule out collapsing along the flow. In Section 11 of [73] he introduced a class of singularity models for Ricci flow, so called κ-solutions and studied their properties. In particular, he showed gradient estimates (Section 11.2 of [73]) and a compactness theorem (Section 11.7 of [73]). In Section 12 of [73] he established the Canonical Neighbourhood Theorem, which gives a complete description of the high curvature regions, indeed these regions are modeled on κ-solutions. In Section 3 and 4 of [75] he used the Canonical Neighbourhood Theorem to analyze the solution at the first singular time and to perform surgery. In constrast to R. Hamilton's approach the surgeries are performed at the singular time. In G. Perelman's approach it is not expected that there is a definite time span between the singularities. However, it can be shown that in a surgery region some definite time elapses before another surgery happens and this region is not affected by other regions of high curvature due to pseudo-locality of curvature (see Section 10 of [73]). Finally, to show that the a-priori assumption on the κ-noncollapsing is preserved through surgery a delicate argument is

necessary, see Lemma 5.2 and Lemma 5.3 of [75]. One uses a modification of the reduced volume in the presence of surgeries to rule out collapsing.
The next surgery result was obtained by G. Huisken and C. Sinestrari [54]. They used Mean Curvature flow with surgery to classify two-convex hypersurfaces in $\mathbb{R}^{n+1}$, where $n \geq 3$. Their work relies on convexity estimates for mean curvature flow, see Theorem 1.1 and Corollary 1.2 of [53], similar in spirit to the Hamilton–Ivey estimate, and a cylindrical estimate, see Theorem 5.3 of [54]. The surgery algorithm is performed in the spirit of R. Hamilton's work in 1997; ie. the surgery is performed at certain curvature threesholds and there is a definite time interval between surgeries. This construction also works for hypersurfaces, which are merely immersed. The construction uses results on noncollapsing for mean curvature flow of mean convex surfaces due to B. White [82].
A different geometric notion for noncollapsing for embedded hypersurfaces by comparison with balls contained in the region bounded by the embedded hypersurface was introduced by W. Sheng and X.-J. Wang [76]. They showed that this notion of δ-noncollapsing is preserved by mean curvature flow. B. Andrews [2] gave a remarkable proof of this result: He used a maximum principle for a two-point function to show that the noncollapsing is preserved along the flow; his argument was inspired by an earlier argument of G. Huisken [51] on noncollapsing for curve shortening flow. S. Brendle [15] refined B. Andrews' work by combining the approach via the two-point function with a Stamppachia iteration technique to obtain a sharp noncollapsing estimate for embedded surfaces. This estimate was crucial in the classification of mean convex surfaces in $\mathbb{R}^3$ due S. Brendle and G. Huisken [20]. The additional difficulty compared to work of G. Huisken and C. Sinestrari mentioned above lies in the fact that there is no cylindrical estimate for the flow in this particular dimension.
Later R. Haslhofer and B. Kleiner used B. Andrews' work on noncollapsing to rederive the convexity estimates for mean-convex mean curvature flow [44] and to construct a surgery algorithm [45] for embedded two-convex hypersurfaces in the spirit of G. Perelman's approach. Their program relies on compactness arguments for ancient solutions.
In 2015 S. Brendle and G. Huisken [21] studied fully nonlinear flows, which preserve two-convexity in general ambient manifolds. They implemented a surgery algorithm for two-convex hypersurfaces in ambient manifolds of positive flag curvature to classify two-convex hypersurfaces in these manifolds. Their result uses a new delicate pointwise curvature estimates, see Theorem 6.2 of [21].
Let us mention work in a different direction: B. Kleiner and J. Lott [57] constructed Ricci flow through singularities in the sense that they send the surgery parameter $\epsilon \to 0$. This was used in subsequent work by R. Bamler and B. Kleiner to understand the diffeomorphism groups of three-manifolds [5, 4].
Finally, let us mention work of L. Bessiers,G. Besson, S. Maillot, M. Boileau and J. Porti on Ricci flow on three-manifolds [9]. They implement a surgery algorithm, which they call Ricci flow with bubbling, where the surgery is performed at finite levels of curvature. In later work of the first three authors this was used to study manifolds of uniformly positive scalar curvature [8].
Their surgery construction was used by R. Haslhofer [43] and Y. Li [58] to give applications to the ADM mass in General Relativity.

1.4. **Open questions.**

Finally, let us mention some open questions surrounding the content of this thesis:
In our construction of List flow with surgery we do not adress the long-time behaviour of the flow yet. For Ricci flow this behaviour was studied by G. Perelman, see Section 13 of [73] and Section 6 and 7 of [75], and in a series of works by R. Bamler [6]. It is an interesting question, what happens to the function u as $t \to \infty$. Based on our estimates we conjecture that u converges to a constant function and the long-time behaviour is given by the long-time behaviour of Ricci flow. In particular, one conjectures that there exists some large time $T^* > 0$, such that the flow is smooth for all $t > T^*$.
One possible topic for further investigation is to study the extended Ricci flow system (1) in higher dimensions. More precisely, one wants to prove an analogue of the Hamilton–Ivey estimate in higher dimensions. Work of S. Brendle on Ricci flow in higher dimensions, see [18] and [17], suggests that a positivity condition on the curvature tensor such as positive isotropic curvature is a natural assumption to start with.
Another ongoing project is to understand the surgery algorithm for Ricci flow coupled with harmonic map heat flow into targets of nonpositive sectional curvature and possible applications to stationary vacuum solutions of the Einstein equations.
Finally, we mention some ideas related to the Bartnik metric extension conjecture for static metrics in General Relativity.
Let M be a smooth noncompact three-manifold with inner boundary ∂M diffeomorphic to the sphere $\mathbb{S}^2$. A pair (g, N) of an asymptotically flat Riemannian metric g and a smooth positive function N, which asymptotes to one at infinity, solving the coupled elliptic system

$$\begin{aligned} N\,\mathrm{Rc} &= \nabla^2 N \\ \Delta N &= 0 \end{aligned} \tag{4}$$

is called static solution to the vacuum Einstein equations on M. As explained in Section 2.3 there is a conformal change of metric, such that stationary points of List flow are solutions to the system (4). As boundary data one prescribes a Riemannian metric γ and a smooth function H on $\mathbb{S}^2$. The Bartnik conjecture asks for which boundary data (γ, H) there exists an unique (up to diffeomorphism) asymptotically flat solution of the coupled elliptic system (4), which is smooth up to the boundary, and for which the induced metric $g|_{\partial M}$ satisfies $g|_{\partial M} = \gamma$ and the mean curvature of ∂M in M is given by the smooth function H.
Partial progess was obtained by M. Anderson [1]. C. Mantoulidis and R. Schoen constructed examples of asymptotically flat manifolds with minimal inner boundary which indicate possible obstructions [63]. The importance of the Bartnik conjecture stems from its connections to the concepts of quasi-local mass for initial data sets in General Relativity.
A possible route of attack to this problem is the following: Study List flow on manifolds with boundary data prescribed as above and understand the singularity formation in this case. The hope is that the flow converges to the desired static extension.

Structure of this work.

Let us explain the structure of this work.
In the second section we introduce our notation for the geometric objects and explain relations of the extended Ricci flow system (1) to the static vacuum Einstein equations and Ricci flow on warped products.

In the third section we remind the reader of the evolution equations for the coupled system: The curvatures of the metric and the derivatives of the maps $u(t) : M \to N$ all satisfy reaction-diffusion equations with diffusion operator given by the Laplace–Beltrami operator. We also collect some first consequences obtained by applying the scalar maximum principle to the evolution equations and state the interior estimates. Moreover, we prove an improved decay estimate for the energy density in List flow.

Starting from the fourth section we focus on three-dimensional solutions. The consequences of the evolution equation are used to prove the crucial improved bound on the Hessian for solutions. This bound implies that the norm squared of the Hessian does not grow with the square of curvature as expected from the scaling, but only linear in the curvature. The proof proceeds by constructing a test function and exploits the fact that the energy density is bounded and that the full curvature tensor is controlled by the Ricci curvature in three dimensions.

The improved estimate on the Hessian is used to prove the Hamilton–Ivey pinching estimate, which implies that the curvature pinches towards nonnegative. This implies that singularity models have nonnegative sectional curvature and that the scalar curvature controls the full curvature tensor.

From the sixth section on we focus on List flow. In the sixth section we prove focus on List flow and prove a convergence theorem for three-manifolds of positive Ricci curvature and large scalar curvature. One of the difficulties here is to show that the roundness can be preserved, since neither the Ricci curvature nor the modified Ricci curvature satisfy an equation with good structure.

In section seven to nine we explain how to adapt the surgery construction due to R. Hamilton and G. Perelman to the setting of List flow:
In section seven we explain how to adapt the Canonical Neighbourhood Theorem to our setting. We benefit from the fact that our singularity models are essentially given by κ-solutions to three-dimensional Ricci flow.
In section eight we discuss the surgery construction. It turns out that the deformation of the function u in the surgery is delicate, because one has to preserve the a-priori bound on the energy density. In Section nine we explain how to preserve the noncollapsing through the surgery procedure and finally construct List flow with surgery.

In section ten we prove a finite time extinction result for List flow on three-manifolds, whose prime decomposition does not contain non-aspherical factors.

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2. Preliminaries

In this section we remind the reader of our notations for the geometric objects and explain the relation of the extended flow system (1) to Ricci flow on warped products (Section 2.2) and static solutions to the Einstein equations (Section 2.3).

2.1. **Notation.**

Suppose (M, g) is an m-dimensional smooth Riemannian manifold with Levi-Civita connection ∇. Its volume element is denoted by dvol_g. We denote the tangent bundle of M by TM and its cotangent bundle by T^*M.
Given four vector fields $X, Y, Z, W \in \Gamma(TM)$ our convention for the Riemann curvature tensor is given by

$$\mathrm{Rm}(X, Y, Z, W) = g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\right).$$

Note that our convention for the Riemann curvature tensors differs from the convention used by R. Hamilton, see for example [36], and by S. Brendle in his book [14].
The sectional curvature of a two-plan Π spanned by linearly independent vectors $v_p, w_p \in T_pM$ at the point $p \in M$ is given by

$$K(p, \Pi) = \frac{\mathrm{Rm}_p(v_p, w_p, w_p, v_p)}{g_p(v_p, v_p) g_p(w_p, w_p) - g_p(v_p, w_p)^2}.$$

We say that (M, g) has non-positive sectional curvature if $K(p, \Pi) \leq 0$ for all $p \in M$ and all two-planes Π in T_pM.
The Ricci curvature is given by the trace over the first and fourth slot of the Riemann curvature tensor:

$$\mathrm{Rc}(X, Y) = \mathrm{tr}_g \mathrm{Rm}(\cdot, X, Y, \cdot); \text{ in abstract index notation } \mathrm{Rc}_{ab} = \mathrm{Rm}_{cab}{}^{c}.$$

The scalar curvature is the metric trace of the Ricci curvature:

$$\mathrm{R} = \mathrm{tr}_g \mathrm{Rc}; \text{ in abstract index notation } \mathrm{R} = g^{ab} \mathrm{Rc}_{ab}.$$

For Riemannian three-manifolds we define the Einstein tensor E by $E = \mathrm{R}\, g - 2\,\mathrm{Rc}$. We denote its eigenvalues by $\sigma_1, \sigma_2, \sigma_3$ and order them by $\sigma_1 \geq \sigma_2 \geq \sigma_3$. The eigenvalues σ_i are twice the sectional curvatures. Observe that the scalar curvature is the trace of the Einstein tensor, i.e. $\mathrm{R} = \mathrm{tr}\, E = \sigma_1 + \sigma_2 + \sigma_3$.
The Riemannian metric g induces an inner product (and hence a norm) on all tensor bundles $T^k_l(M) := \otimes^k T^*M \bigotimes \otimes^l TM$. On symmetric covariant two-tensors (i.e. section of the bundle $T^2_0(M)$) we have for example the inner product

$$\langle \omega, \eta \rangle_{T^2_0, g} = g^{ac} g^{bd} \omega_{ab} g_{cd}.$$

In the following we will usually drop the subscript indicating the associated bundle and the subscript indicating the metric.

In our computations we need to use the commutator relations for covariant derivatives of tensor fields. For example, if ω is a covariant two-tensor field, that is $\omega \in T^*M \otimes T^*M$, then we have in abstract index notation

$$\nabla_a \nabla_b \omega_{cd} = \nabla_b \nabla_a - \mathrm{Rm}_{abed}\, \omega_{cd} - \mathrm{Rm}_{abce}\, \omega_{ce}.$$

Suppose (N, γ) is an n-dimensional smooth Riemannian manifold, we denote its Levi-Civita connection by ∇^N and its Riemann curvature tensor by Rm^N. Moreover, let $u : M \to N$ be a smooth map. This smooth map induces the pullback vector bundle $u^*TN \xrightarrow{\pi} M$. This vector bundle has base space M, the total space u^*TN is given by

$$u^*TN = \bigcup_{p \in M} \{p\} \times T_{u(p)}N,$$

and the projection $\pi : u^*TN \to M$ is given by $\pi(p, v) = p$ for $(p, v) \in (u^*TN)_p$. The Riemannian metric γ on N induces a bundle metric $u^*\gamma$ on the pullback bundle u^*TN given by

$$(u^*\gamma)(p)(v_p, w_p) = \gamma(\nabla u(p, v_p), \nabla u(p, w_p)) \text{ for } v_p, w_p \in T_pM.$$

The differential ∇u is a section of the vector bundle $T^*M \otimes u^*TN \xrightarrow{\pi} M$ with bundle metric $g^{-1} \otimes u^*\gamma$, where g^{-1} denotes the induced metric on the cotangent bundle T^*M. We denote it in abstract index notation by $\nabla_a u^\kappa$. The velocity term $\mathrm{tr}_N(\nabla u \otimes \nabla u)$ is a section of the vector bundle $\mathrm{Sym}^2\, T^*M$ (where $\mathrm{Sym}^2\, E$ denotes the second symmetric power of the vector bundle E) and given in abstract index notation by $\gamma_{\kappa\lambda} \nabla_a u^\kappa \nabla_b u^\lambda$ or in shorthand notation $\nabla_a u^\kappa \nabla_b u^\kappa$.
The Hessian $\nabla^2 u$ is a section of the vector bundle $T^*M \otimes T^*M \otimes u^*TN$. The Laplace–Beltrami operator with respect to the metrics g on M and γ on the target manifold N is denoted by $\Delta_{g,\gamma}$. It is the trace of the Hessian, that is

$$\Delta_{g,\gamma} u = \mathrm{tr}_g\, \nabla^2 u.$$

In abstract index notation we denote it by $g^{ab} \nabla_a \nabla_b u^\kappa$ or with the shorthand $\nabla_a \nabla_a u^\kappa$.
The energy density $|\nabla u|^2$ is defined by

$$|\nabla u|^2 = \langle \nabla u, \nabla u \rangle_{T^*M \otimes u^*TN}$$

where $\langle \cdot, \cdot \rangle_{T^*M \otimes u^*TN}$ denotes the bundle metric on $T^*M \otimes u^*TN$. In abstract index notation this reads

$$|\nabla u|^2 = g^{ab} \gamma_{\kappa\lambda} \nabla_a u^\kappa \nabla_b u^\lambda$$

Moreover we observe that $|\nabla u|^2 = \mathrm{tr}_g(\mathrm{tr}_N(\nabla u \otimes \nabla u))$.
For the norm squared of the Hessian we have

$$|\nabla^2 u|^2 = \langle \nabla^2 u, \nabla^2 u \rangle_V$$

where $\langle \cdot, \cdot \rangle_V$ denotes the bundle metric on $V = T^*M \otimes T^*M \otimes u^*TN$. In abstract index notation this reads

$$|\nabla^2 u|^2 = g^{ac} g^{bd} \gamma_{\kappa\lambda} \nabla_a \nabla_b u^\kappa \nabla_c \nabla_d u^\lambda.$$

In our computations we also need to use commutator relations for covariant derivatives on vector bundles, which are constructed from the pullback bundle u^*TN. We have for example for $\omega \in T^*M \otimes T^*M \otimes u^*TN$ in abstract index notation the commutator rule

$$\begin{aligned}\nabla_a \nabla_b \omega_{cd}{}^{\kappa} &= \nabla_b \nabla_a \omega_{cd}{}^{\kappa} - \mathrm{Rm}_{abce}\, \omega_{ed}{}^{\kappa} - \mathrm{Rm}_{abde}\, \omega_{ce}{}^{\kappa} - (\mathrm{Rm}^{u^*TN})_{ab\kappa\lambda} \omega_{cd}{}^{\lambda} \\ &= \nabla_b \nabla_a \omega_{cd}{}^{\kappa} - \mathrm{Rm}_{abce}\, \omega_{ed}{}^{\kappa} - \mathrm{Rm}_{abde}\, \omega_{ce}{}^{\kappa} - \mathrm{Rm}^N_{\mu\nu\kappa\lambda}\, \nabla_a u^{\mu} \nabla_b u^{\nu} \omega_{cd}{}^{\lambda}.\end{aligned}$$

For our results we need to assume that the curvature of the target manifold is finite in some sense:

Definition 2.1 (Controlled curvature)**.**
The target manifold (N, γ) has controlled curvature, if the curvature and the first derivative of curvature are uniformly bounded, that is

$$\sup_{q \in N} |\,\mathrm{Rm}^N(q)|_{\gamma} + \sup_{q \in N} |\nabla \mathrm{Rm}^N(q)|_{\gamma} =: d_1 < \infty.$$

In particular, if the target manifold (N, γ) is closed, or if (N, γ) is Euclidean space $(\mathbb{R}, \delta)$, the circle $(\mathbb{S}^1, g_{\mathbb{S}^1})$, or hyperbolic space $(\mathbb{H}^2, g_{\mathbb{H}^2})$, then it has controlled curvature.
Let us mention how to rescale the flow:
Given a solution $\{g(t), u(t)\}_{t \in [0,T)}$ to Ricci flow coupled to harmonic map heat flow, and $\lambda > 0$ is a constant, we define

$$g_{\lambda}(s) = \lambda g\left(\frac{s}{\lambda}\right) \text{ and } u_{\lambda}(s) = u\left(\frac{s}{\lambda}\right).$$

Then the pair $\{g_{\lambda}(s), u_{\lambda}(s)\}_{s \in [0, \lambda T)}$ is also a solution to Ricci flow coupled to harmonic map heat flow.
For the special case of List flow there is another symmetry involving the coupling constant: If $\{g(t), u(t)\}_{t \in [0,T)}$ solves List flow with coupling constant $\alpha > 0$, then $\{g(t), \tilde{u}(t)\}_{t \in [0,T)}$ with $\tilde{u}(t) = \sqrt{\alpha} u(t)$ solves List flow with coupling 1. In particular, if

$$c_0 = \sup_{p \in M} |\nabla u|^2(p, 0) \text{ and } s_0 = \sup_{p \in M} |\nabla^2 u|^2(p, 0)$$

denote the initial bound on the energy density and the Hessian, then the natural combinations to appear are αc_0 and αs_0, and hence by metric rescalings as above the combination $\alpha^2 c_0^2 + \alpha s_0$ is natural; compare with the estimates in Section 6.
Our notation for geodesic balls and parabolic neighbourhoods is the following:
Suppose $p_0 \in M$ and let $r > 0$ be a positive radius. We denote by $B_r(p_0)$ the geodesic ball of radius r around the point p, it is given by

$$B_g(p, r) = \{p \in M | d_g(p_0, p) < r\}.$$

If $t_0 \in \mathbb{R}$, $\rho > 0$ and $\tau > 0$ then $\mathcal{P}(p_0, t_0, \rho, \tau)$ denotes the backwards parabolic neighbourhood of temporal duration τ and spatial extension ρ, which is based at the spacetime point $(p_0, t_0) \in M \times \mathbb{R}$. It is given by

$$\mathcal{P}(t_0, p_0, \tau, \rho) = \left\{(t, p) \in \mathbb{R} \times M | t_0 - \tau \le t \le t_0 \text{ and } d_{g_{t_0}}(p_0, p) < \rho\right\}.$$

2.2. **Ricci flow on multiply warped products.**

In this section we explain the relation beetween Ricci flow of multiply warped products and the extended Ricci flow system (1).
Suppose (M, g) is a Riemannian manifold and the fibers $(N_1, h_1), \ldots, (N_k, h_k)$ are Ricci-flat Riemannian manifolds of dimension $\dim N_i = s_i$ for $1 \leq i \leq k$. Moreover, suppose $f_i : M \to \mathbb{R}$ are smooth functions.
We consider on the product manifold

$$L = M \times N_1 \times \cdots \times N_k$$

the multiply warped product metric $\tilde{g}$ given by

$$\tilde{g} = g + \exp(2f_1)h_1 + \cdots + \exp(2f_k)h_k.$$

The following formulae for the Ricci curvature of a multiply warped product are well-known (see for example [32]):
Let X, Y be vector fields tangential to the base manifold M, then we have

$$\mathrm{Rc}_{\tilde{g}}(X, Y) = \mathrm{Rc}_g(X, Y) - \sum_{i=1}^{k} s_i(\nabla_g^2 f_i)(X, Y) - \sum_{i=1}^{k} s_i \nabla f_i(X) \nabla f_i(Y).$$

If X is a vector field tangential to base manifold M and V a vector field tangent to some fiber N_j we have

$$\mathrm{Rc}_{\tilde{g}}(X, V) = 0.$$

If V_i is a vector field tangential to a fiber N_i and V_j is a vector field tangential to a fiber N_j with $i \neq j$ we have

$$\mathrm{Rc}_{\tilde{g}}(V_i, V_j) = 0.$$

Finally, if V, W are vector fields tangent to the same fiber N_i we have

$$\mathrm{Rc}_{\tilde{g}}(V, W) = \mathrm{Rc}_{h_i}(V, W) - \exp(2f_i) \left(\Delta_g f_i + s_i |\nabla f_i|^2 + \sum_{k=1, k \neq i}^{l} s_k \langle \nabla f_i, \nabla f_k \rangle \right) h_i(V, W).$$

The Ricci flow preserves the warped product structure, i.e. we have

$$\tilde{g}(t) = g(t) + \exp(2f_1(t))h_1 + \cdots + \exp(2f_k(t))h_k$$

for a one-parameter family $\{g(t)\}$ of Riemannian metrics on M and smooth functions $f_i : [0, T) \times M \to \mathbb{R}$, where $1 \leq i \leq k$. The Ricci flow equation $\partial_t \tilde{g}(t) = -2 \,\mathrm{Rc}_{\tilde{g}(t)}$ is equivalent to the system

$$\partial_t g(t) = -2 \,\mathrm{Rc}_{g(t)} + 2 \sum_{j=1}^{k} \nabla f_j \otimes \nabla f_j + 2 \sum_{j=1}^{k} s_j \nabla_g^2 f_j,$$

$$\partial_t f_i(t) = \Delta_{g(t)} f_i(t) + s_i |\nabla f_i|^2 + \sum_{j=1, j \neq i}^{k} s_j \langle \nabla f_i, \nabla f_j \rangle.$$

We consider the vector field X tangential to M given by

$$X = \sum_{i=1}^{k} s_i \nabla f_i \in \Gamma(TM)$$

and compute the Lie derivatives

$$\mathcal{L}_X g = 2 \sum_{i=1}^{k} s_i \nabla^2 f_i,$$

$$\mathcal{L}_X \nabla f_i = s_i |\nabla f_i|^2 + \sum_{j=1, j\neq i}^{k} s_j \langle \nabla f_i, \nabla f_j \rangle.$$

Pulling back by flow of the vector field X we arrive at the system

$$\begin{aligned} \partial_t g(t) &= -2 \operatorname{Rc}_{g(t)} + 2 \sum_{j=1}^{k} s_j \nabla f_j(t) \otimes \nabla f_j(t), \\ \partial_t f_i(t) &= \Delta_{g(t)} f_i(t) \end{aligned} \tag{5}$$

If we collect the maps $f_1, \ldots, f_k$ into a map $f : M \to \mathbb{R}^k$ given by

$$f = (f_1, \ldots, f_k)$$

and introduce on $\mathbb{R}^k$ the flat metric

$$\gamma = \sum_{j=1}^{k} s_j \, \mathrm{d}x^j \otimes \mathrm{d}x^j,$$

then the system (5) is given by

$$\begin{aligned} \partial_t g(t) &= -2 \operatorname{Rc}_{g(t)} + 2 \operatorname{tr}_N \nabla f(t) \otimes \nabla f(t), \\ \partial_t f(t) &= \Delta_{g(t),\gamma} f(t) \end{aligned} \tag{6}$$

This system is the Ricci flow coupled to harmonic map heat flow into the flat target $\mathbb{R}^k$. We record the above in a proposition:

Proposition 2.2.
The Ricci flow of a multiply warped product manifold over the base manifold M with k Ricci flat fibers is given (after pullback by diffeomorphisms) by Ricci flow coupled to harmonic map heat flow on the base manifold M into the flat target manifold $\mathbb{R}^k$.

We remark that this relation is well-known in the case $k = 1$ and $N_1 = \mathbb{S}^1$: see the computation in R. Buzano's (formerly R. Mueller) thesis, Lemma A.2 and Lemma A.3 in the appendix of [69], or Remark 1.2 in work of J. Lott and N. Sesum [62]. Indeed, in this case the flow reduces to List flow (2).

2.3. **Static solutions to the Einstein equations.**

In this section we explain the relation of List flow (2) to static vacuum solutions in General Relativity.
Suppose $(L^4, \bar{g})$ is a four-dimensional Lorentzian manifold, that is a four-dimensional smooth manifold equipped with a smooth tensor field $\bar{g} \in \operatorname{Sym}^2 T^*M$, such that the restriction $\bar{g}|_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate bilinear form of signature $(-+++)$.
Given an energy-momentum tensor $T \in \operatorname{Sym}^2 T^*M$ the Lorentzian manifold $(L^4, \bar{g})$ solves the Einstein equations, if

$$\mathrm{Rc}_{\bar{g}} - \frac{1}{2}\mathrm{R}_{\bar{g}}\,\bar{g} = 8\pi T \tag{7}$$

If $T = 0$ we say that the Lorentzian manifold $(L^4, \bar{g})$ is a vacuum solution to the Einstein equations. A vacuum solution satisfies to the Einstein equations satifies $\mathrm{Rc}_{\bar{g}} = 0$.

If there exists a timelike Killing vector field K (i.e. $\bar{g}(K, K) < 0$ and $\mathcal{L}_K \bar{g} = 0$), which is hypersurface orthogonal ($K^\flat \wedge dK^\flat = 0$), the Lorentzian manifold $(L^4, \bar{g})$ is called static. Under some additional mild technical assumptions this implies the decomposition

$$L^4 = I \times M^3 \text{ and } \bar{g} = -N^2\, \mathrm{d}t \otimes \mathrm{d}t + \hat{g},$$

where $I \subset \mathbb{R}$, the tensor field $\hat{g}$ is a Riemannian metric on M^3 and N is smooth positive function on M, which is called the lapse function. Note, that in this setting the metric $\bar{g}$ is a Lorentzian warped product metric. The two most important examples of a static manifolds in General Relativity are Minkowski space and the one-parameter family of Schwarzschild spacetimes.
For Minkowski space the smooth manifold L^4 is given by $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ and the Lorentzian metric $\bar{g}$ is given by

$$\bar{g} = -\mathrm{d}t \otimes \mathrm{d}t + \delta_{\mathbb{R}^3},$$

where $\delta_{\mathbb{R}^3}$ denotes the Euclidean metric on $\mathbb{R}^3$. Hence the lapse function N is given by the constant function $N = 1$ and $\hat{g}$ is the flat metric on euclidean space $\mathbb{R}^3$. The timelike Killing vector field K is given by $K = \partial_t$.
For the Scharzschild spacetime and a parameter $m \geq 0$ (which is called the mass) the smooth manifold L^4_m is given by $L^4_m = \mathbb{R} \times \big((2m, \infty) \times \mathbb{S}^2\big)$ and the Lorentzian metric $\bar{g}_m$ is given by

$$\bar{g}_m = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t \otimes \mathrm{d}t + \frac{1}{1 - \frac{2m}{r}}\mathrm{d}r \otimes \mathrm{d}r + r^2 g_{\mathbb{S}^2},$$

where $g_{\mathbb{S}^2}$ denotes the standard metric of scalar curvature 2 on the two-sphere $\mathbb{S}^2$.
The vacuum Einstein equations $\mathrm{Rc}_{\bar{g}} = 0$ decompose in this setting into a coupled elliptic system given by

$$\begin{aligned} \Delta_{\hat{g}} N &= 0, \\ N\,\mathrm{Rc}_{\hat{g}} &= \nabla^2_{\hat{g}} N, \end{aligned} \tag{8}$$

called the static vacuum Einstein equations. Minkowski space and the Schwarzschild space-time are vacuum solutions to the Einstein equations, since both are Ricci flat.
Sometimes it is more convenient to use a different form of these equations:
One defines $g = N^2\hat{g}$ and sets $u = \log N$. The function u is well-defined since we assumed that $N > 0$ and $u \to -\infty$ as $N \to 0$. By using the transformation formula for the Ricci curvature and the Laplace–Beltrami operator under conformal changes

$$\begin{aligned}\mathrm{Rc}_g &= \mathrm{Rc}_{\hat{g}} - \left(\frac{1}{N}\nabla^2_{\hat{g}}N - \frac{2}{N^2}\nabla N \otimes \nabla N\right) - \frac{1}{N}\left(\Delta_{\hat{g}}N\right)\hat{g},\\ \Delta_g f &= \frac{1}{N^2}\left(\Delta_{\hat{g}}f + \hat{g}(\nabla f, \nabla N)\right),\end{aligned}$$

one deduces the transformed static equations

$$\begin{aligned}\Delta_g u &= 0,\\ \mathrm{Rc}_g &= 2\,\nabla u \otimes \nabla u.\end{aligned} \tag{9}$$

The technical advantage of this system is that the coupling is only in the first derivatives, not in the second derivatives as in the system (8).
We observe that stationary points of List flow on three-manifolds correspond to solutions of the conformally transformed static equations (9), which arise from static solutions to vacuum Einstein equations on a Lorentzian four-manifold.

3. Evolution equations for the extended Ricci flow system

Suppose M is a closed manifold and (N,γ) is a Riemannian manifold with controlled curvature (see Definition 2.1) and $I \subset \mathbb{R}$ an interval. A one-parameter family $\{g(t),u(t)\}_{t\in I}$ consisting of a one-parameter family $\{g(t)\}_{t\in I}$ of Riemannian metrics on M and a one parameter-family $\{u(t)\}_{t\in I}$ of maps $u(t): M \to N$ evolves by Ricci flow coupled to harmonic map heat flow, if the following system is satisfied:

$$\begin{aligned} \partial_t g(t) &= -2\operatorname{Rc}_{g(t)} + 2\alpha \operatorname{tr}_N(\nabla u \otimes \nabla u), \\ \partial_t u(t) &= \Delta_{g(t),\gamma} u(t). \end{aligned} \tag{10}$$

This flow was introduced into the literature by R. Buzano (formerly R. Mueller) in his thesis [69]. The special case $(N,\gamma) = (\mathbb{S}^1, g_{\mathbb{S}^1})$ was introduced earlier by B. List in his thesis [59] and is sometimes called List flow.

For a given smooth Riemannian metric g_0 and a given smooth map $u_0 : M \to N$ there exists by the DeTurck trick (see [30]) adapted to our setting (as explained in work of R. Buzano, see Proposition 2.1 in [69]) a time $T > 0$, a time interval $[0,T)$ and a smooth one-parameter family $\{g(t),u(t)\}_{t\in[0,T)}$, which solves the system (10) and attains its initial data, i.e. $g(0) = g_0$ and $u(0) = u_0$.

We remark that B. List showed for the special case $(N,\gamma) = (\mathbb{S}^1, g_{\mathbb{S}^1})$ the short-time existence in the class of complete Riemannian manifolds with bounded curvature, bounded function u, bounded gradient ∇u and bounded Hessian $\nabla^2 u$, see Theorem 3.22 in [59].

Moreover, we have the following criterium for the maximal extension of the flow:

Proposition 3.1 (Blow-up criterium for the flow; R. Buzano, Theorem 3.12 in [69])**.** *Suppose $\{g(t),u(t)\}_{t\in[0,T)}$ solves Ricci flow coupled to harmonic map heat flow and $T < \infty$. Suppose that the time $T < \infty$ is maximal. Then the curvature of $(M,g(t))$ has to become unbounded for $t \to T$ in the sense that*

$$\limsup_{t\to T} \left(\sup_{p\in M} |\operatorname{Rm}(t,p)|_{g(t)} \right) = \infty.$$

In the following we will recall the evolution equations for geometric quantities such as the volume element, the Ricci curvature and the scalar curvature. Since we work in the following in three dimensions, we omit the evolution equation for the full Riemann curvature tensor. Moreover, we recall the evolution equations for the first two derivatives of the flow. These equations have been computed by B. List [59] and R. Buzano [69], however for the convenience of the reader we provide proofs.

Proposition 3.2 (Evolution for geometric quantites under deformation of the metric)**.** *Suppose a one-parameter family of Riemannian metrics $\{g(t)\}$ evolves by the equation*

$$\partial_t g(t) = s(t),$$

where $s(t)$ denotes a family of symmetric two-tensors on M. Then we have the evolution equations

$$\begin{aligned}\partial_t \operatorname{dvol} &= \frac{1}{2} \operatorname{tr} s,\\ \partial_t \operatorname{Rc} &= -\frac{1}{2}\Delta_L s - \frac{1}{2}\nabla^2 \operatorname{tr} s - \operatorname{div}^* \operatorname{div} s,\\ \partial_t \operatorname{R} &= -\langle \operatorname{Rc}, s\rangle - \Delta \operatorname{tr} s + \operatorname{div}\operatorname{div} s,\end{aligned}$$

where Δ_L denotes the Lichnerowicz Laplacian on symmetric two tensors, div *the divergence operator on symmetric two-tensors and one-forms and* div^* *the adjoint of the divergence operator. In abstract index notation they are given by*

$$\begin{aligned}(\Delta_L s)_{ab} &= \Delta s_{ab} + 2\operatorname{Rm}_{cabd} s_{cd} - \operatorname{Rc}_{ac} s_{cb} - \operatorname{Rc}_{bc} s_{ac},\\ (\operatorname{div} t)_b &= \nabla_a t_{ab},\\ (\operatorname{div}\omega) &= \nabla_a \omega_a,\\ (\operatorname{div}^* t)_{ab} &= -\frac{1}{2}(\nabla_a t_b + \nabla_b t_a).\end{aligned}$$

Proof.
The proofs of these equations are well-known. We refer the reader to the the books [14] and [68]. □

In the following proposition we apply the variation formulae to Ricci flow coupled to harmonic map heat flow:

Proposition 3.3 (Evolution of geometric quantities)**.**
Suppose $(g(t), u(t))_{t\in[0,T)}$ is a solution to Ricci flow coupled to harmonic map heat flow. Then we have the following evolution equations

$$\begin{aligned}\partial_t \operatorname{dvol} =& -\left(\operatorname{R} - \alpha|\nabla u|^2\right),\\ \partial_t \operatorname{Rc} =& \Delta \operatorname{Rc} + 2\langle \operatorname{Rm}, \operatorname{Rc} - \alpha \operatorname{tr}_N(\nabla u \otimes \nabla u)\rangle - 2\operatorname{Rc}^2 + 2\alpha \Delta_{g,\gamma} u \cdot \nabla^2 u - 2\alpha(\nabla^2 u)^2\\ &+ 2\alpha \sum_{k=1}^n \operatorname{Rm}^N(\nabla u(\cdot), \nabla u(e_k), \nabla u(e_k), \nabla u(\cdot)),\\ \partial_t \operatorname{R} =& \Delta \operatorname{R} + 2|\operatorname{Rc}|^2 - 4\alpha\langle \operatorname{Rc}, \operatorname{tr}_N(\nabla u \otimes \nabla u)\rangle + 2\alpha|\Delta_{g,\gamma} u|^2 - 2\alpha|\nabla^2 u|^2\\ &+ 2\alpha \sum_{i,j=1}^n \operatorname{Rm}^N(\nabla u(e_i), \nabla u(e_j), \nabla u(e_j), \nabla u(e_i)).\end{aligned}$$

where Rc^2 and $(\nabla^2 u)^2$ denote the operator square of the Ricci tensor and the Hessian. In abstract index notation they are given by

$$\begin{aligned}(\operatorname{Rc}^2)_{ab} &= \operatorname{Rc}_{ac}\operatorname{Rc}_{cb},\\ (\nabla^2 u)^2_{ab} &= \nabla_c\nabla_a u^\kappa \nabla_c \nabla_b u^\kappa,\\ \langle \operatorname{Rm}, \operatorname{Rc}\rangle_{ab} &= \operatorname{Rm}_{cabd}\operatorname{Rc}_{cd}.\end{aligned}$$

Proof.
This was proven for List flow in Lemma 2.4 and 2.5 of [59] and for Ricci flow coupled to harmonic map heat flow in Proposition 2.3 of [69]. We present the proof for the convenience of the reader. For the evolution of the volume element one directly deduces

$$\partial_t \,\mathrm{dvol} = \frac{1}{2} \operatorname{tr}\left(-2\,\mathrm{Rc} + 2\alpha \operatorname{tr}_N(\nabla u \otimes \nabla u)\right) = -\left(\mathrm{R} - \alpha|\nabla u|^2\right).$$

To compute the evolution equation for the Ricci curvature we insert the velocity into the variation formula for the Ricci curvature from Proposition 3.3:

$$\begin{aligned}\partial_t \mathrm{Rc}_{ab} =& -\frac{1}{2}\Delta_L(-2\,\mathrm{Rc}_{ab} + 2\alpha\nabla_a u^\kappa \nabla_b u^\kappa) - \frac{1}{2}\nabla_a\nabla_b(-2\,\mathrm{R} + 2\alpha|\nabla u|^2)\\ &+ \frac{1}{2}\left(\nabla_a\nabla_c(-2\,\mathrm{Rc}_{cb} + 2\alpha\nabla_c u^\kappa\nabla_b u^\kappa) + \nabla_b\nabla_c(-2\,\mathrm{Rc}_{ac} + 2\alpha\nabla_a u^\kappa \nabla_c u^\kappa)\right)\\ =&\Delta_L \mathrm{Rc}_{ab} - \alpha\Delta_L(\nabla_a u^\kappa\nabla_b u^\kappa) + \nabla_a\nabla_b \mathrm{R} - \alpha\nabla_a\nabla_b|\nabla u|^2\\ &- \nabla_a\nabla_c \mathrm{Rc}_{cb} - \nabla_b\nabla_c \mathrm{Rc}_{ac} + \alpha\nabla_a\nabla_c(\nabla_c u^\kappa\nabla_b u^\kappa) + \alpha\nabla_b\nabla_c(\nabla_a u^\kappa\nabla_c u^\kappa)\end{aligned}$$

We rework the last two terms of the above equation:

$$\begin{aligned}&\nabla_a\nabla_c(\nabla_c u^\kappa \nabla_b u^\kappa)\\ =&\nabla_a(\nabla_c\nabla_c u^\kappa\nabla_b u^\kappa + \nabla_c u^\kappa\nabla_c\nabla_b u^\kappa)\\ =&\nabla_a\nabla_c\nabla_c u^\kappa\nabla_b u^\kappa + \nabla_c\nabla_c u^\kappa\nabla_a\nabla_b u^\kappa + \nabla_a\nabla_c u^\kappa\nabla_c\nabla_b u^\kappa + \nabla_c u^\kappa\nabla_a\nabla_c\nabla_b u^\kappa\\ =&\nabla_a\nabla_c\nabla_c u^\kappa\nabla_b u^\kappa + \nabla_a\nabla_b u^\kappa\Delta u^\kappa + \nabla_a\nabla_c u^\kappa\nabla_c\nabla_b u^\kappa + \nabla_c u^\kappa\nabla_a\nabla_c\nabla_b u^\kappa\end{aligned}$$

$$\begin{aligned}&\nabla_b\nabla_c(\nabla_a u^\kappa\nabla_c u^\kappa)\\ =&\nabla_b(\nabla_c\nabla_a u^\kappa\nabla_c u^\kappa + \nabla_a u^\kappa\nabla_c\nabla_c u^\kappa)\\ =&\nabla_b\nabla_c\nabla_a u^\kappa\nabla_c u^\kappa + \nabla_c\nabla_a u^\kappa\nabla_b\nabla_c u^\kappa + \nabla_b\nabla_a u^\kappa\nabla_c\nabla_c u^\kappa + \nabla_a u^\kappa\nabla_b\nabla_c\nabla_c u^\kappa\\ =&\nabla_b\nabla_c\nabla_a u^\kappa\nabla_c u^\kappa + \nabla_c\nabla_a u^\kappa\nabla_b\nabla_c u^\kappa + \nabla_a\nabla_b u^\kappa\Delta u^\kappa + \nabla_a u^\kappa\nabla_b\nabla_c\nabla_c u^\kappa\end{aligned}$$

On the other hand we observe

$$\begin{aligned}&\Delta(\nabla_a u^\kappa\nabla_b u^\kappa)\\ =&\nabla_c\nabla_c(\nabla_a u^\kappa\nabla_b u^\kappa)\\ =&\nabla_c(\nabla_c\nabla_a u^\kappa\nabla_b u^\kappa + \nabla_a u^\kappa\nabla_c\nabla_b u^\kappa)\\ =&\nabla_c\nabla_c\nabla_a u^\kappa\nabla_b u^\kappa + \nabla_c\nabla_a u^\kappa\nabla_c\nabla_b u^\kappa + \nabla_c\nabla_a u^\kappa\nabla_c\nabla_b u^\kappa + \nabla_a u^\kappa\nabla_c\nabla_c\nabla_b u^\kappa\end{aligned}$$

$$
\begin{aligned}
&\nabla_a\nabla_b\,|\nabla u|^2\\
=&\nabla_a\nabla_b(\nabla_c u^\kappa\nabla_c u^\kappa)\\
=&\nabla_a(\nabla_b\nabla_c u^\kappa\nabla_c u^\kappa+\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa)\\
=&\nabla_a\nabla_b\nabla_c u^\kappa\nabla_c u^\kappa+\nabla_b\nabla_c u^\kappa\nabla_a\nabla_c u^\kappa+\nabla_a\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa+\nabla_c u^\kappa\nabla_a\nabla_b\nabla_c u^\kappa.
\end{aligned}
$$

By collecting terms we arrive at

$$
\begin{aligned}
&\nabla_a\nabla_c(\nabla_c u^\kappa\nabla_b u^\kappa)+\nabla_b\nabla_c(\nabla_a u^\kappa\nabla_c u^\kappa)-\Delta(\nabla_a u^\kappa\nabla_b u^\kappa)-\nabla_a\nabla_b\,|\nabla u|^2\\
=&2\nabla_a\nabla_b u^\kappa\Delta u^\kappa-2\nabla_a\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa+(\nabla_a\nabla_c\nabla_c u^\kappa-\nabla_c\nabla_c\nabla_a u^\kappa)\nabla_b u^\kappa\\
&+(\nabla_b\nabla_c\nabla_c u^\kappa-\nabla_c\nabla_c\nabla_b u^\kappa)\nabla_a u^\kappa\\
&+\nabla_c u^\kappa\,(\nabla_a\nabla_c\nabla_b u^\kappa+\nabla_b\nabla_c\nabla_a u^\kappa-\nabla_a\nabla_b\nabla_c u^\kappa-\nabla_a\nabla_b\nabla_c u^\kappa)\\
=&2\nabla_a\nabla_b u^\kappa\Delta u^\kappa-2\nabla_a\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa+(\nabla_a\nabla_c\nabla_c u^\kappa-\nabla_c\nabla_c\nabla_a u^\kappa)\nabla_b u^\kappa\\
&+(\nabla_b\nabla_c\nabla_c u^\kappa-\nabla_c\nabla_c\nabla_b u^\kappa)\nabla_a u^\kappa\\
=&2\nabla_a\nabla_b u^\kappa\Delta u^\kappa-2\nabla_a\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa\\
&-\left(\mathrm{Rm}_{accd}\,\nabla_d u^\kappa+\mathrm{Rm}_{ac\kappa\lambda}\,\nabla_c u^\lambda\right)\nabla_b u^\kappa-\left(\mathrm{Rm}_{bccd}\,\nabla_d u^\kappa+\mathrm{Rm}_{bc\kappa\lambda}\,\nabla_c u^\lambda\right)\nabla_c u^\kappa\\
=&2\nabla_a\nabla_b u^\kappa\Delta u^\kappa-2\nabla_a\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa-\mathrm{Rm}_{accd}\,\nabla_d u^\kappa\nabla_b u^\kappa+\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\lambda\nabla_b u^\kappa\\
&-\mathrm{Rc}_{bd}\,\nabla_d u^\kappa\nabla_a u^\kappa+\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_b u^\mu\nabla_c u^\nu\nabla_c u^\lambda\nabla_a u^\kappa\\
=&2\nabla_a\nabla_b u^\kappa\Delta u^\kappa-2\nabla_a\nabla_c u^\kappa\nabla_b\nabla_c u^\kappa-\mathrm{Rm}_{accd}\,\nabla_d u^\kappa\nabla_b u^\kappa-\mathrm{Rc}_{bd}\,\nabla_d u^\kappa\nabla_a u^\kappa\\
&+2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\kappa\nabla_b u^\lambda.
\end{aligned}
$$

Note that we used the skew-symmetry in the last two arguments of of the Riemann curvature tensor to deduce that the second line in the first equality vanishes. This could also be deduced from the observation that the last two terms in the initial expression are symmetric in the indices a und b.
If we recollect all terms we observe that the Ricci terms from the previous computation cancel with the Ricci terms from the Lichnerowicz Laplacian of $\mathrm{tr}_N\,(\nabla u\otimes\nabla u)$ and we obtain

$$
\begin{aligned}
\partial_t\,\mathrm{Rc}_{ab}=&\Delta\,\mathrm{Rc}_{ab}+2\,\mathrm{Rm}_{cabd}(\mathrm{Rc}_{cd}-\alpha\nabla_c u^\kappa\nabla_d u^\kappa)-2\,\mathrm{Rc}_{ac}\,\mathrm{Rc}_{cb}\\
&+2\alpha\nabla_a\nabla_b u^\kappa\Delta u^\kappa-2\alpha\nabla_c\nabla_a u^\kappa\nabla_c\nabla_b u^\kappa+2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\kappa\nabla_b u^\lambda.
\end{aligned}
$$

This is the desired diffusion-reaction equation for the Ricci curvature under Ricci flow coupled to harmonic map heat flow.
In the next step we compute the reaction-diffusion equation for the scalar curvature R. There are two ways to compute this equation; one can either directly use the variation formula from Proposition 3.2 or one can proceed by observing that the scalar curvature is the metric trace of the Ricci curvature and hence use the evolution equation for the Ricci

curvature with the evolution equation for the inverse metric. We will use the latter method:

$$
\begin{aligned}
\partial_t \mathrm{R} =&(\partial_t g^{ab}) \mathrm{Rc}_{ab} + g^{ab}\partial_t \mathrm{Rc}_{ab}\\
=& \left(2\,\mathrm{Rc}^{ab} - 2\alpha\nabla^a u^\kappa \nabla^b u^\kappa\right) \mathrm{Rc}_{ab}\\
&+ g^{ab}\left(\Delta \mathrm{Rc}_{ab} + 2\,\mathrm{Rm}_{cabd}\,\mathrm{Sc}_{cd} - 2\,\mathrm{Rc}_{ac}\,\mathrm{Rc}_{cb} + 2\alpha\Delta u\ \nabla_a\nabla_b u - 2\alpha\nabla_c\nabla_a u \nabla_c\nabla_b u\right)\\
&+ g^{ab}2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu \nabla_c u^\nu \nabla_c u^\kappa \nabla_b u^\lambda\\
=&2|\mathrm{Rc}|^2 - 2\alpha\langle \mathrm{Rc}, \mathrm{tr}_N(\nabla u\otimes\nabla u)\rangle + \Delta\mathrm{R} + 2\langle \mathrm{Rc},\mathrm{Sc}\rangle - 2|\mathrm{Rc}|^2 + 2\alpha|\Delta u|^2 - 2\alpha|\nabla^2 u|^2\\
&+ 2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_c u^\mu \nabla_d u^\nu \nabla_d u^\kappa \nabla_c u^\lambda\\
=&\Delta\mathrm{R} + 2|\mathrm{Rc}|^2 - 4\alpha\langle \mathrm{Rc}, \mathrm{tr}_N(\nabla u\otimes\nabla u)\rangle + 2\alpha|\Delta u|^2 - 2\alpha|\nabla^2 u|^2\\
&+ 2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_c u^\mu \nabla_d u^\nu \nabla_d u^\kappa \nabla_c u^\lambda.
\end{aligned}
$$

The last line is exactly the claimed formula in abstrac index notation.

□

In the next proposition we study the evolution equations for the first spatial derivatives of the family $\{u(t)\}_{t\in[0,T)}$ of smooth maps $u(t): M \to N$.

Proposition 3.4 (Evolution of the first derivatives of the map u)**.**
Suppose $(g(t), u(t))_{t\in[0,T)}$ is a solution to Ricci flow coupled to harmonic map heat flow. Then the derivatives of the function u evolve by

$$
\begin{aligned}
\partial_t \nabla u =& \Delta\nabla u - \mathrm{Rc}(\nabla u,\cdot) + \sum_{j=1}^n \mathrm{Rm}^N\left(\nabla u(e_i), \nabla u(e_j), \nabla u(e_j), \cdot\right),\\
\partial_t\,\mathrm{tr}_N(\nabla u\otimes\nabla u) =& \Delta\,\mathrm{tr}_N(\nabla u\otimes\nabla u) - 2\nabla_c\nabla_a u^\kappa \nabla_c\nabla_b u^\kappa - \mathrm{Rc}_{ad}\,\nabla_d u^\kappa \nabla_b u^\kappa\\
&- \mathrm{Rc}_{bd}\,\nabla_d u^\kappa\nabla_a u^\kappa + 2\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\kappa\nabla_b u^\lambda,\\
\partial_t|\nabla u|^2 =& \Delta|\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha|\,\mathrm{tr}_N(\nabla u\otimes\nabla u)|^2\\
&+ \sum_{i,j=1}^n \mathrm{Rm}^N\left(\nabla u(e_i), \nabla u(e_j), \nabla u(e_j), \nabla u(e_i)\right),
\end{aligned}
$$

Let us discuss the structure of the reaction-diffusion equation for the energy density in more detail, since it plays a major role in the following sections:
On a fixed Riemannian background (M, g) the energy density $|\nabla u|_g^2$ of a solution $u: M \to \mathbb{R}$ to the heat equation $(\partial_t - \Delta_g)u = 0$ evolves by

$$(\partial_t - \Delta_g)|\nabla u|_g^2 = -2|\nabla^2 u|_g^2 - 2\,\mathrm{Rc}_g(\nabla u, \nabla u).$$

The first term is the Bochner term. If the Ricci curvature is positive, the second term is negative definite and hence there is exponential decay of the energy density by the maximum principle. However, if the Ricci curvature has a mixed sign, then the second term cannot be easily controlled.

If one considers a one-paramter family of Riemannian metrics $\{g(t)\}_{t\in[0,T)}$ evolving by Ricci flow and a solution $u : M \to \mathbb{R}$ of the heat equation $(\partial_t - \Delta_{g(t)})u = 0$ with respect to the evolving metric $g(t)$, then the energy density evolves by

$$(\partial_t - \Delta_{g(t)})|\nabla u|^2_{g(t)} = -2|\nabla^2 u|^2_{g(t)}$$

and the term involving the Ricci curvature is absent. While the supremum of the energy density does not grow in time, the maximum principle does not imply decay.
Finally, in List flow the coupling term $2\alpha\nabla u \otimes \nabla u$ introduces the term $-\alpha|\nabla u|^4$ into the evolution equation for the energy density and hence we expect decay by the maximum principle (see Proposition 3.9).
Finally, the last term in the evolution equation for the energy density in Proposition 3.4 comes from the curvature of the target. It is already present in Harmonic Map Heat flow, i.e. for a time indepedent Riemannian metric on M.

Proof.
These evolution equations were proven in Lemma 2.7 in [59] for List flow and in Proposition 2.4 of [69] for Ricci flow coupled to harmonic map heat flow. We present the proofs for the convenience of the reader:
We compute for the evolution of the differential:

$$\begin{aligned}\partial_t\nabla_a u^\kappa &= \nabla_a\partial_t u^\kappa = \nabla_a\nabla_c\nabla_c u^\kappa\\ &= \nabla_c\nabla_c\nabla_c u^\kappa - \mathrm{Rm}_{accd}\,\nabla_d u^\kappa - \mathrm{Rm}_{ac\kappa\lambda}\,\nabla_c u^\lambda\\ &= \Delta\nabla_a u^\kappa - \mathrm{Rc}_{cd}\,\nabla_d u^\kappa - \mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\lambda.\end{aligned}$$

Here we have used the commutation rule for covariant derivatives of tensor fields derived from maps into the target (N, γ) in the second line and third line (see the preliminaries in Section 2.1).
For the evolution of the quantity $\mathrm{tr}_N(\nabla u \otimes \nabla u)$ we compute

$$\begin{aligned}\partial_t(\nabla_a u^\kappa\nabla_b u^\kappa) =&(\partial_t\nabla_a u^\kappa)\nabla_b u^\kappa + \nabla_a u^\kappa(\partial_t\nabla_b u^\kappa)\\ =&(\Delta\nabla_a u^\kappa)\nabla_b u^\kappa - \mathrm{Rc}_{ad}\,\nabla_d u^\kappa\nabla_b u^\kappa - \mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\lambda\nabla_b u^\kappa\\ &+ (\Delta\nabla_b u^\kappa)\nabla_a u^\kappa - \mathrm{Rc}_{bd}\,\nabla_d u^\kappa\nabla_a u^\kappa - \mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_b u^\mu\nabla_c u^\nu\nabla_c u^\lambda\nabla_a u^\kappa.\end{aligned}$$

We observe that

$$\Delta(\nabla_a u^\kappa\nabla_b u^\kappa) = (\Delta\nabla_a u^\kappa)\nabla_b u^\kappa + \nabla_a u^\kappa(\Delta\nabla_b u^\kappa) + 2\nabla_c\nabla_a u^\kappa\nabla_c\nabla_b u^\kappa$$

Moreover, the terms involving the Riemann curvature tensor of the manifold (N, γ) recombine by the symmetries of the curvature tensor. We deduce

$$\begin{aligned}\partial_t(\nabla_a u^\kappa\nabla_b u^\kappa) =&\Delta(\nabla_a u^\kappa\nabla_b u^\kappa) - 2\nabla_c\nabla_a u^\kappa\nabla_c\nabla_b u^\kappa - \mathrm{Rc}_{ad}\,\nabla_d u^\kappa\nabla_b u^\kappa - \mathrm{Rc}_{bd}\,\nabla_d u^\kappa\nabla_a u^\kappa\\ &+ 2\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_c u^\nu\nabla_c u^\kappa\nabla_b u^\lambda.\end{aligned}$$

For the evolution of the energy density $|\nabla u|^2$ we compute

$$\begin{aligned}\partial_t|\nabla u|^2 =&\partial_t\left(g^{ab}\nabla_a u^\kappa \nabla_b u^\kappa\right)\\ =&\left(\partial_t g^{ab}\right)\nabla_a u^\kappa\nabla_b u^\kappa + 2g^{ab}\nabla_a u^\kappa \partial_t\left(\nabla_b u^\kappa\right)\\ =&2\left(\mathrm{Rc}^{ab}\,\nabla_a u^\kappa \nabla_b u^\kappa - \alpha\nabla^a u^\kappa\nabla^b u^\kappa\right)\nabla_a u^\kappa\nabla_b u^\kappa\\ &+ 2g^{ab}\nabla_a u^\kappa \Delta\nabla_b u^\kappa - g^{ab}\nabla_a u^\kappa\,\mathrm{Rc}_{bd}\,\nabla_d u^\kappa + 2\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_b u^\nu\nabla_b u^\kappa\nabla_a u^\lambda\\ =&\Delta|\nabla u|^2 - 2|\nabla^2 u|^2 - 2\alpha|\operatorname{tr}_N\left(\nabla u\otimes\nabla u\right)|^2\\ &+\sum_{i,j=1}^{n}\mathrm{Rm}^N\left(\nabla u(e_i),\nabla u(e_j),\nabla u(e_j),\nabla u(e_i)\right)\end{aligned}$$

This finishes the proof of the proposition. □

The reaction-diffusion equations satisfied by the Ricci and scalar curvature involve second spatial derivatives of the map u. Hence we need to understand the reaction-diffusion equation satisfied by the second derivatives to understand the evolution of curvature:

Proposition 3.5 (Evolution of the second derivatives of the map u)**.**
Suppose $(g(t),u(t))_{t\in[0,T)}$ is a solution to Ricci flow coupled to harmonic map heat flow. Then the second derivatives of the function u evolve by

$$\begin{aligned}\partial_t\nabla^2 u =&\Delta\nabla^2 u + \mathrm{Rm} * \nabla^2 u + \alpha\nabla^2 u * \nabla u * \nabla u\\ &+ (\mathrm{Rm}^N * \nabla^2 u * (\nabla u)^{*2} + (\nabla\,\mathrm{Rm}^N) * (\nabla u)^{*4}),\end{aligned}$$

where $*$ denotes expression obtained from the arguments by suitable metric contractions. Moreover, if the target manifold (N,γ) has controlled curvature (see Definition 2.1) we deduce

$$\begin{aligned}\partial_t|\nabla^2 u|^2 \le&\Delta|\nabla^2 u|^2 - 2|\nabla^3 u|^2 + C|\,\mathrm{Rm}\,||\nabla^2 u|^2 + \alpha C|\nabla u|^2|\nabla^2 u|^2\\ &+ Cd_1\left(|\nabla^2 u|^2|\nabla u|^2 + |\nabla u|^4|\nabla^2 u|\right),\end{aligned}$$

where d_1 denotes the bound on the curvature and the first derivatives of curvature as in Definition 2.1 and C denotes a universal constant.

One should observe that the terms in the last bracket in the above equations is zero for List flow or more generally for any flat target manifold (N,γ).

Proof.
The proof of this evolution equation can be found in Proposition 3.5 of [69]. □

It is often convenient to consider the velocity in Ricci flow coupled to harmonic map heat flow

$$\mathrm{Sc} = \mathrm{Rc} - \alpha\operatorname{tr}_N\left(\nabla u\otimes\nabla u\right),$$

which is a modified version of Ricci curvature. Moreover, its trace, which is a modified version of scalar curvature, is given by

$$\mathrm{S} = \mathrm{R} - \alpha |\nabla u|^2.$$

The four-tensor

$$\mathrm{Sm} = \mathrm{Rm} - \frac{\alpha}{n-2}\left[\mathrm{tr}_N(\nabla u \otimes \nabla u)\right] \owedge g + \frac{\alpha}{(2n-1)(n-2)}|\nabla u|^2 g \owedge g$$

has the trace $\mathrm{tr}\,\mathrm{Sm} = \mathrm{Sc}$. Hence one may think of this tensor as a modified version of the Riemann curvature tensor. However, it will not play a role in the following.
The evolution equations of the velocity Sc and the trace of the velocity S have a simpler structure compared to the evolution equations for the Ricci and scalar curvature:

Proposition 3.6 (Evolution of combined quantities)**.**
Suppose $(g(t), u(t))_{t\in[0,T)}$ is a solution to Ricci flow coupled to harmonic map heat flow. Then the combined curvature Sc *and its trace* S *evolve by*

$$\begin{aligned}\partial_t \mathrm{Sc} &= \Delta_L \mathrm{Sc} + 2\alpha \Delta_{g,\gamma} u \cdot \nabla^2 u,\\ \partial_t \mathrm{S} &= \Delta \mathrm{S} + 2|\mathrm{Sc}|^2 + 2\alpha |\Delta_{g,\gamma} u|^2.\end{aligned}$$

Proof.
This was proved for List flow in Lemma 2.11 of [59] and for Ricci flow coupled to harmonic map heat flow in Theorem 2.5 of [69]. We compute for the modified Ricci curvature Sc:

$$\begin{aligned}\partial_t \mathrm{Sc} = &\,\partial_t\left(\mathrm{Rc} - \alpha\,\mathrm{tr}_N(\nabla u \otimes \nabla u)\right)\\ =&\,\Delta \mathrm{Rc} + 2\langle \mathrm{Rm}, \mathrm{Rc}\rangle - 2\,\mathrm{Rc}^2 + 2\alpha \Delta_{g,\gamma} u \cdot \nabla^2 u - 2\alpha(\nabla^2 u)^2 - 2\alpha\langle \mathrm{Rm}, \alpha\,\mathrm{tr}_N(\nabla u \otimes \nabla u)\rangle\\ &+ 2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\nabla_a u^\mu \nabla_c u^\nu \nabla_c u^\kappa \nabla_b u^\lambda\\ &- \alpha\Delta\,\mathrm{tr}_N(\nabla u \otimes \nabla u) + 2\alpha(\nabla^2 u)^2 - \mathrm{Rc}_{ad}\nabla_d u^\kappa \nabla_b u^\kappa - \mathrm{Rc}_{bd}\nabla_d u^\kappa \nabla_b u^\kappa\\ &- 2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\nabla_a u^\mu \nabla_c u^\nu \nabla_c u^\kappa \nabla_b u^\lambda\\ =&\,\Delta\left(\mathrm{Rc} - \alpha\,\mathrm{tr}_N(\nabla u \otimes \nabla u)\right) + 2\langle \mathrm{Rm}, \mathrm{Rc} - \alpha\,\mathrm{tr}_N(\nabla u \otimes \nabla u)\rangle\\ &- \mathrm{Rc}_{ad}(\mathrm{Rc}_{bd} - \alpha\nabla_d u^\kappa \nabla_b u^\kappa) - \mathrm{Rc}_{bd}(\mathrm{Rc}_{ad} - \alpha \nabla_d u^\kappa \nabla_a u^\kappa) + 2\alpha\Delta_{g,\gamma} u \cdot \nabla^2 u\\ =&\,\Delta \mathrm{Sc} + 2\langle \mathrm{Rm}, \mathrm{Sc}\rangle - \mathrm{Rc}_{ad}\mathrm{Sc}_{db} - \mathrm{Rc}_{bd}\mathrm{Sc}_{da} + 2\alpha\Delta_{g,\gamma} u \cdot \nabla^2 u.\end{aligned}$$

This is the desired formula.
To obtain the evolution equation for the modified scalar curvature S we take the trace of the above evolution equation to arrive at

$$\begin{aligned}\partial_t \mathrm{S} =&\,(\partial_t g^{ab})\mathrm{Sc}_{ab} + g^{ab}\partial_t \mathrm{Sc}_{ab}\\ =&\,(2\,\mathrm{Rc}^{ab} - 2\alpha\nabla^a u^\kappa \nabla^b u^\kappa)\mathrm{Sc}_{ab} + g^{ab}(\Delta_L \mathrm{Sc}_{ab} + 2\alpha\Delta u \nabla_a \nabla_b u)\\ =&\,2|\mathrm{Sc}|^2 + \Delta \mathrm{S} + 2\alpha|\Delta u|^2.\end{aligned}$$

We used the auxillary computation

$$\begin{aligned} g^{ab}\Delta_L W_{ab} &= g^{ab}\left(\Delta W_{ab} + 2\,\mathrm{Rm}_{cabd}\,W_{cd} - \mathrm{Rc}_{ac}\,W_{cb} - \mathrm{Rc}_{cb}\,W_{ac}\right) \\ &= \Delta W + 2\langle \mathrm{Rc}, W\rangle - \langle \mathrm{Rc}, W\rangle - \langle \mathrm{Rc}, W\rangle = \Delta W \end{aligned}$$

for any symmetric two-tensor W. □

In our arguments we often use the scalar maximum principle for parabolic equations.

Theorem 3.7 (Scalar maximum principle)**.**
Let M be a closed manifold and $\{g(t)\}_{[0,T]}$ a one-parameter family of Riemannian metrics. Suppose a smooth function $f : [0,T] \times M \to \mathbb{R}$ satisfies the parabolic equation

$$(\partial_t f)(t,p) = (\Delta_{g(t)} f)(t,p) + \langle X(t,p), (\nabla f)(t,p)\rangle_{g(t)} + c(t,p) f(t,p),$$

where $X : [0,T] \times M \to TM$ denotes a time-dependent vector field and $c : [0,T] \times M \to \mathbb{R}$ denotes a smooth function. Then non-negativity of the function f is preserved in the following sense:
Suppose we have initially for all $p \in M$ the inequality

$$f(0,p) \geq 0,$$

then we have the inequality

$$f(t,p) \geq 0$$

for all $t \in [0,T]$ and all $p \in M$.

In particular, suppose we have

$$(\partial_t - \Delta) f(t,p) \leq 0$$

for all $(t,p) \in (0,T) \times M$. Then the values of f are bounded above by its initial data:

$$\sup_{(t,p)\in[0,T]\times M} f(t,p) \leq \sup_{p \in M} f(0,p).$$

The following assertions are direct consequences of the scalar maximum principle applied to the reaction-diffusion equations derived in this section:

Proposition 3.8.
The modified scalar curvature S *is bounded below by its initial data, that is*

$$\mathrm{S}(t,p) \geq \inf_{p\in M} \mathrm{S}(0,p) =: \mathrm{S}_0\,.$$

Actually, a refined argument shows

$$\mathrm{S}(t,p) \geq \frac{\mathrm{S}_0}{1 - \frac{2}{n}\,\mathrm{S}_0\, t}.$$

Proof.
This was proved Lemma 2.12 of [59] for List flow and in Corollary 2.7 of [69] for Ricci flow coupled to harmonic map heat flow. We present the proof for the convenience of the reader: From the evolution equation for the modified scalar curvature S, see Proposition 3.6, we deduce

$$(\partial_t - \Delta)\,\mathrm{S} = 2|\,\mathrm{Sc}\,|^2 + 2\alpha|\Delta_{g,\gamma}u|^2 \geq 0,$$

since both terms are non-negative. Thus the first assertion follows by the scalar maximum principle, Theorem 3.7.
For the second assertion we estimate more carefully

$$(\partial_t - \Delta)\,\mathrm{S} = 2|\,\mathrm{Sc}\,|^2 + 2\alpha|\Delta_{g,\gamma}u|^2 \geq 2|\,\mathrm{Sc}\,|^2 \geq \frac{2}{n}\,\mathrm{S}^2,$$

where we used the trace inequality for symmetric two-tensors. If we subtract the solution to the ODE $\varphi'(t) = \frac{2}{n}\varphi^2(t)$ from the quantity S we obtain the test function

$$h(t,p) = \mathrm{S}(t,p) - \frac{\mathrm{S}_0}{1 - \frac{2}{n}\,\mathrm{S}_0\,t},$$

which satisfies

$$(\partial_t - \Delta)h \geq \frac{2}{n}\,\mathrm{S}^2 - \frac{2}{n}\left(\frac{\mathrm{S}_0}{1 - \frac{2}{n}\,\mathrm{S}_0\,t}\right)^2 = \frac{2}{n}\left(\mathrm{S} + \frac{\mathrm{S}_0}{1 - \frac{2}{n}\,\mathrm{S}_0\,t}\right)h.$$

Then the scalar maximum principle, Theorem 3.7, implies the second assertion. □

Proposition 3.9.
Suppose the target (N,γ) has nonpositive sectional curvature. Then the energy density is bounded above by its initial data, that is

$$|\nabla u|^2(t,p) \leq \sup_{p\in M} |\nabla u|^2(0,p) =: c_0.$$

Actually, we have the more refined estimate

$$|\nabla u|^2(t,p) \leq \frac{c_0}{1 + \frac{2}{\dim M}\alpha c_0 t}.$$

Proof.
This was proved in Proposition 2.4 of [69] for Ricci flow coupled to harmonic map heat flow. We present the proof for the convenience of the reader: For the first assertion we deduce from the evolution equation for the energy density 3.4 and the nonpositivity of the sectional curvature of (N,γ) the inequality

$$\begin{aligned}(\partial_t - \Delta)|\nabla u|^2 = &- 2|\nabla^2 u|^2 - 2\alpha|\,\mathrm{tr}_N\,(\nabla u \otimes \nabla u)\,|^2 \\ &+ \sum_{i,j=1}^{n} \mathrm{Rm}^N\,(\nabla u(e_i), \nabla u(e_j), \nabla u(e_j), \nabla u(e_i)) \leq 0.\end{aligned}$$

Then the first assertion follows by the scalar maximum principle, Theorem 3.7.
For the second assertion we observe the more refined estimate

$$(\partial_t - \Delta)|\nabla u|^2 \leq -\frac{2}{\dim M}\alpha|\nabla u|^4,$$

since we have

$$\frac{1}{\dim M}|\nabla u|^4 \leq |\operatorname{tr}_N(\nabla u \otimes \nabla u)|^2 \leq |\nabla u|^4.$$

The proof follows by computing the evolution equation for

$$h(t,p) = |\nabla u|^2(t,p) - \frac{c_0}{1 + \frac{2}{\dim M}\alpha c_0 t}$$

and arguing as in the previous proposition. □

Corollary 3.10 (Improved decay for List flow)**.**
Suppose $\{g(t), u(t)\}_{t\in[0,T)}$ is a solution to List flow. Then we have the decay estimate

$$|\nabla u|^2(t,p) \leq \frac{c_0}{1 + 2\alpha c_0 t}.$$

Proof.
We observe

$$(\partial_t - \Delta)|\nabla u|^2 \leq -2\alpha|\nabla u|^4$$

and the corollary follows by an adapted ODE comparison argument as in the proof of Proposition 3.9. □

Finally, we record the interior estimates for List flow, which were proved by B. List in Chapter 6 of his thesis [59].

Theorem 3.11 (Interior estimates for List flow)**.**
Suppose that M is a closed m-dimensional manifold and $\{g(t), u(t)\}_{t\in[0,\tau]}$ a solution to List flow with

$$\sup_M |\operatorname{Rm}_{g(t)}| \leq \tau^{-1} \textit{ for all } t \in [0,\tau].$$

Then for any $l \in \mathbb{N}$ there exists a constant $C = C(m,l)$ depending on the dimension m of the manifold and on the number of derivatives l, such that we have the estimate

$$\sup_{p\in M} |\nabla^l \operatorname{Rm}_{g(t)}|^2 \leq C\tau^{-2}t^{-l} \textit{ and } \sup_{p\in M} |\nabla^{2+l} u|^2 \leq C\tau^{-2}t^{-l}$$

for all $t \in (0,\tau]$. Moreover, for $t \in [\tau/2, \tau]$ we deduce

$$\sup_{p\in M} |\nabla^l \operatorname{Rm}_{g(t)}|^2 \leq C\tau^{-(2+l)} \textit{ and } \sup_{p\in M} |\nabla^{2+l} u|^2 \leq C\tau^{-(2+l)}.$$

Proof.
The proof can be found in Theorem 6.15 and Corollary 6.16 of B. List's thesis [59]. □

3.1. **Improved decay for the energy density in List flow.**

The decay of the energy density can be improved by exploiting that the modulus $|u|^2$ of the function u is bounded along the flow by the maximum principle. Such an estimate is potentially useful for preserving weaker decay estimates in the surgery procedure. We first prove a global estimate, which will be refined in Section 8 by exploiting the local geometry:

Lemma 3.12 (Improved decay of the energy density)**.**
Suppose $\{g(t), u(t)\}$ is a solution to List flow on a closed manifold. There exists $\beta > 0$ with $\alpha < \beta \leq \alpha + 1$, such that we have the estimate

$$|\nabla u|^2(t,p) \leq \frac{c_0}{1 + 2\beta c_0 t}.$$

Since $\beta > \alpha$, this is an improved version of the energy density decay estimate, Corollary 3.10.

Proof.
We consider the test function

$$f(t,p) = \frac{1}{\Lambda d^2 - |u|^2} \left(1 + 2\beta c_0 t\right) |\nabla u|^2(t,p)$$

where $\Lambda \geq 2$ and β are dimensionless constants to be fixed later. The constant d is defined by

$$d = \sup_{p \in M} |u|(0,p).$$

We have by the Scalar Maximum Principle (Theorem 3.7) the estimate

$$|u|(t,p) \leq d.$$

This implies that the denominator $\Lambda d^2 - |u|^2$ is positive for all $(p,t) \in M \times [0,T)$ by the requirement $\Lambda \geq 2$.
We compute

$$\begin{aligned}
(\partial_t - \Delta) f = &- \frac{2|\nabla u|^2}{\left(\Lambda d^2 - |u|^2\right)^2} \left(1 + 2\beta c_0 t\right) |\nabla u|^2 + \frac{1}{\Lambda^2 d^2 - |u|^2} 2\beta c_0 |\nabla u|^2 \\
&+ \frac{1}{\Lambda^2 d^2 - |u|^2} (1 + 2\beta c_0 t) \left(-2|\nabla^2 u|^2 - 2\alpha |\nabla u|^4\right) \\
&- \frac{2}{(\Lambda^2 d^2 - |u|^2)^2} (1 + 2\beta c_0 t) \left\langle \nabla |u|^2, \nabla |\nabla u|^2 \right\rangle.
\end{aligned}$$

We estimate the last term by the Cauchy-Schwarz inequality and Young's inequality

$$\begin{aligned}
&-\frac{2}{(\Lambda d^2-|u|^2)^2}(1+2\beta c_0 t)\left\langle \nabla |u|^2, \nabla |\nabla u|^2\right\rangle \\
&\leq \frac{8}{(\Lambda d^2-|u|^2)^2}(1+2\beta c_0 t)|u||\nabla u|^2|\nabla^2 u| \\
&\leq 2\frac{1}{\Lambda d^2-|u|^2}(1+2\beta c_0 t)\left(\sqrt{2}|\nabla^2 u|\right)\left(2\sqrt{2}\frac{|u|}{\Lambda d^2-|u|^2}|\nabla u|^2\right) \\
&\leq \frac{1}{\Lambda d^2-|u|^2}(1+2\beta c_0 t)\left(2|\nabla^2 u|^2+8\frac{|u|^2}{(\Lambda d^2-|u|^2)^2}|\nabla u|^4\right).
\end{aligned}$$

We may absorb the first term into the firs term in the second line of the above evolution equation and the second term into the first term in the first line of the above evolution equation. This implies the evolution inequality

$$\begin{aligned}
(\partial_t-\Delta)f \leq& \frac{1}{\Lambda d^2-|u|^2}\left(1+2\beta c_0 t\right)\left(8\frac{|u|^2}{(\Lambda d^2-|u|^2)^2}-2\frac{1}{\Lambda d^2-|u|^2}-2\alpha\right)|\nabla u|^4 \\
&+\frac{1}{\Lambda^2 d^2-|u|^2}2\beta c_0|\nabla u|^2.
\end{aligned}$$

If we choose $\Lambda \geq 10$ and estimate

$$8\frac{|u|^2}{\Lambda d^2-|u|^2}-2\leq \frac{8d^2}{(\Lambda-1)d^2}-2\leq \frac{8}{\Lambda-1}-2<0$$

we observe that the term in the above bracket is negative.
We rewrite the above evolution equation

$$(\partial_t-\Delta)f \leq \frac{1}{\Lambda d^2-|u|^2}|\nabla u|^2\left[2\beta c_0+\left(8\frac{|u|^2}{\Lambda d^2-|u|^2}-2-2\alpha\right)|\nabla u|^2\right].$$

We now argue as follows: Suppose there exists a constant $L>0$, such that

$$f(p,t)\leq L$$

for all $(p,t)\in M\times[0,T)$. Assume that (p_*,t_*) is a space-time point, where the function f attains a new maximum, that is

$$f(p_*,t_*)=L \text{ and } 0\leq(\partial_t-\Delta)f(p_*,t_*).$$

The second relation implies that

$$\begin{aligned}
0\leq& \frac{1}{\Lambda d^2-|u|^2(p_*,t_*)}|\nabla u|^2(p_*,t_*) \\
&\times\left[2\beta c_0+(1+2\beta c_0 t_*)\left(8\frac{|u|^2(p_*,t_*)}{\Lambda d^2-|u|^2(p_*,t_*)}-2-2\alpha\right)|\nabla u|^2(p_*,t_*)\right].
\end{aligned}$$

and hence

$$|\nabla u|^2(p_*,t_*)\leq \frac{c_0}{1+2\beta c_0 t_*}\frac{\beta}{\alpha+1-\frac{4|u|^2(p_*,t_*)}{\Lambda d^2-|u|^2(p_*,t_*)}}.$$

This implies that

$$L = f(p_*, t_*) \le c_0 \frac{\beta}{\alpha + 1 - \frac{4|u|^2(p_*,t_*)}{\Lambda d^2 - |u|^2(p_*,t_*)}} \frac{1}{\Lambda d^2 - |u|^2(p_*, t_*)}$$
$$\le c_0 \frac{\beta}{(\alpha+1)(\Lambda d^2 - |u|^2(p_*, t_*)) - 4|u|^2(p_*, t_*)}.$$

We may estimate the denominator by

$$\begin{aligned} D &= (\alpha+1)(\Lambda d^2 - |u|^2(p_*, t_*)) - 4|u|^2(p_*, t_*) \\ &\ge (\alpha+1)((\Lambda - 1)d^2) - 4d^2 \\ &= (\alpha+1)d^2\left(\Lambda - \left(1 + \frac{4}{\alpha+1}\right)\right). \end{aligned}$$

This implies

$$L \le c_0 \frac{\beta}{(\alpha+1)d^2\left(\Lambda - \left(1 + \frac{4}{\alpha+1}\right)\right)}.$$

Since $f \le L$ by choice of L we deduce the estimate

$$\begin{aligned} |\nabla u|^2(t,p) &\le \frac{L}{1 + 2\beta c_0 t}(\Lambda d^2 - |u|^2(t,p)) \\ &\le \frac{c_0}{1+2\beta c_0 t} \frac{\beta}{\alpha+1} \frac{\Lambda}{\Lambda - \left(1 + \frac{4}{\alpha+1}\right)}. \end{aligned}$$

This implies our estimate, if we can show that there exists $\beta > \alpha$, such that

$$\frac{\beta}{\alpha+1} \frac{\Lambda}{\Lambda - \left(1 + \frac{4}{\alpha+1}\right)} \le 1.$$

The second assertion is true, if we set

$$\beta := (\alpha+1)\left[1 - \frac{1}{\Lambda}\left(1 + \frac{4}{\alpha+1}\right)\right].$$

Moreover, we observe

$$\beta - \alpha = 1 - \frac{\alpha+1}{\Lambda}\left(1 + \frac{4}{\alpha+1}\right) > 0,$$

whenever $\Lambda > \alpha + 5$. Hence, if we set $\Lambda \ge \max\{10, \alpha+6\}$ the improved decay estimate follows for some $\beta > \alpha$. Moreover, for the upper bound on β we observe $\beta - \alpha \to 1$ as $\Lambda \to \infty$.

□

4. An improved bound on the Hessian

In this section we consider a three-dimensional closed manifold M and a Riemannian manifold (N, γ) of controlled curvature (see Definition 2.1) and non-positive sectional curvature. Moreover, suppose $\{g(t), u(t)\}_{t\in[0,T)}$ is a solution to Ricci flow coupled with harmonic map heat flow.

If we apply the scalar maximum principle, see Theorem 3.7, to the evolution equation for the energy density, see Proposition 3.4, we obtain that the energy density is bounded by its initial value, that is

$$\sup_{(t,p)\in[0,T)\times M} |\nabla u|^2(t,p) \leq \sup_{p\in M} |\nabla u|^2(0,p) =: c_0,$$

as it was already recorded in Proposition 3.9. Moreover, if $c_0 = 0$, then the function u is constant and the coupled system reduces to Ricci flow; therefore we may assume without loss of generality that $c_0 > 0$.

The evolution equations for the Ricci and scalar curvature under Ricci flow coupled to harmonic map heat flow involve first and second derivatives of the function u. To prove a-priori estimates on the curvature we need to control the second derivatives, the first derivatives are already controlled by the above observation. By scaling considerations and the interior estimates for the flow one may expect that the norm of the second derivatives (i.e. $|\nabla^2 u|^2$) is bounded in terms of the square of scalar curvature R^2. However, to the best of the author's knowledge this does not suffice to prove the a-priori estimates on curvature in the following sections.

By using the bound on the energy density discussed above, we may improve on this estimate in the sense that the norm of the Hessian $|\nabla^2 u|^2$ is bounded in terms of the product $c_0\,\mathrm{R}$ of the initial energy density and the scalar curvature. For the proof of this bound we apply the scalar maximum principle, Theorem 3.7, to an auxillary function.

We consider the auxillary function $f : [0,T) \times M \to \mathbb{R}$ defined by

$$f(t,p) = |\nabla^2 u|^2(t,p) \left(\Lambda + |\nabla u|^2(t,p)\right),$$

where $\Lambda > 0$ is a constant only depending on the initial data, which is to be determined in the course of the computation. The choice of this test function is inspired by earlier work of W.-X. Shi [77] on interior estimates for Ricci flow and work of K. Ecker and G. Huisken [33] on interior estimates for mean curvature flow.

We compute the time derivative

$$\partial_t f = \left(\partial_t |\nabla^2 u|^2\right)\left(\Lambda + |\nabla u|^2\right) + |\nabla^2 u|^2 \partial_t |\nabla u|^2$$

and the spatial derivative

$$\Delta f = \left(\Delta |\nabla^2 u|^2\right)\left(\Lambda + |\nabla u|^2\right) + 2\langle \nabla |\nabla^2 u|^2, \nabla |\nabla u|^2\rangle + |\nabla^2 u|^2 \Delta |\nabla u|^2$$

This implies

$$(\partial_t - \Delta) f = \left[(\partial_t - \Delta)|\nabla^2 u|^2\right]\left(\Lambda + |\nabla u|^2\right) + |\nabla^2 u|^2 \left[(\partial_t - \Delta)|\nabla u|^2\right] - 2\langle \nabla |\nabla^2 u|^2, \nabla |\nabla u|^2\rangle$$

and by the evolution equation for the norm of the Hessian from Proposition 3.5 the evolution inequality

$$\begin{aligned}(\partial_t - \Delta) f \leq & -2|\nabla^3 u|^2(\Lambda + |\nabla u|^2) - 2|\nabla^2 u|^4 \\ & + C|\operatorname{Rc}|\,|\nabla^2 u|^2(\Lambda + |\nabla u|^2) + \alpha C|\nabla^2 u|^2|\nabla u|^2(\Lambda + |\nabla u|^2) \\ & + Cd_1\left(|\nabla^2 u||\nabla u|^4 + |\nabla^2 u|^2|\nabla u|^2\right)(\Lambda + |\nabla u|^2) - 2\langle \nabla|\nabla^2 u|^2, \nabla|\nabla u|^2\rangle,\end{aligned} \tag{11}$$

where d_1 is a constant depending only on the curvature Rm^N of the target manifold and the derivative $\nabla^N \operatorname{Rm}^N$ of the curvature of the target manifold (compare with Definition 2.1). For List flow we have $d_1 = 0$.
In the following we use the first two terms to estimate the other terms:
We first estimate the gradient term. The Cauchy–Schwarz inequality and Young's inequality imply the inequality

$$-2\langle \nabla|\nabla^2 u|^2, \nabla|\nabla u|^2\rangle \leq 8|\nabla u|^3|\nabla u|^2|\nabla u| \leq \Lambda|\nabla^3 u|^2 + \frac{16}{\Lambda}c_0|\nabla^2 u|^2 \leq \Lambda|\nabla^3 u|^2 + \frac{1}{4}|\nabla^2 u|^2,$$

where we fixed the constant Λ to be $\Lambda = 64c_0$.
The first term on the right-hand side of this inequality can be absorbed into the first term on the right-hand side (i.e. the quadratic term in the third derivatives) of the evolution inequality (11), while the second term can be absorbed into the second term on the right-hand side (i.e. the quartic term in the second derivatives) of the evolution inequality (11).
To estimate the term involing the Ricci curvature we observe that $\Lambda \geq c_0 \geq |\nabla u|^2$ by choice of Λ and hence Young's inequality implies

$$C|\operatorname{Rc}|\,|\nabla^2 u|^2(\Lambda + |\nabla u|^2) \leq 2C|\operatorname{Rc}||\nabla^2 u^2|\Lambda \leq 4C^2\Lambda^2|\operatorname{Rc}|^2 + \frac{1}{4}|\nabla^2 u|^2.$$

One can absorb the second term into the second term (i.e. the quartic term in the second derivatives) in the evolution inequality (11). The first term involving the Ricci curvature cannot be absorbed yet.
We also observe by Young's inequality

$$\alpha C|\nabla^2 u|^2|\nabla u|^2(\Lambda + |\nabla u|^2) \leq 2\alpha C c_0 \Lambda|\nabla^2 u|^2 \leq 4C^2\alpha^2 c_0^2\Lambda^2 + \frac{1}{4}|\nabla^2 u|^4.$$

The second term can be absorbed into the second term (i.e. the quartic term in the second derivatives) in the evolution inequality (11), while we keep the first term.
Finally, we have to use Young's inequality with $p = 2$ and $p = 4$ to estimate the terms arising from the curvature of the target manifold. We have

$$\begin{aligned}Cd_1\left(|\nabla^2 u||\nabla u|^4 + |\nabla^2 u|^2|\nabla u|^2\right)(\Lambda + |\nabla u|^2) \leq & 2Cd_1\Lambda\left(|\nabla^2 u|c_0^2 + |\nabla^2 u|^2 c_0\right) \\ \leq & \frac{1}{16}|\nabla^2 u|^4 + Cd_1^{\frac{4}{3}}c_0^4 + \frac{1}{16}|\nabla^2 u|^2 + Cd_1^2 c_0^4,\end{aligned}$$

where C is an universal constant. The two terms involving the Hessian can be absored into the second term in the evolution inequality (11).

With the above estimates we may rewrite the evolution inequality (11) into the form

$$(\partial_t - \Delta) f \leq C c_0^2 |\operatorname{Rc}|^2 + C(\alpha^2 + d_2) c_0^4 - |\nabla^2 u|^4,$$

where C is a universal constant and d_2 a constant only depending on d_1, more precisely $d_2 = d_1^2 + d_1^{\frac{4}{3}}$. We have in particular $d_2 = 0$ for List flow.
In order to apply the maximum principle we need to absorb the curvature term. This is achieved by substracting a suitable portion of the modified scalar curvature S:

$$\begin{aligned}(\partial_t - \Delta)(f - C c_0^2 \operatorname{S}) &\leq C c_0^2 |\operatorname{Rc}|^2 + C(\alpha^2 + d_2) c_0^4 - |\nabla^2 u|^4 - 2C c_0^2 |\operatorname{Sc}|^2 - 2C c_0^2 |\Delta_{g,\gamma} u|^2 \\ &\leq C(\alpha^2 + d_2) c_0^4 - |\nabla^2 u|^4 - C c_0^2 |\operatorname{Rc}|^2 + 4\alpha C c_0^2 \langle \operatorname{Rc}, \operatorname{tr}_N(\nabla u \otimes \nabla u)\rangle \\ &\leq 2C(\alpha^2 + d_2) c_0^4 - |\nabla^2 u|^4,\end{aligned}$$

where we used the Cauchy-Schwarz inequality and Young's inequality once more.
Suppose now that we have the bound

$$[f - C c_0^2 \operatorname{S}]| \leq L, \tag{12}$$

where the constant $L > 0$ is to be determined later; we need

$$L \geq \sup_{p \in M} f(0,p) + C c_0^2 \sup_{p \in M} (-\operatorname{S}(0,p)),$$

otherwise the bound is not satisfied at $t = 0$. Suppose (t_*, p_*) with $t_* > 0$ is the first space-time point at which the above bound is violated. In particular, the function obtains a new maximum at (t_*, p_*) and hence we have

$$\begin{aligned}&0 \leq (\partial_t - \Delta)(f - C c_0^2 \operatorname{S})(t_*, p_*) \leq 2C(\alpha^2 + d_2) c_0^4 - |\nabla^2 u|^4 (t_*, p_*), \\ &L = (f - C c_0^2 \operatorname{S})(t_*, p_*).\end{aligned}$$

These two relations imply

$$\begin{aligned}L = (f - C c_0^2 \operatorname{S})(t_*, p_*) =& |\nabla^2 u|^2 (t_*, p_*)(\Lambda + |\nabla u|^2 (t_*, p_*)) - C c_0^2 \operatorname{S}(t_*, p_*) \\ \leq& \sqrt{2C(\alpha^2 + d_2)} c_0^2 C c_0 - C c_0^2 \operatorname{S}(t_*, p_*) \\ \leq& \sqrt{2C(\alpha^2 + d_2)} C c_0^3 - C c_0^2 \operatorname{S}_0 \\ \leq& C\left(\alpha + \sqrt{d_2}\right) c_0^3 - C c_0^2 \operatorname{S}_0,\end{aligned}$$

where we used the inequality $\operatorname{S}(t_*, p_*) \geq \operatorname{S}_0$, since the infimum of the modified scalar curvature is non-decreasing along the flow by Proposition 3.8.
We deduce by the scalar maximum principle, Theorem 3.7, that

$$f - C c_0^2 \operatorname{S} \leq \max \left\{ \sup_{p \in M} f(0,p) + C c_0^2 \sup_{p \in M} (-\operatorname{S}(0,p)), C\left(\alpha + \sqrt{d_2}\right) c_0^3 - C c_0^2 \operatorname{S}_0 \right\},$$

where C is an universal constant.
Since the infimum of the quanity S is non-decreasing along the flow, we conlude that

$$f(t,p) \leq \sup_{p \in M} f(0,p) + C\left(\alpha + \sqrt{d_2}\right) c_0^3 + C c_0^2 \operatorname{S}(t,p) + C c_0^2 \sup_{p \in M} (-\operatorname{S}(0,p)).$$

By definition of the auxillary function f and the constant Λ the above computations imply the following result:

Proposition 4.1 (Improved bound for the norm of the Hessian)**.**
Suppose $\{g(t),u(t)\}_{t\in[0,T)}$ *is a solution to Ricci flow coupled to harmonic map heat flow on a closed three-manifold into a target manifold* (N,γ) *of controlled curvature and nonpositive sectional curvature. Then the norm of the Hessian satisfies the improved estimate*

$$|\nabla^2 u|^2(t,p) \le 2s_0 + C\left(\alpha+\sqrt{d_2}\right)c_0^2 + Cc_0\,\mathrm{R}(t,p) + Cc_0\,\mathrm{R}_0^-, \tag{13}$$

where C *is a universal constant and* d_2 *depends only on the curvature* Rm^N *and the first derivatives* $\nabla^N \mathrm{Rm}^N$ *of the target manifold, in particular* $d_2 = 0$ *for List flow. The constant* c_0 *depends on the initial energy density, the constant* s_0 *depends on the initial data for the Hessian and the constant* R_0^- *on the negative part of the initial scalar curvature, more precisely*

$$c_0 = \sup_{p\in M}|\nabla u|^2(p,0),\ \ s_0 = \sup_{p\in M}|\nabla^2 u|^2(p,0)\ \textit{and}\ \mathrm{R}_0^- = \max\left\{0, \sup_{p\in M}\left(-\mathrm{R}(p,0)\right)\right\}.$$

5. The Hamilton–Ivey estimate

In this section we a prove a pinching estimate on the sectional curvatures of the evolving metrics. This estimate tells us that the curvatures pinch towards nonnegative, whenever the curvature becomes large. As a consequence of this estimate one deduces that singularity models have nonnegative sectional curvature. The corresponding estimate for Ricci flow is independently due to R. Hamilton [40] and T. Ivey [55]. The proof uses the parabolic Tensor Maximum principle introduced by R. Hamilton in 1982:

Theorem 5.1 (Tensor maximum principle; R. Hamilton; Theorem 9.1 in [36])**.** *Suppose $\{g(t)\}_{[0,T)}$ is a one-parameter family of Riemannian metrics on a closed manifold M. Let $\{M(t)\}$ be a one-parameter family of symmetric tensor fields evolving by*

$$\partial_t M(t) \geq \Delta_{g(t)} M(t) + \langle b(t), \nabla M(t)\rangle_{g(t)} + Q(t; M(t)), \tag{14}$$

where $b : [0,T) \times M \to \Gamma(TM)$ is a time-dependet vector field and Q is an gradient expression. Suppose that $M(0) \geq 0$ in the sense of quadratic forms. If we have for any zero-eigenvector ξ of $M(t)$ the inequality $Q(t; \xi \otimes \xi) \geq 0$, then the condition $M(t) \geq 0$ is preserved for all $t \in [0,T)$.

In 1986 R. Hamilton [37, Section 4 and 5] introduced a refined version of the Tensor Maximum Principle, the so called PDE-ODE principle: If for any point $p \in M$ the solution to an ODE, which is associated to equation (14), preserves the nonnegativity, then so does the PDE (14). It is an interesting technical question to understand whether one prove the forthcoming estimates by an adaption of the PDE-ODE principle to our setting.

We introduce as in Section 24 of [40, p. 105] the pinching function $f : (1,\infty) \to (-1,\infty)$ defined by

$$f(x) = x \log x - x,$$

which has superlinear growth at infinity. The pinching function f satisfies

$$f(1) = -1,\ f'(x) = \log x > 0 \text{ and } f''(x) = \frac{1}{x^2} > 0 \text{ on } (1,\infty).$$

Hence f is a bijective, strictly increasing and convex function and it posseses an inverse function

$$f^{-1} : (-1,\infty) \to (1,\infty),$$

which is stricly increasing and concave. Moreover, since $f(x) \sim x \log x$ for $x >> 1$ we observe the limit

$$\lim_{x\to\infty} \frac{f^{-1}(x)}{x} = 0.$$

Recall that we denote the eigenvalues of the Einstein tensor $E = \mathrm{R}\, g - 2\,\mathrm{Rc}$ by σ_1, σ_2 and σ_3, ordered such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. These eigenvalues are twice the sectional curvatures.

We have the following theorem:

Theorem 5.2.
Suppose $(g(t), u(t))_{t\in[0,T)}$, $T < \infty$, is a solution to Ricci flow coupled with harmonic map heat flow (10) on a closed three-manifold M into a Riemannian manifold (N, γ) of controlled curvature and nonpositive sectional curvature. Then there exists a constant $N > 0$, such that we have initially

$$\sigma_3 + N f^{-1}\left(\frac{\mathrm{R}}{N}\right) \geq 0, \tag{15}$$

and this condition is preserved along the flow.

Let us outline the strategy of the proof:
In the first step we rescale to simplify the computations, in the second step we derive the evolution equation for the Einstein tensor $E = \mathrm{R}\, g - 2\,\mathrm{Rc}$ and its $(1,1)$-variant F. The eigenvalues of F are twice the sectional curvatures. In the third step we introduce the pinching function into our computation and use Proposition 4.1 to estimate the second derivatives. In the final step we show that the reaction term in the evolution equation for $F + f^{-1}(\mathrm{R})\delta$ is nonnegative, whenever this tensor acquires an eigenvector with eigenvalue zero; the Tensor Maximum Principle allows us to conclude.

The first step of the proof is to rescale:
Since the manifold M is closed, there exists a large constant $N > 0$, such that we have initially

$$\sigma_3(0) \geq -\frac{1}{6}N,\ c_0 \leq \frac{N}{1000L_1} \text{ and } s_0 \leq \frac{N^2}{800\alpha},$$

where $L_1 > 0$ is a scaling invariant constant given by

$$L_1 = \max\left\{1, 8(C + \alpha + d_1)\right\},$$

where C is the constants from the improved bound on the Hessian, see Proposition 4.1 and d_1 is the constant depending on the curvature of the target from Proposition 3.5. This implies in particular the estimates

$$\mathrm{R}(0) \geq -\frac{1}{2}N, \mathrm{S}(0) \geq -\frac{3}{4}N \text{ and } N f^{-1}\left(\frac{\mathrm{R}}{N}\right) \geq -N.$$

We rescale the initial metric by the factor $\frac{1}{N}$. Thus in the following we aim to prove that the estimate

$$\sigma_3 + f^{-1}(\mathrm{R}) \geq 0$$

is preserved under the flow under the additional assumptions

$$\sigma_3(0) \geq -\frac{1}{6},\ c_0 \leq \frac{1}{1000L_1} \text{ and } s_0 \leq \frac{1}{1000\alpha}.$$

Recall that the Einstein tensor E is defined by

$$E = \mathrm{R}\, g - 2\,\mathrm{Rc}\,.$$

It is a $(0,2)$-tensor and we will consider the $(1,1)$-tensor F obtained by raising the second index, in abstract index notation

$$F_a{}^c = g^{bc} E_{ab}.$$

Then $F_p : T_pM \to T_pM$ is a self-adjoint linear map and thus has eigenvalues $\sigma_1 \geq \sigma_2 \geq \sigma_3$, which are twice the sectional curvatures at the point $p \in M$.

In the next step we first compute the evolution equation for E and then the evolution equation for F. We start by observing

$$(\partial_t - \Delta)E = [(\partial_t - \Delta)\,\mathrm{R}]\, g + \mathrm{R}\,\partial_t g - 2\,(\partial_t - \Delta)\,\mathrm{Rc}$$

and the evolution equations for the metric, the scalar curvature and the Ricci curvature (see Proposition 3.3) imply

$$\begin{aligned}(\partial_t - \Delta)E_{ab} = &\left(2|\,\mathrm{Rc}\,|^2 - 4\alpha\langle \mathrm{Rc}, \mathrm{tr}_N(\nabla u \otimes \nabla u)\rangle + 2\alpha|\Delta_{g,\gamma} u|^2 - 2\alpha|\nabla^2 u|^2\right) g_{ab}\\ &+ \mathrm{R}\,(-2\,\mathrm{Rc}_{ab} + 2\alpha \nabla_a u^\kappa \nabla_b u^\kappa)\\ &- 4\,\mathrm{Rm}_{cabd}(\mathrm{Rc}_{cd} - \alpha \nabla_c u^\kappa \nabla_d u^\kappa) + \mathrm{Rc}_{ab}\,\mathrm{Rc}_{cb}\\ &- 4\Delta_{g,\gamma} u^\kappa \cdot \alpha \nabla_a \nabla_b u^\kappa + 4\alpha \nabla_c \nabla_a u^\kappa \nabla_c \nabla u^\kappa\\ &+ 2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_c u^\kappa \nabla_d u^\nu \nabla_d u^\kappa \nabla_a u^\lambda - 4\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu \nabla_c u^\nu \nabla_c u^\kappa \nabla_b u^\lambda,\end{aligned}$$

where the terms in the first line come from the evolution of the scalar curvature, the terms in the second line from the evolution of the metric, the terms in the third line from the evolution of the Ricci tensor. Finally, the terms in the last line involve the curvature of the target and they arise from the evolution of the scalar curvature and the evolution of the Ricci curvature; these two terms are not present in List flow.

For the tensor F we obtain the following evolution equation

(16)

$$\begin{aligned}(\partial_t - \Delta)F_a{}^c &= \left(\partial_t g^{bc}\right) E_{ab} + g^{bc}(\partial_t - \Delta)E_{ab}\\ &= \left(2\,\mathrm{Rc}^{cb} - 2\alpha \nabla^b u^\kappa \nabla^c u^\kappa\right)(\mathrm{R}\, g_{ab} - 2\,\mathrm{Rc}_{ab})\\ &\quad + \left(2|\,\mathrm{Rc}\,|^2 - 4\alpha\langle \mathrm{Rc}, \mathrm{tr}_N(\nabla u \otimes \nabla u)\rangle + 2\alpha|\Delta_{g,\gamma} u|^2 - 2\alpha|\nabla^2 u|^2\right)\delta_a{}^c\\ &\quad - 2\,\mathrm{R}\,\mathrm{Rc}_a{}^c + 2\alpha\,\mathrm{R}\,\nabla_a u^\kappa \nabla^c u^\kappa\\ &\quad - 4 g^{bc}\,\mathrm{Rm}_{dabe}\,\mathrm{Rc}_{de} + 4\alpha g^{bc}\,\mathrm{Rm}_{dabe}\,\nabla_d u^\kappa \nabla_e u^\kappa + 4\,\mathrm{Rc}_{ad}\,\mathrm{Rc}_d{}^c\\ &\quad - 4\alpha \Delta_{g,\gamma} u^\kappa \nabla_a \nabla^c u^\kappa + 4\alpha \nabla_d \nabla_a u^\kappa \nabla_d \nabla^c u^\kappa\\ &\quad + 2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\left(\nabla_d u^\mu \nabla_e u^\nu \nabla_e u^\kappa \nabla_d u^\lambda g_{ab} - 2\nabla_a u^\mu \nabla_e u^\nu \nabla_e u^\kappa \nabla_b u^\lambda\right) g^{bc}\end{aligned}$$

We recall the decomposition for the Riemann curvature tensor in terms of the scalar curvature and the Ricci curvature in three dimensions:

$$\mathrm{Rm}_{aij}{}^c = \mathrm{Rc}_a{}^c\, g_{ij} + \mathrm{Rc}_{ij}\,\delta_a{}^c - \mathrm{Rc}_{aj}\,\delta_i{}^c - \mathrm{Rc}_i{}^c\, g_{aj} - \frac{1}{2}\,\mathrm{R}\,(\delta_a{}^c g_{ij} - g_{aj}\delta_i{}^c) \tag{17}$$

If we combine the evolution inequality (16) for the tensor F with the curvature decomposition, equation (17) we obtain
(18)
$$\begin{aligned}(\partial_t-\Delta)F_a{}^c =&2\left(\mathrm{R}^2-|\mathrm{Rc}|^2\right)\delta_a{}^c-6\,\mathrm{R}\,\mathrm{Rc}_{aj}\,\mathrm{Rc}^{jc}\\&+4\alpha\,\mathrm{Rc}_a{}^c\,|\nabla u|^2-4\alpha\,\mathrm{Rc}_d{}^c\,\nabla^d u^\kappa\nabla_a u^\kappa-2\alpha\,\mathrm{R}\,|\nabla u|^2\delta_a{}^c\\&+2\alpha|\Delta_{g,\gamma}u|^2\delta_a{}^c-2\alpha|\nabla^2 u|^2\delta_a{}^c-4\alpha\Delta_{g,\gamma}u^\kappa\cdot\nabla_a\nabla^c u^\kappa+4\alpha\nabla_d\nabla_a u^\kappa\nabla_d\nabla^c u^\kappa\\&+2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\left(\nabla_d u^\mu\nabla_e u^\nu\nabla_e u^\kappa\nabla_d u^\lambda g_{ab}-2\nabla_a u^\mu\nabla_e u^\nu\nabla_e u^\kappa\nabla_b u^\lambda\right)g^{bc}.\end{aligned}$$

In the next step we compute the derivatives of the pinching function f^{-1}. We deduce by the chain rule

$$\begin{aligned}\partial_t f^{-1}(\mathrm{R}) =&(f^{-1}(\mathrm{R}))'\left(\Delta\,\mathrm{R}+2|\mathrm{Rc}|^2-4\alpha\langle\mathrm{Rc},\mathrm{tr}_N(\nabla u\otimes\nabla u)\rangle+2\alpha|\Delta_{g,\gamma}u|^2-2\alpha|\nabla^2 u|^2\right)\\&+2\alpha(f^{-1}(\mathrm{R}))'\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_b u^\nu\nabla_b u^\kappa\nabla_a u^\lambda,\\ \Delta f^{-1}(\mathrm{R}) =&(f^{-1}(\mathrm{R}))'\Delta\,\mathrm{R}+(f^{-1}(\mathrm{R}))''|\nabla\,\mathrm{R}|^2\end{aligned}$$

and this implies by the concavity of f^{-1} the estimate
(19)
$$\begin{aligned}(\partial_t-\Delta)f^{-1}(\mathrm{R}) \geq&(f^{-1}(\mathrm{R}))'\left(2|\mathrm{Rc}|^2-4\alpha\langle\mathrm{Rc},\mathrm{tr}_N(\nabla u\otimes\nabla u)\rangle+2\alpha|\Delta_{g,\gamma}u|^2-2\alpha|\nabla^2 u|^2\right)\\&+2\alpha(f^{-1}(\mathrm{R}))'\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_b u^\nu\nabla_b u^\kappa\nabla_a u^\lambda.\end{aligned}$$

Thus combining equation (18) with the inequality (19) we obtain the evolution equation

$$\begin{aligned}&(\partial_t-\Delta)\left(F_a{}^c+f^{-1}(\mathrm{R})\delta_a{}^c\right)\\ \geq&2\,\mathrm{R}^2\,\delta_a{}^c+2|\mathrm{Rc}|^2\delta_a{}^c\left((f^{-1})(\mathrm{R})-1\right)-6\,\mathrm{R}\,\mathrm{Rc}_a{}^c+8\,\mathrm{Rc}_{ad}\,\mathrm{Rc}^{dc}\\&+4\alpha\nabla_d\nabla_a u^\kappa\nabla_d\nabla^c u^\kappa-4\alpha\Delta_{g,\gamma}u^\kappa\nabla_a\nabla_c u^\kappa\\&-2\alpha|\nabla^2 u|^2(1+(f^{-1})'(\mathrm{R}))\delta_a{}^c+2\alpha|\Delta_{g,\gamma}u|^2(1+(f^{-1})'(\mathrm{R}))\delta_a{}^c\\&+4\alpha\,\mathrm{Rc}_a{}^c\,|\nabla u|^2-4\alpha\,\mathrm{Rc}_d{}^c\,\nabla_d u^\kappa\nabla_a u^\kappa-2\alpha\,\mathrm{R}\,|\nabla u|^2\delta_a{}^c\\&-4\alpha(f^{-1}(\mathrm{R}))'\langle\mathrm{Rc},\mathrm{tr}_N(\nabla u\otimes\nabla u)\rangle\delta_a{}^c\\&-4\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_a u^\mu\nabla_d u^\nu\nabla_d u^\kappa\nabla_b u^\lambda g^{bc}\\&+2\alpha\,\mathrm{Rm}^N_{\mu\nu\kappa\lambda}\,\nabla_d u^\mu\nabla_e u^\nu\nabla_e u^\kappa\nabla_d u^\lambda\delta_a{}^c\left(1+(f^{-1})'(\mathrm{R})\right)\\ =:&(I)+(II).\end{aligned}$$

The terms in the first line of the inequality, indicated by (I) are already present in Ricci flow, while the other terms, indicated by (II) arise from the coupling and from the curvature of the target. The terms in the last two lines are not present in List flow.

We estimate the terms in (II) pointwise at a point $(t,p) \in [0,T) \times M$ by

$$\begin{aligned}(II)(p,t) \geq &- 8\alpha|\nabla^2 u|^2(p,t)\delta_a{}^c - 4\alpha|\nabla^2 u|^2(p,t)\left(1+(f^{-1})'(\mathrm{R}(p,t))\right)\\ &- 8\alpha c_0 \max_{g(v,v)=1} \mathrm{Rc}(p,t)(v,v) - 2\alpha c_0 \,\mathrm{R}(p,t)\delta_a{}^c\\ &- 4\alpha c_0 (f^{-1})'(\mathrm{R}(p,t)) \max_{g(v,v)=1} \mathrm{Rc}(p,t)(v,v)\\ &- 4\alpha c_0^2\left(\sup_{q\in N}|\mathrm{Rm}^N(q)|\right) - 2\alpha c_0^2\left(\sup_{q\in N}|\mathrm{Rm}^N(q)|\right)[1+(f^{-1})'(\mathrm{R}(p,t))]\\ \geq &- 4\alpha|\nabla^2 u|^2(p,t)\left[3+(f^{-1})'(\mathrm{R}(p,t))\right]\\ &- 8\alpha c_0 \max_{g(v,v)=1} \mathrm{Rc}(p,t)(v,v) - 2\alpha c_0\,\mathrm{R}(p,t)\delta_a{}^c - 2\alpha c_0^2 d_1[3+(f^{-1})'(\mathrm{R}(p,t))].\end{aligned}$$

By the improved bound on the norm of the Hessian, see Proposition 4.1, we deduce the inequality

$$\begin{aligned}(II)(p,t) \geq &- 4\alpha(2s_0 + C(\alpha+\sqrt{d_2})c_0^2 + Cc_0\,\mathrm{R}(p,t) + Cc_0\,\mathrm{R}_0^-)\left(3+(f^{-1})'(\mathrm{R})\right)\delta_a{}^c\\ &- 8\alpha c_0 \max_{g(v,v)=1} \mathrm{Rc}(p,t)(v,v) - 2\alpha c_0\,\mathrm{R}(p,t)\delta_a{}^c - 2\alpha c_0^2 d_1\left[3+(f^{-1})'(\mathrm{R}(p,t))\right],\end{aligned}$$

where the constants C and d_2 are the constants from Proposition 4.1 and s_0 denotes the norm squared of the initial data of the Hessian.

Suppose now that $(p_*,t_*) \in M \times (0,T)$ is the first time with

$$\langle\langle F(v) + f^{-1}(\mathrm{R})v, v\rangle(p_*,t_*) = 0$$

for some eigenvector $v \in T_{p_*}M$. If we diagonalize the tensor F at (p_*,t_*) we obtain the eigenvalues $\sigma_1 \geq \sigma_2 \geq \sigma_3$, which are proportional to the sectional curvatures. Moreover, the largest eigenvalue of the Ricci curvature at (p_*,t_*) is given by $\frac{1}{2}(\sigma_1+\sigma_2)$ and the scalar curvature is given by $\mathrm{R}(p_*,t_*) = \sigma_1+\sigma_2+\sigma_3$. In this diagonalization the Ricci flow term (I) evaluated at (p_*,t_*) in direction v is given by

$$\langle (I)(p_*,t_*)(v), v\rangle = \sigma_3^2 + \sigma_1\sigma_2 + (f^{-1})'(\sigma_1+\sigma_2+\sigma_3)\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3\right]$$

compare with Section 24 in [40, p. 105].

On the other hand, by the above computations we obtain for the term (II) evaluated at (p_*,t_*) in direction v the formula

$$\begin{aligned}&(II)(p_*,t_*)\\ \geq &- 4\alpha(2s_0 + C(\alpha+\sqrt{d_2})c_0^2 + Cc_0(\sigma_1+\sigma_2+\sigma_3) + Cc_0\,\mathrm{R}_0^{-1})\left[3+(f^{-1})'(\mathrm{R})\right]\\ &- 8\alpha c_0(\sigma_1+\sigma_2) - 2\alpha c_0(\sigma_1+\sigma_2+\sigma_3) - 2\alpha c_0^2 d_1(3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)),\\ \geq &-\left(8\alpha s_0 + 4\alpha C(\alpha+\sqrt{d_2})c_0^2 + 4\alpha Cc_0\,\mathrm{R}_0^- + 2\alpha d_1 c_0^2\right)\left[3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)\right]\\ &- \alpha c_0(\sigma_1+\sigma_2)\left(4\alpha\left[3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)\right]+10\right)\\ &- \alpha c_0\sigma_3\left(4\alpha C\left[3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)\right]+2\right).\end{aligned}$$

where we regrouped the terms in the last inequality according to the appearance of curvature eigenvalues.
Since we have $\sigma_3 < 0$ and $(f^{-1})'(\sigma_1+\sigma_2+\sigma_3) > 0$ in our setting, we may discard the last term, since it is positive anyway. The may estimate the second term (II) by our choice of the constant L_1 by

$$\begin{aligned}\langle (II)(p_*,t_*)v,v\rangle \geq &-\frac{1}{50}\left(3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)+10\right)\\ &-\frac{1}{1000}(\sigma_1+\sigma_2)\left(3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)+10\right).\end{aligned}$$

Moreover, by the choice of the function f one deduces

$$(f^{-1})'(\sigma_1+\sigma_2+\sigma_3) = \frac{-\sigma_3}{\sigma_1+\sigma_2},$$

whenever $\sigma_3 + f^{-1}(\sigma_1+\sigma_2+\sigma_3) = 0$.
Since we have $\sigma_3 < -1$ by construction, we defined $n = -\sigma_3 > 1$.
One derives for the term (I) the expression

$$\begin{aligned}(I) &= \sigma_3^2 + \sigma_1\sigma_2 + \frac{n}{\sigma_1+\sigma_2}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_1\sigma_2-\sigma_1 n-\sigma_2 n\right)\\ &= \frac{1}{\sigma_1+\sigma_2}\left(n(\sigma_1^2+\sigma_1\sigma_2+\sigma_2^2)+(\sigma_1+\sigma_3)\sigma_1\sigma_2+n^3\right)\end{aligned}$$

compare with Section 24 of [40, p. 105].
In the following we show that $(I)+(II) > 0$, whenever $\sigma_3 + f^{-1}(\sigma_1+\sigma_2+\sigma_3) = 0$. By Hamilton's Tensor Maximum Principle (see Theorem 5.1) this implies that the estimate (15) is preserved along the flow.
There are two cases to consider, first $\sigma_1 > 0$ and $\sigma_2 \geq 0$ and second $\sigma_1 > 0$ and $\sigma_2 < 0$.
For the first case, that is $\sigma_1 > 0$ and $\sigma_2 \geq 0$, we observe

$$\begin{aligned}&\frac{1}{50}(\sigma_1+\sigma_2)\left(3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)+10\right)\\ &-\frac{1}{1000}(\sigma_1+\sigma_2)^2\left(3+(f^{-1})'(\sigma_1+\sigma_2+\sigma_3)+10\right)\\ &=\frac{1}{50}\left(3(\sigma_1+\sigma_2)+n+10(\sigma_1+\sigma_2)\right)-\frac{1}{1000}\left(3(\sigma_1+\sigma_2)^2+n(\sigma_1+\sigma_2)+10(\sigma_1+\sigma_2)^2\right).\end{aligned}$$

This implies that

$$\begin{aligned}(\sigma_1+\sigma_2)\left[(I)+(II)\right] \geq &\left(n(\sigma_1^2+\sigma_1\sigma_2+\sigma_2^2)+(\sigma_1+\sigma_3)\sigma_1\sigma_2+n^3\right)\\ &-\frac{1}{50}(3(\sigma_1+\sigma_2)+n)-\frac{1}{200}\left(5(\sigma_1+\sigma_2)^2+n(\sigma_1+\sigma_2)\right).\end{aligned} \tag{20}$$

and we have to show that this expression is non-negative.
Since we have $n > 1$ by construction we deduce

$$\frac{1}{8}n^3 - \frac{1}{50}n > 0$$

and hence we may absorb the linear term in n into the cubic term n^3. For the remaining negative terms we distinguish two cases: First $\sigma_1 \leq 1$, then we have $n > \sigma_1 \geq \sigma_2$ and hence we have the estimates

$$\frac{1}{4}n^3 - \frac{1}{50}(3(\sigma_1+\sigma_2)) \geq \frac{1}{4}n^3 - \frac{6}{50}n^2 > 0,$$
$$\frac{1}{4}n^3 - \frac{1}{200}(5(\sigma_1+\sigma_2)^2 + n(\sigma_1+\sigma_2)) \geq \frac{1}{4}n^3 - \frac{22}{200} > 0.$$

Second $\sigma_1 > 1$, then $\sigma_1 + \sigma_2 > 1$ we have the estimates

$$\frac{1}{4}n(\sigma_1^2 + \sigma_1\sigma_2\sigma_2^2) - \frac{1}{40}(\sigma_1+\sigma_2)^2 \geq \frac{1}{8}(\sigma_1+\sigma_2)^2 - \frac{1}{40}(\sigma_1+\sigma_2)^2 > 0,$$
$$\frac{1}{4}n\sigma_1^2 - \frac{3}{50}(\sigma_1+\sigma_2) - \frac{1}{200}n(\sigma_1+\sigma_2) \geq \frac{1}{8}\sigma_1^2 - \frac{6}{50}\sigma_1 + \frac{1}{8}n\sigma_1^2 - \frac{1}{100}n\sigma_1 > 0.$$

Thus we deduce that $(I) + (II) > 0$ for the first case $\sigma_1 > 0$ and $\sigma_2 \geq 0$.
Let us discuss the second case, $\sigma_1 > 0$ and $\sigma_2 < 0$, which is more involved.
We set $m = -\sigma_2 > 0$ and we observe that $m \leq n$. By rewriting the term (I) in a more suitable fashion and by discarding some positive terms in the term (II) we observe the inequality

$$\begin{aligned} &(\sigma_1 - m)\,[(I) + (II)] \\ &\geq (\sigma_1^2 - \sigma_1 m + m^2)(n-m) + n^3 - m^3 - \frac{3}{17}\sigma_1 - \frac{1}{50}n - \frac{1}{40}\sigma_1^2 - \frac{1}{200}n\sigma_1 \end{aligned} \tag{21}$$

As before we observe that

$$\frac{1}{4}n^3 - \frac{1}{50} > 0.$$

For the remaining negative terms in the expression (21) we distinguish two cases: The first case is $\sigma_1 \leq 3n$ and the second case is $\sigma_1 > 3n$. In the first case we observe that

$$\frac{1}{2}n^3 - \frac{3}{17}\sigma_1 - \frac{1}{50}n - \frac{1}{40}\sigma_1^2 - \frac{1}{200}n\sigma_1 \geq \frac{1}{2}n^3 - \frac{9}{17}n - \left(\frac{9}{40} + \frac{3}{200}\right)n^2 > 0.$$

In the second case we have $\sigma_1 > 3n$. This implies that $\mathrm{R} = \sigma_1 - m - n \geq n$. Moreover, since we have $n = f^{-1}(\sigma_1 - m - n)$ we deduce $n \geq f^{-1}(\sigma_1 - 2n)$ since the function f^{-1} is strictly increasing. Thus we have $f(n) \geq \sigma_1 - 2n$, since the function f is monotone. This implies the bound $\sigma_1 \leq f(n) + 2n = n\log n + n$. With these preparations we estimate

$$\begin{aligned} &n^3 - \frac{3}{17}\sigma_1 - \frac{1}{50}n - \frac{1}{40}\sigma_1^2 - \frac{1}{200}n\sigma_1 \\ \geq& n^3 - \frac{3}{17}n(\log n + 1) - \frac{1}{40}n^2(\log n + 1)^2 - \frac{1}{200}n^2(\log n + 1) \\ \geq& n^3 - \frac{6}{17}n\sqrt{n} - \frac{4}{40}n^3 - \frac{2}{200}n^2\sqrt{n} > 0, \end{aligned}$$

where we used the inequalities $\log n \leq \sqrt{n}$ and $1 \leq \sqrt{n}$ for $n > 1$. This finishes the second case $\sigma_1 > 0$ and $\sigma_2 \leq 0$.

By the above computations we have shown that the reaction term is positive, whenever there is a zero eigenvector of the tensor $F + f^{-1}(\mathrm{R})\delta$. Then Theorem 5.2 follows by Hamilton's tensor maximum principle, see Theorem 5.1.

At the end of this section we state some consequences:
One consequence of the Hamilton–Ivey estimate is that the curvatures are controlled by the pinching function:

Remark 5.3.
By the Hamilton-Ivey estimate the scalar curvature controls the eigenvalues σ_i (twice the sectional curvatures) of the solution in the following sense:

$$-Nf^{-1}\left(\frac{\mathrm{R}}{N}\right) \leq \sigma_i \leq \mathrm{R} + Nf^{-1}\left(\frac{\mathrm{R}}{N}\right).$$

As a consequence of the Hamilton-Ivey estimate we also deduce the following corollary:

Corollary 5.4.
Suppose $\{g(t), u(t)\}_{t\in[0,T)}$ is a solution to Ricci flow coupled to harmonic map heat flow on a closed three-manifold into a target (N,γ) of controlled curvature and nonpositive sectional curvature. Suppose moreover that $T < \infty$ is maximal. Then the scalar curvature blows up as $t \to T$ in the sense that

$$\limsup_{t\to T}\left(\sup_{p\in M} \mathrm{R}(t,p)\right) = \infty.$$

6. A convergence theorem for List flow on three-manifolds

In this section we aim to prove a convergence theorem for List flow on three-manifolds.

Theorem 6.1.
Let (M^3, g_0) be a Riemannian manifold and $u_0 : M \to \mathbb{R}$ a smooth function, such that

$$\mathrm{Rc} > 0, \mathrm{Rc} > \epsilon \,\mathrm{R}\, g \textit{ and} \,\mathrm{R} \geq \frac{1}{\epsilon^2} D(\alpha c_0 + \sqrt{\alpha s_0})$$

initially, where D is an universal constant and

$$c_0 = \sup_{p \in M} |\nabla u_0|^2(p) \textit{ and } s_0 = \sup_{p \in M} |\nabla^2 u_0|^2(p).$$

Then List flow preserves these conditions, the solution extincts at some finite time $T < \infty$. Moreover, as $t \to T$ the metrics

$$\tilde{g}(t) = \frac{1}{4(T-t)} g(t)$$

converge in C^∞ to a metric of constant sectional curvature and the function u converges in C^∞ to some constant u_∞ with $\inf_{p \in M} u_0(p) \leq u_\infty \leq \sup_{p \in M} u_0(p)$.

Observe that the above theorem reduces to Hamilton's convergence theorem for Ricci flow on three-manifolds with positive Ricci curvature, if the function u_0 is constant (and hence $c_0 = s_0 = 0$). Our proof closely follows the strategy outlined by R. Hamilton in his 1982 work on Ricci flow on manifolds with positive Ricci curvature [36], more precisely Theorem 6.3 and Theorem 6.5 play the role of Theorem 9.4. and Theorem 10.1 in R. Hamilton's work [36].
In the first step, we use the Scalar Maximum Principle to establish that scalar curvature above a certain threeshold (depending on the initial data for the function u) is preserved by List flow.

Proposition 6.2.
There exists a large positive constant $D_1 > 0$ given by

$$D_1 = D_{1,F}(\alpha c_0 + \sqrt{\alpha s_0})$$

where the scaling-free constant $D_{1,F}$ depends only on the constant C in Proposition 4.1 on the improved bound for the Hessian with the following property:
For any $D \geq D_1$ the lower scalar curvature bound

$$\mathrm{R} \geq D$$

is preserved along any smooth solution of List flow on three-manifolds.

Proof.
We recall the evolution equation for the scalar curvature for List flow:

$$(\partial_t - \Delta)\,\mathrm{R} = 2|\,\mathrm{Rc}\,|^2 - 4\alpha\langle \mathrm{Rc}, \nabla u \otimes \nabla u\rangle + 2\alpha|\Delta u|^2 - 2\alpha|\nabla^2 u|^2.$$

We estimate by the Cauchy–Schwarz inequality, Young's inequality and the trace inequality for symmetric two-tensors:

$$\begin{aligned}(\partial_t - \Delta)\,\mathrm{R} &\geq 2|\,\mathrm{Rc}\,|^2 - 4\alpha|\,\mathrm{Rc}\,|c_0 - 2\alpha|\nabla^2 u|^2 \\ &\geq |\,\mathrm{Rc}\,|^2 - 4\alpha^2 c_0^2 - 2\alpha|\nabla^2 u|^2 \\ &\geq \frac{1}{3}\,\mathrm{R}^2 - 4\alpha^2 c_0^2 - 2\alpha|\nabla^2 u|^2.\end{aligned}$$

Since we assume $\mathrm{R}_0 \geq 0$, we have $\mathrm{R}_0^- \geq 0$ and thus by the improved bound on the Hessian, Proposition 4.1, the estimate

$$(\partial_t - \Delta)\,\mathrm{R} \geq \frac{1}{3}\,\mathrm{R}^2 - 4\alpha^2 c_0^2 - 2\alpha(s_0 + \alpha C c_0^2 + C c_0\,\mathrm{R}),$$

where C is the constant from Proposition 4.1. We deduce by Young's inequality

$$\begin{aligned}(\partial_t - \Delta)\,\mathrm{R} &\geq \frac{1}{6}\,\mathrm{R}^2 - 4\alpha^2 c_0^2 - 4\alpha s_0 - 2\alpha^2 C c_0^2 - \frac{4}{3}\alpha^2 C^2 c_0^2 \\ &= \frac{1}{6}\left(\mathrm{R}^2 - D^2\right) = \frac{1}{6}(\mathrm{R} + D)(\mathrm{R} - D),\end{aligned}$$

where the constant D is given by

$$D^2 = \alpha^2 c_0 \left(24 + 12C + \frac{4}{3}C^2\right) + 24\alpha s_0.$$

We deduce for the function $f = \mathrm{R} - D$ the evolution inequality

$$(\partial_t - \Delta) f \geq \frac{1}{6}(\mathrm{R} + D) f.$$

By the scalar maximum principle this implies the claim. □

We remark that by applying the improved bound on the Hessian (Propositon 4.1) and the Tensor Maximum Principle to the evolution equations of the Ricci curvature Rc or the Einstein tensor $E = \mathrm{R}\,g - 2\,\mathrm{Rc}$ one can show that Ricci curvature or sectional curvature above a certain threeshold is preserved under the flow.
The next estimate, which is an analogue of Theorem 9.4 in [36] for List flow, is a roundness estimate. We show that the roundness of the solution is preserved, if it is not to close to the optimal value. Indeed, if one starts List flow on a round sphere with a nontrivial function u the solution will become less round initially.

Theorem 6.3 (Roundness estimate)**.**
Suppose (M, g_0, u_0) *is a closed manifold of positive Ricci curvature and let* $0 < \epsilon < \frac{1}{3}$*, such that*

$$\mathrm{Rc}(0) \geq \epsilon\,\mathrm{R}(0) g(0).$$

Then there exists a universal constant D*, such that if*

$$\mathrm{R}(0) \geq \frac{1}{\epsilon^2(1 - 3\epsilon)} D(\alpha c_0 + \sqrt{\alpha s_0})$$

initially, then we have

$$\mathrm{R} \geq \frac{1}{\epsilon^2(1-3\epsilon)} D(\alpha c_0 + \sqrt{\alpha s_0}) \text{ and } \mathrm{Rc} \geq \epsilon \,\mathrm{R}\, g$$

along the flow.

Before we begin the proof, we recall the evolution equations for the Ricci curvature and scalar curvature for three-dimensional solutions.
The scalar curvature under List flow evolves by

$$(\partial_t - \Delta)\,\mathrm{R} = 2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R})$$

where $\Omega(\mathrm{R})$ denotes the terms arising from the coupling and is given by

$$\Omega(\mathrm{R}) = -4\langle \mathrm{Rc}, \nabla u \otimes \nabla u\rangle + 2|\Delta u|^2 - 2|\nabla^2 u|^2.$$

The Ricci curvature under List flow evolves in three dimensions by

$$(\partial_t - \Delta)\,\mathrm{Rc}_{ab} = Q_{ab} + \alpha\Omega(\mathrm{Rc})_{ab}$$

where Q denotes the quadratic curvature terms given by

$$Q_{ab} = (2|\,\mathrm{Rc}\,|^2 - \mathrm{R}^2)g_{ab} + 3\,\mathrm{R}\,\mathrm{Rc}_{ab} - 6\,\mathrm{Rc}_{ac}\,\mathrm{Rc}_{cb}$$

and $\Omega(\mathrm{Rc})$ denotes the terms arising from the coupling given by

$$\Omega(\mathrm{Rc})_{ab} = 2\Delta u \nabla_a \nabla_b u - 2\nabla_a\nabla_c u \nabla_c \nabla_b u - 2\,\mathrm{Rm}_{cabd}\,\nabla_c u \nabla_d u.$$

Note that we not choose the simplify the term involving the Riemann curvature tensor by the curvature decomposition, since we will not be able to exploit this in the computations.

Lemma 6.4.
For $\mathrm{R} > 0$ *we have the evolution equation*

$$\begin{aligned}(22)\qquad &(\partial_t - \Delta)\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right)\\ =&\frac{2}{\mathrm{R}}\nabla_c \mathrm{R}\,\nabla_c\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right) + \frac{1}{\mathrm{R}^2}\left[\mathrm{R}(Q_{ab} + \alpha\Omega(\mathrm{Rc})_{ab}) - (2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R}))\,\mathrm{Rc}_{ab}\right],\end{aligned}$$

where the terms $\Omega(\mathrm{R})$ *and* $\Omega(\mathrm{Rc})$ *are given in the above discussion.*

Proof.
We observe

$$\begin{aligned}
\partial_t\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right) &= \frac{1}{\mathrm{R}}\partial_t\,\mathrm{Rc}_{ab} - \frac{1}{\mathrm{R}^2}(\partial_t\,\mathrm{R})\,\mathrm{Rc}_{ab},\\
\nabla_c\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right) &= \frac{1}{\mathrm{R}}\nabla_c\,\mathrm{Rc}_{ab} - \frac{1}{\mathrm{R}^2}\nabla_c\,\mathrm{R}\,\mathrm{Rc}_{ab},\\
\Delta\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right) &= \frac{1}{\mathrm{R}}\Delta\,\mathrm{Rc}_{ab} - \frac{2}{\mathrm{R}^2}\nabla_c\,\mathrm{R}\,\nabla_c\,\mathrm{Rc}_{ab} + \frac{2}{\mathrm{R}^3}|\nabla\,\mathrm{R}\,|^2\,\mathrm{Rc}_{ab} - \frac{1}{\mathrm{R}^2}(\Delta\,\mathrm{R})\,\mathrm{Rc}_{ab},\\
\frac{2}{\mathrm{R}}\nabla_c\,\mathrm{R}\,\nabla_c\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right) &= \frac{2}{\mathrm{R}^2}\nabla_c\,\mathrm{R}\,\nabla_c\,\mathrm{Rc}_{ab} - \frac{2}{\mathrm{R}^3}|\nabla\,\mathrm{R}\,|^2\,\mathrm{Rc}_{ab}\,.
\end{aligned}$$

This implies by the evolution equations for the Ricci curvature and the scalar curvature under List flow on three-manifolds

$$\begin{aligned}(\partial_t - \Delta)\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right) =& \frac{1}{\mathrm{R}^2}\left(\mathrm{R}(\partial_t - \Delta)\,\mathrm{Rc}_{ab} - ((\partial_t - \Delta)\,\mathrm{R})\,\mathrm{Rc}_{ab}\right) + \frac{2}{\mathrm{R}}\nabla_c \mathrm{R}\,\nabla_c\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right)\\ =& \frac{2}{\mathrm{R}}\nabla_c \mathrm{R}\,\nabla_c\left(\frac{\mathrm{Rc}_{ab}}{\mathrm{R}}\right)\\ &+ \frac{1}{\mathrm{R}^2}\left[\mathrm{R}(Q_{ab} + \alpha\Omega(\mathrm{Rc})_{ab}) - (2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R}))\,\mathrm{Rc}_{ab}\right].\end{aligned}$$

This is the desired formula. □

Proof of Theorem 6.3.
We want to apply the tensor maximum principle to the tensor M defined by

$$M_{ab} = \frac{\mathrm{Rc}_{ab}}{\mathrm{R}} - \epsilon g_{ab}$$

By the previous Lemma we deduce the evolution equation

$$\begin{aligned}(\partial_t - \Delta)M_{ab} =& \frac{2}{\mathrm{R}}\nabla_c \mathrm{R}\,\nabla_c M_{ab} + \frac{1}{\mathrm{R}^2}\left[\mathrm{R}(Q_{ab} + \alpha\Omega(\mathrm{Rc})_{ab}) - (2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R}))\,\mathrm{Rc}_{ab}\right]\\ &+ 2\epsilon(\mathrm{Rc}_{ab} - \alpha\nabla_a u \nabla_b u).\end{aligned}$$

The reaction term is given by

$$\begin{aligned}\mathrm{R}^2 N_{ab} =& \mathrm{R}(Q_{ab} + \alpha\Omega(\mathrm{Rc})) - (2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R}))\,\mathrm{Rc}_{ab} + 2\epsilon\,\mathrm{R}^2(\mathrm{Rc}_{ab} - \alpha\nabla_a u\nabla_b u)\\ =& \left[\mathrm{R}\,Q_{ab} - 2|\,\mathrm{Rc}\,|^2\,\mathrm{Rc}_{ab} + 2\epsilon\,\mathrm{R}^2\,\mathrm{Rc}_{ab}\right] + \alpha\left[\mathrm{R}\,\Omega(\mathrm{Rc})_{ab} - \Omega(\mathrm{R})\,\mathrm{Rc}_{ab} - 2\epsilon\,\mathrm{R}^2\,\nabla_a u \nabla_b u\right]\\ =:& \; Q^\epsilon(\mathrm{Rc})_{ab} + \alpha\Omega^\epsilon_{ab}.\end{aligned}$$

We consider the case when M acquires a zero eigenvector and show that the reaction term is nonnegative in this case. Suppose $\lambda \geq \mu \geq \nu$ and ξ is the corresponding eigenvalue, such that $M_{ab}\xi^a = 0$, ie. $\nu = \epsilon(\lambda + \mu + \nu)$. By work of R. Hamilton (see the proof of Theorem 9.4 in [36, p. 281]) we have

$$Q^\epsilon_{ab}\xi^a\xi^b = \nu^2(\lambda + \mu - 2\nu) + (\lambda + \mu)(\lambda - \mu)^2.$$

On the other hand we estimate by the improved bound on the Hessian, Proposition 4.1:

$$\begin{aligned}
\Omega^{\epsilon}_{ab}\xi^a\xi^b =& \mathrm{R}\,\Omega(\mathrm{Rc})_{ab}\xi^a\xi^b - \Omega(\mathrm{R})\,\mathrm{Rc}_{ab}\,\xi^a\xi^b - 2\epsilon\,\mathrm{R}^2\,\nabla_a u\nabla_b u\xi^a\xi^b \\
=& \mathrm{R}(2\Delta u\nabla_a\nabla_b u - 2\nabla_a\nabla_c u\nabla_c\nabla_b u - 2\,\mathrm{Rm}_{cabd}\,\nabla_c u\nabla_d u)\xi^a\xi^b \\
&+ (-4\langle \mathrm{Rc}, \nabla u\otimes\nabla u\rangle + 2|\Delta u|^2 - 2|\nabla^2 u|^2)\,\mathrm{Rc}_{ab}\,\xi^a\xi^b \\
&- 2\epsilon\,\mathrm{R}^2\,\nabla_a u\nabla_b u\xi^a\xi^b \\
\geq& -\mathrm{R}(4|\nabla^2 u|^2 + 10|\,\mathrm{Rc}\,|c_0) - (4|\,\mathrm{Rc}\,|c_0 - 2|\nabla^2 u|^2)\nu - 2\nu\,\mathrm{R}\,c_0^2 \\
\geq& -14\lambda|\nabla^2 u|^2 - 34c_0|\,\mathrm{Rc}\,|\lambda - 2c_0\nu\,\mathrm{R} \\
\geq& -14(2s_0 + \alpha C c_0^2 + C c_0\,\mathrm{R})\lambda - 68c_0\lambda^2 - 6c_0\lambda^2 \\
\geq& -C(s_0 + \alpha c_0^2)\lambda - C c_0\lambda^2,
\end{aligned}$$

where C denotes an universal constant, and we used the identity $\nu = \epsilon(\lambda+\mu+\nu)$ to replace the dependence on ϵ.

It suffices to show that

$$\nu^2(\lambda+\mu-2\nu) + (\lambda+\mu)(\lambda-\mu)^2 - \alpha C\lambda(\alpha c_0^2 + s_0) - \alpha C\lambda^2 c_0 \geq 0$$

whenever $\nu = \epsilon(\lambda+\mu+\nu)$.

By the relation $\nu = \epsilon(\lambda+\mu+\nu)$ we observe

$$\lambda+\mu = \left(\frac{1}{3} - 1\right)\nu$$

and this implies the bounds

$$\begin{aligned}
\lambda \leq \lambda+\mu &= \left(\frac{1}{\epsilon} - 1\right)\nu \leq \frac{1}{\epsilon}\nu, \\
\lambda+\mu-2\nu &= \left(\frac{1}{\epsilon} - 3\right)\nu =: \delta\nu.
\end{aligned}$$

Thus we observe

$$\begin{aligned}
&\nu^2(\lambda+\mu-2\nu) + (\lambda+\mu)(\lambda-\mu)^2 - \alpha C\lambda(\alpha c_0^2 + s_0) - \alpha C\lambda^2 c_0 \\
\geq& \delta\nu^3 - \alpha C(\alpha c_0^2 + s_0)\frac{1}{\epsilon}\nu - \alpha C c_0\frac{1}{\epsilon^2}\nu \\
\geq& \frac{\delta}{2}\nu\left(\nu^2 - C\frac{\alpha^2 c_0^2 + \alpha s_0}{\delta\epsilon}\right) + \frac{\delta}{2}\nu^2\left(\nu - C\frac{\alpha c_0}{\epsilon^2\delta}\right) \\
\geq& 0,
\end{aligned}$$

provided

$$\nu \geq \max\left\{\sqrt{C}(\alpha c_0 + \sqrt{\alpha s_0})\frac{1}{\delta\epsilon}, C\frac{\alpha c_0}{\epsilon^2\delta}\right\}.$$

By inserting the expression for δ, we may deduce that the above condition is in particular fulfilled if

$$\nu \geq \frac{1}{\epsilon(1-3\epsilon)} D_2(\alpha c_0 + \sqrt{\alpha s_0}),$$

where D_2 is an universal constant depending on the universal constant C from the estimate on the Hessian, Proposition 4.1. □

The next step is to prove a pinching theorem for the traceless part of the Ricci curvature. More precisely, we want to show that the eigenvalues approach each other, if the curvature becomes large.

Theorem 6.5.
There exists $0 < \sigma \leq \epsilon^2$ and a constant $\Lambda = \Lambda(g_0, c_0, s_0, \alpha, \epsilon)$ given by

$$\Lambda = \sup_{p \in M} \mathrm{R}(0,p) + C(\alpha c_0 + \sqrt{\alpha s_0})^\sigma$$

where C is a scaling-free constant, such that we have for all $t \in [0,T)$ the pinching estimate

$$|\,\mathrm{Rc}\,|^2 - \frac{1}{3}\mathrm{R}^2 \leq \Lambda\,\mathrm{R}^{2-\sigma}\,. \tag{23}$$

For the proof we need to compute several evolution equations. We set $\gamma = 2-\sigma$ and restrict $0 < \sigma < 1$ and thus $1 < \gamma < 2$.

Lemma 6.6.
We have the evolution equation

$$(\partial_t - \Delta)|\,\mathrm{Rc}\,|^2 = -2|\nabla\,\mathrm{Rc}\,|^2 + 4\,\mathrm{tr}\,\mathrm{Rc}^3 + 2\langle \mathrm{Rc}, Q\rangle + \alpha\Omega(|\,\mathrm{Rc}\,|^2), \tag{24}$$

where $\Omega(|\,\mathrm{Rc}\,|^2)$ denotes the terms arising from the coupling given by

$$\Omega(|\,\mathrm{Rc}\,|^2) = 2\langle \mathrm{Rc}, \Omega(\mathrm{Rc})\rangle - 4\langle \mathrm{Rc}(\nabla u, \cdot), \mathrm{Rc}(\nabla u, \cdot)\rangle.$$

Proof.
We observe

$$\begin{aligned}
\partial_t|\,\mathrm{Rc}\,|^2 =& \partial_t(g^{ac}g^{bd}\,\mathrm{Rc}_{ab}\,\mathrm{Rc}_{cd}) = 2(\partial_t g^{ac})g^{bd}\,\mathrm{Rc}_{ab}\,\mathrm{Rc}_{cd} + 2g^{ab}g^{bd}\,\mathrm{Rc}_{cd}\,\partial_t\,\mathrm{Rc}_{ab},\\
\Delta|\,\mathrm{Rc}\,|^2 =& 2\langle \mathrm{Rc}, \Delta\,\mathrm{Rc}\rangle + 2|\nabla\,\mathrm{Rc}\,|^2,
\end{aligned}$$

and this implies by the evolution equation for the inverse of the metric and the Ricci curvature the equation

$$\begin{aligned}
(\partial_t - \Delta)|\,\mathrm{Rc}\,|^2 =& -2|\nabla\,\mathrm{Rc}\,|^2 + 2g^{bd}(2\,\mathrm{Rc}^{ac} - 2\alpha\nabla^a u\nabla^c u)\,\mathrm{Rc}_{ab}\,\mathrm{Rc}_{cd} + 2\,\mathrm{Rc}_{ab}(\partial_t - \Delta)\,\mathrm{Rc}_{ab}\\
=& -2|\nabla\,\mathrm{Rc}\,|^2 + 4\,\mathrm{tr}\,\mathrm{Rc}^3 + 2\langle \mathrm{Rc}, Q\rangle\\
&+ 2\alpha\,\mathrm{Rc}_{ab}\,\Omega(\mathrm{Rc})_{ab} - 4\alpha\nabla_a u\nabla_c u\,\mathrm{Rc}_{ab}\,\mathrm{Rc}_{bc}\,.
\end{aligned}$$

This is the claimed formula. □

Lemma 6.7.
For $\mathrm{R} > 0$ *and for any* $\gamma > 0$ *we have the evolution equation*

$$(25)\quad \begin{aligned}(\partial_t - \Delta)\left(\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}\right) =& \frac{2(\gamma-1)}{\mathrm{R}}\left\langle \nabla \mathrm{R}, \nabla\left(\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}\right)\right\rangle - \frac{2}{\mathrm{R}^{\gamma+2}}\,|\mathrm{R}\nabla\operatorname{Rc} - \nabla \mathrm{R}\otimes\operatorname{Rc}|^2\\ &- \frac{(2-\gamma)(\gamma-1)}{\mathrm{R}^{\gamma+2}}|\operatorname{Rc}|^2|\nabla \mathrm{R}|^2\\ &+ \frac{1}{\mathrm{R}^{\gamma+1}}\left(4\operatorname{tr}\operatorname{Rc}^3 + 2\langle Q, \operatorname{Rc}\rangle - 2\gamma|\operatorname{Rc}|^4 + \alpha\Psi\right)\end{aligned}$$

where the term Ψ *denotes the term arising from the coupling given by*

$$\Psi = \Omega(|\operatorname{Rc}|^2) - 2\gamma|\operatorname{Rc}|^2\Omega(\mathrm{R}).$$

Proof.
We compute

$$\begin{aligned}\partial_t\left(\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}\right) =& \frac{1}{\mathrm{R}^\gamma}\partial_t|\operatorname{Rc}|^2 - \gamma\frac{|\operatorname{Rc}|^2}{\mathrm{R}^{\gamma+1}}\partial_t \mathrm{R},\\ \nabla\left(\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}\right) =& \frac{1}{\mathrm{R}^\gamma}\nabla|\operatorname{Rc}|^2 - \gamma\frac{|\operatorname{Rc}|^2}{\mathrm{R}^{\gamma+1}}\nabla \mathrm{R},\\ \Delta\left(\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}\right) =& \frac{1}{\mathrm{R}^\gamma}\Delta|\operatorname{Rc}|^2 - \frac{2\gamma}{\mathrm{R}^{\gamma+1}}\langle\nabla \mathrm{R}, \nabla|\operatorname{Rc}|^2\rangle\\ &+ \gamma(\gamma+1)\frac{|\operatorname{Rc}|^2}{\mathrm{R}^{\gamma+2}}|\nabla \mathrm{R}|^2 - \gamma\frac{|\operatorname{Rc}|^2}{\mathrm{R}^{\gamma+1}}\Delta \mathrm{R}.\end{aligned}$$

This implies

$$\begin{aligned}(\partial_t - \Delta)\left(\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}\right) =& \frac{2\gamma}{\mathrm{R}^{\gamma+1}}\langle\nabla \mathrm{R}, \nabla|\operatorname{Rc}|^2\rangle - \gamma(\gamma+1)\frac{|\operatorname{Rc}|^2}{\mathrm{R}^{\gamma+2}}|\nabla \mathrm{R}|^2\\ &+ \frac{1}{\mathrm{R}^\gamma}\left(-2|\nabla\operatorname{Rc}|^2 + 4\operatorname{tr}\operatorname{Rc}^3 + 2\langle\operatorname{Rc}, Q\rangle + \alpha\Omega(|\operatorname{Rc}|^2)\right)\\ &- \gamma\frac{|\operatorname{Rc}|^2}{\mathrm{R}^\gamma}(2|\operatorname{Rc}|^2 + \alpha\Omega(\mathrm{R}))\\ =& \frac{2\gamma}{\mathrm{R}^{\gamma+1}}\langle\nabla \mathrm{R}, \nabla|\operatorname{Rc}|^2\rangle - \gamma(\gamma+1)\frac{|\operatorname{Rc}|^2}{\mathrm{R}^{\gamma+2}}|\nabla \mathrm{R}|^2 - \frac{2}{\mathrm{R}^\gamma}|\nabla\operatorname{Rc}|^2\\ &+ \frac{1}{\mathrm{R}^{\gamma+1}}\left(4\operatorname{tr}\operatorname{Rc}^3 + 2\langle Q, \operatorname{Rc}\rangle - 2\gamma|\operatorname{Rc}|^4\right)\\ &+ \alpha\frac{1}{\mathrm{R}^{\gamma+1}}\left([\Omega(|\operatorname{Rc}|^2) - 2\gamma|\operatorname{Rc}|^2\Omega(\mathrm{R})\right).\end{aligned}$$

The gradient terms are now reworked exactly as in Lemma 10.3 of [36] and we deduce the result. ∎

Lemma 6.8.
For $\mathrm{R} > 0$ *and for any* $\gamma > 0$ *we have the evolution equation*

$$(\partial_t - \Delta)\,\mathrm{R}^{2-\gamma} = \frac{2(\gamma-1)}{\mathrm{R}}\langle \nabla \mathrm{R}, \nabla \mathrm{R}^{2-\gamma}\rangle - \frac{(2-\gamma)(\gamma-1)}{\mathrm{R}^{\gamma+2}}\,\mathrm{R}^2\,|\nabla \mathrm{R}|^2 + (2-\gamma)\,\mathrm{R}^{1-\gamma}(2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R})). \tag{26}$$

Proof.
We recall the evolution equation for the scalar curvature under List flow:

$$(\partial_t - \Delta)\,\mathrm{R} = 2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R}).$$

Moreover, we observe

$$\begin{aligned}
\partial_t \mathrm{R}^{2-\gamma} &= (2-\gamma)\,\mathrm{R}^{1-\gamma}\,\partial_t \mathrm{R},\\
\nabla \mathrm{R}^{2-\gamma} &= (2-\gamma)\,\mathrm{R}^{1-\gamma}\,\nabla \mathrm{R},\\
\Delta \mathrm{R}^{2-\gamma} &= (2-\gamma)\,\mathrm{R}^{1-\gamma}\,\Delta \mathrm{R} + (2-\gamma)(1-\gamma)\,\mathrm{R}^{-\gamma}\,|\nabla \mathrm{R}|^2,\\
\frac{2(\gamma-1)}{\mathrm{R}}\langle \nabla \mathrm{R}, \nabla \mathrm{R}^{2-\gamma}\rangle &= 2(\gamma-1)(2-\gamma)\,\mathrm{R}^{-\gamma}\,|\nabla \mathrm{R}|^2.
\end{aligned}$$

This implies by the evolution equation for the scalar curvature under List flow the evolution equation

$$\begin{aligned}
(\partial_t - \Delta)\,\mathrm{R}^{2-\gamma} =& \frac{2(\gamma-1)}{\mathrm{R}}\langle \nabla \mathrm{R}, \nabla \mathrm{R}^{2-\gamma}\rangle - \frac{(2-\gamma)(\gamma-1)}{\mathrm{R}^{\gamma+2}}\,\mathrm{R}^2\,|\nabla \mathrm{R}|^2\\
&+ (2-\gamma)\,\mathrm{R}^{1-\gamma}(\partial_t - \Delta)\,\mathrm{R}\\
=& \frac{2(\gamma-1)}{\mathrm{R}}\langle \nabla \mathrm{R}, \nabla \mathrm{R}^{2-\gamma}\rangle - \frac{(2-\gamma)(\gamma-1)}{\mathrm{R}^{\gamma+2}}\,\mathrm{R}^2\,|\nabla \mathrm{R}|^2\\
&+ (2-\gamma)\,\mathrm{R}^{1-\gamma}(2|\,\mathrm{Rc}\,|^2 + \alpha\Omega(\mathrm{R})).
\end{aligned}$$

This is our desired evolution equation. □

Lemma 6.9.
We define for $\mathrm{R} > 0$ *the test function*

$$f = \frac{|\,\mathrm{Rc}\,|^2}{\mathrm{R}^\gamma} - \frac{1}{3}\,\mathrm{R}^{2-\gamma}.$$

The test function f *satisfies the evolution equation*

$$\begin{aligned}
(\partial_t - \Delta)f =& \frac{2(\gamma-1)}{\mathrm{R}}\langle \nabla \mathrm{R}, \nabla f\rangle - \frac{2}{\mathrm{R}^{\gamma+2}}\,|\mathrm{R}\,\nabla\,\mathrm{Rc} - \nabla\,\mathrm{R} \otimes \mathrm{Rc}|^2\\
&- \frac{(2-\gamma)(\gamma-1)}{\mathrm{R}^{\gamma+2}}\left(|\,\mathrm{Rc}\,|^2 - \frac{1}{3}\,\mathrm{R}^2\right)|\nabla \mathrm{R}|^2\\
&+ \frac{2}{\mathrm{R}^{\gamma+1}}\left[(2-\gamma)|\,\mathrm{Rc}\,|^2\left(|\,\mathrm{Rc}\,|^2 - \frac{1}{3}\,\mathrm{R}^2\right) - 2P + \alpha\Sigma\right],
\end{aligned} \tag{27}$$

where the term P, which is quartic in the curvature, is given by

$$P = |\operatorname{Rc}|^4 - \operatorname{R}\left(\operatorname{tr}\operatorname{Rc}^3 + \frac{1}{2}\langle Q, \operatorname{Rc}\rangle\right)$$

and the term Σ arising from the coupling is given by

$$\Sigma = \Psi - \frac{1}{3}(2-\gamma)\operatorname{R}^2\Omega(\operatorname{R}),$$

where Ψ denotes the coupling terms from Lemma 6.7 and $\Omega(\operatorname{R})$ are the coupling terms for the scalar curvature.

Proof.
This follows directly by combining Lemma 6.7 and 6.8. □

The following algebraic Lemma due to R. Hamilton carries directly over to our setting:

Lemma 6.10 (cf. Lemma 10.7 in [36])**.**
If $\operatorname{R} > 0$ and $\operatorname{Rc} \geq \epsilon \operatorname{R} g$, then

$$P \geq \epsilon^2 |\operatorname{Rc}|^2 \left(|\operatorname{Rc}|^2 - \frac{1}{3}\operatorname{R}^2\right).$$

Proof of Theorem 6.5.
We choose $\delta > 0$ so small, that $\delta \leq \epsilon^2$ and let $\gamma = 2 - \delta$. By Lemma 6.10 we have the evolution inequality

$$(\partial_t - \Delta) f \leq \frac{2(\gamma-1)}{\operatorname{R}}\langle \nabla \operatorname{R}, \nabla f\rangle + \frac{2}{\operatorname{R}^{\gamma+1}}\left[-\epsilon |\operatorname{Rc}|^2\left(|\operatorname{Rc}|^2 - \frac{1}{3}\operatorname{R}^2\right) + \alpha\Sigma\right].$$

We analyze the reaction term:

$$\begin{aligned}\frac{2}{\operatorname{R}^{\gamma+1}}\left[-\epsilon |\operatorname{Rc}|^2\left(|\operatorname{Rc}|^2 - \frac{1}{3}\operatorname{R}^2\right) + \alpha\Sigma\right] &= -2\epsilon\frac{|\operatorname{Rc}|^2}{\operatorname{R}} f + 2\alpha\frac{\Sigma}{\operatorname{R}^{\gamma+1}} \\ &\leq -2\epsilon^2 \operatorname{R} f + 2\alpha\frac{\Sigma}{\operatorname{R}^{3-\sigma}}.\end{aligned}$$

By the estimate

$$\alpha\Sigma \leq C(\alpha c_0 + \sqrt{\alpha s_0})\operatorname{R}^3$$

we deduce

$$(\partial_t - \Delta) f \leq \langle X, \nabla f\rangle - 2\epsilon \operatorname{R} f + 2C(\alpha c_0 + \sqrt{\alpha s_0})\operatorname{R}^\sigma$$

where we introduced

$$X = \frac{2(\gamma-1)}{\operatorname{R}}\nabla \operatorname{R}.$$

At time $t = 0$ we have the bound

$$f = \frac{|\operatorname{Rc}|^2 - \frac{1}{3}\operatorname{R}^2}{\operatorname{R}^{2-\sigma}} \leq \frac{\operatorname{R}^2}{\operatorname{R}^{2-\sigma}} \leq \operatorname{R}^\sigma \leq \sup_{p\in M}\operatorname{R}^\sigma(0).$$

Suppose the function f attains at (t_*, p_*) a new maximum κ, then we have

$$0 \leq (\partial_t - \Delta) f(t_*, p_*) \leq -2\epsilon \,\mathrm{R}(t_*, p_*) f(t_*, p_*) + 2\alpha C(c_0 + \sqrt{s_0})\,\mathrm{R}^\sigma$$
$$\rightsquigarrow \kappa = f(t_*, p_*) \leq \frac{\alpha}{\epsilon} C(c_0 + \sqrt{s_0})\,\mathrm{R}^{\sigma-1}(t_*, p_*),$$

however since $\sigma - 1 < 0$ the last term is controlled by the lower bound on the scalar curvature, which is preserved by Proposition 6.2. Hence we deduce

$$\begin{aligned}\kappa &\leq \frac{\alpha}{\epsilon} C(c_0 + \sqrt{s_0}) \left(\frac{1}{\epsilon^2(1-3\epsilon)}\right)^{\sigma-1} D^{\sigma-1}(c_0 + \sqrt{s_0})^{\sigma-1} \\ &\leq \alpha \tilde{C} \epsilon^{1-2\sigma}(1-3\epsilon)^{1-\sigma}(c_0 + \sqrt{s_0})^\sigma \\ &\leq \alpha \tilde{C}(c_0 + \sqrt{s_0})^\sigma\end{aligned}$$

where $\tilde{C} = C \geq C D^{\sigma-1}$.
Thus the theorem follows if we choose $\delta \leq \epsilon^2$ and the constant $\Lambda = \Lambda(g_0, c_0, s_0, \alpha, \epsilon)$ by

$$\Lambda = \sup_{p \in M} \mathrm{R}(0, p) + \alpha C(c_0 + \sqrt{s_0})^\sigma$$

where C is a scaling-free constant.

□

With the roundness estimate, Theorem 6.3, and the pinching estimate, Theorem 6.5, at hand, we can procede to prove the convergence of the flow as $t \to T$.
There have been different approaches to obtain the convergence of the metric given the above pinching estimates.
In R. Hamilton's 1982 work [36] he derived a global gradient estimate for the the scalar curvature and used tensor interpolation inequalities to estimate the higher derivatives of curvature. This is then used to show longtime existence and convergence of the normalized Ricci flow on $(0, \infty)$. We will use a more direct approach relying on the interior derivative estimates for Ricci flow, see Section 5.4 of S. Brendle's book [14].

Proposition 6.11 (cf. Theorem 5.23 in [14])**.**
The rescaled metrics $\{\tilde{g}(t)\}_{t \in [0,T)}$ *given by*

$$\tilde{g}(t) = \frac{1}{4(T-t)} g(t)$$

converge in C^∞ *to a limit metric of constant sectional curvature.*

Proof.
We explain how to modify the arguments in S. Brendle's book [14]:
Lemma 5.13 follows from the pinching theorem 6.5 and Lemma 5.14 follows from the fact that our estimates imply $\mathrm{S} > 0$ and in this setting $T < \infty$ by Proposition 3.8. The Lemmata 5.15, 5.16 and 5.17, which assert that the sectional curvatures approach each other as $t \to T$, go through unchanged, if one replaces the interior estimates for Ricci flow by the corresponding interior estimates for List flow. In Lemma 5.18 additonal terms linear in the scalar curvature appear in the first inequality arise, however these are controlled

once the scalar curvature becomes large. For Lemma 5.19 we invoke our pinching estimate from Theorem 6.5 together with the estimate on the scalar curvature from Lemma 5.18. In Lemma 5.20 and Proposition 5.22 are diverge with lower order compared to the curvature terms and hence the statement remains true. Finally, Lemma 5.21 and the proof of Theorem 5.23 only involve the metric and hence go through unchanged. □

Finally, we have to discuss the behaviour of the function u as $t \to T$:
We observe with respect to the rescaled metric $\tilde{g}(t)$ the estimates

$$\begin{aligned}|\nabla u|^2_{\tilde{g}(t)} &= 4(T-t)|\nabla u|^2_{g(t)} \le 4(T-t)c_0 \longrightarrow 0 \text{ as } t \to T,\\ |\nabla^2 u|^2_{\tilde{g}(t)} &= 16(T-t)^2|\nabla^2 u|^2_{g(t)}\\ &\le 16(T-t)^2(s_0+\alpha C c_0^2) + 16(T-T)Cc_0\,\mathrm{R}_{\tilde{g}(t)} \longrightarrow 0 \text{ as } t \to T.\end{aligned}$$

Thus the velocity $\partial_t u$ satisfies

$$\begin{aligned}|\partial_t u| = |\Delta_{g(t)} u| &\le \frac{1}{4(T-t)}|\Delta_{\tilde{g}(t)} u| \le \frac{3}{4(T-t)}|\nabla^2 u|_{\tilde{g}(t)}\\ &\le C(s_0+\alpha c_0) + Cc_0\,\mathrm{R}_{\tilde{g}(t)}\,\frac{1}{(T-t)^{1/2}}.\end{aligned}$$

The expression on the right-hand side is integrable on $[t,T]$, since $\mathrm{R}_{\tilde{g}(t)} \to 6$. In particular $u(t)$ converges to a limit function u_∞ as $t \to T$, by the above gradient estimates this function is constant.
This finishes the proof of Theorem 6.1.

Remark 6.12.
A similar convergence theorem holds true for Ricci flow coupled to harmonic map heat flow into targets of controlled curvature and nonpositive sectional curvature, the details will appear elsewhere.

7. The Canonical Neighbourhood Theorem

From now on we will consider a solution to List flow on a closed oriented three-manifold M^3 with initial metric g_0 and initial function u_0. We assume without loss of generality that the function u_0 is non-constant.
The goal of this section is to establish an analogue of Perelman's canonical Neighbourhood Theorem, see Theorem 12.1 in [73], for the extended Ricci flow system known as List flow.

7.1. **Preliminary observations.**

We start by recalling some definitions and results on noncollapsing in List flow. The definition of κ-noncollapsing only involves the geometry and is therefore identical to G. Perelman's notion:

Definition 7.1 (κ-noncollapsing at scale ρ; Definition 4.2 in [73])**.**
A Riemannian metric g is κ-noncollapsed at scale ρ, if for all radii $0 < r < \rho$ and all points $p_0 \in M$ the following holds:
Any metric ball $B_r(p_0)$ satisfying the curvature bound

$$\sup_{x \in B_r(x_0)} |\operatorname{Rm}|_g \leq r^{-2}$$

has volume

$$\operatorname{vol}_g(B_r(p_0)) \geq \kappa r^n.$$

In his thesis B. List proved that solutions to List flow are noncollapsed:

Theorem 7.2 (Noncollapsing for List flow, Theorem 7.2 in [59])**.**
Suppose M is a closed manifold and $T < \infty$. Then a solution $(g(t), u(t))_{t \in [0,T]}$ of List flow is noncollapsed on $[0, T] \times M$.

The proof of B. List closely follows G. Perelmans's argument in Theorem 4.1 in [73] and uses a modified version of the $\mathcal{W}$-functional, which is adapted to List flow. The modified $\mathcal{W}$-functional is given by

$$\mathcal{W}[g, u, f; \tau] := (4\pi\tau)^{-\frac{n}{2}} \int_M \left[\tau(\mathrm{R} - \alpha|\nabla u|^2 + |\nabla f|^2) + f - n\right] \exp(-f) \, \mathrm{d}\mu_g$$

where $\tau \in (0, \infty)$ and $f \in C^\infty(M)$. The one has the monotonicity $\partial_t \mathcal{W} \geq 0$ along any solution of List flow on a closed manifold (c.f. Theorem 4.4 in [59]).
Let us remark that G. Perelman gave another approach to noncollapsing along Ricci flow by using the reduced volume, compare Section 8 of [73]. For List flow this approach was worked out by R. Buzano (formely R. Mueller) in Chapter 6 of his thesis [69], actually his results apply to Ricci flow coupled to harmonic map heat flow.
In the next step we introduce the singularity models for our flow. The following definition closely follows the related definition for Ricci flow due to G. Perelman, see Section 11 of [73]:

Definition 7.3 ($(\kappa, \underline{u}, \overline{u})$-solution)**.**
Suppose M is a three-manifold without boundary. A solution $(g(t), u(t))_{t\in(-\infty,0]}$ to List flow is called $(\kappa, \underline{u}, \overline{u})$-solution if the following criteria are satisfied:

- *the metric $g(t)$ is complete for each $t \in (-\infty, 0]$ and the solution has bounded curvature on bounded time intervals,*
- *the solution has nonnegative sectional curvature and positive scalar curvature,*
- *the solution is κ-noncollapsed on all scales,*
- *we have $u(t,p) = u_0$ on $(-\infty, 0]$ for some $u_0 \in \mathbb{R}$ with $\underline{u} \leq u_0 \leq \overline{u}$.*

Since the function u is constant on a $(\kappa, \underline{u}, \overline{u})$-solution, we deduce that for every $(\kappa, \underline{u}, \overline{u})$-solution $\{g(t), u(t)\}_{t\in(-\infty,0]}$ to List flow one obtains a κ-solution to Ricci flow given by the one-parameter family $\{g(t)\}_{t\in(-\infty,0]}$.
We deduce in particular by the Harnack inequality for Ricci flow, see R. Hamilton [39] (under the assumption of positive curvature operator) and S. Brendle [13] (under the assumption of PIC2), for any $(\kappa, \underline{u}, \overline{u})$-solution the inequality

$$\partial_t \mathrm{R} + 2\langle \nabla \mathrm{R}, v\rangle + 2\,\mathrm{Rc}(v,v) \geq 0$$

which holds for any $p \in M$ and any $v \in T_pM$. Moreover, we have for $t_1 < t_2$ the integrated Harnack inequality

$$\mathrm{R}(p_1, t_1) \leq \exp\left(\frac{d^2_{g(t_1)}(p_1, p_2)}{2(t_2 - t_1)}\right) \mathrm{R}(p_2, t_2).$$

The Harnack inequality and the noncollapsing implies a longrange curvature estimate (which in turn implies gradient estimates) on $(\kappa, \underline{u}, \overline{u})$-solutions, see Chapter 11 of [73]:

Theorem 7.4 (Longrange curvature estimate, cf. Perelman)**.**
For any $\kappa > 0$ there exists a function $\omega : [0,\infty) \to (0,\infty)$ depending only on κ, such that on any $(\kappa, \underline{u}, \overline{u})$-solution we have for any points $x, y \in M$, $t \in (-\infty, 0]$ the longrange curvature estimate

$$\mathrm{R}(x,t) \leq \mathrm{R}(y,t)\omega\left(\mathrm{R}(y,t) d^2_{g(t)}(x,y)\right). \tag{28}$$

Moreover, there exists a constant $\eta = \eta(\kappa)$ only depending on the noncollapsing, such that for any point $x \in M$ and any time $t \in (-\infty, 0]$ we have the gradient estimates

$$|\nabla \mathrm{R}|(x,t) \leq \eta\, \mathrm{R}(x,t)^{\frac{3}{2}} \text{ and } |\nabla^2 \mathrm{R}|(x,t) \leq \eta\, \mathrm{R}(x,t)^2. \tag{29}$$

Proof.
This follows as in Section 11.7 of Perelman's work [73] since a $(\kappa, \underline{u}, \overline{u})$-solution to List flow gives a κ-solution to Ricci flow.

□

As in Ricci flow one also has the following universal noncollapsing theorem. In particular, one can choose the η from the previous theorem uniformly.

Theorem 7.5 (Universal noncollapsing, cf. G. Perelman [73]).
There exists a constant $\kappa_0 > 0$, such that any $(\kappa, \underline{u}, \overline{u})$ is either κ_0-noncollapsed; or is quotient of the round sphere $\mathbb{S}^3$; or is a noncompact quotient of the standard cylinder $\mathbb{S}^2 \times \mathbb{R}$.

In the following we introduce modified notions of the geometric models of high-curvature regions in Ricci flow, which are adapted to our setting.
Fix a smooth three-manifold M and a Riemannian metric g.

Definition 7.6 ((ϵ, c)-neck).
Suppose (M, g) is a Riemannian manifold and u a smooth function. An open subset $U \subset M$ is called (ϵ, c)-neck, if the tuple $((M, g), u))$ is a after rescaling ϵ-close to the c-standard cylinder, that is the standard cylinder $\mathbb{S}^2_r \times I$, where $r = \frac{1}{2}$, $I \subset \mathbb{R}$ an open interval of length greater than $2\epsilon^{-1}$ and $u_{St} = c$.

Definition 7.7 (parabolic (ϵ, c)-neck).
Suppose $(M, g(t), u(t))$ is a solution to List flow for $t \in [a, b]$. An spacetime subset $U \times [a, b] \subset M \times [a, b]$, where $U \subset M$ is open, is called parabolic (ϵ, c)-neck, if it is after parabolic rescaling and time-shifting ϵ-close to the List flow given by the shrinking cylinder on $[-1, 0]$ with $t = 0$-slice given by the standard cylinder $\mathbb{S}^2 \times I$, where I is an interval of length greater than $2\epsilon^{-1}$ and $u(t) = c$ for all $t \in [a, b]$.

We remark that in the literature often the term *strong* instead of *parabolic* is used.

Definition 7.8 (ϵ-tube).
A List tuple $(M = \mathbb{S}^2 \times I, g, u)$, where $I \subset \mathbb{R}$ is an open interval, is called an ϵ-tube, if every point $p \in I \times \mathbb{S}^2$ is contained in an (ϵ, c)-neck and the scalar curvature stays bounded on both ends.

Definition 7.9 (ϵ-horn).
A List tuple $(M = \mathbb{S}^2 \times I, g, u)$, where $I \subset \mathbb{R}$ is an open interval, is called an ϵ-horn, if every point $p \in I \times \mathbb{S}^2$ is contained in an (ϵ, c)-neck and the scalar curvature stays bounded in one end and tends to infinity on the other end.

Definition 7.10 (doubled ϵ-horn).
A List tuple $(M = \mathbb{S}^2 \times I, g, u)$, where $I \subset \mathbb{R}$ is an open interval, is called an doubled ϵ-horn, if every point $p \in I \times \mathbb{S}^2$ is contained in an (ϵ, c)-neck and the scalar curvature tends to infinity in both ends.

In the following two definitions the smooth manifold C is either the open ball B^3 or the manifold $\mathbb{RP}^3 \backslash \overline{B^3}$.

Definition 7.11 (ϵ-cap).
A List tuple (C, g, u) is called ϵ-cap, if there exists a compact set $K \subset C$, such that all points $p \in C \backslash K$ are contained in an (ϵ, c)-neck for some $\underline{c} \leq c \leq \overline{c}$ and the scalar curvature stays bounded on the end.

Definition 7.12 (capped ϵ-horn)**.**
A List tuple (C, g, u) is called capped ϵ-horn, if there exists a compact set $K \subset C$, such that all points $p \in C \backslash K$ are contained in an (ϵ, c)-neck and the scalar curvature tends to infinity on the end.

As in Ricci flow we obtain a precise structure theorem for κ-solutions:

Theorem 7.13 (Structure theorem for κ-solutions, cf. G. Perelman, Section 1.5 in [75])**.**
Fix $\epsilon > 0$. Then there exist positive constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ with the following property: Suppose that $(M, g(t), u(t))$ is a $(\kappa, \underline{u}, \overline{u})$-solution to List flow on $(-\infty, 0]$. Then for each $(p_0, t_0) \in M \times (-\infty, 0]$ there exists a neighbourhood B of x_0, such that

$$B_{g(t_0)}(p_0, C_1^{-1}\,\mathrm{R}(p_0, t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(p_0, C_1\,\mathrm{R}(p_0, t_0)^{-\frac{1}{2}})$$

and

$$C_2^{-1}\,\mathrm{R}(p_0, t_0) \leq \mathrm{R}(p, t_0) \leq C_2\,\mathrm{R}(x_0, t_0).$$

Moreover, the neighbourhood B satisfies atleast one of the below conditions:

- *B is contained in the final time slice of an parabolic (ϵ, c)-neck with center at (p_0, t_0),*
- *B is an ϵ-cap,*
- *B is a closed manifold diffeomorphic to $\mathbb{S}^3 \diagup \Gamma$,*

Proof.
Since $(\kappa, \underline{u}, \overline{u})$-solutions to List flow on $(-\infty, 0]$ are κ-solutions to Ricci flow, this is immediate from the corresponding results for Ricci flow. For a detailed exposition, see Theorem 6.20 and Corollary 6.21 in S. Brendle's work [18].
Note that at first one obtains by Theorem 6.20 that B is contained in an (ϵ, c)-neck centered at p_0 and then this can upgraded to the stronger version with the final time slice of an parabolic (ϵ, c)-neck above. □

7.2. **Statement and Proof of the Canonical Neighbourhood Theorem.**

To perform surgery we need to understand the structure of the solution in regions of high curvature in a precise manner. For Ricci flow this was shown by G. Perelman, see Theorem 12.1 of [73]. Our statement and the idea of the proof closely follow Section 12 of G. Perelman's work [73], while for the details we follow Section 7 of S. Brendle's work [18].

Theorem 7.14 (Canonical Neighbourhood Theorem for List flow)**.**
Suppose (M^3, g_0) is a closed Riemannian manifold and $u_0 : M \to \mathbb{R}$ a smooth function. Let $\{g(t), u(t)\}_{t \in [0,T)}$ be a smooth solution to List flow with initial data $g(0) = g_0$ and $u(0) = u_0$. For a given small constant $\epsilon > 0$ and a given large constant $A_0 > 0$, there exists a positive constant $\hat{r} > 0$ with the following property:
Suppose $(t_0, p_0) \in (0, T) \times M$ is a space time point with $Q = \mathrm{R}(t_0, p_0) \geq \hat{r}^{-2}$, then the parabolic neighbourhood

$$\begin{aligned}&\mathcal{P}\left(t_0, p_0, A_0 Q^{-1}, A_0 Q^{-\frac{1}{2}}\right)\\ &= \left\{(t, p) \in [0, T) \times M \,\middle|\, 0 \leq t_0 - t \leq A_0 Q^{-1} \text{ and } d_{g(t_0)}(p, p_0) \leq A_0 Q^{-\frac{1}{2}}\right\}\end{aligned}$$

around (t_0, p_0) *is after scaling by the factor* Q *and shifting to the origin* ϵ*-close to the corresponding subset of an* $(\kappa_0, \underline{u}, \overline{u})$*-solution.*

The proof is by contradicicition and consists of several steps.
If the assertion is not true, then there exists a sequence $(p_j, t_j) \in M \times [0, T]$ with the following properties:

- the scalar curvature satisfies $Q_j = \mathrm{R}(p_j, t_j) \geq j^2$,
- after scaling by the factor Q_j and shifting the parabolic neighbourhood

$$\begin{aligned}&\mathcal{P}\left(p_j, t_j, A_0 Q_j^{-\frac{1}{2}}, A_0 Q_j^{-1}\right)\\ &= \left\{(p,t) \in M \times [0,T) \,\middle|\, d_{g(t_j)}(p, p_j) \leq A_0 Q_j^{-\frac{1}{2}} \text{ and } 0 \leq t_0 - t \leq A_0 Q_j^{-1}\right\}\end{aligned}$$

  is not ϵ-close to the subset of any $(\kappa_0, \underline{u}, \overline{u})$ solution.

Step 1: Point picking argument
By a point picking argument we may additionally arrange the following condition:
If $(\tilde{p}, \tilde{t}) \in M \times [0, T)$ is a point in space-time, such that $\tilde{t} \leq t_j$ and $\tilde{Q} = \mathrm{R}(\tilde{p}, \tilde{t}) \geq 4Q_j$, then the parabolic neighbourhood

$$\begin{aligned}&\mathcal{P}\left(p_j, t_j, A_0 Q_j^{-\frac{1}{2}}, A_0 Q_j^{-1}\right)\\ &= \left\{(p,t) \in M \times [0,T) \,\middle|\, d_{g(t_j)}(p, p_j) \leq A_0 Q_j^{-\frac{1}{2}} \text{ and } 0 \leq t_0 - t \leq A_0 Q_j^{-1}\right\}\end{aligned}$$

is after scaling by the factor $\tilde{Q}$ and shifting ϵ-close to an corresponding subset of some $(\kappa_0, \underline{u}, \overline{u})$-solution.

Step 2: Curvature estimates in the good regions
Any point $(p, t) \in M \times [0, T)$, which fulfills $t \leq t_j$ and $\mathrm{R}(p, t) \geq 4Q_j$ is contained in the region, where we have an approximation by $(\kappa, \underline{u}, \overline{u})$-solutions. By the longrange curvature estimate, Theorem 7.4, this implies for some $\eta > 0$ the estimates

$$|\nabla \mathrm{R}|(x,t) \leq 2\eta \mathrm{R}(x,t)^{\frac{3}{2}} \text{ and } |\nabla^2 \mathrm{R}|(x,t) \leq 2\eta \mathrm{R}(x,t)^2.$$

Step 3: Longrange curvature estimate
We define as in the work of G. Perelman the function $D : (0, \infty) \to [0, \infty]$ by

$$D(\rho) = \limsup_{j \to \infty} \sup \left\{Q_j^{-1} \mathrm{R}(p, t_j) \,\middle|\, p \in B_{g(t_j)}(p_j, \rho Q_j^{-\frac{1}{2}})\right\}$$

By the curvature estimates in the good region (Step 2) one obtains that $D(\rho) \leq c_0$ for some $c_0 > 0$ and $\rho < c_1$ for $c_1 = c_1(\eta)$.

We claim that $D(\rho) < \infty$ for all $\rho > 0$. This is the longrange curvature estimate we want to prove. Again, we assume to the contrary that this is false. Let us define

$$\rho^* = \sup\{D(\rho) < \infty | \rho > 0\}$$

and observe that $0 < \rho^* < \infty$ by our assumptions. In the next step we want to rescale our solution around (p_j, t_j) by the factor Q_j and pass to the limit $j \to \infty$ using an adapted version of Hamilton's compactness theorem due to B. List, see Theorem 8.2 in [59], for this we need uniform curvature bounds and a lower bound on the volume. The curvature bound for $\rho < \rho^*$ follows by construction of ρ^*, while the lower volume bound is implied by the noncollapsing theorem for List flow, as recorded in Theorem 7.2.
Since the remaining parts of the argument are purely geometric in nature, we may argue identically as in the corresponding step in the Ricci flow proof. In the end this yields a contradiction to the assumption $\rho^* < \infty$. Hence we have the longrange curvature estimate

$$\sup\left\{Q_j^{-1}\,\mathrm{R}(t_j, p)\,\middle|\, p \in B_{g(t_j)}(p_j, \rho Q_j^{-\frac{1}{2}})\right\} \leq C \text{ for all } j \in \mathbb{N}.$$

Step 4: Construction of a limit
In this step we construct a smooth complete limit manifold as follows: We consider the smooth manifold M and define the rescaled metric g_j by $g_j = Q_j g$. This implies in particular that $\mathrm{R}_{g_j}(p_j) = 1$.
We want to show that the sequence of pointed Riemannian manifolds (M, g_j, p_j) converges in the Cheeger–Gromov sense to a pointed smooth limit manifold $(M_\infty, g_\infty, p_\infty)$.
First, observe that we have the necessary control on the curvature: By the previous step we have uniform bounds on the curvature on bounded distance. The gradient estimates from Step 2 in the good region and the interior estimates for List flow, Theorem 6.15 in [59], allow us to upgrade this information to estimates on the covariant derivative of the curvature on bounded distance. If we combine this with the noncollapsing estimate for List flow, Theorem 7.2, we may apply Hamilton's compactness theorem for List flow (Theorem 8.2 in [59]) to obtain a converging subsequence.
Finally, by the Hamilton-Ivey type pinching estimate, Theorem 5.2 we deduce that the limit manifold has nonnegative sectional curvature.
Moreover, we have

$$|\mathrm{d}u|^2_{g_j}(p) = Q_j^{-1}|\mathrm{d}u|^2_g(p) \leq Q_j^{-1}c_0$$

for all $p \in M$. This implies that $|\mathrm{d}u_\infty|^2_{g_\infty} = 0$ and thus u_∞ is a constant function.

Step 5: Backwards extension in time:
The argument for the backwards extension works exactly as in the Ricci flow proof, since the function u is constant. That is we obtain a complete smooth solution to List flow on $(T^*, 0]$

for some $T^* \in [-\infty, 0)$, where the function u is constant along the solution. Moreover, the solution has bounded curvature and nonnegative sectional curvature.

Step 6: Construction of ancient solution:
The flow constructed above has $T^* = -\infty$. Again, since the function u is constant we may use the Harnack inequality for Ricci flow and monotonicity of the scalar curvature under the flow. The argument goes through as in the Ricci flow case.

In the previous steps we have constructed a sequence of Riemannian manifolds, which converges after rescaling in the Cheeger–Gromov sense to an ancient κ-solution. This is in contradiction to the existence of the spacetime points (p_j, t_j) and this finishes the proof.
If we combime the Canonical Neighbourhood Theorem, 7.14 with the description of $(\kappa, \underline{u}, \overline{u})$-solutions from Theorem 7.13, we obtain the following Corollary:

Corollary 7.15.
Suppose $\{g(t), u(t)\}_{t\in[0,T)}$ is a solution to List flow on $[0, T)$ on a closed three-manifold M in with initial data g_0 and u_0. Given $\epsilon > 0$, there exists $\hat{r} > 0$ with the following property: If $(p_0, t_0) \in M \times [0, T)$ is a point, such that $Q = \mathrm{R}(p_0, t_0) \geq \hat{r}^{-2}$, then there exists a neighbourhood B of the point p, such that

$$B_{g(t_0)}\left(p_0, (2C_1)^{-1}Q^{-\frac{1}{2}}\right) \subset B \subset B_{g(t_0)}\left(p_0, 2C_1 Q^{-\frac{1}{2}}\right)$$

and

$$(2C_2)^{-1}Q \leq \mathrm{R}(p, t_0) \leq 2C_2 \,\mathrm{R}\, Q \text{ for all } p \in B.$$

Moreover, B is one of the following

- *B is contained in the final time slice of a parabolic $(2\epsilon, c)$-neck centered at (p_0, t_0),*
- *B is a 2ϵ-cap,*
- *B is a closed manifold diffeomorphic to $\mathbb{S}^3 \diagup \Gamma$.*

Here $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ are the constants from the Structure Theorem on $(\kappa, \underline{u}, \overline{u})$-solutions, see Theorem 7.13.
Finally, we have the derivative estimate

$$|\nabla \mathrm{R}\,| \leq 2\eta \,\mathrm{R}^{\frac{3}{2}} \text{ and } |\nabla^2 \mathrm{R}\,| \leq 2\eta \,\mathrm{R}^2$$

at the point $(p_0, t_0) \in M \times [0, T)$, where η denotes an universal constant.

8. The surgery construction

In this section we use the Canonical Neighbourhood Theorem established in the previous section to understand the high curvature regions at the singular time and to identify the regions, where we want to perform surgery. While this follows the same way as in Perelman's work on Ricci flow, see for example Section 67 in work of B. Kleiner and J. Lott [56] for an detailed exposition, the surgery is more involved, since we have to control the energy density of the smooth function u.

8.1. **The structure of the solution at the first singular time.**

Suppose a smooth solution to List flow is defined on $[0, T)$ and goes singular as $t \to T$. By the Hamilton–Ivey estimate, Theorem 5.2 we deduce that $\sup_{p\in M} \mathrm{R}(p,t) \to \infty$ as $t \to T$. We define following G. Perelman (cf. Section 3 of [75]) the set

$$\Omega = \left\{ p \in M \,\middle|\, \limsup_{t\to T} \mathrm{R}(p,t) < \infty \right\}.$$

This set is open and we distinguish two cases: In the first case we have $\Omega = \emptyset$: Then the scalar curvature has to be uniformly high and every point is contained in a canonical neighbourhood, that is every point admits a neighbourhood, which is either an $(2\epsilon, c)$-neck, an 2ϵ-cap; or a closed manifold diffeomorphic to $\mathbb{S}^3 \diagup \Gamma$.

In the second case we have $\Omega \neq \emptyset$. Then the metrics $g(t)$ converge on Ω by the interior estimates for List flow, Theorem 3.11, to a smooth metric $g(T)$. Choose a small positive number $\rho < r/2$ with the property that the Canonical Neighbourhood Theorem holds at every point $p \in \Omega$ with $\mathrm{R}(p,T) \geq 4\rho^{-2}$. Following Perelman we define $\Omega_\rho \subset \Omega$ by

$$\Omega_\rho = \left\{ p \in \Omega | \, \mathrm{R}(p,T) \leq \rho^{-2} \right\}.$$

Then every point in the part $\Omega \backslash \Omega_\rho$ with high, but finite curvature, lies by the Structure Theorem for Canonical Neighbourhoods, Corollary 7.15, in one of the following alternatives:

- on a 2ϵ-tube with boundary components in Ω_ρ,
- on a 2ϵ-cap with boundary in Ω_ρ,
- on a 2ϵ-horn with boundary in Ω_ρ,
- on a doubled 2ϵ-horn,
- on a capped 2ϵ-horn,
- on a closed manifold diffeomorphic to $\mathbb{S}^3 \backslash \Gamma$.

As in G. Perelman's work we leave unchanged

- all 2ϵ-tubes with boundary components in Ω_ρ,
- all 2ϵ-caps with boundary in Ω_ρ,

We discard

- all doubled 2ϵ-horns,
- all capped 2ϵ-horns,
- all closed manifold diffeomorphic to $\mathbb{S}^3 \backslash \Gamma$.

We perform surgery on each 2ϵ-horn with boundary in Ω_ρ.

Let us describe the surgery algorithm in more detail:
In the following we will explain how to perform surgery on a parabolic (δ, c)-neck of radius h, which sits inside an ϵ-horn. The existence of such an parabolic (δ, c)-neck will be established later, see Proposition 9.9. We know from work of R. Hamilton that a final time slice of a parabolic (δ, c)-neck has a nice parametrization by the standard cylinder, see Section 3 of [41]. In the following z denotes the coordinate along the axis of such an final timeslice of a (δ, c)-neck. We may assume without loss of generality that the end of the horn, where the curvature becomes unbounded is contained in the right half cylinder $\{z \geq 0\}$.

R. Hamilton introduced the surgery procedure in Section 4 of [41]. The idea is to first bend the neck a little bit inwards by a conformal follow, then interpolate between the neck metric and a standard metric and finally cap off the resulting metric. The conformal factor for this deformation is given by

$$f_{C,D}(z) = \begin{cases} 0 & z \leq 0, \\ C\exp(-\frac{D}{z}) & z \in (0,3], \\ \text{ is strictly convex on} & z \in [3, 3.9], \\ -\frac{1}{2}\log(16 - z^2) & z \in [3.9, 4) \end{cases}.$$

The metric is deformed in the surgery procedure as follow:

$$\tilde{g} = \begin{cases} \bar{g} & z \leq 0, \\ \exp(-2f_{C,D})\bar{g} & z \in [0,2], \\ \varphi\exp(-2f_{C,D})\bar{g} + (1-\varphi)\exp(-2f_{C,D})h^2 g_0 & z \in [2,3] \\ h^2\exp(-2f_{C,D})g_0 & z \in [3,4] \end{cases}$$

where h denotes the radius of the neck in which we perform surgery and φ is a suitable cutoff function.
We deform the function u towards a constant function in the surgery region. This is intricated, since we want to preserve both the bounds on the function u and the bound on the energy density. The deformations of the function u take place in the region $[-L, 0]$, where the metric is not changed.
We deform the function u by

$$\tilde{u}(z) = \begin{cases} u(z) & z \in [-L, -2 - z_0] \\ F(u(z)) & z \in [-2 - z_0, -1] \\ u_0 & z \in [-1, 4], \end{cases}$$

where u_0 is a constant to be defined below.
The deformation $F(u)$ depends on the size of the gradient in the parabolic δ-neck compared to the smooth bound at that time.

If the gradient in the parabolic δ-neck is small compared to the a-priori bound on the gradient, then we deform by

$$\tilde{u}(z) = \begin{cases} u(z) & z \in [-2 - z_0, -2] \\ \varphi u(z) + (1-\varphi)u_0 & z \in [-2, -1] \\ u_0 & z \in [-1, 4], \end{cases} \tag{30}$$

in the region $z \in [-L, -1]$ for a suitable cutoff-function φ. The constant u_0 is defined by

$$u_0 = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2 \times \{z=-2\}} u \,\mathrm{d}\mu_g$$

and $\mathbb{S}^2 \times \{z = -2\}$ is the two-sphere sitting at $z = -2$ in the parametrization of the neck (cf. Chapter 3 of [41]).
If the gradient in the parabolic δ-neck is comparable to the a-priori bound on the gradient, then we deform by

$$\tilde{u}(z) = \begin{cases} u(z) - \eta(z) & z \in [-2 - z_0, -2] \\ \varphi(u(z) - \eta(-z)) + (1-\varphi)u_0 & z \in [-2, -1] \\ u_0 & z \in [-1, 4], \end{cases} \tag{31}$$

in the region $z \in [-L, -1]$ for a suitable cutoff function φ and a deformation function $\eta : [-2 - z_0, -1] \to \mathbb{R}$, which is explained in Proposition 8.5. In this case the constant u_0 is defined by

$$u_0 = \frac{1}{|\mathbb{S}^2|} \int_{\mathbb{S}^2 \times \{z=-2\}} u \,\mathrm{d}\mu_g - \eta(z).$$

8.2. **Preserving a-priori estimates on curvature and energy density.**

To allow for repeated surgeries it is necessary to preserve the initial assumptions on curvature:

Proposition 8.1.
A surgery in a δ-neck described above preserves the a-priori estimates on the curvature, that is the Hamilton–Ivey estimate described in Theorem 5.2.

Proof.
We argue separately for the curvature and for the function u.
The change of curvature under a δ-cutoff surgery was described by R. Hamilton in Section 4.2 of [41] for four-manifolds of positive isotropic curvature. In section 4.3 of the same work he showed how to preserve the pinching estimates through surgery. G. Perelman remarked in section 4.4 of [75] that the preservation of the pinching through surgery is also possible in thre-dimensional Ricci flow, this was worked out for example by H.-D. Cao and X.-P. Zhu in Lemma 7.3.4 of [25]. Since we have the identical pinching function and since we perform identical geometric changes on the neck, there is nothing new to show.

□

In the rest of this subsection we want to establish that the a-priori bounds on the function u and on the energy density $|\nabla u|^2$ are preserved:

Theorem 8.2.
In the surgery procedure the a-priori estimates on the function u, the energy density $|\nabla u|^2$ and the Hessian $|\nabla^2 u|^2$ are preserved.
To be more precise: The estimate

$$\inf_{p\in M} u(p,0) \leq u \leq \sup_{p\in M} u(p,0)$$

for the function u, the estimate

$$|\nabla u|^2 \leq \frac{c_0}{1+2\alpha c_0 t}$$

for the energy density and the estimate

$$|\nabla^2 u|^2(t,p) \leq 2s_0 + C\alpha c_0^2 + Cc_0 \,\mathrm{R}(t,p) + Cc_0 \,\mathrm{R}_0^-,$$

for the Hessian from Proposition 4.1 are preserved in the surgery procedure.

The proof of this theorem is given in several steps:

(1) Analysis of the heat equation on a family of shrinking cylinders,
(2) Asymptotics for the function u on a parabolic δ-neck (see Proposition 8.3),
(3) Analysis of changes under the surgery (see Proposition 8.4 and Proposition 8.5).

The analysis in the first two steps is inspired by recent work of S. Brendle and K. Choi on uniqueness of noncompact convex ancient solutions to mean curvature flow in three dimensions [19] and work of S. Brendle on noncompact ancient solutions to Ricci flow in three dimensions [16]. In both works a similar mode decomposition is used to deduce special asymptotics for the solutions.

We consider the shrinking cylinder $\mathbb{S}^2 \times \mathbb{R}$ on $(-\infty, 0)$ with metric

$$g(t) = (-2t)g_{\mathbb{S}^2} + \mathrm{d}z \otimes \mathrm{d}z,$$

such that $g(t)$ has scalar curvature 2 at $t = -1$. We consider a solution u of the heat equation

$$\partial_t u(p,t) = \Delta_{g(t)} u(p,t)$$

on the round cylinder $\mathbb{S}^2 \times \mathbb{R}$, which is defined for $t \in [-L/2, -1]$ and $z \in [-L/3, -L/3]$ with $L >> 1$. We assume the bound $|\nabla u|^2_{g(t)} \leq L^2$ in this region.
We have the coordinates (θ, z) on $\mathbb{S}^2 \times \mathbb{R}$ and we use separation of variables: We denote by $Y_{l,m} : \mathbb{S}^2 \to \mathbb{R}$ with $l \geq 0$ and $-l \leq m \leq l$ denote the spherical harmonics with eigenvalues

$$-\Delta_{\mathbb{S}^2, g_{\mathbb{S}^2}} Y_{l,m} = l(l+1)Y_{l,m}.$$

We have for every $t \in (-\infty, 0)$ the expansion

$$u(\theta, z, t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{m,l}(z,t) Y_{l,m}(\theta).$$

The Laplace–Beltrami operator decomposes as

$$\Delta_{g(t)} = \partial_z^2 + \frac{1}{(-2t)}\Delta_{\mathbb{S}^2}$$

and this implies for the modes the parabolic equation

$$\begin{aligned}\partial_t u_{l,m}(z,t) &= \partial_z^2 u_{l,m}(z,t) + \frac{1}{(-2t)}\Delta_{\mathbb{S}^2} u_{l,m}(z,t)\\ &= \partial_z^2 u_{l,m}(z,t) - \frac{l(l+1)}{(-2t)} u_{l,m}.\end{aligned}$$

We remark that the bound $|\nabla u|^2_{g(t)} \leq L^2$ gives the bounds

$$|\partial_z u| \leq CL \text{ and } |\nabla^{\mathbb{S}^2} u|_{\mathbb{S}^2} \leq CL\sqrt{(-t)}. \tag{32}$$

We consider the cases $l = 0$ and $l \geq 1$ separately. For $l \geq 1$ we introduce the rescaled modes $v_{l,m}$ by

$$v_{l,m}(z,t) = (-t)^{-\frac{l(l+1)}{2}} u_{m,l}.$$

The rescaled modes satisfy the parabolic equation

$$\partial_t v_{l,m}(z,t) = \partial_z^2 v_{l,m}(z,t).$$

Then equation (32) implies that these modes are bounded by

$$|v_{l,m}|(z,t) \leq CL^2(-t)^{-\frac{l(l+1)}{2}+\frac{1}{2}},$$

it is important to observe that the exponent of $(-t)$ is negative for $l \geq 1$.
By using the representation formula for the one-dimensional heat equation as on page 58 of [16] we deduce

$$\left|\sum_{l=1}^{\infty}\sum_{m=-l}^{l} u_{m,l}(z,t)Y_{l,m}(\theta)\right| \leq CL^{-1/2}.$$

in the region $z \in [-100, 100]$ for the time interval $t \in [-2, -1]$.
For the case $m = 0$ we have to argue differently: The mode $u_{0,0}$ satisfies

$$\partial_t u_{0,0}(z,t) = \partial_z^2 u_{0,0}(z,t).$$

We recall the bound

$$|u_{0,0}|(z,t) \leq CL^2\sqrt{(-t)}.$$

Then the second derivatives $w = \partial_z^2 u_{0,0}(z,t)$ satisfy the parabolic equation

$$\partial_t w = \partial_z^2 w.$$

By parabolic interior estimates we deduce

$$|w(t,z)|(z,t) \leq CL^2\frac{1}{\sqrt{-t}}.$$

By using the representation formula for the one-dimensional heat equation as on page 58 of [16] we deduce

$$|w(z,t)| \le CL^{-1/2}.$$

in the region $z \in [-100, 100]$ for the time interval $t \in [-2, -1]$. This implies that that there exist $a, b \in \mathbb{R}$, such that

$$|u_{0,0}(z,t) - a - bz| \le CL^{-1/2}.$$

in the region $z \in [-100, 100]$ for the time interval $t \in [-2, -1]$.
By combining the above two estimates we deduce

$$|u((z,\theta),t) - a - bz| \le CL^{-1/2}.$$

Since we perform surgery on the final time slice of a parabolic δ-neck we deduce the following consequence:

Proposition 8.3.
Suppose we have a parabolic δ-neck. Suppose its final time slice is rescaled to scalar curvature one. Then there exist constants $a, b \in \mathbb{R}$, such that the function u satisfies in the region $z \in [-100, 100]$ the estimate

$$|u(z,\theta) - a - bz| \le C\delta.$$

Proof.
We observe: the angular modes decay on the family of shrinking cylinders by the previous computations, and the second derivatives of u in z-direction are small. Thus on the cylinder we have the claimed relation. Since we are δ-close to the family of shrinking cylinders, this introduces another small error. $\square$

For notational simplicity we set $d_0 = \frac{c_0}{1+2\alpha c_0 T}$ at the surgery time T.
Observe that by the smooth bound on the energy density we have $b^2 \le (1+10\delta)d_0$. We distinguish the cases of small gradient given by $b^2 \le \frac{1}{1+10\delta}d_0$ and the case of large gradient given by $\frac{1}{1+10\delta}d_0 \le b^2 \le (1+10\delta)d_0$.

Proposition 8.4.
If we have $b^2 \le \frac{1}{1+10\delta}d_0$, then we perform the deformation (30) and the a-priori bounds are preserved through surgery.

Proof.
In the region $z \in [-2-z_0, -2]$ the function u is unchanged and hence the bounds on u and the bounds on the energy density are preserved.
In the region $z \in [-1, 4]$ the function $\tilde{u}$ is constant. Thus the bound on the energy density is clearly preserved; the lower und upper bound on u is preserved, since the constant u_0 is defined by a mean value and the integrand satisfies the bounds.
The bound on the function u in the region $z \in [-2, -1]$ is preserved, since the test function φ satisfies $0 \le \varphi(z) \le 1$ for all $z \in [-2, -1]$ and since u and u_0 satisfy the bound.

It is left to bound the gradient in the region $z \in [-2,-1]$. We observe

$$\nabla \tilde{u} = (u - u_0)\nabla\varphi + \varphi\nabla u$$

and hence

$$|\nabla \tilde{u}|^2 = |u-u_0|^2|\nabla\varphi|^2 + 2(u-u_0)\varphi\langle\nabla\varphi,\nabla u\rangle + |\varphi|^2|\nabla u|^2.$$

We define $\varphi : [-2,-1] \to [0,1]$ by

$$\varphi(z) = -(z+1),$$

where z denotes the height coordinate on the neck $\mathbb{S}^2 \times I$. We have $\varphi(-2) = 1$ and $\varphi(-1) = 0$. Since we are on a δ-neck the metric is close to the round metric and we have $|\nabla\varphi| = 1+\delta$.
By the mode decomposition result we observe that

$$\begin{aligned}|u(z,\theta)-u_0| &\le |u(z,\theta)-u(-2,\theta)| + |u(-2,\theta)-u_0| \\ &\le (|b|(z+2)+\delta)+\delta = |b|(z+2)+2\delta.\end{aligned}$$

This implies

$$\begin{aligned}|\nabla\tilde{u}|^2 &\le |u-u_0|^2|\varphi|^2 + 2|u-u_0||\varphi||\nabla\varphi||\nabla u| + |\varphi|^2|\nabla u|^2 \\ &\le (|b|(z+2)+2\delta)^2(1+\delta)^2 + 2\left(|b|(z+2)+2\delta\right)(z+1)(1+\delta)(|b|+\delta) + (z+1)^2(|b|+\delta)^2 \\ &\le \left(b^2(z+2)^2+4\delta\right)(1+\delta)^3 + 2(1+\delta)^2(z+1)\left(b^2(z+2)+C\delta\right) + (1+\delta)(z+1)^2(b^2+\delta) \\ &\le (1+\delta)^3\left(b^2\left[(z+2)^2+2(z+1)(z+2)+(z+1)^2\right]+C\delta\right] \\ &\le (1+\delta)^3[(2z+3)^2b^2+C\delta].\end{aligned}$$

We conclude that

$$|\nabla\tilde{u}|^2 \le (1+5\delta)b^2$$

since $(2z+3)^2 \le 1$ for $z \in [-2,-1]$. Hence the bound on the gradient is preserved in this region, since the above error is offset by our assumption.
In the next step we have to discuss the Hessian:
We observe in the interpolation region

$$\nabla^2\tilde{u} = (u-u_0)\nabla^2\varphi + \varphi\nabla^2 u + \nabla u \otimes \nabla\varphi + \nabla\varphi \otimes \nabla u.$$

This implies

$$
\begin{aligned}
|\nabla^2\tilde{u}|^2 =&|u-u_0|^2|\nabla^2\varphi|^2+\varphi^2|\nabla^2 u|^2+2|\nabla u|^2|\nabla\varphi|^2+2\varphi(u-u_0)\langle\nabla^2 u,\nabla^2\varphi\rangle\\
&+(u-u_0)\langle\nabla^2\varphi,\nabla u\otimes\nabla\varphi+\nabla\varphi\otimes\nabla u\rangle+\varphi\langle\nabla^2 u,\nabla u\otimes\nabla\varphi+\nabla\varphi\otimes\nabla u\rangle\\
\le&|u-u_0|^2|\nabla^2\varphi|^2+\varphi^2|\nabla^2 u|^2+2|\nabla u|^2|\nabla\varphi|^2+2|\varphi||u-u_0||\nabla^2\varphi||\nabla^2 u|\\
&+2|u-u_0||\nabla^2\varphi||\nabla u||\nabla\varphi|+2|\varphi||\nabla^2 u||\nabla\varphi||\nabla u|\\
\le&2\delta(b^2(z+2)^2+\delta)+(z+1)^2|\nabla^2 u|^2+2(1+\delta)(b^2+\delta)\\
&+2\delta(1+z)(|b|(z+2)+\delta)|\nabla^2 u|+2\delta(|b|(z+2)+\delta)(|b|+\delta)(1+\delta)\\
&+2(1+z)(1+\delta)(|b|(z+2)+\delta)|\nabla^2 u|\\
\le&(1+z)^2|\nabla^2 u|^2+(2+\delta)(1+z)(1+\delta)(|b|(z+2)+\delta)|\nabla^2 u|\\
&+2(1+\delta)(b^2+\delta)+4\delta.
\end{aligned}
$$

However, from the mode decomposition we deduce $|\nabla^2 u|\le\delta$ in this region. Hence

$$
|\nabla^2\tilde{u}|^2\le C_1\delta+C_2(b+\delta)+3(b^2+\delta).
$$

However, this is smaller than the bound we need to preserve. Indeed, the bound we want to preserve contains terms proportional to c_0^2.
This finishes the proof of the proposition and the case of small gradient. □

Hence it is left to treat the case of large gradient:

Proposition 8.5.
If we have $\frac{1}{1+10\delta}d_0\le b^2\le(1+10\delta)d_0$, then we perform the deformation (31) and the a-priori bounds are preserved through surgery.

Proof.
In the region $z\in[-2-z_0,-2]$ we define for $w_0=2+z_0$

$$
\tilde{u}(z,\theta)=u(z,\theta)-b\exp\left(-\frac{D}{z-w_0}\right)
$$

where the constant D is to be determined. We compute and rearrange

$$
\begin{aligned}
|\nabla\tilde{u}|^2 =&|\nabla u|^2+\left|\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right|^2-2\left\langle\nabla u,\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right\rangle\\
=&|\nabla u|^2+\left|\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right|^2\\
&-2\left\langle(\nabla u-a-b(z+w_0)),\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right\rangle\\
&-2b\left\langle\nabla z,\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right\rangle.
\end{aligned}
$$

We estimate

$$
\begin{aligned}
|\nabla u|^2 &\leq d_0,\\
\left|\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right|^2 &\leq b^2\frac{D^2}{(z+w_0)^4}\exp\left(-\frac{2D}{z+w_0}\right),\\
2\left|\left\langle(\nabla u - a - b(z+w_0)),\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right\rangle\right| &\leq C\delta\frac{D}{(z+w_0)^2}\exp\left(-\frac{D}{z+w_0}\right),\\
2b\left\langle\nabla z,\nabla\left(b\exp\left(-\frac{D}{z+w_0}\right)\right)\right\rangle &\geq \frac{3}{2}b^2\frac{D}{(z+w_0)^2}\exp\left(-\frac{D}{z+w_0}\right).
\end{aligned}
$$

This implies in the region $[-w_0,-2]$ the estimate

$$
|\nabla\tilde{u}|^2 \leq d_0 - \frac{D}{(z+w_0)^2}\exp\left(-\frac{D}{z+w_0}\right)\left(b^2 - \frac{D}{(z+w_0)^2}\exp\left(-\frac{D}{z+w_0}\right)\right).
$$

We may choose $D = D(\delta)$ and z_0 such that the term in the brackets is sufficently positive, such that we obtain the estimate

$$
|\nabla\tilde{u}|^2(z=-2) \leq d_0\frac{1}{1+10\delta}.
$$

Hence the a-priori assumption on the gradient is preserved in the region $z\in[-2-z_0,-2]$. In the interpolation region $z\in[-2,-1]$ there is a small increase in the energy density, but the factor $\frac{1}{1+10\delta}$, which we gained by the deformation is enough to offset this error.
The argument to show that the bound on the Hessian is preserved is similar to the proof of the previous proposition and we omit it.
The a-priori assumptions in the region $z\in[-1,4]$ are again satisfied, since the function u_0 is constant in this region and defined by a mean value of a function, which satisfies the bounds on u.

□

8.3. **The standard solution.**
In the previous subsection we explained how to deform the metric by adding a geometric cap on one side of the δ-neck. Moreover, we observe that in the cap region the function u is constant.
If one restarts the flow after surgery, then the standard solution models the evolution of the surgery cap:
Consider for $t<0$ the family $(\mathbb{S}^2\times\mathbb{R},g(t))$ of shrinking cylinders, where the metric $g(t)$ is given by

$$
g(t) = \frac{1}{2}(1-2t)g_{\mathbb{S}^2} + \mathrm{d}z\otimes\mathrm{d}z
$$

such that the scalar curvature is normalized to be

$$
\mathrm{R}_{g(t)} = \frac{1}{1-2t}.
$$

Moreover, suppose that $u(t)=u_0$ for some constant $u_0\in\mathbb{R}$ (and hence the solution is really a solution to Ricci flow). This family models a parabolic neck in List flow. The standard solution is obtained as follows: At the singular time $t = 0$ one performs surgery by first removing a half-cylinder and then gluing in a cap in the form of a rotationally symmetric metric with positive curvature (as in the previous subsection). The function u is extended by u_0 to the cap. The resulting manifold is isometric to $(\mathbb{R}^3, g(0), u(0))$, where the metric $g(0)$ is rotationally symmetric and $u(0)$ is the constant u_0. The standard solution is obtained by evolving the metric $g(0)$ by List flow. Since the function u is constant, the standard solution obtained in this way coincides with the standard solution constructed by G. Perelman in Section 3 of [75]. There he proved the following properties of the standard solution:

Theorem 8.6 (Properties of the standard solution, G. Perelman, Section 3 of [75]).
For each $c \in \mathbb{R}$ there exists a complete solution $(\mathbb{R}^3, g(t), u(t))_{t\in[0,1)}$ to List flow with the following properties:

1. *The function $u(t)$ is constant and we have $u(t) = c$ for all $t \in [0, 1)$.*
2. *The initial manifold $(\mathbb{R}^3, g(0))$ is isometric to the standard cylinder with scalar curvature 1 outside of a compact set $K \subset \mathbb{R}^3$. The compact set K is isometric to the cap used in the surgery procedure in the previous subsection.*
3. *The metric $(\mathbb{R}^3, g(t))$ is rotationally symmetric for each $t \in [0, 1)$.*
4. *The manifold $(\mathbb{R}^3, g(t))$ is asymptotic to a cylinder with scalar curvature $\frac{1}{1-2t}$ at infinity.*
5. *The scalar curvature of the solution is bounded below at time $t \in [0, 1)$ by th expression $\frac{1}{K_{Standard}(1-2t)}$, where $K_{Standard}$ denotes a universal constant.*
6. *The manifold $(\mathbb{R}^3, g(t))$ has nonnegative sectional curvature for each $t \in [0, 1)$.*
7. *The flow $(\mathbb{R}^n, g(t), u(t))$ is κ-noncollapsed for some universal constant $\kappa > 0$.*
8. *The scalar curvature at different points is controlled in the following sense: There exists a function $\omega : [0, \infty) \to (0, \infty)$, such that we have for all $t \in [0, 1)$ and all $p, q \in \mathbb{R}^3$ the estimate*
$$\mathrm{R}(p, t) \leq \mathrm{R}(q, t)\omega(\mathrm{R}(q, t)d^2_{g(t)}(p, q)).$$

The standard solution also satisfies a canonical neighbourhood property for late times:

Proposition 8.7 (cf. G. Perelman, [75]).
For a given small constant $\hat{\epsilon} > 0$ and a given large constant A_0, there exists a time $\alpha \in [0, 1)$ with the following property: If $(p_0, t_0) \in \mathbb{R}^3 \times [0, 1)$ with $t_0 \in [\alpha, 1)$, then the parabolic neighbourhood
$$\mathcal{P}\left(p_0, t_0, A_0 \,\mathrm{R}(p_0, t_0)^{-\frac{1}{2}}, -A_0 \,\mathrm{R}(p_0, t_0)^{-1}\right)$$
in the standard solution is after scaling by the factor $\mathrm{R}(p_0, t_0)$ and shifting $\hat{\epsilon}$-close to a corresponding subset of a noncompact $(\kappa, \underline{u}, \overline{u})$-solution to List flow.

Proof.
Since the function u is constant along the standard solution, the proof for the standard solution of Ricci flow applies. For an exposition, see Theorem 9.2 in S. Brendle's work [18]. □

Finally, we have a similar description as in Theorem 7.13 for neighbourhoods in the standard solution to List flow:

Corollary 8.8.
For a given $\epsilon > 0$, there exist positive constants $C_1 = C_1(\hat{\epsilon})$ and $C_2 = C_2(\hat{\epsilon})$ with the following property:
For each space-time point (p_0, t_0) on the standard solution, there exists a neighbourhood B of the point p_0, such that we have for all $p \in B$ the relations

$$B_{g(t_0)}(p_0, C_1^{-1}\,\mathrm{R}(p_0,t_0)^{-\frac{1}{2}}) \subset B \subset B_{g(t_0)}(p_0, C_1\,\mathrm{R}(p_0,t_0)^{-\frac{1}{2}})$$

and

$$C_2^{-1}\,\mathrm{R}(p_0,t_0) \leq \mathrm{R}(p,t_0) \leq C_2\,\mathrm{R}(p_0,t_0).$$

Moreover, the neighbourhood B satisfies one of the following conditions:

- *B is contained in the final time slice of parabolic ϵ-neck with center at (p_0, t_0). At sufficently late times, that is $t_0 \leq \frac{1}{\mathrm{R}(p_0,t_0)}$, the neighbourhood B is disjoint from the surgery cap, which was glued in at time $t = 0$.*
- *B is an ϵ-cap.*

Moreover, we have at every point (p,t) the gradient estimates

$$|\nabla \mathrm{R}|(p,t) \leq \eta\,\mathrm{R}^{\frac{3}{2}}(p,t) \text{ and } |\nabla^2 \mathrm{R}|(p,t) \leq \eta\,\mathrm{R}^2(p,t).$$

Proof.
There are two cases:
Either t_0 is very close to 1. Then any region in the standard solution is close to a $(\kappa, \underline{u}, \overline{u})$-solution by Proposition 8.7 and the statement follows from the Structure Theorem for $(\kappa, \underline{u}, \overline{u})$-solutions, see Theorem 7.13.
Else if t_0 is bounded away from 1, then the claim follows since the standard solution is asymptotic to the round cylinder. ∎

9. Construction of List flow with surgery

In this section we define List flow with surgery — the definition is analog to the notion of Ricci flow with surgery. The main goal of this section is to show that List flow with surgery exists for any initial metric g_0 and any initial smooth function u_0 on a closed connected oriented three-manifold.
The main difficulty in the existence proof for List flow with surgery is to show that the solution is noncollapsed in the sense of Definition 7.1 in the presence of surgeries.
We follow the arguments in Section 4 and 5 of G. Perelman's work [75] and explain the necessary modifications compared to Ricci flow. For an exposition of the arguments see also Section 10 of S. Brendle's work [18] for and section 68 to 80 in work of B. Kleiner and J. Lott [56].

9.1. **Setup.**

We fix a closed connected oriented three-manifold with a Riemannian metric g_0 and a smooth function u_0. By rescaling we may assume without loss of generality that $|\operatorname{Rm}_{g_0}|_{g_0} \leq 1$ and $c_0 \leq 1$. We fix the constants $\underline{u}$ and $\overline{u}$ by requiring

$$\underline{u} \leq \inf_{p\in M} u_0(p) \text{ and } \overline{u} \geq \sup_{p\in M} u_0(p).$$

Furthermore, we fix some constants:
There is an universal constant $\kappa_0 > 0$, such that all $(\kappa, \underline{u}, \overline{u})$-solutions are κ_0-noncollapsed or are quotients of the round sphere $\mathbb{S}^3$ or noncompact quotients of the cylinder $\mathbb{S}^2 \times \mathbb{R}$, see Theorem 7.5.
Then there exists $\eta > 0$, such that the gradient estimates

$$|\nabla \mathrm{R}|(x,t) \leq \eta \,\mathrm{R}(x,t)^{\frac{3}{2}} \text{ and } |\nabla^2 \mathrm{R}|(x,t) \leq \eta \,\mathrm{R}(x,t)^2.$$

hold for all κ-solutions (by Theorem 7.4) and for the standard solution (by Corollary 8.8).
The second constant we fix is the accuracy parameter $\epsilon > 0$.
Moreover, we fix the constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$, such that the conclusions of Theorem 7.13 and Corollary 8.8 for the local description of κ-solutions and the standard solution hold.
We define List flow with surgery as follows:

Definition 9.1 (List flow with surgery)**.**
A List flow with surgery on the time interval $[0,\infty)$ with initial data (M, g_0, u_0) given by a oriented, connected, closed three-manifold M, a Riemannian metric g_0 on M and a smooth function u_0 on M consists of the following data:

- *A partition of the time interval $[0,\infty)$ into a countably infinite union of disjoint intervals $[t_l^-, t_l^+)$ with $l \in \mathbb{N}_0$. In other words,*

$$[0,\infty) = \bigcup_{l\in\mathbb{N}_0} [t_l^-, t_l^+) \text{ such that } t_0^- = 0, t_L^+ = T \text{ and } t_l^- = t_{l-1}^+.$$

  We call the times t_l^- for $l \geq 1$ singular times.

- *For each $l \in \mathbb{N}$, there is a smooth oriented closed manifold M_l (which is possibly disconnected with finitely many connected components, possibly empty) and a smooth solution $(g_l(t), u_l(t))$ of List flow for $t \in [t_l^-, t_l^+)$ on M_l.*
- *For $l = 0$ we have that $(M_0, g_0(0))$ is isometric to (M, g_0) and $u_0(0) = u_0$.*
- *The solutions $(g_l(t), u_l(t))_{t\in[t_l^-,t_l^+)}$ go singular as $t \to t_l^+$ and we set*

$$\Omega_l = \left\{ p \in M_l \,\middle|\, \limsup_{t\to t_l^+} \mathrm{R}(p,t) < \infty \right\}.$$

- *There exist $\epsilon > 0$, called the accuracy parameter, and nonincreasing functions $r : [0,\infty) \to (0,\infty)$, called the curvature scale, $\delta : [0,\infty) \to (0,\infty)$, called the neck accuracy, and $h : [0,\infty) \to (0,\infty)$, called the neck radius, such that we have the estimates $\delta(t) \le \epsilon$, and $h(t) \le \delta(t)r(t)$.*
- *The manifold $(M_l, g_l(t_l^-))$ is obtained from $(\Omega_{l-1}, g_{l-1}(t_{l-1}^+))$ by performing surgery on finitely many $\delta(t_{l-1}^-)$-necks, discarding all double 4ϵ-horns, all capped 4ϵ-horns, and removing all connected components diffeomorphic to $\mathbb{S}^3\backslash\Gamma$.*

Additionally one requires that the following conditions are satisfied:

- *The Canonical Neighbourhood property holds with accuracy 4ϵ on all curvature scales less than $r(t)$: Suppose (p_0, t_0) has scalar curvature $\mathrm{R}(p_0,t_0) \ge r^{-2}(t)$. Then there exists a neighbourhood B of p_0, such that*

  $$B_{g(t_0)}(p_0, (8C_1)^{-1}) \subset B \subset B_{g(t_0)}(p_0, 8C_1)$$

  and for all $p \in B$ we have

  $$(8C_2)^{-1}\,\mathrm{R}(p_0,t_0) \le \mathrm{R}(p,t_0) \le 8C_2\,\mathrm{R}(p_0,t_0).$$

  Moreover, B is either contained in a final time slice of a parabolic 4ϵ-neck centered at (p_0,t_0) or in a 4ϵ-cap.
- *For any spacetime point (p_0,t_0) with $\mathrm{R}(p,t) \ge r(t)^{-2}$ we have the gradient estimates*

  $$|\nabla \mathrm{R}| \le 4\eta\,\mathrm{R}^{\frac{3}{2}} \text{ and } |\nabla^2 \mathrm{R}| \le 4\eta\,\mathrm{R}^2.$$

- *In any surgery on a $\delta(t)$-neck, we find a point (p_0,t_0) at the center of the neck with scalar curvature $\mathrm{R}(p_0,t_0) = h^{-2}(t_0)$ and the parabolic neighbourhood*

  $$\mathcal{P}(p_0, t_0, \rho = \delta^{-1}(t)h(t), \tau = -\delta^{-1}(t)h^2(t))$$

  is free of surgeries.

Remark 9.2.
If we have that the solution extincts in finite time, for example if $\mathrm{S} > 0$ initially, or if M is a homotopy three-sphere, see Section 11, then we may replace the countably infinite collection of time intervals by a finite collection of time intervals, and one may choose the curvature scale r, the neck accuracy δ and the neck radius h independent of time.

Remark 9.3.
The reason we cannot take δ and r independent of time in general comes from the proof of the noncollapsing through surgeries (compare Theorem 9.7), where the constant depends on the time elapsed. If there is a finite time of existence, then the estimates in Theorem 9.7 are uniform in time.

To establish the existence of List flow with surgery we need to justify the a-priori assumptions, in particular the Canonical Neighbourhood Theorem in the presence of surgeries and preserve all a-priori assumptions in the presence of surgeries. For Ricci flow this was established in Sections 4 and 5 in G. Perelman's work [75].
We first record a consequence of the previous section:

Proposition 9.4.
Suppose $(M, g(t))$ is a List flow with surgery parameters $(\epsilon, r(t), \delta(t), h(t))$. Then the flow satisfies the Hamilton-Ivey pinching estimate from Theorem 5.2; the bound $\underline{u} \leq u \leq \overline{u}$ on the function u; and the bound energy density $|\nabla u|^2$ from Corollary 3.10.

Proof.
We deduce the first claim from Proposition 8.1 and the second and third claim from Proposition 8.2: Indeed, we know by the definition of List flow with surgery, Definition 9.1, that we can find in any surgery on a parabolic δ-neck a point (p_0, t_0) at the center of the neck with scalar curvature $\mathrm{R}(p_0, t_0) = h^{-2}(t_0)$, such that the parabolic neighbourhood $\mathcal{P}(p_0, t_0, \delta^{-1}(t)h(t), -\delta^{-1}(t)h^2(t))$ is free of surgeries. □

The other parts of the construction are more involved. Let us assume for the moment that the List flow with surgery exists for a finite time interval and has surgery parameters (ϵ, r, δ, h). Let us explain the argument:

(1) The List flow with surgery satisfies by definition the Canonical Neighbourhood property with accuracy 4ϵ on all curvature scales less than r.
(2) With help of the Canonical Neighbourhood assumption one controls the noncollapsing in the presence of surgeries (cf. Theorem 9.7): For fixed $\epsilon > 0$ there exists κ and a function $\bar{\delta}(\cdot) : (0, \infty) \to (0, \infty)$ such that the List flow with surgery with parameters (ϵ, r, δ, h) is κ-noncollapsed on all scales less than ϵ, whenever $\delta \leq \bar{\delta}(r)$.
(3) In the next step one shows that the Canonical Neighbourhood assumption for a List flow with surgery parameters (ϵ, r, δ, h) is satisfied with better accuracy 2ϵ on larger scales $2\hat{r}$ (compare Theorem 9.8) provided we fix the surgery parameters r and δ. The proof is similar to the proof of the Canonical Neighbourhood Theorem (cf. Theorem 7.14): One uses the accuracy ϵ from Step 1 to deduce gradient estimates and Step 2 to enforce the κ-noncollapsing.
(4) In the fourth step (compare Proposition 9.9) we find for given $\epsilon, \hat{r}$ and $\hat{\delta}$ a neck radius h, such that we find in any 4ϵ-horn of the List flow with surgery with parameters $(\epsilon, \hat{r}, \hat{\delta}, h)$ a nice parabolic δ-neck.
(5) The previous steps assure the following:
We start the flow from the initial data, up to the first singular time the flow satisfies the Canonical Neighbourhood assumption with accuracy 2ϵ on scales less than $2\hat{r}$.

We perform surgery at the first singular time at necks with curvature level h^{-2}, which exist by Step 4. The restarted flow satisfies after the first singular time up to the second singular time the Canonical Neighbourhood assumption again with accuracy 2ϵ at scales less than $2\hat{r}$ by Step 3. At the second time we again find necks with curvature level h^{-2} (by Step 4) at which we perform surgery. Then we can repeat this procedure. Furthermore, the volume is reduced in any surgery by an definite amount proportional to h^3 and hence there is no accumulation of surgery times.

9.2. **Noncollapsing in the presense of surgeries.**
In smooth Ricci and List flow there are two ways to show noncollapsing along the flow: One way is to use the monotonicity of the $\mathcal{W}$-functional, the other way is to use the reduced volume. For Ricci flow with surgery G. Perelman used a modified version of the reduced volume to show noncollapsing in the presence of surgeries. To the best of the authors knowledge a proof of noncollapsing in the presence of surgeries using the $\mathcal{W}$-functional is not known.

Definition 9.5.
Curves in space-time, which stay in the region unaffected by surgery, are called admissible. Curves on the boundary of the set of all-admissable curves are called barely admissible.

The $\mathcal{L}$-length of a curve $\gamma : [t_0 - \tau, t_0] \to M$ on a solution to List flow was defined by R. Buzano in Chapter 6 of his thesis [69]. It is defined by

$$\mathcal{L}(\gamma) = \int_{t_0-\tau}^{t_0} \sqrt{t_0 - \tau} \left(\mathrm{S}(\gamma(t), t) + |\dot{\gamma}(t)|^2 \right) \, \mathrm{d}t. \tag{33}$$

Moreover, let us introduce the $\mathcal{L}_+$-length of a curve $\gamma : [t_0 - \tau, t_0] \to M$ as in Section 5 of G. Perelman's work [73]:

$$\mathcal{L}_+(\gamma) = \int_{t_0-\tau}^{t_0} \sqrt{t_0 - \tau} \left(\mathrm{S}_+(\gamma(t), t) + |\dot{\gamma}(t)|^2 \right) \, \mathrm{d}t. \tag{34}$$

where the positive part S_+ of the modified scalar curvature S is given by $\mathrm{S}_+(p,t) = \max\{\mathrm{S}(p,t), 0\}$.
For the proof of the noncollapsing through surgeries we need to rule out that the l-distance is realised by such a barely admissable curve, thus we show that such curves have large length.

Proposition 9.6 (Barely admissible curves are long, cf. G. Perelman, Lemma 5.3)**.**
Fix the parameters ϵ, r, L. Then there exists a real number $\bar{\delta} = \bar{\delta}(r, L) > 0$ with the following property: Suppose we have a List flow with surgery with parameters (ϵ, r, δ, h), where $\delta \le \bar{\delta}$. Given a space time point (x_0, t_0) with $\mathrm{R}(x_0, t_0) \le r^{-2}$, a surgery time $T_0 < t_0$ and a barely admissible curve $\gamma : [T_0, t_0] \to M \times \mathbb{R}$ with $\gamma(t_0) = x_0$ for which $\gamma(T_0)$ lies on the boundary of a surgical cap at time T_0 we have the estimate

$$\mathcal{L}_+(\gamma) = \int_{T_0}^{t_0} \sqrt{t_0 - t} \left(\mathrm{S}_+(t, \gamma(t)) + |\dot{\gamma}(t)|^2 \right) \, \mathrm{d}t \ge L.$$

Proof.
If one replaces the l-distance for Ricci flow by the l-distance for List flow (which was developed by R. Buzano in Chapter 5 of this thesis [69]) the Ricci flow proof goes through. For an exposition see Lemma 10.12 of S. Brendle's work [18]. □

Theorem 9.7 (Noncollapsing of the flow is preserved through surgery, cf. G. Perelman [75], Lemma 5.2)**.**
Fix a small constant $\epsilon > 0$. Then there exists a positive number κ and a positive function $\tilde{\delta}(\cdot)$ with the following property: Suppose we have a List flow with surgery on $[0, T)$ with parameters ϵ, r, δ, h, where the accuracy δ satisfies $\delta \leq \tilde{\delta}(r)$. Then the List flow with surgery is κ-noncollapsed on all scales less than ϵ.

Proof.
For the proof we fix a space-time point (p_0, t_0) and a small radius $r_0 \leq \epsilon$, such that $\mathrm{R}(p_0, t_0) \leq r_0^{-2}$ for all points $(p, t) \in \mathcal{P}(p_0, t_0, r_0, -r_0^2)$ for which the flow is defined. We distinguish three cases:

(1) The curvature at (p_0, t_0) is large, that is $\mathrm{R}(p_0, t_0) \geq r^{-2}$, where r denotes the scale from the Canonical Neighbourhood assumption in the definition of a List flow with surgery with parameters (ϵ, r, δ, h). Then the Canonical Neighbourhood assumption implies closeness to a uniformly noncollapsed solution.
(2) The curvature at (p_0, t_0) is small, that is $\mathrm{R}(p_0, t_0) < r^{-2}$, and the parabolic neighbourhood $\mathcal{P}(p_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4})$ does contain surgeries. Then there is a point (p, t) in the parabolic neighbourhood $\mathcal{P}(p_0, t_0, r_0/2, -r_0^2/4)$, which lies on a surgery cap. However, surgery caps are noncollapsed and this implies a lower bound on the volume.
(3) The curvature at (p_0, t_0) is small, that is $\mathrm{R}(p_0, t_0) < r^{-2}$, and the parabolic neighbourhood $\mathcal{P}(p_0, t_0, \frac{r_0}{2}, -\frac{r_0^2}{4})$ is free of surgeries. If the time t_0 is bounded above by some time T, then by Proposition 9.6 any barely admissible curve γ has length $\mathcal{L}_+ \geq 24\sqrt{t_0}$ and hence the reduced distance of every barely admissible curve l is greather than 6.
The next step of the argument is as follows: For $t < t_0$ we denote by $l(p, t)$ the reduced distance from (p_0, t_0) in List flow, this is the infimum of the reduced length for admissible curves from (p, t) to (p_0, t_0). The claim is that $\inf_p l(p, t) \leq 3/2$ for all $t < t_0$. We know by work of R. Buzano, see Chapter 5 of [69], that if $l(p, t) < 6$, then the reduced length is attained by a strictly admissble curve. In List flow the reduced distance satisfies the PDE
$$\partial_t l \geq \Delta l + \frac{1}{t_0 - t}\left(l - \frac{3}{2}\right)$$
whenever $l(p, t) < 6$. Then the maximum principle implies the claim. The rest of the argument using the reduced volume now follows as in Ricci flow. For an exposition of the detailed argument, see Proposition 10.9 of S. Brendle's work [18]. □

9.3. **Finalizing the surgery parameters.**
With the preparations at hand one may establish the Canonical Neighbourhood Theorem in the presence of surgeries:

Theorem 9.8 (Canonical Neighbourhood Theorem through surgeries; cf. G. Perelman [75], Section 5)**.**
Fix a small constant $\epsilon > 0$*. Then there exist positive numbers* $\tilde{r}$ *and* $\tilde{\delta}$ *with the following property: Suppose we have a List flow with surgery with parameters* $(\epsilon, \tilde{r}, \tilde{\delta}, h)$ *defined on the time interval* $[0, T)$*. Then the flow satisfies the Canonical Neighbourhood property from Theorem 7.14 with accuracy* 2ϵ *on all scales less than* $2\tilde{r}$*.*

Proof.
The proof is essentially a reproof of the Canonical Neighbourhood Theorem respecting the presence of surgeries. For an exposition of the Ricci flow proof see S. Brendle [18], Theorem 10.10. □

Finally, one needs to establish that one can always find a parabolic δ-neck in a 4ϵ-horn:

Proposition 9.9 (Finalization of neck radius for List flow with surgery; cf. G. Perelman [75], Lemma 4.3)**.**
Given an accuracy ϵ*, a curvature threshold* $\hat{r}$ *and a neck fineness* $\hat{\delta}$ *there exists a neck radius* $\hat{h} \in (0, \hat{\delta}\hat{r})$ *with the following property:*
Suppose we have a List flow with surgery with parameters $(\epsilon, \hat{r}, \hat{\delta}, \hat{\varepsilon})$ *defined on a time interval* $[0, T)$*, which goes singular as* $t \to T$*. Suppose* x *is a point, which lies in an* 4ϵ*-horn* $(\Omega, g(T))$ *at curvature* $\mathrm{R}(x, T) = h^{-2}$*. Then the parabolic neighbourhood*

$$P(x, T, \hat{\delta}^{-1}h, -\hat{\delta}^{-1}h^2)$$

is free of surgeries. Moreover, this parabolic neighbourhood is a δ*-neck.*

Proof.
Since the Ricci flow proof uses rescaling arguments, geometric properties of manifolds and the construction of κ-solutions it goes through with minimal modifications. For an exposition, see Proposition 11.1 in S. Brendle's work [18]. □

The above arguments assumed that there is finite time of extinction T, such that the surgery parameters can be chosen independent of time. For the general case one proceeds as in Proposition 5.1 and Lemma 5.2 of G. Perelman's work [75] (the point of Lemma 5.2 is to localize the argument of the proof of noncollapsing through surgeries (cf. Theorem 9.7) in time).
Combining all the above results we deduce:

Theorem 9.10 (Existence of List flow with surgery)**.**
Suppose M^3 *is a closed smooth three-manifold,* g_0 *a smooth Riemannian metric and* u_0 *a smooth function. Then there exists a List flow with surgery on* M^3 *with* $g(0) = g_0$ *and* $u(0) = u_0$*.*

10. Finite time extinction for List flow

In this section we will show finite extinction time of List's flow on a large class of closed three-manifolds. For Ricci flow this was first shown by G. Perelman [74] by using regularized curve shortening flow and later by T. Colding and W. Minicozzi by using minimal immersions [28], see also the survey article [29]. We adapt the latter approach. In our computation there is an unfavourable term, which we barely control and this term leads to a logarithmic divergence for the width.

Suppose $(g(t), u(t))_{t\in[0,T)}$ is a smooth solution to List flow on a closed three-manifold M^3. We may assume without loss of generality $\mathrm{S}_0 < 0$ and $c_0 > 0$. Indeed, if $c_0 = 0$ the result follows from the finite time extinction result for Ricci flow; if $\mathrm{S}_0 > 0$ finite time extinction follows from Proposition 3.8 (and the observation that S is nondecreasing under surgery, because the scalar curvature is nondecreasing and the energy density $|\nabla u|^2$ is nonincreasing) and if $\mathrm{S}_0 = 0$, then either $\mathrm{S} > 0$ immediately or $\mathrm{Sc} = 0$ and $\Delta_{g,\gamma} u = 0$ and hence the manifold is Ricci flat and u is constant.

Moreover, we denote by $(\Sigma_t)_{t\in[0,T)}$ a family of branched minimal immersions of the two-sphere $\mathbb{S}^2$ into M^3. We denote the area element of Σ_t by $\mathrm{d}\mu$.

From the evolution of the volume element of the Riemannian metric $g(t)$ under List flow, see Proposition 3.3 we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{d}\mu = -\frac{1}{2}\,\mathrm{tr}_\Sigma(2\,\mathrm{Sc})\;\mathrm{d}\mu = -\left(\mathrm{Sc}(e_1,e_1) + \mathrm{Sc}(e_2,e_2)\right)\;\mathrm{d}\mu = -\left(\mathrm{S} - \mathrm{Sc}(\nu,\nu)\right)\;\mathrm{d}\mu$$

where $\{e_1, e_2\}$ denotes an orthonormal frame on Σ and ν the unit normal of Σ in the ambient manifold M. This implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{area}(\Sigma) &= -\int_\Sigma \left(\mathrm{S} - \mathrm{Sc}(\nu,\nu)\right)\;\mathrm{d}\mu \\
&= -\int_\Sigma \left(\mathrm{R} - \mathrm{Rc}(\nu,\nu)\right)\;\mathrm{d}\mu + \alpha\int_\Sigma \left(|\nabla u|^2 - (\nabla u)(\nu)^2\right)\;\mathrm{d}\mu \\
&\leq -\int_\Sigma \left(\mathrm{R} - \mathrm{Rc}(\nu,\nu)\right)\;\mathrm{d}\mu + \alpha\int_\Sigma |\nabla u|^2\;\mathrm{d}\mu.
\end{aligned}$$

Since Σ_t is a minimal immersion we deduce by the Gauss equation

$$-\,\mathrm{Rc}(\nu,\nu) = \kappa_\Sigma - \frac{1}{2}\left(\mathrm{R} - |h|^2\right),$$

where κ_Σ denotes the Gauss curvature of Σ and h the scalar-valued second fundamental form of $\Sigma \hookrightarrow M$. This implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{area}(\Sigma) &\leq -\int_\Sigma \kappa_\Sigma - \frac{1}{2}\int_\Sigma \mathrm{R}\;\mathrm{d}\mu - \int_\Sigma |h|^2\;\mathrm{d}\mu + \alpha\int_\Sigma |\nabla u|^2\;\mathrm{d}\mu \\
&\leq -\int_\Sigma \kappa_\Sigma - \frac{1}{2}\int_\Sigma \mathrm{R}\;\mathrm{d}\mu + \alpha\int_\Sigma |\nabla u|^2\;\mathrm{d}\mu
\end{aligned}$$

Moreover, we have by the Theorem of Gauss–Bonnet with branch points

$$\int_\Sigma \kappa_\Sigma = 4\pi + \sum_{i=1}^N b_i \geq 4\pi$$

where $\{p_i\}_{i=1}^N$ are the branch points of branching order b_i. Since $b_i \geq 0$ the term involving the branching order only helps in our computation. Hence we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{area}(\Sigma) \leq -4\pi - \frac{1}{2}\int_\Sigma \mathrm{R}\ \mathrm{d}\mu + \alpha \int_\Sigma |\nabla u|^2\ \mathrm{d}\mu.$$

We recombine $\mathrm{S} = \mathrm{R} - \alpha|\nabla u|^2$ to arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{area}(\Sigma) \leq -4\pi - \frac{1}{2}\int_\Sigma \mathrm{S}\ \mathrm{d}\mu + \frac{1}{2}\alpha \int_\Sigma |\nabla u|^2\ \mathrm{d}\mu.$$

The integrands are bounded in time by the parabolic maximum principle, see Proposition 3.8 and 3.9:

$$\mathrm{S} \geq -\frac{3}{2}\frac{1}{t - \frac{3}{2\,\mathrm{S}_0}} \text{ and } |\nabla u|^2 \leq \frac{1}{2\alpha}\frac{1}{t + \frac{1}{2\alpha c_0}}$$

With the choice $C = \min\left\{-\frac{3}{2}\frac{1}{\mathrm{S}_0}, \frac{1}{2\alpha c_0}\right\}$ we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{area}(\Sigma) \leq -4\pi + \frac{1}{t + C}\operatorname{area}\Sigma.$$

We remark that the constant in front of the second term may not be any worse for the following argument.

This implies for the width in the sense of forward difference quotients

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) \leq -4\pi + \frac{1}{t+C}W(t).$$

By integrating this differential inequality from $t = 0$ to $t = T$ one obtains

$$W(T)(T+C)^{-1} \leq W(0)C^{-1} - 4\pi\left(\log(T+C) - \log C\right).$$

Since $\log(T + C) \to \infty$ as $T \to \infty$ the right-hand side becomes negative for T large and hence this inequality implies $W(T) < 0$ for T large. However, since the width is positive in our setting, we deduce that the flow extincts in finite time.

The technical details (construction of the minimal immersions and the validity of the computation through the surgery) follow as in the survey article [29] of T. Colding and W. Minicozzi. Hence we have established:

Theorem 10.1.
Suppose M^3 is a closed orientable three-manifold, whose prime decomposition has only non-aspherical factors. Then for any initial metric g_0 and any smooth function u_0 on M^3 the solution $(g(t), u(t))_{t\in[0,\infty)}$ to List flow with surgery becomes extinct in finite time.

Corollary 10.2.
On any closed orientable simply-connected three-manifold (which is diffeomorphic to the sphere $\mathbb{S}^3$ *by the resolution of the Poincaré conjecture) the solution* $(g(t), u(t))_{t\in[0,\infty)}$ *to List flow with surgery extincts in finite time.*

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