The Valuation of Option Contracts subject to Counterparty Risk

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1 Introduction

Derivatives play an increasingly important role as hedging and investment instruments for both financial and non-financial corporations. Especially the trading volume in over-the-counter (OTC) derivatives has experienced a tremendous increase over the last decades, since these contracts can be designed to meet the investors' specific needs. Between 2000 and 2017 alone, the notional amount of outstanding OTC derivatives contracts increased from $94 trillion to $542 trillion according to the Bank for International Settlements.[1] The global financial crisis of 2007–2009 and the bankruptcy of Lehman Brothers Holdings Inc. drew attention to OTC markets, since the majority of the derivatives involved in the emergence of this financial turmoil were traded in OTC markets.

As a result of the global financial crisis, the credit risk of OTC derivatives became a more important issue in finance industry. In contrast to exchange traded markets, OTC markets lack the advantage of a central clearing house ensuring that the counterparties fulfill their obligations. The risk that the promised payments are not made is called counterparty or default risk. Derivatives subject to counterparty risk are called vulnerable derivatives. Since the counterparty risk cannot be ignored, it should be considered in the valuation of OTC derivatives.

This dissertation addresses the valuation of European and American options which are traded on OTC markets. Both European and American options exhibit unilateral counterparty risk, since these contracts constitute an obligation only for the option writer. For vulnerable European options, the valuation models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) prevail in the literature. Based on an extended Black-Scholes world, they use the structural approach of Merton (1974) to price European options subject to counterparty risk. In the following, we combine these models in a general model which incorprates their key characteristics. Moreover, we extend the mentioned models to a stochastic interest rate framework. In addition, we set up valuation models for vulnerable American options using the core ideas of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011).

[1] The detailed statistics on OTC markets are found in Bank of International Settlements (2018) or can be retrieved from the BIS Statistics Explorer provided on the website of the Bank for International Settlements.

The remainder of this dissertation is organized as follows: In Chapter 2, we give an overview of the existing literature on European and American options subject to counterparty risk. Chapter 3 deals with the valuation of vulnerable European options in an extended Black-Scholes world. In particular, the models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) are presented and discussed. Moreover, we develop a general model which includes the previously mentioned models as special cases. Despite the complexity of the general model, an approximate closed form valuation formula is derived. Chapter 4 addresses the valuation of European options subject to both counterparty and interest rate risk. The risk-free interest rate is governed by the Ornstein-Uhlenbeck process suggested by Vasicek (1977). In particular, we extend the valuation models presented in the previous chapter to the considered stochastic interest rate framework and derive the corresponding closed form valuation formulas. Furthermore, we set up again a general model which incoporates the fundamental features of the other models. Despite the general model's complexity, an approximate closed form valuation formula is derived. Chapter 5 is devoted to the valuation of vulnerable American options. We pick up on the fundamental ideas of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) to analyze the properties of the corresponding American options subject to counterparty risk. Furthermore, we set up a general model. The option values are computed using the least squares Monte Carlo simulation approach suggested by Longstaff and Schwartz (2001). Chapter 6 addresses the valuation of American options subject to counterparty and interest rate risk. The risk-free interest rate follows the Ornstein-Uhlenbeck process of Vasicek (1977). Based on this framework, we extend the models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) to be applicable for the valuation of vulnerable American options under stochastic interest rates. Again, we set up a general model which incorporates the features of the other models. The option values are computed using the least squares Monte Carlo simulation approach suggested by Longstaff and Schwartz (2001). Chapter 7 concludes the dissertation and indicates further research fields.

2 Review on Options subject to Counterparty Risk

Counterparty risk is included under the concept of credit risks and constitutes a phenomenon which may occur in over-the-counter (OTC) markets. In general terms, counterparty risk is defined as the risk that a business partner in an OTC derivative transaction is not able to (fully) meet its contractual obligations (see Bielecki & Rutkowski, 2002: 26–27). Depending on the type of the considered OTC derivative, counterparty risk can be unilateral (e.g. option contracts) or bilateral (e.g. futures contracts or swaps). In the context of European and American options, counterparty risk is clearly unilateral, since only the option holder faces the risk that a contractual payment will not be made. In particular, there is only the risk that the option writer (i.e. the counterparty) may not be able to fulfill the option holder's claim if the option is exercised. Options which are subject to counterparty risk are typically refered to as vulnerable options.

2.1 Modelling the Counterparty's Default

Before dealing with the valuation of vulnerable European and American options, we discuss the modelling of the counterparty's default risk. Essentially, two major theoretical approaches have been emerged in the literature to account for the potential default of the counterparty: structural models[2] and intensity models[3]. In the following, the key features of these two approaches will be presented and discussed.

2.1.1 Structural Models

The fundamental idea of the structural default models is based on the seminal work of Merton (1974).[4] Under the assumption of a constant risk-free interest rate, Merton (1974) links the counterparty's default explicitly to its ability to pay back its outstanding liabilities. In particular, the default is triggered if the market value

[2] A profound examination of structural models can be found in Bielecki and Rutkowski (2002: 32–120) and Brigo et al. (2013: 47–65).
[3] Bielecki and Rutkowski (2002: 221–264) as well as Brigo et al. (2013: 65–86) provide a comprehensive analysis of the intensity models.
[4] The structural model of Merton (1974) was originally developed to value zero and coupon bonds subject to credit risk. However, its main ideal can be easily extended and applied to any financial security that faces default risk.

of the counterparty's assets is below the default boundary L_T at the end of the considered time period T (=maturity), i.e. default can only occur at one specific point in time. In the original work of Merton (1974), the default boundary $L_T = \bar{L}$ is a constant which is equal to the counterparty's nominal debt. However, the default boundary L_t can also be a determinstic and time-dependent or a random variable (see Johnson & Stulz, 1987; Hull & White, 1995).

In a first step, we must address the mathematical modelling of the counterparty's assets. In principle, any stochastic process can be used to describe the evolution of the counterparty's assets over time. Typically, it is assumed that the market value of the counterparty's assets follows a continuous-time geometric Brownian motion. The dynamics are given by

$$dV_t = \mu_V \, V_t \, dt + \sigma_V \, V_t \, dW_V \, , \tag{2.1}$$

where μ_V gives the expected instantaneous return of the counterparty's assets, σ_V is the instantaneous return volatility of the counterparty's assets and dW_V represents the standard Wiener process.

Since Merton (1974) assumes that the counterparty's default may occur only at one specific point in time (typically at the maturity of the outstanding liabilities), the default condition is given by

$$V_T < L_T, \tag{2.2}$$

i.e. the default is triggered if the counterparty's assets at time T are below the default boundary L_T.

The future payoff of any financial security F_t subject to default risk depends on whether the counterparty actually is bankrupt or not. Discounting this payoff yields today's price of the considered financial security. In general terms, it is given by

$$F_t = e^{-r(T-t)} \left((1-p) \cdot \mathbb{E}\left[PO_T^{\text{NoDef}} \,|\, V_T \geq L_T \right] + p \cdot \mathbb{E}\left[PO_T^{\text{Def}} \,|\, V_T < L_T \right] \right), \tag{2.3}$$

where p gives the counterparty's default probability and $\mathbb{E}[\,\cdot\,]$ denotes the expectation under the risk-neutral measure regarding the payoff at time T. In particular, $\mathbb{E}\left[PO_T^{\text{NoDef}}\right]$ expresses the expected payoff if the counterparty does not default, whereas $\mathbb{E}\left[PO_T^{\text{Def}}\right]$ gives the expected payoff in case of default.

The original model of Merton (1974) can be easily extended to a stochastic interest rate framework (e.g. Shimko et al., 1993). In this case, the price of the financial security F_t is given by

$$F_t = B_{t,T} \left((1-p) \cdot \mathbb{E}\left[PO_T^{\text{NoDef}} \,|\, V_T \geq L_T \right] + p \cdot \mathbb{E}\left[PO_T^{\text{Def}} \,|\, V_T < L_T \right] \right), \tag{2.4}$$

where $B_{t,T}$ denotes the discount factor of the considered stochastic interest rate framework.

Black and Cox (1976) extend the model of Merton (1974). It is still assumed that the risk-free interest rate is constant over time, but default may now occur at every future point in time. In particular, default is triggered as soon as the value of the counterparty's assets V_t falls below the default boundary L_t for the first time. Therefore, the Black-Cox model is also refered to as the first-time passage model.

Denoting the point in time at which the counterparty defaults by τ, the default condition is now given by

$$V_\tau < L_\tau \quad \text{with } \tau = \inf\{t \geq 0 : \, V_t < L_t\}. \tag{2.5}$$

The payoff of any financial security F_t subject to default risk depends on whether the counterparty actually is bankrupt at any point in time in the future. Discounting the future payoff yields today's price of the considered financial security. In general terms, it is given by

$$\begin{aligned} F_t = (1-p) \cdot e^{-r(T-t)} \cdot \mathbb{E}\left[PO_T^{\text{NoDef}} \,|\, V_T \geq L_T \right] \\ + p \cdot e^{-r(\tau-t)} \cdot \mathbb{E}\left[PO_\tau^{\text{Def}} \,|\, V_\tau < L_\tau \right] \end{aligned} \tag{2.6}$$

where p represents the counterparty's default probability and $\mathbb{E}[\cdot]$ denotes the expectation under the risk-neutral measure regarding the future payoff. In particular, $\mathbb{E}\left[PO_T^{\text{NoDef}}\right]$ denotes the expected payoff at time T if the counterparty does not default, whereas $\mathbb{E}\left[PO_\tau^{\text{Def}}\right]$ gives the expected payoff at the default time τ.

Longstaff and Schwartz (1995) extend the Black-Cox model to the stochastic interest rate framework of Vasicek (1977). In contrast to Black and Cox (1976), however, they assume that the default boundary is constant over time, i.e. $L_t = \bar{L}$. Briys and de Varenne (1997) as well as Schöbel (1999), in turn, extend the model of Longstaff and Schwartz (1995) by allowing the default boundary to change over time. Unlike Longstaff and Schwartz (1995), they are able to derive closed form solutions for the price of both zero and coupon bonds.

The approaches of Briys and de Varenne (1997) as well as of Schöbel (1999) cannot only be used to price zero or coupon bonds subject to credit risk but they can also be applied to price any vulnerable financial security F_t. Under the existence of stochastic interest rates, the current price of the considered financial security F_t is given by

$$\begin{aligned} F_t = (1-p) \cdot B_{t,T} \cdot \mathbb{E}\left[PO_T^{\text{NoDef}} \,|\, V_T \geq L_T\right] \\ + p \cdot B_{t,\tau} \cdot \mathbb{E}\left[PO_\tau^{\text{Def}} \,|\, V_\tau < L_\tau\right] \end{aligned} \tag{2.7}$$

where $B_{t,T}$ denotes the discount factor.

To value vulnerable European or American options using the structural approach, the payoffs PO_T^{NoDef} and PO_T^{Def} as well as the default barrier L_t in Equations (2.3) to (2.7) must be specified in accordance with the desired valuation model.

2.1.2 Intensity Models

In the intensity models, the counterparty's default is not linked to the value of the counterparty's assets or the counterparty's capital structure. Instead, the counterparty's default is described by an exogenous jump process. In particular, the time at which the counterparty defaults is given by the first jump time of a Poisson process with a deterministic or stochastic intensity.

Assuming a Poisson process to model the default risk, the probability that the counterparty defaults over the next dt instants, under the presumption that the default has not occured before time t, is equal to

$$\mathbb{P}\left(\tau \,\epsilon\, [t, t+dt] \,\middle|\, \mathcal{F}_t\right) = \lambda_t \, dt, \tag{2.8}$$

where λ_t is the time-dependent hazard rate and $\mathcal{F}_t$ is the information available at time t. The corresponding cumulated hazard rate is given by

$$\Lambda(t) = \int_0^t \lambda_u \, du. \tag{2.9}$$

In the context of vulnerable European and American options, the probability that the counterparty's default occurs within a given time period $[0,\, t]$ needs to be known. This probability is given by

$$\mathbb{P}\left(\tau \,\epsilon\, [0,\, t] \,\middle|\, \mathcal{F}_0\right) = 1 - e^{\int_0^t \lambda_u \, du}. \tag{2.10}$$

At this point it is important to note that the default event in intensity models is not triggered by a random variable whose behavior is observable in the market. When evaluating vulnerable European or American options based on the intensity model, it must be considered that the counterparty's default risk is typically independent of other stochastic variables (e.g. the price of the option's underlying) within the valuation model. This restriction is required to keep the model mathematically tractable. (Brigo et al., 2013: 65–66).

2.2 Review on European Options subject to Counterparty Risk

Over the last three decades, various valuation models for vulnerable European options have been developed. In the following, we give a comprehensive literature overview of the existing valuation models.

2.2.1 Models with Deterministic Interest Rates

Picking up on the ideas of Merton (1974), Johnson and Stulz (1987) model the effect of default risk on the value of European options. They assume that the short position in the option is the counterparty's sole liability and that the counterparty defaults if its asstes are not sufficient to meet the option holder's claim at maturity. Hence, default may be triggered either by a decline in the counterparty's assets or by an increase in the option value. In case of default, the option holder receives the entire assets of the counterparty potentially reduced by the cost of default. Johnson and Stulz (1987) also allow for the correlation between the counterparty's assets and the option's underlying. However, it is important to note that the Johnson-Stulz model is only suitable if the counterparty's assets are relatively small compared to the expected option payoff and if the counterparty's other liabilities are negligible.

Klein (1996), however, considers this assumption to be inappropriate in most situations and thus extends the Johnson-Stulz model by allowing for other liabilities which rank equally with the option. The counterparty's total liabilities are assumed to be exogenous and, by construction, must include the short position in the option. Since the structural model of Merton (1974) is used, default may only occur at the option's maturity. In particular, the counterparty is in default if its assets are less than the total liabilities. In this case, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. As in the Johnson-Stulz model, Klein (1996) accounts for the correlation between the counterparty's assets and the option's underlying. Based on these assumptions, the default risk can only arise from the potential deterioration of the counterparty's assets, since the total liabilities are fixed.

Klein and Inglis (2001) set up a model which incorporates the features of both Johnson and Stulz (1987) and Klein (1996). In particular, the counterparty's total liabilities are split into two components: the short position in the option (stochastic) and all other equally ranked liabilities (deterministic). Default occurs if the counterparty's assets are less than the sum of the option holder's claim and the market value of the other liabilities at the option's maturity. The payout ratio in default is linked to the counterparty's assets and the correlation between the counterparty's assets and the option's underlying is retained. In this model, default can be caused either by a decline in the counterparty's assets or an increase in the option value making the model applicable in various situations.

Liu and Liu (2011) extend the model of Klein (1996) by assuming that the counterparty's total liabilities are stochastic. Consequently, the counterparty is in default if the assets are not sufficient to meet the total liabilities at the option's maturity. In case of default, the option holder receives a proportion of his claim which depends on the market value of both the counterparty's assets and total liabilities. In this model, the default risk arises either from a decrease in the counterparty's assets or an increase of the counterparty's liabilities. Liu and Liu (2011) also account for all possible correlations between the random variables.

In contrast to the previously presented models, Hull and White (1995) use the structural approach of Black and Cox (1976) to account for the default risk. They assume that all the liabilities of the counterparty are of equal rank. Default occurs if the counterparty's assets fall below a determinsitic boundary prior to the option's maturity. In this case, the option holder receives an exogenously determined proportion of his claim. To keep the model tractable, Hull and White (1995) assume that the counterparty's default risk and the option's underlying are independent.

Rich (1996) assumes that the option's underlying as well as the counterparty's credit quality (e.g. the counterparty's assets) and the default boundary (e.g. the counterparty's liabilities) are characterized by geometric Brownian motions. The correlations between the three stochastic variables are also considered. Since the structural approach of Black and Cox (1976) is applied, the counterparty is in default if the stochastic variable describing the counterparty's credit quality falls below the default boundary for the first time. Rich (1996) assumes that the payout ratio of the option holder's claim in case of the counterparty's default is exogenously given. This assumption is necessary in order to keep the model mathematically tractable.

The model of Hui et al. (2003) extend the models of Hull and White (1995) and Klein (1996). They assume that the counterparty's total liabilities are time-dependent and are governed by the volatility of the counterparty's assets. The counterparty is in default if the market value of the assets falls below the market value of the total liabilities at any point in time prior or at the option's maturity. Furthermore, it is assumed that the option holder receives a exogenously given proportion of his claim if the counterparty defaults.

Hui et al. (2007) can be seen as an extension of Hui et al. (2003), since they assue that the counterparty's liabilities are governed by its own stochastic process. The counterparty is in default if the market value of the assets falls below the market value of the total liabilities at any point in time prior or at the option's maturity. To keep the model mathematically tractable, Hui et al. (2007) assume that payout ratio in case of the counterparty's default is exogenously specified in order to keep the model mathematically tractable.

Liang and Ren (2007) set up a valuation for vulnerable European options which can be seen as an extension of Johnson and Stulz (1987) and Hull and White (1995). In particular, they assume that the short position is the counterparty's only liability and that default occurs as soon as the value of the counterparty's assets falls below the intrinsic value of the option. Hence, default may occur also prior to the option's maturity. In contrast to other valuation models based on the Black-Cox approach, Liang and Ren (2007) assume that the payout ratio to the option holder in case of default is endogenously determined.

2.2.2 Stochastic Interest Rate Models

Klein and Inglis (1999) set up a valuation model for vulnerable European options under stochastic interest rates. In particular, they extend the model of Klein (1996) by assuming that the risk-free interest rate follows the Ornstein-Uhlenbeck process of Vasicek (1977). The counterparty's liabilities are ranked equally and are assumed to be constant. If the assets at the option's maturity are less than the total liabilities, the counterparty defaults and the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Furthermore, they account for correlations between all stochastic variables.

Yoon and Kim (2015) also extend the model of Klein (1996) to a stochastic interest rate framework. In particular, it is assumed that the risk-free interest rate is characterized by the model of Hull and White (1990). The counterparty's liabilities are ranked equally and are assumed to be fixed. Like in the original model, the counterparty's default may only occur at the option's maturity. In case of default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Moreover, the correlations between the counterparty's assets, the option's underlying the risk-free interest rate are considered.

Cao and Wei (2001) also deal with the valuation of vulnerable European options under stochastic interest rates. In particular, they assume that the risk-free interest rate is governed by the Ornstein-Uhlenbeck process suggested by Vasicek (1977). In contrast to Klein and Inglis (1999), however, it is assumed that the counterparty's liabilities consist of a zero bond and a short position in the option where both of them have different maturities. Furthermore, Cao and Wei (2001) assume that the counterparty may default prior to the option's maturity.

Liao and Huang (2005) also deal with the valuation of vulnerable European options under stochastic interest rates. In particular, they assume that the risk-free interest rate is governed by the Ornstein-Uhlenbeck process of Vasicek (1977). In contrast to Klein and Inglis (1999), Liao and Huang (2005) assume that the counterparty may also default prior to maturity. Additionally, the correlations between the counterparty's assets, the option's underlying and the interest rate are considered.

In contrast to the other valuation models, Kang and Kim (2005) use the intensity model to value European options subject to counterparty and interest rate risk. They assume that the risk-free interest rate follows the Ornstein-Uhlenbeck process suggested by Vasicek (1977). The counterparty's default is triggered by the first jump of a Poisson process, where the default intensity is assumed to be constant. In case of default, the recovery rate is exogenously given in order to keep the model mathematically tractable.

Su and Wang (2012) also deal with the valuation of European options subject to counterparty and interest rate risk using the intensity model. The risk-free interest rate is governed by the Ornstein-Uhlenbeck process suggested by Vasicek (1977) and the counterparty's default is triggered by the first jump of a Poisson process. In contrast to Kang and Kim (2005), however, the default intensity is assumed to be stochastic. In case of default, the payout ratio of the option holder's claim is exogenously specified.

Jarrow and Turnbull (1995) propose a third approach for the valuation of European options subject to counterparty and interest rate risk. Based on a foreign currency analogy in which the stochastic term structure of risk-free interest rates and the maturity-specific stochastic credit spreads are given, they use arbitrage-free valuation to compute the price of the vulnerable European options. Again, the payout ratio of the option holder's claim in case of default is assumed to be exogenously given.

2.2.3 Stochastic Volatility Models

Yang et al. (2014) extend the model of Klein (1996) to a stochastic volatility framework. In particular, it is assumed that only the return volatility of the option's underlying is stochastic being governed by an Ornstein-Uhlenbeck process. The counterparty's assets follow a geometric Brownian motion. Like in the orginal model of Klein (1996), the counterparty's liabilities are fixed and default may only occur at the option's maturity. In case of the counterparty's default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Furthermore, the mutual correlations between the counterparty's assets, the option's underlying the the risk-free interest rate are considered.

Following the main ideas of Klein (1996), Lee et al. (2016) set up a valuation model for vulnerable European options under the assumption of stochastic volatility. In particular, they assume that both the option's underlying and the counterparty's assets follow the dynamics suggested by Heston (1993). Like in the orginal model of Klein (1996), the counterparty's liabilities are fixed and default may only occur at the option's maturity. In case of the counterparty's default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Furthermore, the correlations between the counterparty's assets, the option's underlying the the risk-free interest rate are considered.

Wang et al. (2017) also extend the model of Klein (1996) to a stochastic volatility framework. In particular, they decompose the stochastic volatility into the long-term and short-term volatility. It is assumed that the short-term volatility is described by a mean reverting stochastic process, whereas the long-term volatility is assumed to be constant. Like in the orginal model of Klein (1996), the counterparty's liabilities are fixed and default may only occur at the option's maturity. In case of default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Furthermore, the correlations between the counterparty's assets, the option's underlying the the risk-free interest rate are considered.

Wang (2017a) sets up a valuation model for European options subject to counterparty risk in a stochastic volatility framework. The return volatility of both the option's underlying and the counterparty's assets are modeled by Generalized Autoregressive Conditional Heteroscedasticity processes, respectively. Furthermore, the correlation between the returns of the option's underlying and the counterparty's assets is assumed to be stochastic. Like in the model of Klein (1996), the level of the counterparty's liabilities is fixed and default may only occur at maturity. In case of default, the payout ratio is linked to the value of the counterparty's assets.

Using the intensity model, Wang (2017b) develops a valuation model for vulnerable European options in a stochastic volatility framework. The return volatility of the option's underlying is modeled by a Generalized Autoregressive Conditional Heteroscedasticity process. The counterparty's default is triggered by the first jump of a Poisson process, where the default intensity is assumed to be stochastic.

2.2.4 Jump-Diffusion Models

Xu et al. (2012) as well as Xu et al. (2016) extend the model of Klein (1996) by assuming that both the option's underlying and the counterparty's assets follow jump-diffusion processes, respectively. Like in the orginal valuation model of Klein (1996), the counterparty's liabilities are fixed and default may only occur at the option's maturity. In case of the counterparty's default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Furthermore, the correlation between the counterparty's assets and the option's underlying are considered.

Tian et al. (2014) also follow the ideas of Klein (1996) and provide a valuation model for vulnerable European options in which both the option's underlying and the counterparty's assets are governed by jump-diffusion processes, respectively. The authors account for the correlation between the two stochastic variables. In contrast to Xu et al. (2012, 2016), Tian et al. (2014) divide the jumps into an idiosyncratic and a systematic component for both stochastic variables. Like in the orginal model of Klein (1996), the counterparty's liabilities are fixed and default may only occur at the option's maturity. In case of the counterparty's default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets.

Wang (2016), in turn, extends the model of Liu and Liu (2011) by assuming that the option's underlying as well as the counterparty's assets and liabilities follow jump-diffusion processes. Wang (2016) also picks up on the idea of Wang et al (2014) and assumes that the jumps for all three stoachstic variables consist of an idiosyncratic and a systematic component. The counterparty is in default if the value of the counterparty's assets falls below the value of the counterparty's liabilities.

In contrast to the other valuation models, Fard (2015) uses the intensity model to deal with the valuation of vulnerable European options whose underlying follows a jump-diffusion model. In particular, the counterparty's default is triggered by the first jump of a Poisson process, where the default intensity is assumed to be stochastic. Aditionally, the correlations between the option's underlying and the counterparty's default risk is considered.

2.2.5 Incomplete Markets

Hung and Liu (2005) set up a valuation for vulnerable European options when the market is incomplete based on the structural approach of Merton (1974). They pick up on the idea of Klein (1996) assuming that default occurs if the value of the counterparty's assets are less than the fixed level of the counterparty's liabilities at the option's maturity. In contrast to Klein (1996), Hung and Liu (2005) assume that neither the option's underlying nor the counterparty's assets are traded in the financial market. Hence, closed form valuation formulas cannot be derived. Using the methodology of Cochrane and Saa-Requejo (2000), price bounds for vulnerable European options are computed under deterministic and stochastic interest rates.

Murgoci (2013) also deals with the valuation of European options subject to counterparty risk in an incomplete market based on the ideas of Klein (1996). In contrast to Hung and Liu (2005), Murgoci applies the methodology of Björk and Slinko (2006) to get the price bounds for vulnerable European options. As a result, she finds that her computed price bounds are tighter than those obtained by Hung and Liu (2005).

2.3 Review on American Options subject to Counterparty Risk

Compared to vulnerable European options, fewer models have been set up for American options subject to counterparty risk. In the following, an overview of the existing valuation models will be given.

2.3.1 Models with Deterministic Interest Rates

Hull and White (1995) use the structural approach of Black and Cox (1976) to model the effect of default risk on the value of American options which rank equally with the other liabilities of the counterparty. Default occurs if the counterparty's assets fall below a determinsitic boundary prior to the option's maturity. In this case, the option holder receives an exogenously determined proportion of his claim. To keep the model mathematically tractable, Hull and White (1995) assume that the counterparty's default risk and the price of the option's underlying are independent.

Chang and Hung (2006) adopt the framework of Klein (1996) to evaluate American options subject to counterparty risk. The option's underlying and counterparty's assets follow geometric Brownian motions, respectively. Furthermore, the correlation between the option's underlying and the counterparty's assets is considered. If the counterparty defaults prior to maturity, Chang and Hung (2005) assume that the American option is not necessarily exercised. Instead, the option holder has the opportunity to keep the American option unexercised until maturity despite the counterparty's default. The payout ratio in case of the counterparty's default is endogenously sp within the model.

Klein and Yang (2010) set up a valuation model for vulnerable American options based on the framework of Klein and Inglis (2001). The option's underlying and counterparty's assets follow geometric Brownian motions, respectively. The correlation between the option's underlying and the counterparty's assets is considered. In case of the counterparty's default, Klein and Yang (2010) assume that the American option is only exercised immediately if the option is in the money at that point in time. The payout ratio in case of the counterparty's default is linked to the value of the counterparty's assets.

Klein and Yang (2013) adopt the framework of Klein (1996) to evaluate American options subject to counterparty risk. The option's underlying and counterparty's assets follow geometric Brownian motions, respectively. Furthermore, the correlation between the option's underlying and the counterparty's assets is considered. If the counterparty defaults prior to maturity, Klein and Yang (2013) assume that the American option is only exercised immediately if the option is in the money at that point in time. In case of default, the payout ratio of the option holder's claim is exogenously specified.

2.3.2 Jump-Diffusion Models

Xu et al. (2012) adopt the framework of Klein (1996) to evaluate American options subject to counterparty risk. In contrast to Klein (1996), it is assumed that both the option's underlying and the counterparty's assets follow jump-diffusion processes, respectively. The counterparty's liabilities are fixed and default may only occur at the option's maturity. In case of the counterparty's default, the option holder receives a proportion of his claim which is linked to the value of the counterparty's assets. Furthermore, the correlation between the counterparty's assets and the option's underlying are considered.

2.4 Summary

The vast majority of the existing literature deals with the valuation of vulnerable European options. Predominantly, the counterparty's default is modeled using the structural approaches of Merton (1974), Black and Cox (1976) or an extended version of them, respectively. Intensity models, however, play a subordinate role. The overall literature on the valuation of American options subject to counterparty risk is relatively small. The existing models in the context of vulnerable American options use the structural approach of Black and Cox (1976) or an extended version to account for the counterparty's default.

In the following, the valuation of vulnerable European options will be based on the structural approach of Merton (1974). This approach is rather restrictive with respect to the default time, but it has a better mathematical tractability, i.e. closed form valuation formulas can be derived. Furthermore, the payout ratio in case of the counterparty's default can be endogenously determined within the considered valuation model. Using the approach of Black and Cox (1976), the greater flexibility with respect to the default time comes at the cost of an exogenously given payout ratio for the option holder's claim in case of the counterparty's default.

Due to the early exercise features of American options, we apply the structural approach of Black and Cox (1976) in this context. The higher mathematical complexity of the Black-Cox approach is not problematic, since we will price American options subject to counterparty risk by Monte Carlo simulation.

3 European Options subject to Counterparty Risk

In this chapter, we present and discuss different valuation models for European options subject to counterparty risk. The risk of the counterparty's default is modeled using the structural approach suggested by Merton (1974). In this context, the counterparty's default may occur only at the option's maturity and is triggered by the value of the counterparty's assets being below the value of the counterparty's total liabilities.

Based on this theoretical framework, Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) develop valuation models for vulnerable European options. These models differ only with respect to the characterization of the counterparty's total liabilities and therefore with respect to the condition under which the counterparty is in default.[5]

In the following, we set a general valuation model which incorporates all the features and characteristics of the previously mentioned models. Despite the general model's complexity, we derive an approximate closed form solution. Furthermore, we apply Monte Carlo simulation to price vulnerable European options based on the general model. Comparing the approximate closed form with the numerical solution shows that our valuation formula provides accurate values for vulnerable European options in most situations.

Section 3.1 outlines and discusses the assumptions of the considered theoretical framework. In Section 3.2, we derive the partial differential equation that characterizes the price of a European option subject to counterparty risk. Section 3.3 deals with the solution to this partial differential equation. In Section 3.4, the Klein, Klein-Inglis and Liu-Liu model are discussed. Moreover, we develop our general valuation model and derive the corresponding approximate closed form solution. Section 3.5 provides a comparative analysis of the different valuation models based on numerical examples. Section 3.6 gives a summary of the main findings.

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[5] Johnson and Stulz (1987) also set up a valuation model for vulnerable European options based on the theoretical framework considered in this chapter. However, this model is not included into the analysis, since the authors assume that the counterparty does not have any other liabilities beside the short position in the option. Due to this rather strict and unrealistic assumption, the Johnson-Stulz model is not very useful for practical applications.

3.1 Assumptions

The assumptions that characterize the theoretical framework for the valuation of European options subject to counterparty risk are based on Black and Scholes (1973), Merton (1974), Johnson and Stulz (1987), Klein (1996), Klein and Inglis (2001) as well as on Liu and Liu (2011).

1. The price of the option's underlying S_t follows a continuous-time geometric Brownian motion. Assuming that the option's underlying is a dividend-paying stock, its dynamics are given by

$$dS_t = (\mu_S - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S\,, \tag{3.1}$$

   where μ_S indicates the expected instantaneous return of the option's underlying, q denotes the continuous dividend yield, σ_S is the instantaneous return volatility and dW_S represents the standard Wiener process.

2. Likewise, the market value of the counterparty's assets V_t follows a continuous-time geometric Brownian motion. Its dynamics are given by

$$dV_t = \mu_V\, V_t\, dt + \sigma_V\, V_t\, dW_V\,, \tag{3.2}$$

   where μ_V is the expected instantaneous return of the counterparty's assets, σ_V gives the instantaneous return volatility and dW_V is a standard Wiener process. The instantaneous correlation between dW_S and dW_V equals ρ_{SV}.

3. The total liabilities D_t comprise all the obligations of the counterparty's, i.e. debt, short positions in financial securities and accruals. The dynamics follow a continuous-time geometric Brownian motion which is given by

$$dD_t = \mu_D\, D_t\, dt + \sigma_D\, D_t\, dW_D\,, \tag{3.3}$$

   where μ_D is the expected instantaneous return of the counterparty's liabilities, σ_D indicates the instantaneous return volatility and dW_D represents the standard Wiener process. The instantaneous correlation between dW_S and dW_D equals ρ_{SD} and ρ_{VD} between dW_V and dW_D, respectively.

4. The market is perfect and frictionless, i.e. it is free of transaction costs or taxes and the available securities are traded in continuous time.

5. The instantaneous risk-free interest rate r is assumed to be deterministic and constant over time.

6. The expected instantaneous return of the option's underlying as well as of the counterparty's assets and liabilities (μ_S, μ_V and μ_D) are deterministic and constant over time. The same applies for the dividend yield q of the option's underlying.

7. The instantaneous return volatilities of the option's underlying as well as of the counterparty's assets and liabilities (σ_S, σ_V and σ_D) are deterministic and constant over time. The instantaneous correlations ρ_{SV}, ρ_{SD} and ρ_{VD} are also constant and independent of time.

8. All the liabilities of the counterparty (i.e. debt, short positions in options, etc.) are assumed to be of equal rank.

9. Default can only occur at the option's maturity T. The counterparty is in default, if the counterparty's assets V_T are less than the threshold level L:

$$V_T < \bar{L} \quad \text{or} \quad V_T < L(S_T,\, D_T). \tag{3.4}$$

   Depending on the considered valuation model, the threshold level L is characterized in different ways and is either a constant or a function of the stochastic variables S_T and D_T.

10. If the counterparty is in default, the option holder's claim must be determined. In principle, the option holder's claim is equal to the intrinsic value of the European option at its maturity. If the counterparty, however, is in default, the option holder faces a percentage write-down ω on his claim. In this case, the option holder receives

$$(1-\omega)\,\max(S_T - K, 0) \qquad \text{or} \qquad (1-\omega)\,\max(K - S_T, 0) \tag{3.5}$$

   depending on whether the option is a call or a put.

The percentage write-down ω on the claim can be endogenized. Assuming that all the liabilities of the counterparty are ranked equally, the amount payable to the holder of a European call is given by

$$(1-\omega)\,\max\,(S_T-K,0)=\frac{(1-\alpha)\,V_T}{L(S_T,\,D_T)}\,\max\,(S_T-K,0), \tag{3.6}$$

whereas it is given by

$$(1-\omega)\,\max\,(K-S_T,0)=\frac{(1-\alpha)\,V_T}{L(S_T,\,D_T)}\,\max\,(K-S_T,0) \tag{3.7}$$

for the holder of a European put. The parameter α represents the cost of default as a percentage of the counterparty's assets and the ratio $V_T/L(S_T,\,D_T)$ gives the proportion of the option holder's claim which can be paid back.

Based on Assumptions 9 and 10, the counterparty can only default at the option's maturity which is in line with the valuation models of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011). Due to this assumption, the valuation models become mathematically tractable and analytical or approximate analytical solutions can be derived. On the other hand, however, this assumption might be criticized as being too restrictive and not taking into account the real-world circumstances of the default occurring prior to the option's maturity.

As pointed out by Klein and Inglis (2001), the assumption that default can only occur at the option's maturity is less restrictive as it initially seems due to the special treatment of OTC European options if the counterparty defaults. Most OTC European option contracts are concluded in compliance with the standards recommended by the International Swap and Derivatives Association (ISDA). In contrast to other financial instruments subject to counterparty risk, the option holder does not have to determine his claim associated with the considered OTC option immediately at the default date but has the right to wait until the maturity date is reached. Even if the option holder decides not to wait until the option's maturity to determine his claim, Assumptions 9 and 10 can still be valid. Based on the ISDA standardized contract for OTC European options, the option holder's claim at the counterparty's default is equal to the market value of the option at that point in time. This market value, in turn, depends on the expected option payoff at maturity. Another point in favor of assuming that default can only occur at option maturity is the fact that there is typically a time lag between the default event and the point in time, at which the counterparty's assets are distributed among all claim holders. Consequently, the option's maturity is a valid proxy for the date at which it is determined whether the counterparty is in default or not.

3.2 Derivation of the Partial Differential Equation

Following the argument of Hull (2012: 309–312), we derive the partial differential equation governing the price evolution of a vulnerable European option. In the considered theoretical framework (see Section 3.1), the price of a vulnerable European option F_t must be a function of the underlying S_t, the counterparty's assets V_t, the counterparty's liabilities D_t and time t. According to Itô's lemma, the price evolution of a vulnerable European option is given by the following stochastic differential equation:

$$\begin{aligned} dF_t = {} & \frac{\partial F_t}{\partial t}dt + (\mu_S - q)S_t\frac{\partial F_t}{\partial S_t}dt + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2}dt + \sigma_S S_t\frac{\partial F_t}{\partial S_t}dW_S \\ & + \mu_V V_t\frac{\partial F_t}{\partial V_t}dt + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2}dt + \sigma_V V_t\frac{\partial F_t}{\partial V_t}dW_V + \mu_D D_t\frac{\partial F_t}{\partial D_t}dt \\ & + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2}dt + \sigma_D D_t\frac{\partial F_t}{\partial D_t}dW_D + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t \partial V_t}dt \\ & + \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t \partial D_t}dt + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t \partial D_t}dt. \end{aligned} \tag{3.8}$$

To eliminate the Wiener processes dW_S, dW_V and dW_D, a portfolio Π_t consisting of the European option F_t, the underlying S_t, the counterparty's assets V_t and the counterparty's liabilities D_t must be set up.[6] In particular, this portfolio consists of a short position in the European option and long positions in the underlying, the counterparty's assets and liabilities. The amount of shares in the long positions

[6] To construct such a portfolio, it is necessary to assume that option's underlying as well as the counterparty's assets and liabilities are traded securities. This assumption is not questionable for the option's underlying, but it is for both the counterparty's assets and liabilities. As argued by Klein (1996), it is likely that the counterparty's assets and liabilities are not traded directly in the market, but that their market values behave similarly as if they were traded securities.

are equal to $\partial F_t/\partial S_t$, $\partial F_t/\partial V_t$ and $\partial F_t/\partial D_t$, respectively. Hence, the value of the portfolio at time t is given by

$$\Pi_t = -F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t. \tag{3.9}$$

The change in the value of the portfolio over the time interval dt is characterized by the total differential which is equal to

$$d\Pi_t = -dF_t + \frac{\partial F_t}{\partial S_t} dS_t + \frac{\partial F_t}{\partial V_t} dV_t + \frac{\partial F_t}{\partial D_t} dD_t. \tag{3.10}$$

Substituting Equations (3.1) to (3.3) and (3.8) into Equation (3.10) yields

$$\begin{aligned} d\Pi_t = -\frac{\partial F_t}{\partial t} dt + qS_t \frac{\partial F_t}{\partial S_t} - \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt - \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt \\ - \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt - \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt \\ - \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt - \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt. \end{aligned} \tag{3.11}$$

Since the portfolio dynamics are independent of the Wiener processes dW_S, dW_V and dW_D, the portfolio is riskless during the infinitesimal time interval dt. To avoid arbitrage opportunities, the portfolio must earn the same return as other short-term risk-free investments – namely the risk-free interest rate r:

$$r\Pi dt = d\Pi_t. \tag{3.12}$$

We substitute Equations (3.9) and (3.11) into Equation (3.12) which yields

$$\begin{aligned} & r\left(-F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t\right) dt \\ & = \frac{\partial F_t}{\partial t} dt - qS_t \frac{\partial F_t}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt + \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt \\ & \quad + \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt + \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt \\ & \quad + \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt. \end{aligned} \tag{3.13}$$

Rewriting Equation (3.13), the partial differential equation that characterizes the price of a European option whose payoff at time T is contigent upon the price of the option's underlying as well as upon the value of both the counterparty's assets and liabilities is obtained. It is given by

$$\begin{aligned} 0 = {} & \frac{\partial F_t}{\partial t} - rF_t + (r-q)S_t\frac{\partial F_t}{\partial S_t} + rV_t\frac{\partial F_t}{\partial V_t} + rD_t\frac{\partial F_t}{\partial D_t} \\ & + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2} + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2} + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2} + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t \partial V_t} \\ & + \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t \partial D_t} + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t \partial D_t}. \end{aligned} \tag{3.14}$$

To obtain a unique solution of the partial differential equation, we must set up the boundary conditions which specify the value of the European option at the boundaries of S_t, V_t, D_t and t. The key boundary condition specifies the option payoff at maturity. Based on Assumption 10, the boundary condition for the European call is thus equal to

$$F_T = C_T = \begin{cases} S_T - K & \text{if } S_T \geq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(S_T - K) & \text{if } S_T \geq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{3.15}$$

whereas the boundary condition for the corresponding vulnerable European put is given by

$$F_T = P_T = \begin{cases} K - S_T & \text{if } S_T \leq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(K - S_T) & \text{if } S_T \leq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

For both European calls and puts, the first line in the boundary condition refers to the situation in which the option is in the money at maturity and the counterparty does not default, i.e. $S_T - K$ and $K - S_T$ are paid out to the holder of a European call and a European put, respectively. The second line indicates the option payoff if the option expires in the money and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportional payout ratio is given by $((1-\alpha)\, V_T)\, / L(S_T,\, D_T)$, i.e. the value of the counterparty's assets available for distribution is divided by the value of the counterparty's total liabilities. Hence, the holder of a European call receives $((1-\alpha)\, V_T\, (S_T - K))\, / L(S_T,\, D_T)$, whereas the holder of a European put receives $((1-\alpha)\, V_T\, (K - S_T))\, / L(S_T,\, D_T)$. The third line refers to the out-of-the-money scenario, in which the option holder receives nothing irrespective of whether the counterparty defaults or not.

The actual characterization of the boundary conditions depends on the choice of a specific valuation model (see Section 3.4). In particular, the variable $L(S_T,\, D_T)$ must be defined according to the chosen model.

3.3 Solution to the Partial Differential Equation

The partial differential equation given by Equation (3.14) depends on the price of the option's underlying, the counterparty's assets and liabilities, the risk-free interest rate, the dividend yield of the option's underlying as well as on the return volatilities. All these variables and parameters are independent of the risk preferences of the investors.[7] Since the risk preferences of the investors do not enter the partial differential equation, they cannot affect its solution. Consequently, any type of risk preferences can be used when solving the partial differential equation.

Using the approach of Cox and Ross (1976) and Harrison and Pliska (1981), the risk-neutral stochastic processes for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities are equal to

$$dS_t = (r - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S, \tag{3.17}$$

[7] Following the argument of Hull (2012: 311–312), the partial differential equation given by Equation (3.14) would not be independent of risk preferences if it included the expected returns of the option's underlying, the counterparty's assets and the counterparty's liabilities. These parameters depend on risk preferences, since their magnitude represents the level of risk aversion of the investor: the higher the level of the investor's risk aversion, the higher the required expected return.

$$dV_t = r\, V_t\, dt + \sigma_V\, V_t\, dW_V \tag{3.18}$$

and

$$dD_t = r\, D_t\, dt + \sigma_D\, D_t\, dW_D, \tag{3.19}$$

where r denotes the risk-free interest rate and all other variables are defined as before.

Applying Itô's lemma to Equations (3.17) to (3.19), the stochastic processes for $\ln S_t$, $\ln V_t$ and $\ln D_t$ are obtained. They are given by

$$d \ln S_t = \left(r - q - \frac{1}{2}\sigma_S^2 \right) dt + \sigma_S dW_S, \tag{3.20}$$

$$d \ln V_t = \left(r - \frac{1}{2}\sigma_V^2 \right) dt + \sigma_V dW_V \tag{3.21}$$

and

$$d \ln D_t = \left(r - \frac{1}{2}\sigma_D^2 \right) dt + \sigma_D dW_D. \tag{3.22}$$

Rewriting Equations (3.20) to (3.22), the expressions for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities at the option's maturity are obtained. They are equal to

$$S_T = S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\, x_S}, \tag{3.23}$$

$$V_T = V_t\, e^{(r-\frac{1}{2}\sigma_V^2)(T-t)+\sigma_V\sqrt{T-t}\, x_V} \tag{3.24}$$

and

$$D_T = D_t\, e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\, x_D}, \tag{3.25}$$

where the three random variables x_S, x_V and x_D are jointly standard normally distributed and their respective correlations are given by the coefficients ρ_{SV}, ρ_{SD} and ρ_{VD}.

The Feynman-Kač theorem states that the solution to the partial differential equation specified in Equation (3.14) is given by

$$F_t = \mathbb{E}\left[e^{-\int\limits_t^T r_u \, du} g\left(S_T, V_T, D_T\right) \right], \tag{3.26}$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation under the risk-neutral measure and function $g(\,\cdot\,)$ determines the payoff of the considered European option (Musiela & Rutkowski, 2005: 296; Pennacchi, 2008: 209–210). Consequently, the value of a vulnerable European option is equal to the expected payoff at maturity which is discounted at the risk-free interest rate. Since the risk-free interest rate is deterministic and constant over time according to Assumption 5, Equation (3.26) can be rewritten as follows:

$$F_t = e^{-r(T-t)}\, \mathbb{E}\left[g\left(S_T, V_T, D_T\right) \right]. \tag{3.27}$$

Equation (3.27) can be used to set up the pricing equations for vulnerable European calls and puts by specifying the payoff function $g(\,\cdot\,)$ accordingly. In particular, if the payoff function $g(\,\cdot\,)$ is defined according to the boundary condition given by Equation (3.15), the pricing equation for vulnerable European calls is received which is given by

$$\begin{aligned} C_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{[S_T \geq K,\, V_T \geq L(S_T,\, D_T)]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T}{L(S_T,\, D_T)} \left(S_T - K\right) \cdot 1_{[S_T \geq K,\, V_T < L(S_T,\, D_T)]} \right] \Bigg). \end{aligned} \tag{3.28}$$

In the same manner, the pricing equation for vulnerable European puts is obtained if the boundary condition given by Equation (3.16) is used to specify the payoff function $g(\,\cdot\,)$:

$$\begin{aligned} P_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K,\, V_T \geq L(S_T,\, D_T)]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T}{L(S_T,\, D_T)} \left(K - S_T\right) \cdot 1_{[S_T \leq K,\, V_T < L(S_T,\, D_T)]} \right] \Bigg). \end{aligned} \tag{3.29}$$

In both pricing equations, the first line gives the expected payoff if the option is in the money at maturity and the counterparty does not default. The second line, in turn, gives the expected payoff if the option expires in the money and the counterparty is in default. The out-of-the-money scenario is only implicitly specified, since the option payoff is equal to zero in this case.

To derive analytic valuation formulas for both vulnerable European calls and puts based on the above pricing equations, the following major steps must be performed. First, the variable $L(S_T,\, D_T)$ indicating the default condition must be characterized in accordance with the considered valuation model. Subsequently, the expected value expressions in Equations (3.28) and (3.29) are rewritten as integrals, since S_T, V_T and D_T are continuous random variables. Afterwards, the expressions for the market values of the option's underlying, the counterparty's assets and the counterparty's liabilities at the option's maturity specified by Equations (3.23) and (3.25) are inserted and the density function of the corresponding trivariate normal distribution is standardized. Finally, the closed form solutions for vulnerable European options are received after some algebraic transformations (see Section 3.4).

3.4 Valuation Models

Various models to value vulnerable European options have been developed over the last three decades based on the theoretical framework described in Section 3.1. In the following, the most important models are discussed in greater detail. Furthermore, we set up a general valuation model which incorporates the features of the other models.

3.4.1 Absence of Default Risk

Since the counterparty cannot default, the valuation model of Black and Scholes (1973) gives the default-free value of a European option which serves as an upper price limit. The pricing equations given by Equations (3.28) and (3.29) are substantially simplified, since the second summand vanishes completely due to the absence of counterparty risk. The pricing equation for a European call is given by

$$C_t = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)\,\cdot 1_{[S_T \geq K]}\right], \tag{3.30}$$

whereas the pricing equation for a European put is equal to

$$P_t = e^{-r(T-t)} \mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K]} \right]. \tag{3.31}$$

Since the counterparty cannot default, the structure of the pricing equations is rather simple. If the option expires in the money, the payoff of a European call is equal to $S_T - K$, whereas the payoff of the European put is given by $K - S_T$. If the option is out of the money at maturity, the option holder receives nothing.

Computing the expected value expressions in Equations (3.30) and (3.31), the closed-form valuation formulas for default-free European options can be obtained (see Black & Scholes, 1973). For European calls and puts, these valuation formulas are given by

$$C_t = S_t \, e^{-q(T-t)} \, N(a_1) - K \, e^{-r(T-t)} \, N(a_2) \tag{3.32}$$

and

$$P_t = K \, e^{-r(T-t)} \, N(-a_2) - S_t \, e^{-q(T-t)} \, N(-a_1), \tag{3.33}$$

where $N(\cdot)$ represents the cumulative distribution function of the univariate standard normal distribution and where a_1 and a_2 are given as follows:

$$\begin{aligned} a_1 &= \frac{\ln \frac{S_t}{K} + (r - q + \frac{1}{2} \sigma_S^2) \, (T-t)}{\sigma_S \sqrt{T-t}}, \\ a_2 &= \frac{\ln \frac{S_t}{K} + (r - q - \frac{1}{2} \sigma_S^2) \, (T-t)}{\sigma_S \sqrt{T-t}}. \end{aligned}$$

3.4.2 Deterministic Liabilities

In the model of Klein (1996), the counterparty is in default if its assets are not sufficient to meet its total liabilities at the option's maturity. The total liabilities of the counterparty are assumed to be deterministic and must include the short position in the option, since it obliges the option writer to deliver or purchase the option's underlying at maturity.

In particular, Klein (1996) assumes that the market value of the counterparty's total liabilities at the option's maturity is equal to its initial market value. To put it differently, the level of the counterparty's total liabilities is constant over time and therefore the default boundary $L(S_T,\, D_T)$ must be given by

$$L(S_T,\, D_T) = \bar{L} = \bar{D} = D_0. \tag{3.34}$$

Inserting the above expression into Equations (3.28) and (3.29) yields the pricing equations for vulnerable European options based on the model of Klein (1996). In particular, the pricing equations for vulnerable European calls and puts, respectively, are equal to

$$\begin{aligned} C_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{\left[S_T \geq K,\, V_T \geq \bar{D})\right]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{\bar{D}} \cdot 1_{\left[S_T \geq K,\, V_T < \bar{D})\right]} \right] \Bigg) \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} P_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{\left[S_T \leq K,\, V_T \geq \bar{D})\right]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{\bar{D}} \cdot 1_{\left[S_T \leq K,\, V_T < \bar{D})\right]} \right] \Bigg). \end{aligned} \tag{3.36}$$

In both pricing equations, the first line is related to the situation in which the option expires in the money and the counterparty does not default. Hence, the payoff of a European call is equal to $S_T - K$, whereas the payoff of the European put is given by $K - S_T$. The second line gives the payoff if the option is in the money at maturity and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $\left((1-\alpha)\, V_T\right)/\bar{D}$, i.e. the asset value available for distribution is divided by the value of the counterparty's total liabilities. The holder of a European call receives $\left((1-\alpha)\, V_T\, (S_T - K)\right)/\bar{D}$, whereas $\left((1-\alpha)\, V_T\, (K - S_T)\right)/\bar{D}$ is paid out to the holder of a European put. If the option expires out of the money, the option holder receives nothing irrespective of whether the counterparty defaults or not.

Computing the expected values given by Equations (3.35) and (3.36), the closed-form valuation formulas for vulnerable European options based on the Klein model are obtained (see Klein, 1996). They are given by

$$
\begin{aligned}
C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \rho_{SV}) - K\, e^{-r(T-t)}\, N_2(a_2, b_2, \rho_{SV}) \\
& + \frac{(1-\alpha) V_t\, S_t\, e^{(r-q+\rho_{SV}\sigma_S\sigma_V)(T-t)}}{\bar{D}}\, N_2(a_3, b_3, -\rho_{SV}) \\
& - \frac{(1-\alpha) V_t\, K}{\bar{D}}\, N_2(a_4, b_4, -\rho_{SV})
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
P_t = {} & K\, e^{-r(T-t)}\, N_2(-a_2, b_2, -\rho_{SV}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\rho_{SV}) \\
& + \frac{(1-\alpha) V_t\, K}{\bar{D}}\, N_2(-a_4, b_4, \rho_{SV}) \\
& - \frac{(1-\alpha) V_t\, S_t\, e^{(r-q+\rho_{SV}\sigma_S\sigma_V)(T-t)}}{\bar{D}}\, N_2(-a_3, b_3, \rho_{SV}),
\end{aligned}
\tag{3.38}
$$

where $N_2(\,\cdot\,)$ represents the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{S_t}{K} + (r - q + \frac{1}{2}\,\sigma_S^2)\,(T-t)}{\sigma_S\sqrt{T-t}}, \\
a_2 &= \frac{\ln \frac{S_t}{K} + (r - q - \frac{1}{2}\,\sigma_S^2)\,(T-t)}{\sigma_S\sqrt{T-t}}, \\
a_3 &= \frac{\ln \frac{S_t}{K} + (r - q + \frac{1}{2}\,\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V)\,(T-t)}{\sigma_S\sqrt{T-t}}, \\
a_4 &= \frac{\ln \frac{S_t}{K} + (r - q - \frac{1}{2}\,\sigma_S^2 + \rho_{SV}\sigma_S\sigma_V)\,(T-t)}{\sigma_S\sqrt{T-t}}, \\
b_1 &= \frac{\ln \frac{V_t}{\bar{D}} + (r - \frac{1}{2}\,\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)\,(T-t)}{\sigma_V\sqrt{T-t}}, \\
b_2 &= \frac{\ln \frac{V_t}{\bar{D}} + (r + \frac{1}{2}\,\sigma_V^2)\,(T-t)}{\sigma_V\sqrt{T-t}}, \\
b_3 &= -\frac{\ln \frac{V_t}{\bar{D}} + (r + \frac{1}{2}\,\sigma_V^2 + \rho_{SV}\sigma_S\sigma_V)\,(T-t)}{\sigma_V\sqrt{T-t}}, \\
b_4 &= -\frac{\ln \frac{V_t}{\bar{D}} + (r + \frac{1}{2}\,\sigma_V^2)\,(T-t)}{\sigma_V\sqrt{T-t}}.
\end{aligned}
$$

3.4.3 Deterministic Liabilities and Option induced Default Risk

Klein and Inglis (2001) recognize that the short position in the option itself may cause additional financial distress. To account for this potential source of default risk, they extend the model of Klein (1996) by splitting the counterparty's total liabilities into two components. In particular, the counterparty's total liabilities now consist of the short position in the option on the one hand and all the other liabilities on the other.

Klein and Inglis (2001) assume that the market value of the counterparty's other liabilities at the option's maturity is equal to its initial market value. Hence, the level of the counterparty's other liabilities is constant over time. The value of the short position in the option is taken into account separately. Combining these two features, the counterparty's total liabilities are given by either $\bar{D} + S_T - K$ or $\bar{D} + K - S_T$ depending on whether the considered option is a European call or put. Consequently, the default boundary $L(S_T,\, D_T)$ depends on the type of the considered option. For vulnerable European calls and puts, respectively, it is given by

$$L(S_T,\, D_T) = L(S_T) = \bar{D} + S_T - K = D_0 + S_T - K \tag{3.39}$$

and

$$L(S_T,\, D_T) = L(S_T) = \bar{D} + K - S_T = D_0 + K - S_T. \tag{3.40}$$

Inserting the expressions for $L(S_T,\, D_T)$ into Equations (3.28) and (3.29), the pricing equations of the Klein-Inglis model are obtained. For vulnerable European calls and puts, respectively, they are given by

$$\begin{aligned} C_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{\left[S_T \geq K,\, V_T \geq \bar{D} + S_T - K\right]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{\bar{D} + S_T - K} \cdot 1_{\left[S_T \geq K,\, V_T < \bar{D} + S_T - K\right]} \right] \Bigg) \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} P_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{\left[S_T \leq K,\, V_T \geq \bar{D} + K - S_T\right]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{\bar{D} + K - S_T} \cdot 1_{\left[S_T \leq K,\, V_T < \bar{D} + K - S_T\right]} \right] \Bigg). \end{aligned} \tag{3.42}$$

Like in the Klein model, the first line in both pricing equations refers to the situation in which the option expires in the money and the counterparty does not default, i.e. $S_T - K$ and $K - S_T$ are paid out to the holder of a European call and a European put, respectively. The second line indicates the option payoff if the option expires in the money and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by $((1-\alpha)\,V_T)\,/\left(\bar{D}+S_T-K\right)$ for a European call and by $((1-\alpha)\,V_T)\,/\left(\bar{D}+K-S_T\right)$ for a European put. To put it differently, the asset value available for distribution is divided by the value of the counterparty's total liabilities. The holder of a European call receives $((1-\alpha)\,V_T\,(S_T-K))\,/\left(\bar{D}+S_T-K\right)$, whereas the holder of a European put receives $((1-\alpha)\,V_T\,(K-S_T))\,/\left(\bar{D}+K-S_T\right)$. If the option expires out of the money, the option holder receives nothing irrespective of whether the counterparty defaults or not.

In Equations (3.41) and (3.42), the default boundary as well as the expression in the denominator of the second summand of the the pricing equations are non-linear and depend on a stochastic variable – namely on the price of the option's underlying at maturity. To cope with this issue when computing the expected values, both the default boundary and the second summand's denominator must be linearized and approximated. Such an approximation can be achieved by employing a first order Taylor series expansion. Subsequently, the closed form solutions for vulnerable European options based on the Klein-Inglis model are obtained by explicitly computing the expected value expressions given by Equations (3.41) and (3.42) (see Klein & Inglis, 2001). For vulnerable European calls and puts, respectively, the valuation formulas are equal to

$$\begin{aligned} C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \delta_{SV}) - K\, e^{-r(T-t)}\, N_2(a_2, b_2, \delta_{SV}) \\ & + \frac{(1-\alpha)V_t\, S_t\, e^{\left(r-q+(\rho_{SV}-m)\sigma_S\sigma_V+\frac{1}{2}\sigma_V^2(m^2-2\rho_{SV}m)\right)(T-t)-g\,p}}{\bar{D}+S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}-K}\, N_2(a_3, b_3, -\delta_{SV}) \\ & - \frac{(1-\alpha)V_t\, K\, e^{\frac{1}{2}\sigma_V^2(m^2-2\rho_{SV}m)(T-t)-g\,p}}{\bar{D}+S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}-K}\, N_2(a_4, b_4, -\delta_{SV}) \end{aligned} \tag{3.43}$$

and

$$
\begin{aligned}
P_t = {} & K\, e^{-r(T-t)}\, N_2(-a_2, b_2, -\delta_{SV}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\delta_{SV}) \\
& + \frac{(1-\alpha) V_t\, K\, e^{\frac{1}{2}\sigma_V^2 (m^2 - 2\rho_{SV} m)(T-t) - g\,p}}{\bar{D} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t) + \sigma_S \sqrt{T-t}\, p}}\, N_2(-a_4, b_4, \delta_{SV}) \\
& + \frac{(1-\alpha) V_t\, S_t\, e^{\left(r-q+(\rho_{SV}-m)\sigma_S \sigma_V + \frac{1}{2}\sigma_V^2 (m^2 - 2\rho_{SV} m)\right)(T-t) - g\,p}}{\bar{D} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t) + \sigma_S \sqrt{T-t}\, p}}\, N_2(-a_3, b_3, \delta_{SV})\,,
\end{aligned}
\tag{3.44}
$$

where $N_2(\,\cdot\,)$ represents the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{S_t}{K} + (r-q-\frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S \sqrt{T-t}} + \sigma_S \sqrt{T-t}, \\
a_2 &= \frac{\ln \frac{S_t}{K} + (r-q-\frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S \sqrt{T-t}}, \\
a_3 &= \frac{\ln \frac{S_t}{K} + (r-q-\frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S \sqrt{T-t}} + \sigma_S \sqrt{T-t} \\
&\qquad + m\, \sigma_V \sqrt{T-t} + g \\
&\qquad + \delta_{SV} \sqrt{1 - 2\rho_{SV} m + m^2}\, \sigma_V \sqrt{T-t}, \\
a_4 &= \frac{\ln \frac{S_t}{K} + (r-q-\frac{1}{2}\sigma_S^2)(T-t)}{\sigma_S \sqrt{T-t}} + m\, \sigma_V \sqrt{T-t} + g \\
&\qquad + \delta_{SV} \sqrt{1 - 2\rho_{SV} m + m^2}\, \sigma_V \sqrt{T-t}, \\
b_1 &= -\frac{-b - m\,p}{\sqrt{1 - 2\rho_{SV} m + m^2}} + \delta_{SV}\, \sigma_S \sqrt{T-t}, \\
b_2 &= -\frac{-b - m\,p}{\sqrt{1 - 2\rho_{SV} m + m^2}} \\
b_3 &= \frac{-b - m\,p}{\sqrt{1 - 2\rho_{SV} m + m^2}} - \sqrt{1 - 2\rho_{SV} m + m^2}\, \sigma_V \sqrt{T-t}, \\
b_4 &= \frac{-b - m\,p}{\sqrt{1 - 2\rho_{SV} m + m^2}} - \sqrt{1 - 2\rho_{SV} m + m^2}\, \sigma_V \sqrt{T-t} \\
&\qquad - \delta_{SV} \left(m\, \sigma_V \sqrt{T-t} + g\right).
\end{aligned}
$$

The parameter δ_{SV} gives the adjusted correlation between the return of the option's underlying and the counterparty's assets. It is defined as

$$\delta_{SV} = \frac{\rho_{SV} - m}{\sqrt{1 - 2\rho_{SV} m + m^2}}.$$

The parameters b, m and g depend on the type of the considered option due to the first order Taylor series expansion applied in the derivation of the valuation formulas. For vulnerable European calls, they are given by

$$b^{\text{Call}} = \frac{\ln \frac{V_t}{\bar{D} + S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p} - K} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}},$$

$$m^{\text{Call}} = \frac{\sigma_S}{\sigma_V} \frac{S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}}{\bar{D} + S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p} - K},$$

$$g^{\text{Call}} = -\sigma_S\sqrt{T-t}\, \frac{S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}}{\bar{D} + S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p} - K},$$

whereas they are equal to

$$b^{\text{Put}} = \frac{\ln \frac{V_t}{\bar{D} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}},$$

$$m^{\text{Put}} = -\frac{\sigma_S}{\sigma_V} \frac{S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}}{\bar{D} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}},$$

$$g^{\text{Put}} = \sigma_S\sqrt{T-t}\, \frac{S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}p}}{\bar{D} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p}}$$

for vulnerable European puts.

Since a first order Taylor series expansion is used in the derivation to linearize and approximate both the default boundary and the denominator in the expected value's second summand, the valuation formulas given by Equations (3.43) and (3.44) are only analytical approximations and depend on the point of expansion p. In principle, the value for p can be chosen freely, however, it is important to note that this choice might have a decisive impact on the accuracy of the obtained option values.

Figures 3.1 and 3.2 provide insights to the impact of choosing a particular value for the point of expansion p. In these two figures, the option values are depicted as functions of the price of the option's underyling, the time to maturity and the value of the counterparty's assets. These option values are obtained from the approximate closed form solutions given by Equations (3.43) and (3.44) using different values for the point of expansion. The approximate analytical solution and the numerical solution (e.g. Monte Carlo simulation) are almost identical for $p = 1.5$ and $p = -1.5$ in case of vulnerable European calls and puts, respectively. This finding is consistent with Klein and Inglis (2001) who suggest the same choice for p using different numerical examples. Hence, the approximate closed form solutions are quite accurate for a wide range of moneyness, different times to maturity and various values of the counterparty's assets if the point of expansion p is chosen appropriately.

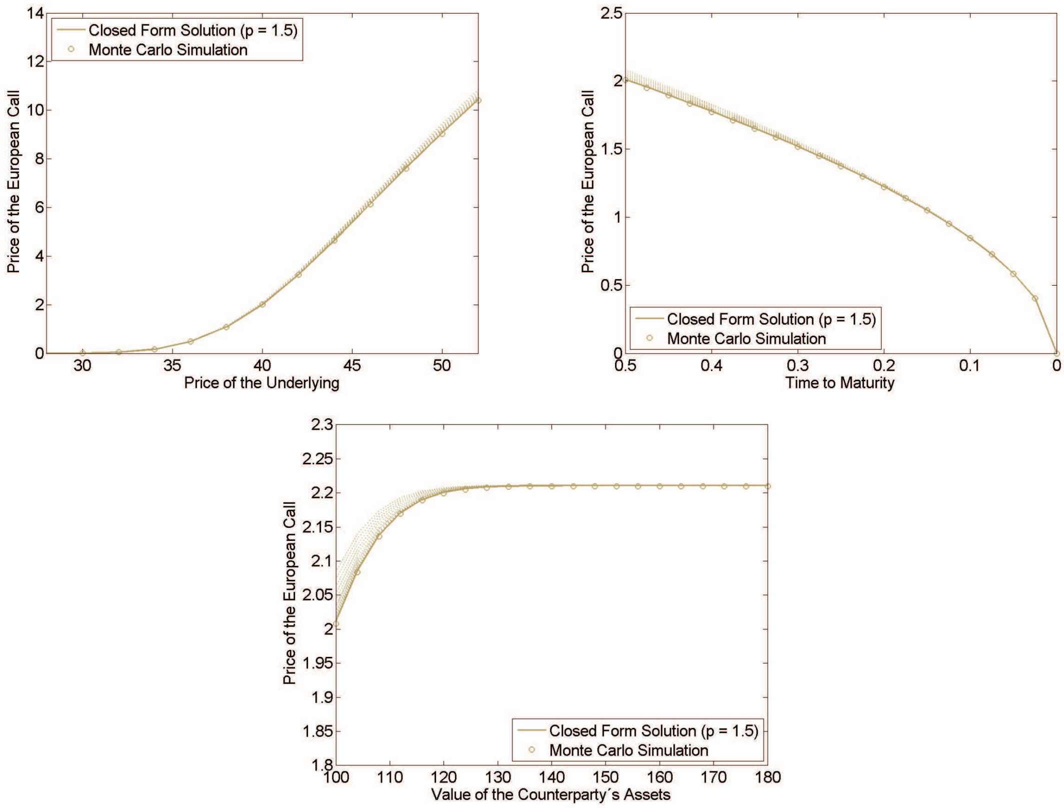

Figure 3.1: European Calls in the Model of Klein and Inglis (2001)

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values (ochre line) are generated using the approximate closed form solution given by Equation (3.43) based on $p = 1.5$. The numerical solution of the Klein-Inglis model (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the point of expansion p ranging from 0 to 4.

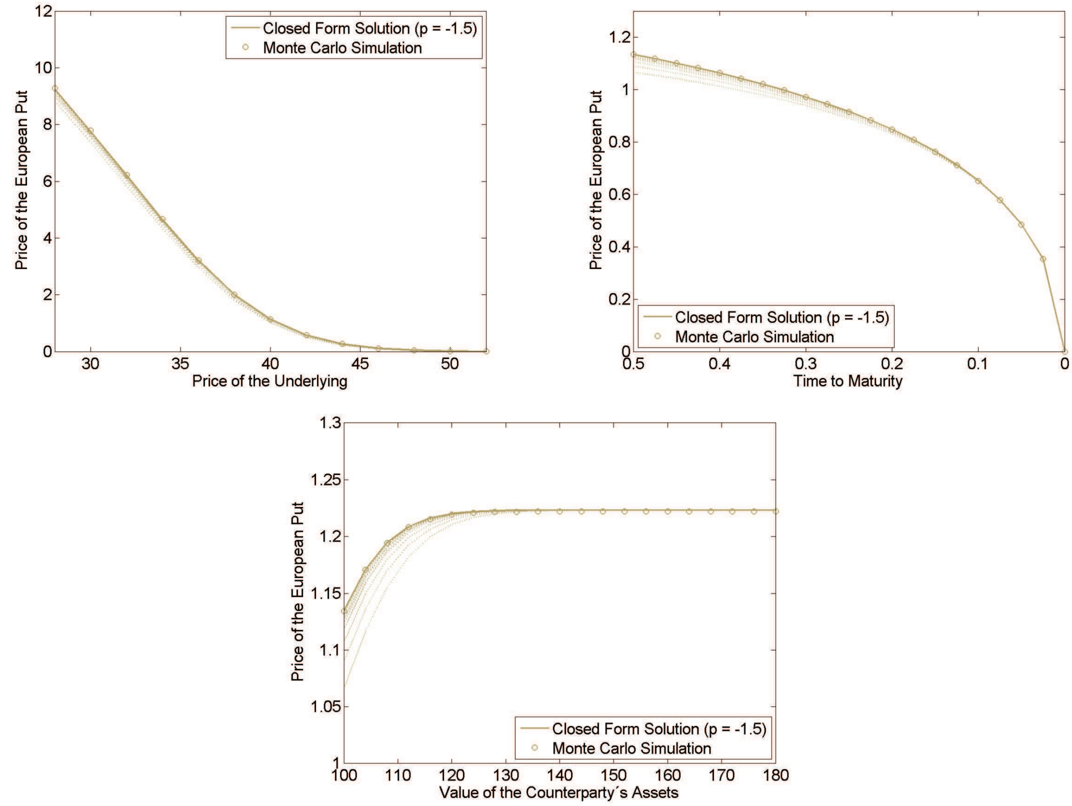

Figure 3.2: European Puts in the Model of Klein and Inglis (2001)

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values (ochre line) are generated using the approximate closed form solution given by Equation (3.44) based on $p = -1.5$. The numerical solution of the Klein-Inglis model (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the point of expansion p ranging from -4 to 0.

In Table 3.1, the option values for vulernable European calls and puts based on the Klein-Inglis model are presented. The first two columns give the values of a vulnerable European call computed by the approximate valuation formula and the numerical solution, respectively. The third column reports the approximation error which is measured as the percentage deviation of the approximate from the numerical solution. Most errors are smaller than $\pm 0.2\%$ with the highest error being $+0.25\%$. Compared to the base case scenario, the magnitude of the approximation error considerably increases for in-the-money options ($S \uparrow$), an increased return volatility of the option's underlying ($\sigma_S \uparrow$), a longer time to maturity ($T \uparrow$) and higher default cost ($\alpha \uparrow$).

	European Call			European Put		
	Approx. CF	Num. Sol.	Approx. Error	Approx. CF	Num. Sol.	Approx. Error
Base Case	2.0110	2.0084	+0.15%	1.1341	1.1342	−0.01%
$S = 45$	5.3869	5.3755	+0.21%	0.1718	0.1721	−0.20%
$S = 35$	0.2912	0.2908	+0.13%	3.9007	3.9121	−0.29%
$V = 105$	2.1011	2.0990	+0.10%	1.1778	1.1777	+0.01%
$V = 95$	1.8847	1.8818	+0.16%	1.0682	1.0687	−0.05%
$\sigma_S = 0.2$	2.4389	2.4344	+0.18%	1.6102	1.6123	−0.13%
$\sigma_S = 0.1$	1.5614	1.5601	+0.09%	0.6496	0.6493	+0.05%
$\sigma_V = 0.2$	1.9603	1.9579	+0.12%	1.1032	1.1035	−0.02%
$\sigma_V = 0.1$	2.0740	2.0715	+0.12%	1.1724	1.1728	−0.03%
$\rho_{SV} = 0.5$	2.1521	2.1501	+0.09%	1.0409	1.0415	−0.06%
$\rho_{SV} = -0.5$	1.8567	1.8537	+0.16%	1.2037	1.2033	+0.04%
$T - t = 1$	3.0009	2.9950	+0.20%	1.3411	1.3423	−0.09%
$T - t = 0.25$	1.3770	1.3758	+0.09%	0.9153	0.9151	0.02%
$\alpha = 0.5$	1.8560	1.8524	+0.20%	1.0634	1.0644	−0.09%
$\alpha = 0$	2.1660	2.1645	+0.07%	1.2047	1.2040	0.06%
$r = 0.08$	2.3553	2.3524	+0.12%	0.9329	0.9330	−0.01%
$r = 0.02$	1.6968	1.6943	+0.15%	1.3584	1.3592	−0.06%
$q = 0.02$	1.8000	1.7975	+0.14%	1.2814	1.2817	−0.02%

Table 3.1: Approx. Error in the Model of Klein and Inglis (2001)

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The approximate closed form solutions that are used to compute the option values are given by Equations (3.43) and (3.44), respectively. The point of expansion are chosen to be $p = 1.5$ in case of a European calll and $p = -1.5$ in case of a European put. The numerical solution is calculated by Monte Carlo simulation ($N = 1\,000\,000$).

In the fourth and fifth columns, the values of a vulnerable European put computed by the approximate valuation formula and numerical solution, respectively, are presented. In the sixth column, the approximation error is given. Again, it is measured as the percentage deviation of the approximate from the numerical solution. Most errors are smaller than $\pm 0.25\%$ with the highest error being -0.29%. Compared to the base case scenario, the magnitude of the approximation error considerably increases for in-the-money and out-of-the-money options ($S\downarrow$ and $S\uparrow$), an increased return volatility of the option's underlying ($\sigma_S \uparrow$), a longer time to maturity ($T\uparrow$) and higher default cost ($\alpha\uparrow$).

To conclude, the magnitude of the approximation errors is relatively low for both vulnerable European calls and puts which indicates that the approximate valuation formulas suggested by Klein and Inglis (2001) work quite well for the given set of parameters. This result is in line with the findings of Klein and Inglis (2001). They perform a similar analysis to verify the accuracy and quality of their approximate analytical solution. Using slightly different parameter values, they find marginally higher but still rather small approximation errors.

3.4.4 Stochastic Liabilities

In contrast to Klein (1996) and Klein and Inglis (2001), Liu and Liu (2011) suggest a valuation model in which the counterparty's total liabilities may vary over time. In particular, it is assumed that the market value of the counterparty's total liabilities follow a geometric Brownian motion (see Equation (3.3)). The market value at the option's maturity is denoted by D_T. It is important to note that the short position in the option is implicitly included in the counterparty's total liabilities, since it is an obligation to the option writer. Unlike in the Klein-Inglis model, however, its impact on the value of the couterparty's total liabilities is not explicitly modeled. Hence, the default boundary $L(S_T,\, D_T)$ in the Liu-Liu model is defined as

$$L(S_T,\, D_T) = L(D_T) = D_T. \tag{3.45}$$

Inserting this expression into Equations (3.28) and (3.29) yields the pricing equations of the Liu-Liu model. The pricing equation for a vulnerable European call equals

$$\begin{aligned} C_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{[S_T \geq K,\, V_T \geq D_T]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{D_T} \cdot 1_{[S_T \geq K,\, V_T < D_T]} \right] \Bigg), \end{aligned} \tag{3.46}$$

whereas the pricing equation for a vulnerable European put is given by

$$P_t = e^{-r(T-t)} \Bigg(\mathbb{E} \left[(K - S_T) \cdot 1_{[S_T \le K,\, V_T \ge D_T]} \right] + \mathbb{E} \left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{D_T} \cdot 1_{[S_T \le K,\, V_T < D_T]} \right] \Bigg). \tag{3.47}$$

The first line in both pricing equations still refers to the situation in which the option expires in the money and the counterparty does not default, i.e. $S_T - K$ and $K - S_T$ are paid out to the holder of a European call and a European put, respectively. The second line gives the payoff if the option is in the money at maturity and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $((1-\alpha)\, V_T) / D_T$, i.e. the asset value available for distribution is divided by the value of the counterparty's total liabilities. The holder of a European call receives $((1-\alpha)\, V_T\, (S_T - K)) / D_T$, whereas $((1-\alpha)\, V_T\, (K - S_T)) / D_T$ is paid out to the holder of a European put. If the option expires out of the money, the option holder receives nothing irrespective of whether the counterparty defaults or not.

In Equations (3.46) and (3.47), the default boundary as well as the denominator of the pricing equations' second summand depend on the value of the counterparty's liabilities which is a stochastic variable. To circumvent this issue, a new variable, the debt ratio, is introduced which is defined as $R_t = V_t / D_t$. Using the debt ratio, the expected value expressions in Equations (3.46) and (3.47) can be computed analytically and the closed-form valuation formulas for vulnerable European options based on the Liu-Liu model are obtained (see Liu & Liu, 2011). For vulnerable European calls and puts, respectively, these valuation formulas are equal to

$$\begin{aligned} C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \delta_{SR}) - K\, e^{-r(T-t)}\, N_2(a_2, b_2, \delta_{SR}) \\ & + \frac{(1-\alpha) V_t\, S_t\, e^{(-q+\sigma_D^2+\rho_{SV}\sigma_S\sigma_V+\rho_{SD}\sigma_S\sigma_D-\rho_{VD}\sigma_V\sigma_D)(T-t)}}{D_t}\, N_2(a_3, b_3, -\delta_{SR}) \\ & - \frac{(1-\alpha) V_t\, K\, e^{(-r+\sigma_D^2-\rho_{VD}\sigma_V\sigma_D)(T-t)}}{D_t}\, N_2(a_4, b_4, -\delta_{SR}) \end{aligned} \tag{3.48}$$

and

$$P_t = K\, e^{-r(T-t)}\, N_2(-a_2, b_2, -\delta_{SR}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\delta_{SR}) \tag{3.49}$$
$$+ \frac{(1-\alpha)V_t\, K\, e^{(-r+\sigma_D^2-\rho_{VD}\sigma_V\sigma_D)(T-t)}}{D_t}\, N_2(-a_4, b_4, \delta_{SR})$$
$$- \frac{(1-\alpha)V_t\, S_t\, e^{(-q+\sigma_D^2+\rho_{SV}\sigma_S\sigma_V+\rho_{SD}\sigma_S\sigma_D-\rho_{VD}\sigma_V\sigma_D)(T-t)}}{D_t}\, N_2(-a_3, b_3, \delta_{SR}),$$

where $N_2(\,\cdot\,)$ represents the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$a_1 = \frac{\ln\frac{S_t}{K} + (r - q + \frac{1}{2}\,\sigma_S^2)\,(T-t)}{\sigma_S\sqrt{T-t}},$$
$$a_2 = \frac{\ln\frac{S_t}{K} + (r - q - \frac{1}{2}\,\sigma_S^2)\,(T-t)}{\sigma_S\sqrt{T-t}},$$
$$a_3 = \frac{\ln\frac{S_t}{K} + (r - q + \frac{1}{2}\,\sigma_S^2 + \sigma_{SV} - \sigma_{SD})\,(T-t)}{\sigma_S\sqrt{T-t}}\sqrt{T-t},$$
$$a_4 = \frac{\ln\frac{S_t}{K} + (r - q - \frac{1}{2}\,\sigma_S^2 + \sigma_{SV} - \sigma_{SD})\,(T-t)}{\sigma_S\sqrt{T-t}},$$
$$b_1 = \frac{\ln\frac{V_t}{D_t} - \frac{1}{2}(\sigma_V^2 - \sigma_D^2 - 2\sigma_{SV} + 2\sigma_{SD})\,(T-t)}{(\sigma_{SV} - \sigma_{SD})\sqrt{T-t}},$$
$$b_2 = \frac{\ln\frac{V_t}{D_t} - \frac{1}{2}(\sigma_V^2 - \sigma_D^2)\,(T-t)}{(\sigma_{SV} - \sigma_{SD})\sqrt{T-t}},$$
$$b_3 = -\frac{\ln\frac{V_t}{D_t} - \frac{1}{2}(\sigma_V^2 - \sigma_D^2 - 2\sigma_{SV} + 2\sigma_{SD})\,(T-t)}{(\sigma_{SV} - \sigma_{SD})\sqrt{T-t}} - \sqrt{\sigma_V^2 + \sigma_D^2 - 2\sigma_{VD}}\,\sqrt{T-t},$$
$$b_4 = -\frac{\ln\frac{V_t}{D_t} - \frac{1}{2}(\sigma_V^2 - \sigma_D^2)\,(T-t)}{(\sigma_{SV} - \sigma_{SD})\sqrt{T-t}} - \sqrt{\sigma_V^2 + \sigma_D^2 - 2\sigma_{VD}}\,\sqrt{T-t}.$$

The parameter δ_{SR} gives the adjusted correlation between the returns of the option's underlying and the counterparty's debt ratio. It is defined as

$$\delta_{SR} = \frac{\rho_{SV}\sigma_V - \rho_{SD}\sigma_D}{\sqrt{\sigma_V^2 + \sigma_D^2 - 2\,\rho_{VD}\sigma_V\sigma_D}}.$$

3.4.5 General Model

Our general model picks up on the ideas of both Klein and Inglis (2001) and Liu and Liu (2011). In particular, we assume that the short position in the option may increase the counterparty's default risk and the market value of the counterparty's other liabilities follows a geometric Brownian motion. At the option's maturity the counterparty's total liabilities are given by $D_T + S_T - K$ in the case of a European call and $D_T + K - S_T$ in the case of a European put, respectively. Consequently, the default boundary $L(S_T,\, D_T)$ depends on the type of the considered option. For vulnerable European calls and puts, respectively, it is given by

$$L(S_T,\, D_T) = D_T + S_T - K \tag{3.50}$$

and

$$L(S_T,\, D_T) = D_T + K - S_T. \tag{3.51}$$

Plugging these expressions into Equations (3.28) and (3.29) yields the pricing equations of the general model. The pricing equation for a vulnerable European call equals

$$\begin{aligned} C_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{[S_T \geq K,\, V_T \geq D_T + S_T - K]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{D_T + S_T - K} \cdot 1_{[S_T \geq K,\, V_T < D_T + S_T - K]} \right] \Bigg), \end{aligned} \tag{3.52}$$

whereas the pricing equation for a vulnerable European put is given by

$$\begin{aligned} P_t = e^{-r(T-t)} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K,\, V_T \geq D_T + K - S_T]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{D_T + K - S_T} \cdot 1_{[S_T \leq K,\, V_T < D_T + K - S_T]} \right] \Bigg). \end{aligned} \tag{3.53}$$

In analogy to the other valuation models, the first line of both pricing equations refers to the situation in which the option expires in the money and the counterparty does not default. Consequently, the corresponding payoff of a European call is equal to $S_T - K$, whereas it is given by $K - S_T$ for a European put. The second line of both pricing equations indicates the payoff if the option expires in the money and the counterparty is in default. In this case, the entire assets of the counterparty (less the default cost α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by $((1-\alpha)\,V_T)\,/\,(D_T + S_T - K)$ for a European call and $((1-\alpha)\,V_T)\,/\,(D_T + K - S_T)$ for a European put, respectively. To put it differently, the asset value available for distribution must be divided by the value of the counterparty's total liabilities to obtain the payout ratio in case of the counterparty's default. Therefore, the holder of a European call receives $((1-\alpha)\,V_T\,(S_T - K))\,/\,(D_T + S_T - K)$, whereas the holder of a European put receives $((1-\alpha)\,V_T\,(K - S_T))\,/\,(D_T + K - S_T)$. If the option is, however, out of the money at maturity the option holder receives nothing irrespective of whether the counterparty is in default or not.

Looking at Equations (3.52) and (3.53), it becomes clearly evident that our general valuation model incorporates the models of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) as special cases. The communalities and differences between these models are summarized as follows:

1. If the counterparty's other liabilities are assumed to be deterministic and constant over time, our general model is reduced to the model of Klein and Inglis (2001) which is represented by Equations (3.41) and (3.42), since then the default condition is given by $V_T < \bar{D} + S_T - K$ and $V_T < \bar{D} + K - S_T$, respectively.

2. If the option holder's claim $S_T - K$ and $K - S_T$, respectively, is removed from the default condition and the market value of the counterparty's other liabilities is still assumed to follow a geometric Brownian motion, our general model collapses to the model of Liu and Liu (2011) which is specified by Equations (3.46) and (3.47), since the default condition is equal to $V_T < D_T$ in this case.

3. If the option holder's claim $S_T - K$ and $K - S_T$, respectively, is removed from the default condition and the counterparty's other liabilities are assumed to be constant over time, our general model is reduced to the model of Klein (1996) specified by Equations (3.35) and (3.36), since the default condition is equal to $V_T < \bar{D}$ in this case.

In Equations (3.52) and (3.53), the default boundary as well as the denominator of the pricing equations' second summand are non-linear and depend on two stochastic variables – namely the price of the option's underlying and the market value of the counterparty's other liabilities. Due to this issue, an exact analytical solution cannot be derived. However, we are able to derive an approximate closed form solution if the returns of the option's underlying and the counterparty's other liabilities are assumed to be uncorrelated ($\rho_{SD} = 0$).

We employ a first order Taylor series expansion to linearize and approximate both the default boundary and the second summand's denominator. After some algebraic transformations, we obtain the approximate valuation formulas for vulnerable European options (see Appendix 1). For vulnerable European calls and puts, respectively, these approximate valuation formulas are equal to

$$
\begin{aligned}
C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \delta_{SV}) - K\, e^{-r(T-t)}\, N_2(a_2, b_2, \delta_{SV}) \\
& + \frac{(1-\alpha)V_t\, S_t\, e^{\left(r-q+(\rho_{SV}-m_1)\sigma_S\sigma_V+\frac{1}{2}\sigma_V^2(m_1^2+m_2^2-2\rho_{SV}m_1-2\rho_{VD}m_2)\right)(T-t)}}{D_t\, e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \\
& \cdot e^{-g_1 p_1 - g_2 p_2}\, N_2(a_3, b_3, -\delta_{SV}) \\
& - \frac{(1-\alpha)V_t\, K\, e^{\frac{1}{2}\sigma_V^2(m_1^2+m_2^2-2\rho_{SV}m_1-2\rho_{VD}m_2)(T-t)}}{D_t\, e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \\
& \cdot e^{-g_1 p_1 - g_2 p_2}\, N_2(a_4, b_4, -\delta_{SV})
\end{aligned}
\tag{3.54}
$$

and

$$
\begin{aligned}
P_t = {} & K\, e^{-r(T-t)}\, N_2(-a_2, b_2, -\delta_{SV}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\delta_{SV}) \\
& + \frac{(1-\alpha)V_t\, K\, e^{\frac{1}{2}\sigma_V^2(m_1^2+m_2^2-2\rho_{SV}m_1-2\rho_{VD}m_2)(T-t)}}{D_t\, e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}} \\
& \cdot e^{-g_1 p_1 - g_2 p_2}\, N_2(-a_4, b_4, \delta_{SV}) \\
& - \frac{(1-\alpha)V_t\, S_t\, e^{\left(r-q+(\rho_{SV}-m_1)\sigma_S\sigma_V+\frac{1}{2}\sigma_V^2(m_1^2+m_2^2-2\rho_{SV}m_1-2\rho_{VD}m_2)\right)(T-t)}}{D_t\, e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + K - S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}} \\
& \cdot e^{-g_1 p_1 - g_2 p_2}\, N_2(-a_3, b_3, \delta_{SV}),
\end{aligned}
\tag{3.55}
$$

where $N_2(\,\cdot\,)$ is the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{S_t}{K} + \left(r - q - \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{T-t}} + \sigma_S\sqrt{T-t}, \\
a_2 &= \frac{\ln \frac{S_t}{K} + \left(r - q - \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{T-t}}, \\
a_3 &= \frac{\ln \frac{S_t}{K} + \left(r - q - \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{T-t}} + \sigma_S\sqrt{T-t} + m_1\,\sigma_V\sqrt{T-t} + g_1 \\
&\qquad + \delta_{SV}\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}\,\sigma_V\sqrt{T-t}, \\
a_4 &= \frac{\ln \frac{S_t}{K} + \left(r - q - \frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S\sqrt{T-t}} + m_1\,\sigma_V\sqrt{T-t} + g_1 \\
&\qquad + \delta_{SV}\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}\,\sigma_V\sqrt{T-t}, \\
b_1 &= -\frac{-b - p_1\,m_1 - p_2\,m_2}{\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}} + \delta_{SV}\sigma_S\sqrt{T-t}, \\
b_2 &= -\frac{-b - p_1\,m_1 - p_2\,m_2}{\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}}, \\
b_3 &= \frac{-b - p_1\,m_1 - p_2\,m_2}{\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}} \\
&\qquad - \sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}\,\sigma_V\sqrt{T-t} \\
&\qquad - \delta_{SV}\left(\sigma_S\sqrt{T-t} + m_1\,\sigma_V\sqrt{T-t} + g_1\right) - \delta_{VD}\left(m_2\,\sigma_V\sqrt{T-t} + g_2\right), \\
b_4 &= \frac{-b - p_1\,m_1 - p_2\,m_2}{\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}} \\
&\qquad - \sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}\,\sigma_V\sqrt{T-t} \\
&\qquad - \delta_{SV}\left(m_1\,\sigma_V\sqrt{T-t} + g_1\right) - \delta_{VD}\left(m_2\,\sigma_V\sqrt{T-t} + g_2\right).
\end{aligned}
$$

The adjusted correlation between the return of the option's underlying and the counterparty's assets is denoted by δ_{SV} and is equal to

$$
\delta_{SV} = \frac{\rho_{SV} - m_1}{\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}},
$$

whereas the adjusted correlation between the return of the counterparty's assets and the counterparty's liabilities is denoted by δ_{VD} and is equal to

$$\delta_{VD} = \frac{\rho_{VD} - m_2}{\sqrt{1 - 2\,\rho_{SV}\,m_1 - 2\,\rho_{VD}\,m_2 + m_1^2 + m_2^2}}.$$

The parameters b, m_1, m_2, g_1 and g_2 depend on the type of the considered option due to the first order Taylor series expansion applied in the derivation of the valuation formulas. For vulnerable European calls, they are given by

$$b^{\mathrm{Call}} = \frac{\ln \frac{V_t}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2}+S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}-K} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}},$$

$$m_1^{\mathrm{Call}} = \frac{\sigma_S}{\sigma_V}\,\frac{S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K},$$

$$m_2^{\mathrm{Call}} = \frac{\sigma_D}{\sigma_V}\,\frac{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K},$$

$$g_1^{\mathrm{Call}} = \frac{-\sigma_S\sqrt{T-t}\,S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K},$$

$$g_2^{\mathrm{Call}} = \frac{-\sigma_D\sqrt{T-t}\,D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K},$$

whereas they are equal to

$$b^{\mathrm{Put}} = \frac{\ln \frac{V_t}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2}+K-S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}} + (r - \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}},$$

$$m_1^{\mathrm{Put}} = -\frac{\sigma_S}{\sigma_V}\,\frac{S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + K - S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}},$$

$$m_2^{\mathrm{Put}} = \frac{\sigma_D}{\sigma_V}\,\frac{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + K - S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}},$$

$$g_1^{\mathrm{Put}} = \frac{\sigma_S\sqrt{T-t}\,S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + K - S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}$$

$$g_2^{\mathrm{Put}} = \frac{-\sigma_D\sqrt{T-t}\,D_t\,e^{(r-q-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2}}{D_t\,e^{(r-\frac{1}{2}\sigma_D^2)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + K - S_t\,e^{(r-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}.$$

for vulnerable European puts.

Since a first order Taylor series expansion is used in the derivation to linearize and approximate both the default boundary and the denominator in the expected value's second summand, the valuation formulas given by Equations (3.54) and (3.55) are analytical approximations and depend on the points of expansion p_1 and p_2. In principle, the values for p_1 and p_2 can be chosen freely, however, it is important to bear in mind that this choice might have a decisive impact on the accuracy of the obtained option values. Consequently, we must analyze to what extent an inappropriate choice for the values of p_1 and p_2 affect the quality of our valuation formulas (see Figures 3.3 and 3.4).

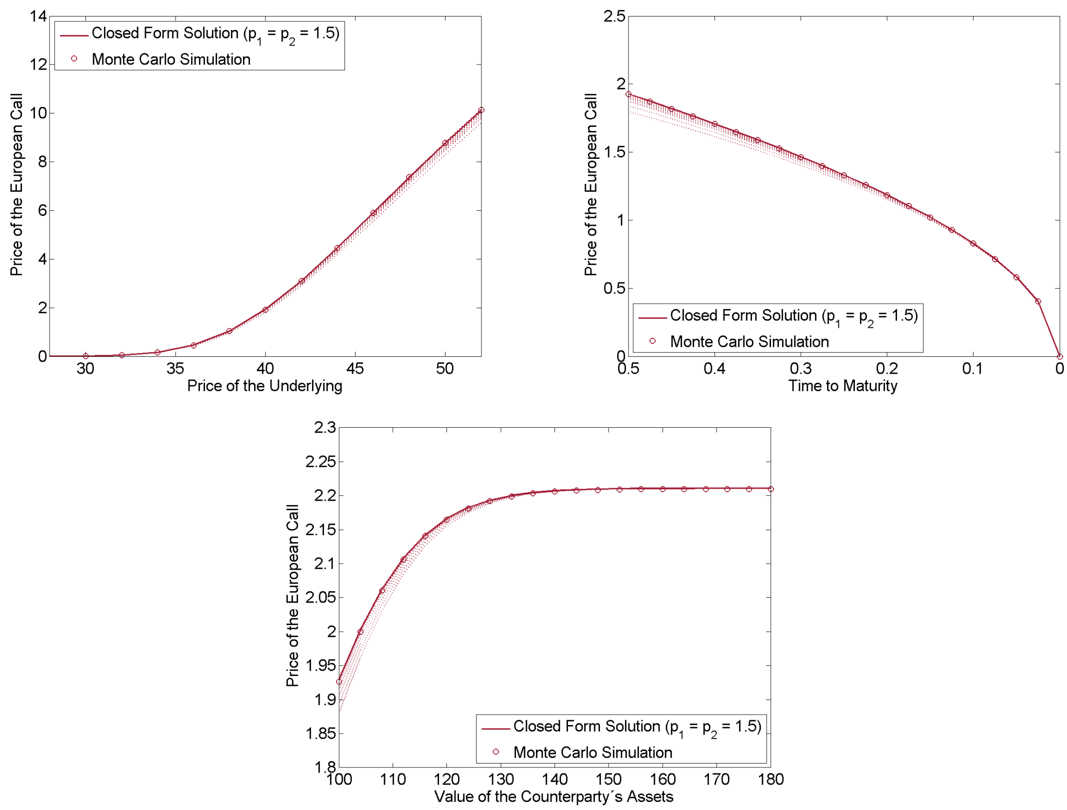

Figure 3.3: European Calls in the General Model

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values (red line) are generated using the approximate closed form solution given by Equation (3.54) based on $p_1 = p_2 = 1.5$. The numerical solution of the general model (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the points of expansion p_1 and p_2 ranging from 0 to 4.

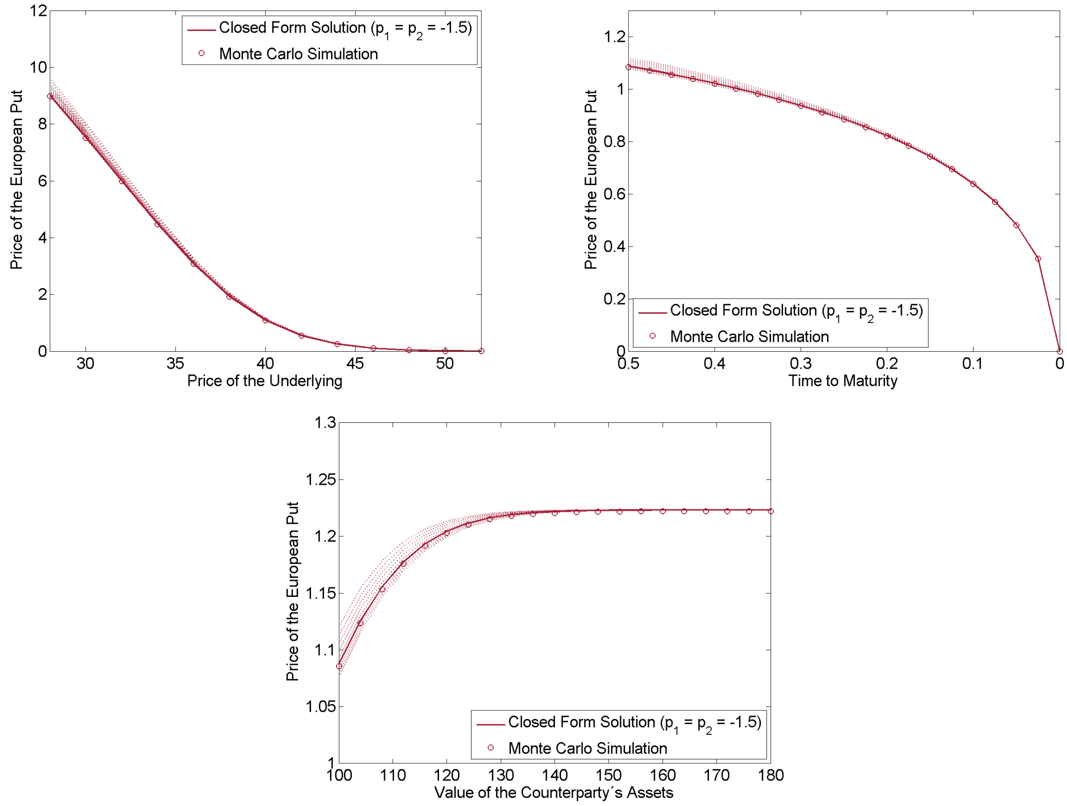

Figure 3.4: European Puts in the General Model

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values (red line) are generated using the approximate closed form solution given by Equation (3.55) based on $p_1 = p_2 = -1.5$. The numerical solution of the general model (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the points of expansion p_1 and p_2 ranging from -4 to 0.

Figures 3.3 and 3.4 provide insights to the impact of choosing a particular value for the points of expansion p_1 and p_2. In these two figures, the option values are depicted as functions of the price of the option's underlying, the time to maturity and the value of the counterparty's assets. These option values are obtained from our approximate closed form solutions given by Equations (3.54) and (3.55) using different values for the points of expansion. The approximate analytical solution and the numerical solution are almost identical for $p_1 = p_2 = 1.5$ in case of vulnerable European calls and for $p_1 = p_2 = -1.5$ in case of vulnerable European puts. The same choice for p_1 and p_2 is also obtained for a large variety of other numerical examples. Hence, the approximate closed form valuation formulas of the general model work quite well for a wide range of parameters if the values for the points of expansion are chosen appropriately.

In Table 3.2, the option values for vulernable European calls and puts based on our general model are presented. The first two columns give the values of a vulnerable European call computed by the approximate valuation formula and Monte Carlo simulation (= numerical solution), respectively. The third column reports the approximation error which is measured as the percentage deviation of the approximate from the numerical solution. Most errors are smaller than $\pm 0.2\%$ with the highest errors being equal to -5.85% and $+7.01\%$. These errors are observed if the correlation between the return of the option's underlying and the counterparty's other liabilities is -0.5 and $+0.5$, respectively. This result is obvious, since the analytical approximation is based on the assumption of independence between these returns. Compared to the base case scenario, the magnitude of the approximation error considerably increases for out-of-the-money options ($S \downarrow$), an increased return volatility of the option's underlying ($\sigma_S \uparrow$), a longer time to maturity ($T \uparrow$) and higher default cost ($\alpha \uparrow$). In the fourth and fifth columns, the values of a vulnerable European put computed by the approximate valuation formula and Monte Carlo simulation (= numerical solution), respectively, are presented. In the sixth column, the approximation error is given. Again, it is defined as the percentage deviation of the approximate solution from the numerical solution. Most errors are smaller than $\pm 0.3\%$ with the highest errors being equal to -5.77% and $+7.92\%$. These errors are observed if the correlation between the return of the option's underlying and the counterparty's other liabilities is 0.5 and -0.5, repsectively. This result is obvious, since the analytical approximation is based on the assumption of independence between these returns. Compared to the base case scenario, the magnitude of the approximation error considerably increases for in-the-money and out-of-the-money options ($S \downarrow$ and $S \uparrow$), an increased return volatility of the counterparty's other liabilities ($\sigma_D \uparrow$), a longer time to maturity ($T \uparrow$) and higher default cost ($\alpha \uparrow$).

To conclude, the size of the approximation errors is relatively low for both vulnerable European calls and puts which indicates that the approximate valuation formulas of our general model work quite well for the given set of parameters. The size of the observed approximation errors is similarly high as in the Klein-Inglis model.

	European Call			European Put		
	Approx. CF	Num. Sol.	Approx. Error	Approx. CF	Num. Sol.	Approx. Error
Base Case	1.9277	1.9261	+0.09%	1.0876	1.0855	+0.19%
$S = 45$	5.1751	5.1790	−0.07%	0.1635	0.1646	−0.71%
$S = 35$	0.2794	0.2782	+0.41%	3.7664	3.7509	+0.41%
$V = 105$	2.0184	2.0164	+0.10%	1.1338	1.1318	+0.17%
$V = 95$	1.8166	1.8152	+0.08%	1.0290	1.0268	+0.21%
$\sigma_S = 0.2$	2.3465	2.3418	+0.20%	1.5484	1.5451	+0.22%
$\sigma_S = 0.1$	1.4932	1.4928	+0.03%	0.6218	0.6210	+0.14%
$\sigma_V = 0.2$	1.8962	1.8945	+0.09%	1.0684	1.0665	+0.18%
$\sigma_V = 0.1$	1.9576	1.9552	+0.12%	1.1059	1.1037	+0.20%
$\sigma_D = 0.2$	1.9143	1.9125	+0.09%	1.0793	1.0758	+0.32%
$\sigma_D = 0.1$	1.9410	1.9389	+0.11%	1.0961	1.0954	+0.07%
$\rho_{SV} = 0.5$	2.0576	2.0575	+0.01%	1.0053	1.0027	+0.26%
$\rho_{SV} = -0.5$	1.7923	1.7902	+0.12%	1.1604	1.1576	+0.24%
$\rho_{VD} = 0.5$	1.9719	1.9696	+0.12%	1.1165	1.1145	+0.18%
$\rho_{VD} = -0.5$	1.9003	1.8984	+0.10%	1.0701	1.0681	+0.19%
$\rho_{SD} = 0.5$	1.9277	1.8015	+7.01%	1.0876	1.1542	−5.77%
$\rho_{SD} = -0.5$	1.9277	2.0474	−5.85%	1.0876	1.0078	+7.92%
$T - t = 1$	2.8399	2.8353	+0.16%	1.2700	1.2663	+0.30%
$T - t = 0.25$	1.3304	1.3294	+0.08%	0.8850	0.8839	+0.13%
$\alpha = 0.5$	1.7296	1.7278	+0.11%	0.9910	0.9884	+0.27%
$\alpha = 0$	2.1258	2.1243	+0.07%	1.1842	1.1827	+0.13%
$r = 0.08$	2.2251	2.2238	+0.06%	0.8827	0.8815	+0.13%
$r = 0.02$	1.6524	1.6502	+0.13%	1.3235	1.3201	+0.25%
$q = 0.02$	1.7254	1.7234	+0.12%	1.2296	1.2267	+0.23%

Table 3.2: Approx. Error in the General Model

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The approximate closed form solutions that are used to compute the option values are given by Equations (3.54) and (3.55), respectively. The points of expansion are chosen to be $p_1 = p_2 = 1.5$ in case of a European call and $p_1 = p_2 = -1.5$ in case of a European put. The numerical solution is calculated by Monte Carlo simulation ($N = 1\,000\,000$).

3.5 Numerical Examples

In this section, we present various numerical examples to compare the results of the different valuation models for European options subject to counterparty risk. Since the full payoff on the option cannot be made if the option writer defaults, it should be expected that vulnerable options will have lower values than otherwise identical non-vulnerable options. Thus, the upper limit for the option values is given by the default-free option price obtained from the Black-Scholes model, in which it is assumed that the counterparty cannot default. Consequently, the value of a vulnerable European option can never be higher than the Black-Scholes option value irrespective of the considered valuation model.

The starting point of the following comparative analysis is a typical market situation for a European option. At time $t = 0$, the option is at the money ($S_0 = 40$, $K = 40$) and expires in six months ($T = 0.5$). The return volatility of the option's underlying equals 15% ($\sigma_S = 0.15$) and its dividend yield is zero ($q = 0$). The risk-free interest rate is assumed to be 5% ($r = 0.05$). The option writer is assumed to be highly levered ($V_0 = 100$, $D_0 = 90$). The return volatility of the counterparty's assets and liabilities is assumed to be 15% ($\sigma_V = 0.15$, $\sigma_D = 0.15$). The correlations between the returns of the option's underlying, the counterparty's assets and liabilities are assumed be zero ($\rho_{SV} = \rho_{VD} = \rho_{SD} = 0$). If the counterparty defaults, deadweight costs of 25% are applied ($\alpha = 0.25$).

Figures 3.5 and 3.6 depict the values of European calls and puts, respectively, as functions of the price of the option's underlying, the option's time to maturity and the value of the counterparty's assets for the valuation models presented in previous section. As expected, the option values obtained from the Klein, Klein-Inglis, Liu-Liu and the general model are always lower than the default-free option value given by the Black-Scholes model. The highest price reduction due to counterparty risk can be observed for our general model followed by the models of Klein and Inglis (2001), Liu and Liu (2011) and Klein (1996).

In the upper left diagram in Figures 3.5 and 3.6, the values of vulnerable European calls and puts, respectively, are plotted against the price of the option's underlying. It is obvious that the price difference between default-free and vulnerable European options increases if the option is deeper in the money. This behavior is applicable for all valuation models, but it is most prominent for the Klein-Inglis and the general model. Furthermore, we observe that the price difference between these two models and the other models increases substantially if the considered European option is further in the money. This observation is attributed to the fact that both the Klein-Inglis and our general model include the option itself directly in the default boundary which additionally increases the counterparty's default risk for in-the-money options.

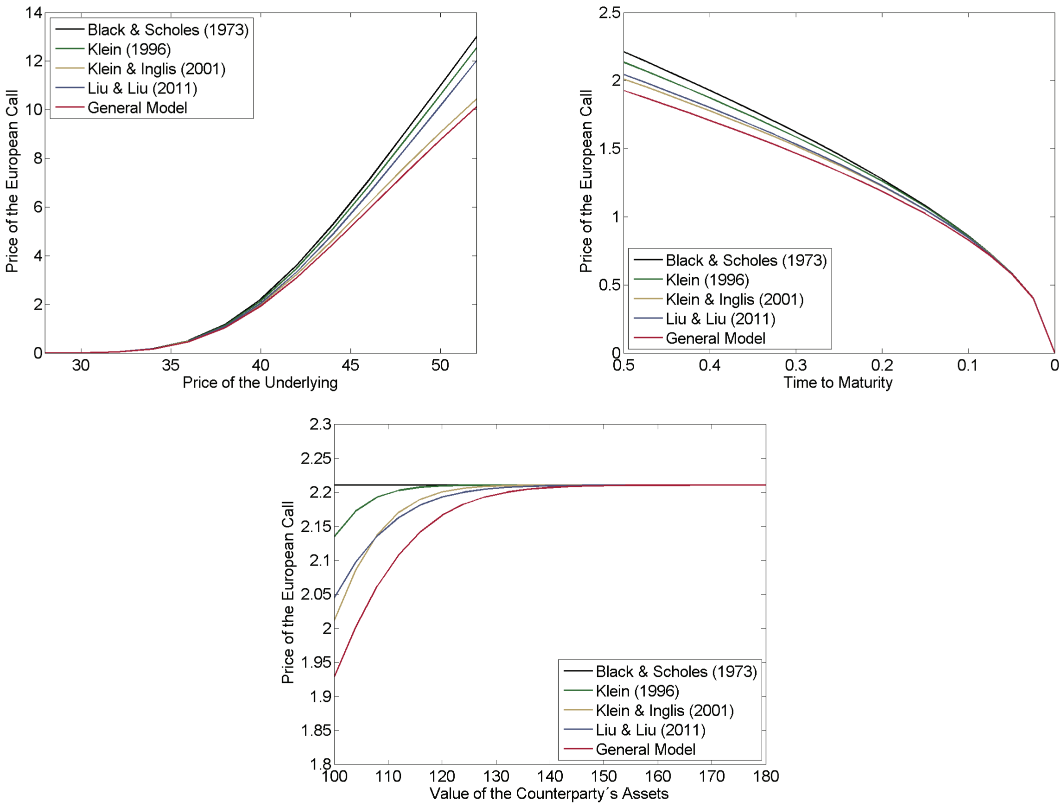

Figure 3.5: European Calls subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_t = 40$, $K = 40$, $V_t = 100$, $D_t = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 3.4. The analytical approximations of the Klein-Inglis and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively.

Referring to the upper right diagram in Figures 3.5 and 3.6, the effect of the time to maturity on the value of vulnerable European options is analyzed. If the time to maturity decreases, the difference between the default-free and the vulnerable European call values is also reduced. This result is not surprising, since the counterparty is less likely to default if the maturity date of the considered European option gets closer.

The lower diagram in Figures 3.5 and 3.6 shows that the prices of a vulnerable European option converge to the default-free option price with increasing values for the counterparty's assets, since the probability that the value of the counterparty's assets hits the default barrier decreases. Our general model has the lowest convergence speed which is most likely explained by the fact that this model is the only one that incorporates three sources of default risk simultaneously: a decrease in the value of the counterparty's assets, an increase in the counterparty's other liabilities and an increase in the option value.

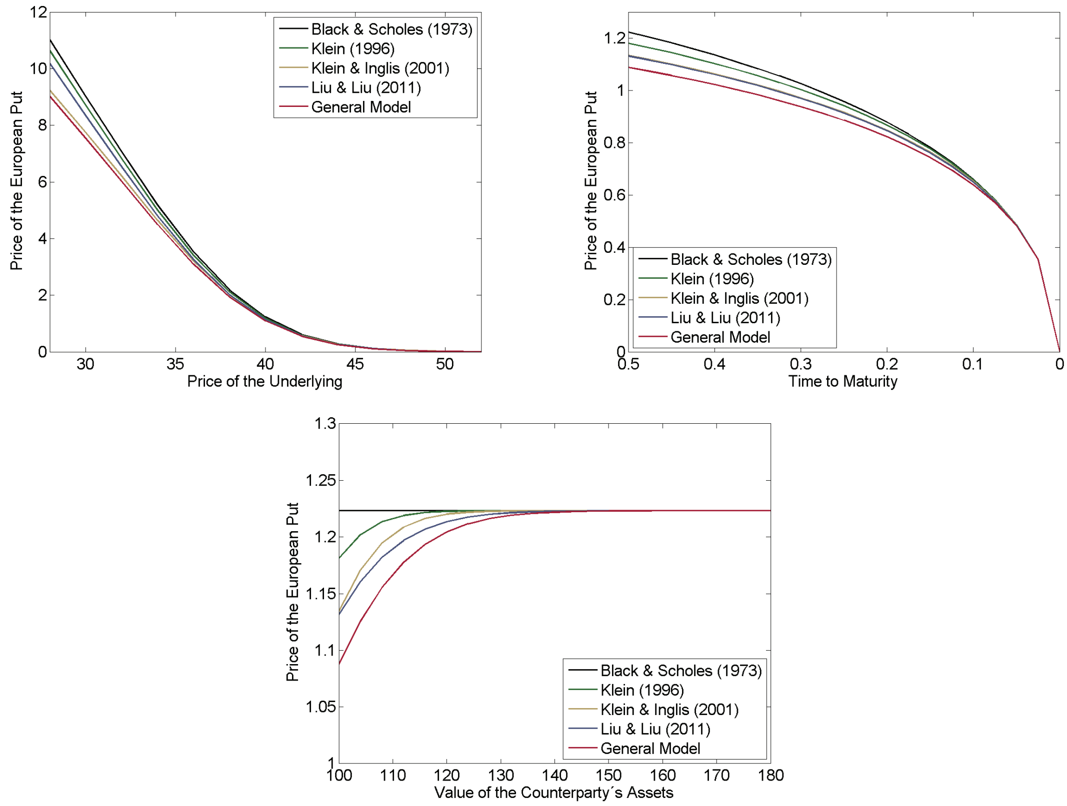

Figure 3.6: European Puts subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_t = 40$, $K = 40$, $V_t = 100$, $D_t = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 3.4. The analytical approximations of the Klein-Inglis and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively.

	General Model	LL2011	KI2001	K1996	BS1973
Base Case	1.9277	2.0446	2.0110	2.1347	2.2108
$S = 45$	5.1751	5.7067	5.3869	5.9582	6.1707
$S = 35$	0.2794	0.2886	0.2912	0.3013	0.3121
$V = 105$	2.0184	2.1084	2.1011	2.1791	2.2108
$V = 95$	1.8166	1.9562	1.8847	2.0516	2.2108
$\sigma_S = 0.2$	2.3465	2.5483	2.4389	2.6606	2.7555
$\sigma_S = 0.1$	1.4932	1.5508	1.5614	1.6192	1.6769
$\sigma_V = 0.2$	1.8962	2.0065	1.9603	2.0776	2.2108
$\sigma_V = 0.1$	1.9576	2.0799	2.0740	2.1897	2.2108
$\sigma_D = 0.2$	1.9143	2.0193	2.0110	2.1347	2.2108
$\sigma_D = 0.1$	1.9410	2.0702	2.0110	2.1347	2.2108
$\rho_{SV} = 0.5$	2.0576	2.1289	2.1521	2.1935	2.2108
$\rho_{SV} = -0.5$	1.7923	1.9396	1.8567	2.0402	2.2108
$\rho_{VD} = 0.5$	1.9719	2.1081	2.0110	2.1347	2.2108
$\rho_{VD} = -0.5$	1.9003	2.0054	2.0110	2.1347	2.2108
$\rho_{SD} = 0.5$	1.9277	1.9396	2.0110	2.1347	2.2108
$\rho_{SD} = -0.5$	1.9277	2.1289	2.0110	2.1347	2.2108
$T - t = 1$	2.8399	3.0730	3.0009	3.2596	3.4367
$T - t = 0.25$	1.3304	1.3865	1.3770	1.4291	1.4540
$\alpha = 0.5$	1.7296	1.9223	1.8560	2.0718	2.2108
$\alpha = 0$	2.1258	2.1670	2.1660	2.1976	2.2108
$r = 0.08$	2.2251	2.3668	2.3553	2.4907	2.5593
$r = 0.02$	1.6524	1.7477	1.6968	1.8076	1.8898
$q = 0.02$	1.7254	1.8254	1.8000	1.9059	1.9739

Table 3.3: European Calls subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_t = 40$, $K = 40$, $V_t = 100$, $D_t = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 3.4. The analytical approximations of the Klein-Inglis and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively. The abbreviations BS1973, K1996, KI2001 and LL2011 stand for the models of Black and Scholes (1973), Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011).

	General Model	LL2011	KI2001	K1996	BS1973
Base Case	1.0876	1.1313	1.1341	1.1811	1.2232
$S = 45$	0.1635	0.1693	0.1718	0.1768	0.1831
$S = 35$	3.7664	3.9993	3.9007	4.1756	4.3245
$V = 105$	1.1338	1.1666	1.1778	1.2057	1.2232
$V = 95$	1.0290	1.0824	1.0682	1.1351	1.2232
$\sigma_S = 0.2$	1.5484	1.6350	1.6102	1.7070	1.7679
$\sigma_S = 0.1$	0.6218	0.6375	0.6496	0.6656	0.6893
$\sigma_V = 0.2$	1.0684	1.1102	1.1032	1.1495	1.2232
$\sigma_V = 0.1$	1.1059	1.1508	1.1724	1.2116	1.2232
$\sigma_D = 0.2$	1.0793	1.1172	1.1341	1.1811	1.2232
$\sigma_D = 0.1$	1.0961	1.1454	1.1341	1.1811	1.2232
$\rho_{SV} = 0.5$	1.0053	1.0637	1.0409	1.1189	1.2232
$\rho_{SV} = -0.5$	1.1604	1.1829	1.2037	1.2159	1.2232
$\rho_{VD} = 0.5$	1.1165	1.1664	1.1341	1.1811	1.2232
$\rho_{VD} = -0.5$	1.0701	1.1096	1.1341	1.1811	1.2232
$\rho_{SD} = 0.5$	1.0876	1.1829	1.1341	1.1811	1.2232
$\rho_{SD} = -0.5$	1.0876	1.0637	1.1341	1.1811	1.2232
$T - t = 1$	1.2700	1.3286	1.3411	1.4093	1.4858
$T - t = 0.25$	0.8850	0.9127	0.9153	0.9408	0.9571
$\alpha = 0.5$	0.9910	1.0636	1.0634	1.1463	1.2232
$\alpha = 0$	1.1842	1.1990	1.2047	1.2159	1.2232
$r = 0.08$	0.8827	0.9163	0.9329	0.9643	0.9908
$r = 0.02$	1.3235	1.3796	1.3584	1.4269	1.4918
$q = 0.02$	1.2296	1.2802	1.2814	1.3366	1.3843

Table 3.4: European Puts subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_t = 40$, $K = 40$, $V_t = 100$, $D_t = 90$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 3.4. The analytical approximations of the Klein-Inglis and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively. The abbreviations BS1973, K1996, KI2001 and LL2011 stand for the models of Black and Scholes (1973), Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011).

Tables 3.3 and 3.4 present the option values for vulnerable European calls and puts, respectively, which are obtained from valuation models presented in Section 3.4. Once again it can be observed that the option values based on the Klein, Klein-Inglis, Liu-Liu and the general valuation model are always lower than the Black-Scholes option values. Furthermore, the option values obtained from our general model differ substantially from those of the other valuation models in most situations. This finding is explained by the construction of the general model's default boundary. The general model is the only one which incorporates three sources of risk simultaneously. First, a decrease in the value of the counterparty's assets might lead to the default of the option writer like in all the other valuation models. Second, the general model accounts for the potential increase in the default risk induced by the option itself (unlike the Klein and the Liu-Liu model). Third, it is assumed that the counterparty's other liabilities are stochastic which creates an additional default risk (unlike the Klein and the Klein-Inglis model). Consequently, the option values based on our general model are the lowest, since it accounts for all possible sources of the counterparty's default risk.

3.6 Summary

In this chapter, the valuation models of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) were presented and discussed. Furthermore, we combied the features of these models in a general valuation model. Therefore, it is the only model which incorporates three sources of financial distress simultaneously: a decline in the value of the counterparty's assets, an increase in the value of the counterparty's other liabilities or an increase in the value of the option itself.

Despite the complexity of the default condition of our general model, we derived an approximate closed form solution for vulnerable European calls and puts. In particular, we approximated the default condition by employing a first order Taylor series expansion and assumed that the returns of the option's underlying and the counterparty's other liabilities are assumed to be uncorrelated. The obtained approximate valuation formula depends on the two points around which the Taylor series is expanded in the derivation. Choosing the points of expansion to be equal to $p_1 = p_2 = 1.5$ in case of a European call and to be equal to $p_1 = p_2 = -1.5$ in case of a European put, respectively, the approximate analytical solution is quite close to the numerical solution for a wide range of parameters.

Based on various numerical examples and graphical illustrations, we compared the results of our general model with those of the alternative models for vulnerable European options. All the considered valuation models have in common that the reduction in the value of a vulnerable European option (compared to a default-free European option) increases if the option is deeper in the money, the time to maturity is longer and if the counterparty's assets are decreased. The option values obtained from the general model are typically the lowest, since it is the only model which accounts for all possible sources of the counterparty's default.

4 European Options subject to Counterparty and Interest Rate Risk

In this chapter, we present and discuss different valuation models for European options subject to counterparty and interest rate risk. The counterparty's default risk is modeled using the structural approach suggested by Merton (1974). In this context, the counterparty's default may occur only at the option's maturity and is triggered by the value of the counterparty's assets being below the value of the counterparty's total liabilities. In addition to that, it is assumed that the risk-free interest rate is stochastic and follows the mean-reverting Ornstein-Uhlenbeck process suggested by Vasicek (1977).

Klein and Inglis (1999) set up a valuation model for vulnerable European options in the stochastic interest rate framework of Vasicek (1977) using the basic idea of Klein (1996). In the following, we extend the valuation models of Klein and Inglis (2001) and Liu and Liu (2011) to the stochastic interest rate framework in the same way as Klein and Inglis (1999) extended the model of Klein (1996).[8]

Furthermore, we set up a general valuation model incorporating the features of the other models. Despite the general model's complexity, we derive an approximate closed form solution. Monte Carlo simulation is used to price vulnerable European options numerically. Comparing the approximate closed form with the numerical solution shows that our valuation formula provides accurate values for vulnerable European options in most situations.

Section 4.1 outlines and discusses the assumptions of the considered stochastic interest rate framework. In Section 4.2, we derive the derivation of the partial differential equation that characterizes the price of a European option subject to counterparty and interest rate risk. Section 4.3 deals with the solution to this partial differential equation. In Section 4.4, we discuss the considered valuation models and derive the respective closed form solutions. Section 4.5 provides a comparative analysis of the different valuation models based on numerical examples. Section 4.6 gives a summary of the main findings.

[8] In Chapter 3, the valuation models of Klein (1996), Klein and Inglis (2001) as well as of Liu and Liu (2011) are presented and discussed in greater details.

4.1 Assumptions

The assumptions characterizing the valuation framework for European options subject to counterparty and interest rate risk are based on the work of Black and Scholes (1973), Merton (1973, 1974), Vasicek (1977), Rabinovitch (1989), Klein (1996), Klein and Inglis (1999, 2001) as well as on Liu and Liu (2011).

1. The price of the option's underlying S_t follows a continuous-time geometric Brownian motion. Assuming that the option's underlying is a dividend-paying stock, its dynamics are given by

$$dS_t = (\mu_S - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S\,, \tag{4.1}$$

where μ_S indicates the expected instantaneous return of the option's underlying, q denotes the continuous dividend yield, σ_S is the instantaneous return volatility and dW_S represents the standard Wiener process.

2. Likewise, the market value of the counterparty's assets V_t follows a continuous-time geometric Brownian motion. Its dynamics are given by

$$dV_t = \mu_V\, V_t\, dt + \sigma_V\, V_t\, dW_V\,, \tag{4.2}$$

where μ_V is the expected instantaneous return of the counterparty's assets, σ_V gives the instantaneous return volatility and dW_V is a standard Wiener process. The instantaneous correlation between dW_S and dW_V equals ρ_{SV}.

3. The total liabilities D_t comprise all the obligations of the counterparty's, i.e. debt, short positions in financial securities and accruals. The dynamics follow a continuous-time geometric Brownian motion which is given by

$$dD_t = \mu_D\, D_t\, dt + \sigma_D\, D_t\, dW_D\,, \tag{4.3}$$

where μ_D is the expected instantaneous return of the counterparty's liabilities, σ_D indicates the instantaneous return volatility and dW_D represents the standard Wiener process. The instantaneous correlation between dW_S and dW_D equals ρ_{SD} and ρ_{VD} between dW_V and dW_D, respectively.

If the counterparty's total liabilities, however, are given by a zero bond only and the risk-free interest rate follows the Ornstein-Uhlenbeck, the expected instantaneous return μ_D as well as the instantaneous return volatility σ_D cannot be chosen arbitrarily anymore. In particular, μ_D and σ_D become time-dependent parameters which are given by the expressions specified in Equation (4.6).[9]

4. The market is perfect and frictionless, i.e. it is free of transaction costs or taxes and the available securities are traded in continuous time.

5. The instantaneous risk-free interest rate r_t is stochastic and follows the Ornstein-Uhlenbeck process suggested by Vasicek (1977). The mean-reverting dynamics of r_t are given by

$$dr_t = \kappa\,(\theta - r_t)\;dt + \sigma_r\,dW_r\,, \tag{4.4}$$

where κ is the speed of reversion, θ represents the long-term mean of the risk-free interest rate, σ_r is the instantaneous volatility of the risk-free interest rate and dW_r represents the standard Wiener process. The instantaneous correlations between dW_r and dW_S, between dW_r and dW_V as well as between dW_r and dW_D are equal to ρ_{Sr}, ρ_{Vr} and ρ_{Dr}, respectively.

In the considered stochastic interest rate framework, a closed form solution for the price of a risk-free zero bond paying one dollar at maturity T can be derived (Vasicek, 1977; Mamon, 2004). Denoting the price at time t of a zero bond by $B_{t,T}$, the analytical bond price formula is given by

$$B_{t,T} = e^{A_{t,T}\,r_t + C_{t,T}} \tag{4.5}$$

where

$$\begin{aligned} A_{t,T} &= \frac{1}{\kappa}\left(1 - e^{-\kappa\,(T-t)}\right) \\ C_{t,T} &= \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right)\left(A_{t,T} - (T-t)\right) - \frac{\sigma_r^2 A_{t,T}^2}{4\kappa} \end{aligned}$$

[9] This issue only affects the extended model of Liu and Liu (2011) as well as the general model, since it is assumed that the counterparty's liabilities are stochastic in these two models exclusively (see Sections 4.4.4 and 4.4.5).

The instantaneous expected return and the return volatility of the risk-free zero bond are time-dependent. In particular, they are given as follows:

$$\mu_B(t) = r_t + \frac{1 - e^{-\kappa(T-t)}}{\kappa}\,\sigma_r, \qquad \sigma_B(t) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}\,\sigma_r. \tag{4.6}$$

6. The expected instantaneous return of the option's underlying as well as of the counterparty's assets and liabilities (μ_S, μ_V and μ_D) are constant over time. The same applies for the dividend yield of the option's underlying.

7. The instantaneous return volatilities of the option's underlying as well as of the counterparty's assets and liabilities (σ_S, σ_V and σ_D) are constant over time. The same applies for the risk-free interest rate's instantanenous volatility σ_r as well as for the instantaneous correlations ρ_{SV}, ρ_{SD}, ρ_{VD}, ρ_{Sr}, ρ_{Vr} and ρ_{Dr}.

8. All the liabilities of the counterparty (i.e. debt, short positions in financial securties, etc.) are assumed to be of equal rank.

9. Default can only occur at the option's maturity T. The counterparty is in default, if the counterparty's assets V_T are less than the threshold level L:

$$V_T < \bar{L} \quad \text{or} \quad V_T < L(S_T,\, D_T). \tag{4.7}$$

Depending on the considered valuation model, the threshold level L is characterized in different ways and is either a constant or a function of the stochastic variables S_T and D_T.

10. If the counterparty is in default, the option holder's claim must be determined. In principle, the option holder's claim is equal to the intrinsic value of the European option at its maturity. In case of the counterparty's default, however, the option holder faces a percentage write-down ω on his claim. In default, the holder of a European call or put receives

$$(1 - \omega)\,\max(S_T - K, 0) \qquad \text{or} \qquad (1 - \omega)\,\max(K - S_T, 0). \tag{4.8}$$

The percentage write-down ω on the option holder's claim in case of the counterparty's default can be endogenized. Assuming that all the liabilities of the counterparty are ranked equally, the amount payable to the holder of a European call is given by

$$(1-\omega)\,\max\left(S_T-K,0\right)=\frac{(1-\alpha)\,V_T}{L(S_T,\,D_T)}\,\max\left(S_T-K,0\right), \tag{4.9}$$

whereas it is given by

$$(1-\omega)\,\max\left(K-S_T,0\right)=\frac{(1-\alpha)\,V_T}{L(S_T,\,D_T)}\,\max\left(K-S_T,0\right) \tag{4.10}$$

for the holder of a European put. The parameter α represents the cost of default as a percentage of the counterparty's assets and the ratio $V_T/L(S_T,\,D_T)$ gives the proportion of the option holder's claim which can be paid back.

Based on Assumptions 9 and 10, the counterparty can only default at the option's maturity which is in line with the valuation models of Klein (1996), Klein and Inglis (1999, 2001) and Liu and Liu (2011). Due to this assumption, the valuation models become mathematically tractable, i.e. analytical or approximate analytical solutions can be derived. However, this assumption might be criticized as being too restrictive and not taking into account the real-world circumstances of the default occurring prior to the option's maturity. Refering to Klein and Inglis (2001), the assumption that default can only occur at the option's maturity is less restrictive as it initially seems due to the special treatment of OTC European options if the counterparty defaults. Most OTC European option contracts are concluded in compliance with the standards recommended by the International Swap and Derivatives Association (ISDA). In contrast to other financial instruments subject to counterparty risk, the option holder does not have to determine his claim associated with the considered OTC option immediately at the default date but has the right to wait until the maturity date is reached. Even if the option holder decides not to wait until the option's maturity to determine his claim, Assumptions 9 and 10 can still be valid. Based on the ISDA standardized contract for OTC European options, the option holder's claim at the counterparty's default is equal to the market value of the option at that point in time. This market value, in turn, depends on the expected option payoff at maturity. Another point in favor of assuming that default can only occur at option maturity is the fact that there is typically a time lag between the default event and the point in time, at which the counterparty's assets are distributed among all claim holders. Consequently, the option's maturity is a valid proxy for the date at which it is determined whether the counterparty is in default or not.

4.2 Derivation of the Partial Differential Equation

Following the argument of Fang (2012), we derive the partial differential equation governing the price evolution of a vulnerable European option under stochastic interest rates. In the considered framework (see Section 4.1), the price of a vulnerable European option F_t must be a function of the underlying S_t, the counterparty's assets V_t, the counterparty's liabilities D_t, the risk-free interest rate r_t and time t. According to Itô's lemma, the price evolution of a vulnerable European option is given by the following stochastic differential equation:

$$
\begin{aligned}
dF_t = {} & \frac{\partial F_t}{\partial t}dt + (\mu_S - q)S_t\frac{\partial F_t}{\partial S_t}dt + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2}dt + \sigma_S S_t\frac{\partial F_t}{\partial S_t}dW_S \\
& + \mu_V V_t\frac{\partial F_t}{\partial V_t}dt + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2}dt + \sigma_V V_t\frac{\partial F_t}{\partial V_t}dW_V + \mu_D D_t\frac{\partial F_t}{\partial D_t}dt \\
& + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2}dt + \sigma_D D_t\frac{\partial F_t}{\partial D_t}dW_D + \kappa(\theta - r_t)\frac{\partial F_t}{\partial r_t}dt + \frac{1}{2}\sigma_r^2\frac{\partial^2 F_t}{\partial r_t^2}dt \\
& + \sigma_r\frac{\partial F_t}{\partial r_t}dW_r + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t \partial V_t}dt + \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t \partial D_t}dt \\
& + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t \partial D_t}dt + \rho_{Sr}\sigma_S\sigma_r S_t\frac{\partial^2 F_t}{\partial S_t \partial r_t}dt \\
& + \rho_{Vr}\sigma_V\sigma_r V_t\frac{\partial^2 F_t}{\partial V_t \partial r_t}dt + \rho_{Dr}\sigma_D\sigma_r D_t\frac{\partial^2 F_t}{\partial D_t \partial r_t}dt.
\end{aligned} \tag{4.11}
$$

To eliminate the four Wiener processes dW_S, dW_V, dW_D and dW_r, a portfolio Π_t which consists of the European option F_t, the underlying S_t, the counterparty's assets V_t, the counterparty's liabilities D_t and the risk-free zero bond $B_{t,T}$ is set up.[10] In particular, this portfolio consists of a short position in the European option

[10] To construct such a portfolio, it is necessary to assume that option's underlying, the counterparty's assets and liabilities as well as the risk-free zero bond are traded securities. This assumption is not questionable for the option's underlying and the risk-free zero bond, but it is for both the counterparty's assets and liabilities. As argued by Klein (1996), it is likely that the counterparty's assets and liabilities are not traded directly in the market, but that their market values behave similarly as if they were traded securities.

and long positions in the underlying, the counterparty's assets and liabilities as well as in the risk-free zero bond. The amount of shares in the long positions are equal to $\partial F_t/\partial S_t$, $\partial F_t/\partial V_t$, $\partial F_t/\partial D_t$ and $\partial F_t/\partial r_t\,\partial r_t/\partial B_{t,T}$, respectively. Hence, the value of the portfolio at time t is given by

$$\Pi_t = -F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}} B_{t,T}. \tag{4.12}$$

The change in the value of the portfolio over the time interval dt is characterized by the total differential which is equal to

$$d\Pi_t = -dF_t + \frac{\partial F_t}{\partial S_t} dS_t + \frac{\partial F_t}{\partial V_t} dV_t + \frac{\partial F_t}{\partial D_t} dD_t + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}} dB_{t,T}. \tag{4.13}$$

Using Itô's lemma, the dynamics of the risk-free zero bond can be set up. The dynamics $dB_{t,T}$ are given by

$$dB_{t,T} = \frac{\partial B_{t,T}}{\partial t} dt + \kappa(\theta - r_t)\frac{\partial B_{t,T}}{\partial r_t} dt + \sigma_r \frac{\partial B_{t,T}}{\partial r_t} dW_r + \frac{1}{2}\sigma_r^2 \frac{\partial^2 B_{t,T}}{\partial r_t^2} dt. \tag{4.14}$$

Under the martingale measure, the dynamics of the risk-free zero bond given by Equation (4.14) can be rewritten as follows (see Fang, 2012):

$$dB_{t,T} = r_t B_{t,T} dt + \sigma_r \frac{\partial B_{t,T}}{\partial r_t} dW_r. \tag{4.15}$$

Substituting Equations (4.1) to (4.3), (4.11) and (4.15) into Equation (4.13) yields the following expression:

$$\begin{aligned} d\Pi_t = &-\frac{\partial F_t}{\partial t} dt + q S_t \frac{\partial F_t}{\partial S_t} dt - \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt - \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt \\ &- \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt - \kappa(\theta - r_t)\frac{\partial F_t}{\partial r_t} dt - \frac{1}{2}\sigma_r^2 \frac{\partial^2 F_t}{\partial r_t^2} dt - \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt \\ &- \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt - \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt - \rho_{Sr}\sigma_S\sigma_r S_t \frac{\partial^2 F_t}{\partial S_t \partial r_t} dt \\ &- \rho_{Vr}\sigma_V\sigma_r V_t \frac{\partial^2 F_t}{\partial V_t \partial r_t} dt - \rho_{Dr}\sigma_D\sigma_r D_t \frac{\partial^2 F_t}{\partial D_t \partial r_t} dt + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}} r_t B_{t,T} dt. \end{aligned} \tag{4.16}$$

Since the dynamics of portfolio Π_t are independent of the four Wiener processes dW_S, dW_V,dW_D and dW_B, the portfolio must be riskless during the infinitesimal time interval dt. Consequently, the portfolio must earn the same return as other short-term risk-free investments, namely the risk-free interest rate r_t, to avoid arbitrage opportunities:

$$r_t \Pi dt = d\Pi_t. \tag{4.17}$$

We substitute Equations (4.12) and (4.16) into Equation (4.17) which yields the following expression:

$$\begin{aligned}
& r_t \left(-F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t + \frac{\partial F_t}{\partial r_t} \frac{\partial r_t}{\partial B_{t,T}} B_{t,T} \right) dt \qquad (4.18) \\
&\quad = -\frac{\partial F_t}{\partial t} dt + q S_t \frac{\partial F_t}{\partial S_t} dt - \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt - \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt - \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt \\
&\qquad - \kappa(\theta - r_t) \frac{\partial F_t}{\partial r_t} dt - \frac{1}{2}\sigma_r^2 \frac{\partial^2 F_t}{\partial r_t^2} dt - \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt \\
&\qquad - \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt - \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt - \rho_{Sr}\sigma_S\sigma_r S_t \frac{\partial^2 F_t}{\partial S_t \partial r_t} dt \\
&\qquad - \rho_{Vr}\sigma_V\sigma_r V_t \frac{\partial^2 F_t}{\partial V_t \partial r_t} dt - \rho_{Dr}\sigma_D\sigma_r D_t \frac{\partial^2 F_t}{\partial D_t \partial r_t} dt + \frac{\partial F_t}{\partial r_t} \frac{\partial r_t}{\partial B_{t,T}} r_t B_{t,T} dt.
\end{aligned}$$

Rewriting Equation (4.18), the partial differential equation that characterizes the price of a European option whose payoff at time T is contigent upon the price of the option's underlying as well as upon the value of both the counterparty's assets and liabilities is obtained. It is given by

$$\begin{aligned}
0 = {} & \frac{\partial F_t}{\partial t} + (r_t - q) S_t \frac{\partial F_t}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} + r_t V_t \frac{\partial F_t}{\partial V_t} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} \qquad (4.19) \\
& + r_t D_t \frac{\partial F_t}{\partial D_t} + \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} + \kappa(\theta - r_t) \frac{\partial F_t}{\partial r_t} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 F_t}{\partial r_t^2} \\
& + \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} + \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} \\
& + \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} + \rho_{Sr}\sigma_S\sigma_r S_t \frac{\partial^2 F_t}{\partial S_t \partial r_t} \\
& + \rho_{Vr}\sigma_V\sigma_r V_t \frac{\partial^2 F_t}{\partial V_t \partial r_t} + \rho_{Dr}\sigma_D\sigma_r D_t \frac{\partial^2 F_t}{\partial D_t \partial r_t} - r_t F_t.
\end{aligned}$$

To obtain a unique solution to the above partial differential equation, we must set up the boundary conditions which specify the value of the European option at the boundaries of S_t, V_t, D_t and t. The key boundary condition specifies the option payoff at maturity. Based on Assumptions 10, the boundary condition for the European call is thus equal to

$$F_T = C_T = \begin{cases} S_T - K & \text{if } S_T \geq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(S_T - K) & \text{if } S_T \geq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

whereas the boundary condition for the vulnerable European put is given by

$$F_T = P_T = \begin{cases} K - S_T & \text{if } S_T \leq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(K - S_T) & \text{if } S_T \leq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{4.21}$$

For both European calls and puts, the first line in the boundary condition refers to the situation in which the option is in the money at maturity and the counterparty does not default, i.e. $S_T - K$ and $K - S_T$ are paid out to the holder of a European call and a European put, respectively. The second line indicates the option payoff if the option expires in the money and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportional payout ratio is given by $((1-\alpha)\,V_T)\,/L(S_T,\, D_T)$, i.e. the value counterparty's assets available for distribution is divided by the value of the counterparty's total liabilities. Hence, the holder of a European call receives $((1-\alpha)\,V_T\,(S_T - K))\,/L(S_T,\, D_T)$, whereas the holder of a European put receives $((1-\alpha)\,V_T\,(K - S_T))\,/L(S_T,\, D_T)$. The third line refers to the out-of-the-money scenario, in which the option holder receives nothing irrespective of whether the counterparty defaults or not.

The actual characterization of the boundary conditions depends on the choice of a specific valuation model (see Section 4.4). In particular, the variable $L(S_T,\, D_T)$ must be defined according to the chosen model.

4.3 Solution to the Partial Differential Equation

The partial differential equation given by Equation (4.19) depends on the price of the option's underlying, the counterparty's assets and liabilities, the risk-free interest rate, the dividend yield of the option's underlying as well as on the return volatilities. All these variables and parameters are indepedent of the risk preferences of the investors.[11] Since the risk preferences of the investors do not enter the partial differential equation, they cannot affect its solution. Consequently, any type of risk preferences can be used when solving the partial differential equation.

Using the approach of Cox and Ross (1976) and Harrison and Pliska (1981), the risk-neutral stochastic processes for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities are equal to

$$dS_t = (r_t - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S, \tag{4.22}$$

$$dV_t = r_t\, V_t\, dt + \sigma_V\, V_t\, dW_V \tag{4.23}$$

and

$$dD_t = r_t\, D_t\, dt + \sigma_D\, D_t\, dW_D, \tag{4.24}$$

where r_t denotes the risk-free interest rate and all other variables are defined as before.

Applying Itô's lemma to Equations (4.22) to (4.24), the stochastic processes for $\ln S_t$, $\ln V_t$ and $\ln D_t$ are obtained. They are given by

$$d\ln S_t = \left(r_t - q - \frac{1}{2}\sigma_S^2\right) dt + \sigma_S dW_S, \tag{4.25}$$

$$d\ln V_t = \left(r_t - \frac{1}{2}\sigma_V^2\right) dt + \sigma_V dW_V \tag{4.26}$$

[11] Following the argument of Hull (2012: 311–312), the partial differential equation given by Equation (4.19) would not be independent of risk preferences if it included the expected returns of the option's underlying, the counterparty's assets and the counterparty's liabilities. These parameters depend on risk preferences, since their magnitude represents the level of risk aversion of the investor: the higher the level of the investor's risk aversion, the higher the required expected return.

and

$$d\ln D_t = \left(r_t - \frac{1}{2}\sigma_D^2\right) dt + \sigma_D dW_D. \tag{4.27}$$

Rewriting Equations (4.25) to (4.27), the expressions for the price of the option's underlying as well as for the values of the counterparty's assets and liabilities at the option's maturity are obtained (see Kim & Kunitomo, 1999). They are equal to

$$S_T = \frac{S_t}{B_{t,T}}\, e^{-0.5\,\bar{\sigma}_S^2 + \bar{\sigma}_S\, x_S}, \tag{4.28}$$

$$V_T = \frac{V_t}{B_{t,T}}\, e^{-0.5\,\bar{\sigma}_V^2 + \bar{\sigma}_V\, x_V} \tag{4.29}$$

and

$$D_T = \frac{D_t}{B_{t,T}}\, e^{-0.5\,\bar{\sigma}_D^2 + \bar{\sigma}_D\, x_D}, \tag{4.30}$$

where the three random variables x_S, x_V and x_D are jointly standard normally distributed and their respective correlations are given by $\bar{\rho}_{SV}$, $\bar{\rho}_{SD}$ and $\bar{\rho}_{VD}$. The adjusted variances, covariances and correlation coefficients in the stochastic interest rate framework of Vasicek (1977) are given as follows:

$$\begin{aligned}
\bar{\sigma}_S^2 = &\left(\sigma_S^2 + \frac{\sigma_r^2}{\kappa^2} + \frac{2\rho_{Sr}\sigma_S\sigma_r}{\kappa}\right)(T-t) \\
&+ \left(e^{-\kappa(T-t)} - 1\right)\left(\frac{2\sigma_r^2}{\kappa^3} + \frac{2\rho_{Sr}\sigma_S\sigma_r}{\kappa^2}\right) - \left(e^{-2\kappa(T-t)} - 1\right)\frac{\sigma_r^2}{2\kappa^3},
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_V^2 = &\left(\sigma_V^2 + \frac{\sigma_r^2}{\kappa^2} + \frac{2\rho_{Vr}\sigma_V\sigma_r}{\kappa}\right)(T-t) \\
&+ \left(e^{-\kappa(T-t)} - 1\right)\left(\frac{2\sigma_r^2}{\kappa^3} + \frac{2\rho_{Vr}\sigma_V\sigma_r}{\kappa^2}\right) - \left(e^{-2\kappa(T-t)} - 1\right)\frac{\sigma_r^2}{2\kappa^3},
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_D^2 = &\left(\sigma_D^2 + \frac{\sigma_r^2}{\kappa^2} + \frac{2\rho_{Dr}\sigma_D\sigma_r}{\kappa}\right)(T-t) \\
&+ \left(e^{-\kappa(T-t)} - 1\right)\left(\frac{2\sigma_r^2}{\kappa^3} + \frac{2\rho_{Dr}\sigma_D\sigma_r}{\kappa^2}\right) - \left(e^{-2\kappa(T-t)} - 1\right)\frac{\sigma_r^2}{2\kappa^3},
\end{aligned}$$

$$\begin{aligned}
\bar{\sigma}_{SV} = &\left(\rho_{SV}\sigma_S\sigma_V + \frac{\sigma_r^2}{\kappa^2} + \frac{\rho_{Sr}\sigma_S\sigma_r}{\kappa} + \frac{\rho_{Vr}\sigma_V\sigma_r}{\kappa}\right)(T-t) \\
&+ \left(e^{-\kappa(T-t)} - 1\right)\left(\frac{\rho_{Sr}\sigma_S\sigma_r}{\kappa^2} + \frac{\rho_{Vr}\sigma_V\sigma_r}{\kappa^2} + \frac{2\sigma_r^2}{\kappa^3}\right) - \left(e^{-2\kappa(T-t)} - 1\right)\frac{\sigma_r^2}{2\kappa^3},
\end{aligned}$$

$$\bar{\sigma}_{SD} = \left(\rho_{SD}\sigma_S\sigma_D + \frac{\sigma_r^2}{\kappa^2} + \frac{\rho_{Sr}\sigma_S\sigma_r}{\kappa} + \frac{\rho_{Dr}\sigma_D\sigma_r}{\kappa}\right)(T-t)$$
$$+ \left(e^{-\kappa(T-t)} - 1\right)\left(\frac{\rho_{Sr}\sigma_S\sigma_r}{\kappa^2} + \frac{\rho_{Dr}\sigma_D\sigma_r}{\kappa^2} + \frac{2\sigma_r^2}{\kappa^3}\right) - \left(e^{-2\kappa(T-t)} - 1\right)\frac{\sigma_r^2}{2\kappa^3},$$

$$\bar{\sigma}_{VD} = \left(\rho_{VD}\sigma_V\sigma_D + \frac{\sigma_r^2}{\kappa^2} + \frac{\rho_{Vr}\sigma_V\sigma_r}{\kappa} + \frac{\rho_{Dr}\sigma_D\sigma_r}{\kappa}\right)(T-t)$$
$$+ \left(e^{-\kappa(T-t)} - 1\right)\left(\frac{\rho_{Vr}\sigma_V\sigma_r}{\kappa^2} + \frac{\rho_{Dr}\sigma_D\sigma_r}{\kappa^2} + \frac{2\sigma_r^2}{\kappa^3}\right) - \left(e^{-2\kappa(T-t)} - 1\right)\frac{\sigma_r^2}{2\kappa^3},$$

$$\bar{\rho}_{SV} = \frac{\bar{\sigma}_{SV}}{\bar{\sigma}_S\,\bar{\sigma}_V}, \qquad \bar{\rho}_{SD} = \frac{\bar{\sigma}_{SD}}{\bar{\sigma}_S\,\bar{\sigma}_D}, \qquad \bar{\rho}_{VD} = \frac{\bar{\sigma}_{VD}}{\bar{\sigma}_V\,\bar{\sigma}_D}.$$

The Feynman-Kač theorem states that the solution to the partial differential equation specified in Equation (4.19) is given by

$$F_t = \mathbb{E}\left[e^{-\int_t^T r_u\,du} g\left(S_T, V_T, D_T\right)\right], \tag{4.31}$$

where $\mathbb{E}[\,\cdot\,]$ denotes the expectation under the risk-neutral measure and function $g(\,\cdot\,)$ determines the payoff of the considered European option (Musiela & Rutkowski, 2005: 296; Pennacchi, 2008: 209–210; Fang, 2012). Consequently, the value of a vulnerable European option is equal to the expected payoff at maturity which is discounted at the risk-free interest rate.

According to Assumption 5, the dynamics of the risk-free interest rate are driven by the Ornstein-Uhlenbeck process suggested by Vasicek (1977). Consequently, Equation (4.31) can be rewritten as

$$F_t = B_{t,T}\,\mathbb{E}\left[g\left(S_T, V_T, D_T\right)\right], \tag{4.32}$$

where $B_{t,T}$ represents the discount factor which is equal to the price of the risk-free zero bond given by Equation (4.5).

Equation (4.32) can be used to set up the pricing equations for vulnerable European calls and puts by specifying the payoff function $g(\,\cdot\,)$ accordingly. In particular, if the payoff function $g(\,\cdot\,)$ is defined according to the boundary condition given by

Equation (4.20), the pricing equation for vulnerable European calls is received which is given by

$$
\begin{aligned}
C_t = B_{t,T} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{[S_T \geq K,\, V_T \geq L(S_T,\, D_T)]}\right] \\
& + \mathbb{E}\left[\frac{(1-\alpha)\, V_T}{L(S_T,\, D_T)} \, (S_T - K) \cdot 1_{[S_T \geq K,\, V_T < L(S_T,\, D_T)]}\right]\Bigg) .
\end{aligned}
\tag{4.33}
$$

In the same manner, the pricing equation for vulnerable European puts is obtained if the boundary condition given by Equation (4.21) is used to specify the payoff function $g(\,\cdot\,)$:

$$
\begin{aligned}
P_t = B_{t,T} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K,\, V_T \geq L(S_T,\, D_T)]}\right] \\
& + \mathbb{E}\left[\frac{(1-\alpha)\, V_T}{L(S_T,\, D_T)} \, (K - S_T) \cdot 1_{[S_T \leq K,\, V_T < L(S_T,\, D_T)]}\right]\Bigg) .
\end{aligned}
\tag{4.34}
$$

In both pricing equations, the first line gives the expected payoff if the option is in the money at maturity and to the counterparty does not default. The second line, in turn, gives the expected payoff if the option expires in the money and the counterparty is in default. The out-of-the-money scenario is only implicitly specified, since the option payoff is equal to zero in this case.

To derive analytic valuation formulas for both vulnerable European calls and puts based on the above pricing equations, the following major steps must be performed. First, the variable $L(S_T,\, D_T)$ indicating the default condition must be characterized in accordance with the considered valuation model. Subsequently, the expected value expressions in Equations (4.33) and (4.34) are rewritten as integrals, since S_T, V_T and D_T are continuous random variables. Afterwards, the expressions for the market values of the option's underlying, the counterparty's assets and the counterparty's liabilities at the option's maturity specified by Equations (4.28) and (4.30) are inserted and the density function of the corresponding trivariate normal distribution is standardized. Finally, the closed form solutions for vulnerable European options are received after some algebraic transformations (see Section 4.4).

4.4 Valuation Models

Various models to value vulnerable European options have been developed over the last three decades assuming a deterministic and constant risk-free interest rate. Klein and Inglis (1999) extend the valuation model of Klein (1996) to the stochastic interest rate framework of Vasicek (1977). In the following, we extend the valuation models of Klein and Inglis (2001) and Liu and Liu (2011) in the same way. Furthermore, we set up a general valuation model incorporating the features of the other models.

4.4.1 Absence of Default Risk

Rabinovitch (1989) extends the model of Black and Scholes (1973) to account for stochastic interest rates driven by the Ornstein-Uhlenbeck process suggested by Vasicek (1977). Consequently, the Rabinovitch model gives the default-free value of a European option which serves as an upper price limit. The pricing equations given by Equations (4.33) and (4.34) are substantially simplified, since the second summand vanishes completely due to the absence of counterparty risk. The pricing equation for a European call is given by

$$C_t = B_{t,T}\,\mathbb{E}\left[(S_T - K)\;\cdot 1_{[S_T \geq K]}\right]. \tag{4.35}$$

whereas the pricing equation for a European put is equal to

$$P_t = B_{t,T}\,\mathbb{E}\left[(K - S_T)\;\cdot 1_{[S_T \leq K]}\right]. \tag{4.36}$$

Since the counterparty cannot default, the structure of the pricing equations is rather simple. If the option expires in the money, the payoff of a European call is equal to $S_T - K$, whereas the payoff of the European put is given by $K - S_T$. If the option is out of the money at maturity, the option holder receives nothing.

Computing the expected values given by Equations (4.35) and (4.36), the closed-form valuation formulas for default-free European options are derived (see Rabinovitch, 1989). For European calls and puts, these valuation formulas are given by

$$C_t = S_t\,e^{-q(T-t)}\,N(a_1) - B_{t,T}\,K\,N(a_2) \tag{4.37}$$

and

$$P_t = B_{t,T}\, K\, N(-a_2) - S_t\, e^{-q(T-t)}\, N(-a_1), \tag{4.38}$$

where $N(\,\cdot\,)$ represents the cumulative distribution function of the univariate standard normal distribution and where a_1 and a_2 are given as follows:

$$\begin{aligned} a_1 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) + \frac{1}{2}\, \bar{\sigma}_S^2}{\bar{\sigma}_S} \\ a_2 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\, \bar{\sigma}_S^2}{\bar{\sigma}_S}. \end{aligned}$$

4.4.2 Deterministic Liabilities

Klein and Inglis (1999) extend the model of Klein (1996) to the stochastic interest rate framework suggested by Vasicek (1977), while the conditions for the counterparty's default are the same as in the original model. The counterparty is in default if its assets are not sufficient to meet its total liabilities at the option's maturity. The total liabilities of the counterparty are assumed to be deterministic and must include the short position in the option, since it obliges the option writer to deliver or purchase the option's underlying at maturity.

In particular, Klein and Inglis (1999) assume that the market value of the counterparty's total liabilities at the option's maturity is equal to its initial market value. To put it differently, the level of the counterparty's total liabilities is therefore constant over time. Consequently, the default boundary $L(S_T,\, D_T)$ must be given by the following expression:

$$L(S_T,\, D_T) = \bar{L} = \bar{D} = D_t. \tag{4.39}$$

Inserting the above expression into Equations (4.33) and (4.34) yields the pricing equations of the extended Klein model. The pricing equation for a vulnerable European call equals

$$\begin{aligned} C_t = B_{t,T} \Bigg(\mathbb{E}\left[(S_T - K) \cdot 1_{\left[S_T \geq K,\, V_T \geq \bar{D}\right]} \right] \qquad\qquad (4.40) \\ + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{\bar{D}} \cdot 1_{\left[S_T \geq K,\, V_T < \bar{D}\right]} \right] \Bigg), \end{aligned}$$

whereas the pricing equation for a vulnerable European put is given by

$$P_t = B_{t,T} \left(\mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K,\, V_T \geq \bar{D}]} \right] + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{\bar{D}} \cdot 1_{[S_T \leq K,\, V_T < \bar{D}]} \right] \right). \tag{4.41}$$

In both pricing equations, the first line is related to the situation in which the option expires in the money and the counterparty does not default. Hence, the payoff of a European call is equal to $S_T - K$, whereas the payoff of the European put is given by $K - S_T$. The second line gives the payoff if the option is in the money at maturity and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $((1-\alpha)\, V_T)\, / \bar{D}$, i.e. the asset value available for distribution is divided by the value of the counterparty's total liabilities. The holder of a European call receives $((1-\alpha)\, V_T\, (S_T - K))\, / \bar{D}$, whereas $((1-\alpha)\, V_T\, (K - S_T))\, / \bar{D}$ is paid out to the holder of a European put. If the option expires out of the money, the option holder receives nothing irrespective of whether the counterparty defaults or not.

Computing the expected values given by Equations (4.40) and (4.41), the closed-form valuation formulas for vulnerable European options based on the model of Klein and Inglis (1999) are obtained (see Klein & Inglis, 1999). They are given by

$$\begin{aligned} C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \bar{\rho}_{SV}) - B_{t,T}\, K\, N_2(a_2, b_2, \bar{\rho}_{SV}) \\ & + \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t) + \bar{\rho}_{SV} \bar{\sigma}_S \bar{\sigma}_V}}{B_{t,T}\, D_t}\, N_2(a_3, b_3, -\bar{\rho}_{SV}) \\ & - \frac{(1-\alpha) V_t\, K}{D_t}\, N_2(a_4, b_4, -\bar{\rho}_{SV}) \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} P_t = {} & B_{t,T}\, K\, N_2(-a_2, b_2, -\bar{\rho}_{SV}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\bar{\rho}_{SV}) \\ & + \frac{(1-\alpha) V_t\, K}{D_t}\, N_2(-a_4, b_4, \bar{\rho}_{SV}) \\ & - \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t) + \bar{\rho}_{SV} \bar{\sigma}_S \bar{\sigma}_V}}{B_{t,T}\, D_t}\, N_2(-a_3, b_3, \bar{\rho}_{SV}), \end{aligned} \tag{4.43}$$

where $N_2(\,\cdot\,)$ is the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) + \frac{1}{2}\,\bar{\sigma}_S^2}{\bar{\sigma}_S}, \\
a_2 &= \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) - \frac{1}{2}\,\bar{\sigma}_S^2}{\bar{\sigma}_S}, \\
a_3 &= \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) + \frac{1}{2}\,\bar{\sigma}_S^2 + \bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V}{\bar{\sigma}_S}, \\
a_4 &= \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) - \frac{1}{2}\,\bar{\sigma}_S^2 + \bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V}{\bar{\sigma}_S}, \\
b_1 &= \frac{\ln \frac{V_t}{B_{t,T}\,D_t} - \frac{1}{2}\,\bar{\sigma}_V^2 + \bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V)}{\bar{\sigma}_V}, \\
b_2 &= \frac{\ln \frac{V_t}{B_{t,T}\,D_t} + \frac{1}{2}\,\bar{\sigma}_V^2}{\bar{\sigma}_V}, \\
b_3 &= -\frac{\ln \frac{V_t}{B_{t,T}\,D_t} + \frac{1}{2}\,\bar{\sigma}_V^2 + \bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V}{\bar{\sigma}_V}, \\
b_4 &= -\frac{\ln \frac{V_t}{B_{t,T}\,D_t} + \frac{1}{2}\,\bar{\sigma}_V^2}{\bar{\sigma}_V}.
\end{aligned}
$$

4.4.3 Deterministic Liabilities and Option induced Default Risk

We extend the model of Klein and Inglis (2001) to the stochastic interest rate framework suggested by Vasicek (1977) in the same way as Klein and Inglis (1999) extended the model of Klein (1996). Like in the original model, we still assume that the short position in the option itself may cause additional financial distress. To account for this potential source of default risk, the counterparty's total liabilities are split into two components. In particular, the total liabilities now consist of the short position in the option on the one hand and all the other liabilities on the other.

Klein and Inglis (2001) assume that the market value of the counterparty's total liabilities at the option's maturity is equal to its initial market value. To put it differently, the level of the counterparty's total liabilities is therefore constant over time. The value of the short position in the option is taken into account separately. Combining these two features, the counterparty's total liabilities are given by either $\bar{D}+S_T-K$ or $\bar{D}+K-S_T$ depending on whether the considered option is a European call or put. Consequently, the default boundary $L(S_T,\, D_T)$ depends on the type of the considered option and is given by the following expressions:

$$L(S_T,\, D_T) = L(S_T) = \bar{D} + S_T - K = D_0 + S_T - K \tag{4.44}$$

and

$$L(S_T,\, D_T) = L(S_T) = \bar{D} + K - S_T = D_0 + K - S_T. \tag{4.45}$$

Inserting the expressions for $L(S_T,\, D_T)$ into Equations (4.33) and (4.34), the pricing equations of the extended model of Klein and Inglis (2001) are obtained. The pricing equation for a vulnerable European call is equal to

$$\begin{aligned} C_t = B_{t,T} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{\left[S_T \geq K,\, V_T \geq \bar{D} + S_T - K\right]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{\bar{D} + S_T - K} \cdot 1_{\left[S_T \geq K,\, V_T < \bar{D} + S_T - K\right]} \right] \Bigg). \end{aligned} \tag{4.46}$$

whereas the pricing equation for a vulnerable European put is given by

$$\begin{aligned} P_t = B_{t,T} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{\left[S_T \leq K,\, V_T \geq \bar{D} + K - S_T\right]} \right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{\bar{D} + K - S_T} \cdot 1_{\left[S_T \leq K,\, V_T < \bar{D} + K - S_T\right]} \right] \Bigg). \end{aligned} \tag{4.47}$$

The first line in both pricing equations refers to the situation in which the option expires in the money and the counterparty does not default, i.e. $S_T - K$ and $K - S_T$ are paid out to the holder of a European call and a European put, respectively. The second line indicates the option payoff if the option expires in the money and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by $((1-\alpha)\, V_T) \,/\, \left(\bar{D} + S_T - K\right)$ for a European call and by $((1-\alpha)\, V_T) \,/\, \left(\bar{D} + K - S_T\right)$ for a European put. The holder of a European call receives $((1-\alpha)\, V_T\, (S_T - K)) \,/\, \left(\bar{D} + S_T - K\right)$, whereas the holder of a European put receives $((1-\alpha)\, V_T\, (K - S_T)) \,/\, \left(\bar{D} + K - S_T\right)$. If the option expires out of the money, the option holder receives nothing irrespective of whether the counterparty defaults or not.

In Equations (4.46) and (4.47), the default boundary as well as the expression in the denominator of the second summand of the the pricing equations are non-linear and depend on a stochastic variable – namely on the price of the option's underlying at maturity. To cope with this issue when computing the expected values, we must be linearize and approximate both the default boundary and the second summand's denominator. We achieve this approximation by employing a first order Taylor series expansion. Subsequently, we obtain the approximate closed form solutions for vulnerable European options based on the extended model of Klein and Inglis (2001) by explicitly computing the expected value expressions given by Equations (4.46) and (4.47) (see Appendix 2). The approximate valuation formula for vulnerable European calls is equal to

$$\begin{aligned} C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \delta_{SV}) - B_{t,T}\, K\, N_2(a_2, b_2, \delta_{SV}) \\ & + \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t)+(\bar{\rho}_{SV}-m)\bar{\sigma}_S\bar{\sigma}_V+\frac{1}{2}\bar{\sigma}_V^2(m^2-2\bar{\rho}_{SV}m)-g\,p}}{B_{t,T}D_t + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}\, K}\, N_2(a_3, b_3, -\delta_{SV}) \\ & - \frac{(1-\alpha) V_t\, B_{t,T}\, K\, e^{\frac{1}{2}\bar{\sigma}_V^2(m^2-2\bar{\rho}_{SV}m)-g\,p}}{B_{t,T}D_t + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}\, K}\, N_2(a_4, b_4, -\delta_{SV}), \end{aligned} \tag{4.48}$$

whereas for a vulnerable European put it is given by

$$\begin{aligned} P_t = {} & B_{t,T}\, K\, N_2(-a_2, b_2, -\delta_{SV}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\delta_{SV}) \\ & + \frac{(1-\alpha) V_t\, B_{t,T}\, K\, e^{\frac{1}{2}\bar{\sigma}_V^2(m^2-2\bar{\rho}_{SV}m)-g\,p}}{B_{t,T}D_t + B_{t,T}\, K - S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}\, N_2(-a_4, b_4, \delta_{SV}) \\ & + \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t)+(\bar{\rho}_{SV}-m)\bar{\sigma}_S\bar{\sigma}_V+\frac{1}{2}\bar{\sigma}_V^2(m^2-2\bar{\rho}_{SV}m)-g\,p}}{B_{t,T}D_t + B_{t,T}\, K - S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}\, N_2(-a_3, b_3, \delta_{SV}), \end{aligned} \tag{4.49}$$

where $N_2(\,\cdot\,)$ represents the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$a_1 = \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S} + \bar{\sigma}_S,$$

$$a_2 = \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S},$$

$$
\begin{aligned}
a_3 &= \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S} + \bar{\sigma}_S + m\,\bar{\sigma}_V + g + \delta_{SV}\,\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}\,\bar{\sigma}_V,\\
a_4 &= \frac{\ln \frac{S_t}{B_{t,T}\,K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S} + m\,\bar{\sigma}_V + g + \delta_{SV}\,\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}\,\bar{\sigma}_V,\\
b_1 &= -\frac{-b - m\,p}{\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}} + \delta_{SV}\,\bar{\sigma}_S,\\
b_2 &= -\frac{-b - m\,p}{\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}}\\
b_3 &= \frac{-b - m\,p}{\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}} - \sqrt{1 - 2\bar{\rho}_{SV} m + m^2}\,\bar{\sigma}_V,\\
b_4 &= \frac{-b - m\,p}{\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}} - \sqrt{1 - 2\bar{\rho}_{SV} m + m^2}\,\bar{\sigma}_V, -\delta_{SV}\,\left(m\,\bar{\sigma}_V + g\right).
\end{aligned}
$$

The parameter δ_{SV} gives the adjusted correlation between the return of the option's underlying and the counterparty's assets. It is defined as

$$
\delta_{SV} = \frac{\bar{\rho}_{SV} - m}{\sqrt{1 - 2\bar{\rho}_{SV} m + m^2}}.
$$

The parameters b, m and g depend on the type of the considered option. For vulnerable European calls, they are given by

$$
\begin{aligned}
b^{\text{Call}} &= \frac{\ln \frac{V_t}{B_{t,T}D_t + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}\,K} - \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V},\\
m^{\text{Call}} &= \frac{\bar{\sigma}_S}{\bar{\sigma}_V}\,\frac{S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}{B_{t,T}D_t + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}\,K},\\
g^{\text{Call}} &= -\bar{\sigma}_S\,\frac{B_{t,T}S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}{B_{t,T}D_t + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}\,K},
\end{aligned}
$$

whereas for vulnerable European puts they are equal to

$$
\begin{aligned}
b^{\text{Put}} &= \frac{\ln \frac{V_t}{B_{t,T}D_t + B_{t,T}\,K - S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}} - \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V},\\
m^{\text{Put}} &= -\frac{\bar{\sigma}_S}{\bar{\sigma}_V}\,\frac{S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}{B_{t,T}D_t + B_{t,T}\,K - S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}},\\
g^{\text{Put}} &= \bar{\sigma}_S\,\frac{S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S p}}{B_{t,T}D_t + B_{t,T}\,K - S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}.
\end{aligned}
$$

Since a first order Taylor series expansion is used in the derivation to linearize and approximate both the default boundary and the denominator in the expected value's second summand, the valuation formulas given by Equations (4.48) and (4.49) are only analytical approximations and depend on the point of expansion p. In principle, the value for p can be chosen freely, however, it is important to note that this choice might have a decisive impact on the accuracy of the obtained option values.

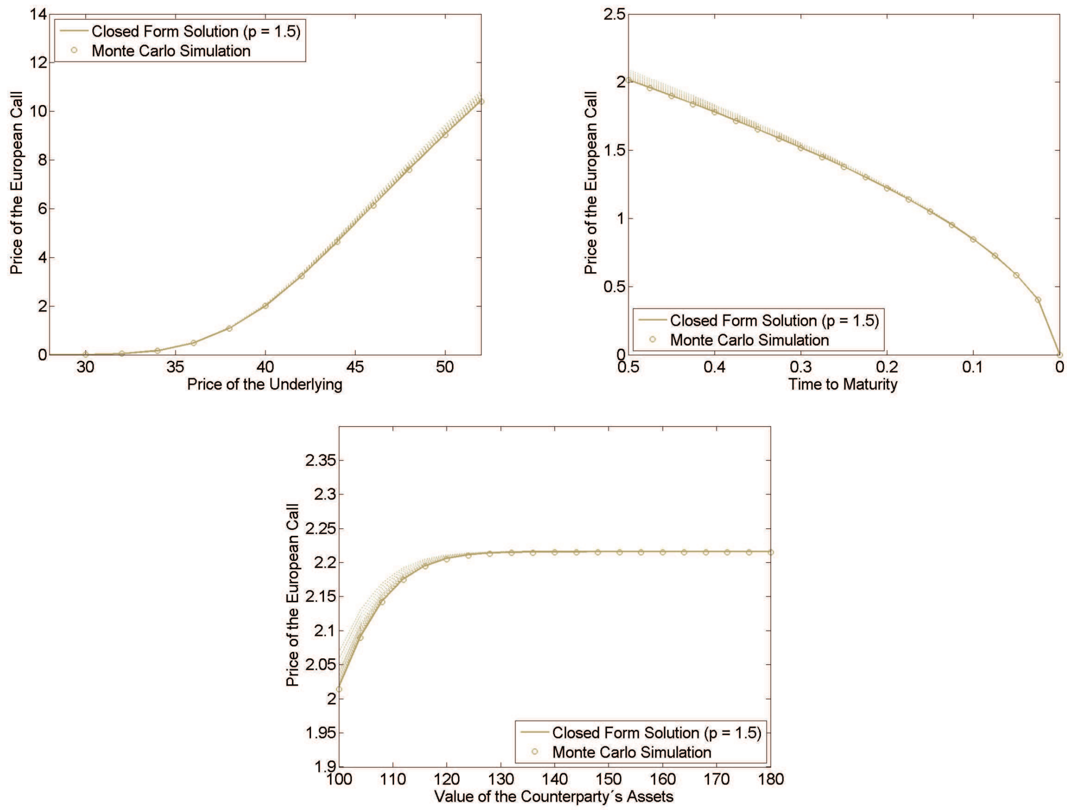

Figure 4.1: European Calls in the Extended Model of Klein and Inglis (2001)

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values (ochre line) are generated using the approximate closed form solution given by Equation (4.48) based on $p = 1.5$. The numerical solution of the extended model of Klein and Inglis (2001) (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the point of expansion p ranging from 0 to 4.

Figures 4.1 and 4.2 provide insights to the impact of choosing a particular value for the point of expansion p. In these two figures, the option values are depicted as functions of the price of the option's underyling, the time to maturity and the value of the counterparty's assets. These option values are obtained from the approximate closed form solutions given by Equations (4.48) and (4.49) using different values for the point of expansion. The approximate analytical solution and the numerical solution are almost identical for $p = 1.5$ and $p = -1.5$ in case of vulnerable European calls and puts, respectively. The same finding is also obtained based on several other numerical examples. Hence, the approximate closed form solutions are quite accurate for a wide range of moneyness, different times to maturity and various values of the counterparty's assets if the point of expansion is chosen appropriately.

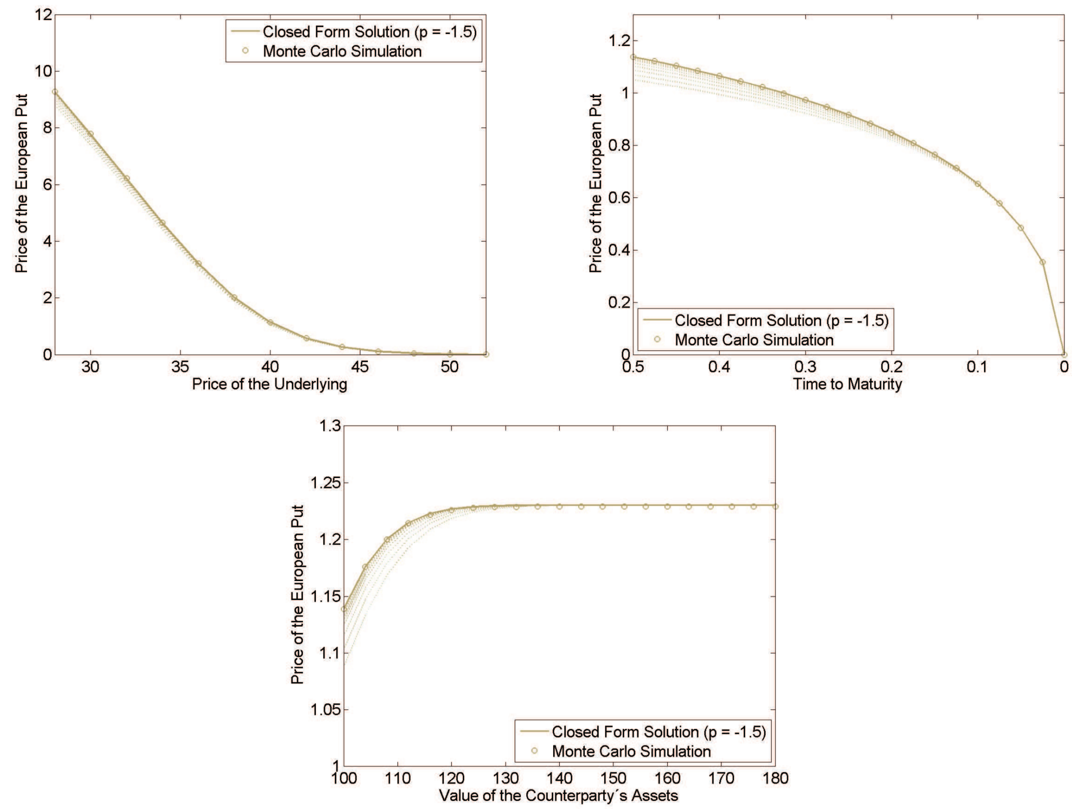

Figure 4.2: European Puts in the Extended Model of Klein and Inglis (2001)

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values (ochre line) are generated using the approximate closed form solution given by Equation (4.49) based on $p = -1.5$. The numerical solution of the extended model of Klein and Inglis (2001) (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the point of expansion p ranging from -4 to 0.

	European Call			European Put		
	Approx. CF	Num. Sol.	Approx. Error	Approx. CF	Num. Sol.	Approx. Error
Base Case	2.0168	2.0143	+0.13%	1.1385	1.1387	−0.02%
$S = 45$	5.3895	5.3781	+0.21%	0.1741	0.1745	−0.23%
$S = 35$	0.2945	0.2941	+0.13%	3.9044	3.9112	−0.17%
$V = 105$	2.1067	2.1047	+0.09%	1.1831	1.1830	0.00%
$V = 95$	1.8908	1.8878	+0.16%	1.0720	1.0726	−0.05%
$T - t = 1$	3.0262	3.0200	+0.20%	1.3607	1.3622	−0.10%
$T - t = 0.25$	1.3781	1.3770	+0.09%	0.9163	0.9161	0.02%
$q = 0.02$	1.8059	1.8034	+0.14%	1.2858	1.2861	−0.02%
$r_0 = 0.08$	2.3197	2.3168	+0.13%	0.9595	0.9596	−0.01%
$r_0 = 0.02$	1.7374	1.7349	+0.15%	1.3357	1.3364	−0.05%
$\kappa = 0.8$	2.0162	2.0137	+0.13%	1.1381	1.1383	−0.02%
$\kappa = 0.2$	2.0175	2.0150	+0.12%	1.1390	1.1392	−0.02%
$\theta = 0.08$	2.0549	2.0523	+0.13%	1.1142	1.1143	−0.01%
$\theta = 0.02$	1.9791	1.9766	+0.13%	1.1632	1.1634	−0.02%
$\sigma_r = 0.08$	2.0258	2.0232	+0.13%	1.1455	1.1457	−0.02%
$\sigma_r = 0.02$	2.0119	2.0094	+0.13%	1.1348	1.1349	−0.02%
$\rho_{Sr} = 0.5$	2.0780	2.0750	+0.14%	1.1860	1.1865	−0.04%
$\rho_{Sr} = -0.5$	1.9535	1.9509	+0.13%	1.0884	1.0885	−0.01%
$\rho_{Vr} = 0.5$	2.0219	2.0192	+0.13%	1.1282	1.1286	−0.03%
$\rho_{Vr} = -0.5$	2.0116	2.0089	+0.13%	1.1491	1.1493	−0.02%

Table 4.1: Approx. Error in the Extended Model of Klein and Inglis (2001)

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The approximate closed form solutions that are used to compute the option values are given by Equations (4.48) and (4.49), respectively. The point of expansion are chosen to be $p = 1.5$ in case of a European calll and $p = -1.5$ in case of a European put. The numerical solution is calculated by Monte Carlo simulation ($N = 1\,000\,000$).

In Table 4.1, the values for vulnerable European calls and puts based on the extended model of Klein and Inglis (2001) are presented. The first two columns give the values of a vulnerable European call computed by the approximate valuation formula and Monte Carlo simulation (= numerical solution), respectively. The third column reports the approximation error which is measured as the percentage deviation of the approximate from the numerical solution. Most errors are smaller than ±0.2% with the highest error being +0.24%. Compared to the base case scenario, the magnitude of the approximation error considerably increases for in-the-money options ($S \uparrow$) and a longer time to maturity ($T \uparrow$). The parameters characterizing the stochastic process of the risk-free interest rate do not influence the quality of the analytical approximation substantially. In the fourth and fifth columns, the values of a vulnerable European put computed by the approximate valuation formula and Monte Carlo simulation (= numerical solution), respectively, are presented. In the sixth column, the approximation error is given. Again, it is measured as the percentage deviation of the approximate from the numerical solution. Most errors are smaller than ±0.2% with the highest error being −0.26%. Compared to the base case scenario, the magnitude of the approximation error considerably increases for in-the-money and out-of-the-money options ($S \downarrow$ and $S \uparrow$), aas well as for a longer time to maturity ($T \uparrow$). Like in the case of vulnerable European calls, the impact of the parameters characterizing the stochastic process of the risk-free interest rate on the quality of the analytical approximation is negligible.

To conclude, the magnitude of the approximation errors is relatively low for both vulnerable European calls and puts which indicates that the approximate valuation formulas of the extended model suggested by Klein and Inglis (2001) work quite well for the given set of parameters.

4.4.4 Stochastic Liabilities

We also extend the model of Liu and Liu (2011) to the stochastic interest rate framework suggested by Vasicek (1977). In contrast to the previous models, it is assumed that the counterparty's total liabilities may vary over time. In particular, it is assumed that the market value of the counterparty's total liabilities follows a geometric Brownian motion (see Equation (4.3)). The market value at the option's maturity is denoted by D_T. It is important to note that the short position in the option is implicitly included in the counterparty's total liabilities, since it is an obligation to the option writer. However, its impact on the value of the couterparty's total liabilities is not explicitly modeled. Since the value of the counterparty's total liabilities is assumed to be stochastic, the default boundary $L(S_T,\, D_T)$ in the extended Liu-Liu model is defined as

$$L(S_T,\, D_T) = L(D_T) = D_T. \tag{4.50}$$

Inserting this expression into Equations (4.33) and (4.34) yields the pricing equations of the extended Liu-Liu model. The pricing equation for a vulnerable European call equals

$$\begin{aligned} C_t = B_{t,T} \Bigg(& \mathbb{E}\left[(S_T - K) \cdot 1_{[S_T \geq K,\, V_T \geq D_T]}\right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{D_T} \cdot 1_{[S_T \geq K,\, V_T < D_T]}\right]\Bigg), \end{aligned} \tag{4.51}$$

whereas the pricing equation for a vulnerable European put is given by

$$\begin{aligned} P_t = B_{t,T} \Bigg(& \mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K,\, V_T \geq D_T]}\right] \\ & + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{D_T} \cdot 1_{[S_T \leq K,\, V_T < D_T]}\right]\Bigg). \end{aligned} \tag{4.52}$$

The first line in both pricing equations still refers to the situation in which the option expires in the money and the counterparty does not default, i.e. $S_T - K$ and $K - S_T$ are paid out to the holder of a European call and a European put, respectively. The second line gives the payoff if the option is in the money at maturity and the counterparty is in default. In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $((1-\alpha)\, V_T)\,/D_T$, i.e. the asset value available for distribution is divided by the value of the counterparty's total liabilities. The holder of a European call receives $((1-\alpha)\, V_T\, (S_T - K))\,/D_T$, whereas $((1-\alpha)\, V_T\, (K - S_T))\,/D_T$ is paid out to the holder of a European put. If the option expires out of the money, the option holder receives nothing irrespective of whether the counterparty defaults or not.

In Equations (4.51) and (4.52), the default boundary and the denominator of the pricing equations' second summand depend on the value of the counterparty's liabilities which is a stochastic variable. To circumvent this issue, we introduce a new variable, the debt ratio, which is defined as $R_t = V_t/D_t$. Using the debt ratio, we analytically compute the expected values in Equations (4.51) and (4.52) and obtain the valuation formulas for vulnerable European options based on the extended model of Liu and Liu (2011) after some algebraic transformations (see Appendix 3). For vulnerable European calls and puts, respectively, they are equal to

$$\begin{aligned} C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \delta_{SR}) - B_{t,T}\, K\, N_2(a_2, b_2, \delta_{SR}) \\ & + \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t)+\bar{\sigma}_D^2+\bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V-\bar{\rho}_{SD}\bar{\sigma}_S\bar{\sigma}_D-\bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D}}{D_t}\, N_2(a_3, b_3, -\delta_{SR}) \\ & - \frac{(1-\alpha) V_t\, B_{t,T}\, K\, e^{\bar{\sigma}_D^2-\bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D}}{D_t}\, N_2(a_4, b_4, -\delta_{SR}) \end{aligned} \tag{4.53}$$

and

$$\begin{aligned} P_t = {} & B_{t,T}\, K\, N_2(-a_2, b_2, -\delta_{SR}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\delta_{SR}) \\ & + \frac{(1-\alpha) V_t\, B_{t,T}\, K\, e^{\bar{\sigma}_D^2-\bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D}}{D_t}\, N_2(-a_4, b_4, \delta_{SR}) \\ & - \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t)+\bar{\sigma}_D^2+\bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V-\bar{\rho}_{SD}\bar{\sigma}_S\bar{\sigma}_D-\bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D}}{D_t}\, N_2(-a_3, b_3, \delta_{SR}), \end{aligned} \tag{4.54}$$

where $N_2(\,\cdot\,)$ is the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$\begin{aligned} a_1 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) + \frac{1}{2}\,\bar{\sigma}_S^2}{\bar{\sigma}_S}, \\ a_2 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\,\bar{\sigma}_S^2}{\bar{\sigma}_S}, \\ a_3 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) + \frac{1}{2}\,\bar{\sigma}_S^2 + \bar{\sigma}_{SV} - \bar{\sigma}_{SD}}{\bar{\sigma}_S}, \\ a_4 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\,\bar{\sigma}_S^2 + \bar{\sigma}_{SV} - \bar{\sigma}_{SD}}{\bar{\sigma}_S}, \\ b_1 &= \frac{\ln \frac{V_t}{D_t} - \frac{1}{2}(\bar{\sigma}_V^2 - \bar{\sigma}_D^2 - 2\bar{\sigma}_{SV} + 2\bar{\sigma}_{SD})}{(\bar{\sigma}_{SV} - \bar{\sigma}_{SD})}, \end{aligned}$$

$$b_2 = \frac{\ln \frac{V_t}{D_t} - \frac{1}{2}(\bar{\sigma}_V^2 - \bar{\sigma}_D^2)}{(\bar{\sigma}_{SV} - \bar{\sigma}_{SD})},$$

$$b_3 = -\frac{\ln \frac{V_t}{D_t} - \frac{1}{2}(\bar{\sigma}_V^2 - \bar{\sigma}_D^2 - 2\bar{\sigma}_{SV} + 2\bar{\sigma}_{SD})}{(\bar{\sigma}_{SV} - \bar{\sigma}_{SD})} - \sqrt{\bar{\sigma}_V^2 + \bar{\sigma}_D^2 - 2\bar{\sigma}_{VD}}\,,$$

$$b_4 = -\frac{\ln \frac{V_t}{D_t} - \frac{1}{2}(\bar{\sigma}_V^2 - \bar{\sigma}_D^2)}{(\bar{\sigma}_{SV} - \bar{\sigma}_{SD})} - \sqrt{\bar{\sigma}_V^2 + \bar{\sigma}_D^2 - 2\bar{\sigma}_{VD}}\,.$$

The parameter δ_{SR} gives the adjusted correlation between the returns of the option's underlying and the counterparty's debt ratio. It is defined as

$$\delta_{SR} = \frac{\bar{\rho}_{SV}\bar{\sigma}_V - \bar{\rho}_{SD}\bar{\sigma}_D}{\sqrt{\bar{\sigma}_V^2 + \bar{\sigma}_D^2 - 2\,\bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D}}.$$

4.4.5 General Model

Our general model picks up on the ideas of both Klein and Inglis (2001) and Liu and Liu (2011) and additionally accounts for the stochastic interest rate framework suggested by Vasicek (1977). In particular, we assume that the short position in the option may increase the counterparty's default risk and the market value of the counterparty's other liabilities follows a geometric Brownian motion. At the option's maturity the counterparty's total liabilities are given by $D_T + S_T - K$ in the case of a European call and $D_T + K - S_T$ in the case of a European put. Hence, the default boundary $L(S_T,\, D_T)$ depends on the type of the considered option. For European calls and puts, respectively, it is given as follows:

$$L(S_T,\, D_T) = D_T + S_T - K, \tag{4.55}$$

$$L(S_T,\, D_T) = D_T + K - S_T. \tag{4.56}$$

Plugging these expressions into Equations (4.33) and (4.34) yields the pricing equations of the general model. For vulnerable European calls, the pricing equation is equal to

$$C_t = B_{t,T}\left(\mathbb{E}\left[(S_T - K)\cdot 1_{[S_T \geq K,\, V_T \geq D_T + S_T - K]}\right] + \mathbb{E}\left[\frac{(1-\alpha)\,V_T\,(S_T - K)}{D_T + S_T - K}\cdot 1_{[S_T \geq K,\, V_T < D_T + S_T - K]}\right]\right), \tag{4.57}$$

whereas for a vulnerable European put it is given by

$$P_t = B_{t,T} \left(\mathbb{E}\left[(K - S_T) \cdot 1_{[S_T \leq K,\, V_T \geq D_T + K - S_T]} \right] \right. \qquad (4.58)$$
$$\left. + \mathbb{E}\left[\frac{(1-\alpha)\, V_T\, (K - S_T)}{D_T + K - S_T} \cdot 1_{[S_T \leq K,\, V_T < D_T + K - S_T]} \right] \right).$$

In analogy to the other valuation models, the first line of both pricing equations refers to the situation in which the option expires in the money and the counterparty does not default. Consequently, the corresponding payoff of a European call is equal to $S_T - K$, whereas it is given by $K - S_T$ for a European put. The second line of both pricing equations indicates the payoff if the option expires in the money and the counterparty is in default. In this case, the entire assets of the counterparty (less the default cost α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by $((1-\alpha)\, V_T) \,/\, (D_T + S_T - K)$ in the case of a European call, whereas it is equal to $((1-\alpha)\, V_T) \,/\, (D_T + K - S_T)$ in the case of a European put. Consequently, the holder of a European call receives $((1-\alpha)\, V_T\, (S_T - K)) \,/\, (D_T + S_T - K)$, whereas the holder of a European put receives $((1-\alpha)\, V_T\, (K - S_T)) \,/\, (D_T + K - S_T)$. If the option is, however, out of the money at maturity, the option holder receives nothing irrespective of whether the counterparty is in default or not.

Looking at Equations (4.57) and (4.58), it becomes clearly evident that our general valuation model incorporates the previously presented valuation models as special cases. The communalities and differences between these models are summarized as follows:

1. If the counterparty's other liabilities are assumed to be deterministic and constant over time, our general model is reduced to the extended model of Klein and Inglis (2001) represented by Equations (4.46) and (4.47), since then the default condition is given by $V_T < \bar{D} + S_T - K$ and $V_T < \bar{D} + K - S_T$, respectively.

2. If the option holder's claim $S_T - K$ and $K - S_T$, respectively, is removed from the default condition and the market value of the counterparty's other

liabilities is still assumed to follow a geometric Brownian motion, our general model collapses to the extended model of Liu and Liu (2011) which is specified by Equations (4.51) and (4.52), since the default condition is equal to $V_T < D_T$ in this case.

3. If the option holder's claim $S_T - K$ and $K - S_T$, respectively, is removed from the default condition and the counterparty's other liabilities are assumed to be constant over time, our general model is reduced to the model of Klein and Inglis (1999) (i.e. the extended model of Klein (1996)) which is specified by Equations (4.40) and (4.41), since the default condition is equal to $V_T < \bar{D}$ in this case.

In Equations (4.57) and (4.58), the default boundary as well as the denominator of the pricing equations' second summand are non-linear and depend on two stochastic variables – namely the price of the option's underlying and the market value of the counterparty's other liabilities. Due to this issue, an exact analytical solution cannot be derived. However, we are able to derive an approximate closed form solution if the returns of the option's underlying and the counterparty's other liabilities are assumed to be uncorrelated ($\rho_{SD} = 0$).

We employ a first order Taylor series expansion to linearize and approximate both the default boundary and the second summand's denominator. After some algebraic transformations, we obtain the approximate valuation formulas for vulnerable European options (see Appendix 4). For vulnerable European calls and puts, respectively, these approximate closed form solutions are equal to

$$
\begin{aligned}
C_t = {} & S_t\, e^{-q(T-t)}\, N_2(a_1, b_1, \delta_{SV}) - K\, e^{-r(T-t)}\, N_2(a_2, b_2, \delta_{SV}) \qquad (4.59) \\
& + \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t) + (\bar{\rho}_{SV} - m_1)\bar{\sigma}_S \bar{\sigma}_V + \frac{1}{2}\bar{\sigma}_V^2 (m_1^2 + m_2^2 - 2\bar{\rho}_{SV} m_1 - 2\bar{\rho}_{VD} m_2)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K} \\
& \qquad \cdot e^{-g_1\, p_1 - g_2\, p_2}\, N_2(a_3, b_3, -\delta_{SV}) \\
& - \frac{(1-\alpha) V_t\, B_{t,T}\, K\, e^{\frac{1}{2}\bar{\sigma}_V^2 (m_1^2 + m_2^2 - 2\bar{\rho}_{SV} m_1 - 2\bar{\rho}_{VD} m_2)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K} \\
& \qquad \cdot e^{-g_1\, p_1 - g_2\, p_2}\, N_2(a_4, b_4, -\delta_{SV})
\end{aligned}
$$

and

$$
\begin{aligned}
P_t = {} & B_{t,T}\, K\, N_2(-a_2, b_2, -\delta_{SV}) - S_t\, e^{-q(T-t)}\, N_2(-a_1, b_1, -\delta_{SV}) \qquad (4.60)\\
& + \frac{(1-\alpha) V_t\, B_{t,T}\, K\, e^{\frac{1}{2}\bar{\sigma}_V^2 (m_1^2 + m_2^2 - 2\bar{\rho}_{SV} m_1 - 2\bar{\rho}_{VD} m_2)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}} \\
& \qquad \cdot e^{-g_1\, p_1 - g_2\, p_2}\, N_2(-a_4, b_4, \delta_{SV}) \\
& - \frac{(1-\alpha) V_t\, S_t\, e^{-q(T-t) + (\bar{\rho}_{SV} - m_1)\bar{\sigma}_S \bar{\sigma}_V + \frac{1}{2}\bar{\sigma}_V^2 (m_1^2 + m_2^2 - 2\bar{\rho}_{SV} m_1 - 2\bar{\rho}_{VD} m_2)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}} \\
& \qquad \cdot e^{-g_1\, p_1 - g_2\, p_2}\, N_2(-a_3, b_3, \delta_{SV}),
\end{aligned}
$$

where $N_2(\,\cdot\,)$ represents the cumulative distribution function of the bivariate standard normal distribution and where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are given as follows:

$$
\begin{aligned}
a_1 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S} + \bar{\sigma}_S, \\
a_2 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S}, \\
a_3 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S} + \bar{\sigma}_S + m_1\, \bar{\sigma}_V + g_1 \\
&\qquad + \delta_{SV} \sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}\, \bar{\sigma}_V, \\
a_4 &= \frac{\ln \frac{S_t}{B_{t,T}\, K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S} + m_1\, \bar{\sigma}_V + g_1 \\
&\qquad + \delta_{SV} \sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}\, \bar{\sigma}_V, \\
b_1 &= -\frac{-b - p_1\, m_1 - p_2\, m_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}} + \delta_{SV}\bar{\sigma}_S, \\
b_2 &= -\frac{-b - p_1\, m_1 - p_2\, m_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}}, \\
b_3 &= \frac{-b - p_1\, m_1 - p_2\, m_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}} \\
&\qquad - \sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}\, \bar{\sigma}_V \\
&\qquad - \delta_{SV}\, (\bar{\sigma}_S + m_1\, \bar{\sigma}_V + g_1) - \delta_{VD}\, (m_2\, \bar{\sigma}_V + g_2)
\end{aligned}
$$

$$b_4 = \frac{-b - p_1\, m_1 - p_2\, m_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}}$$

$$- \sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}\,\bar{\sigma}_V$$

$$- \delta_{SV}\,(m_1\,\bar{\sigma}_V + g_1) - \delta_{VD}\,(m_2\,\bar{\sigma}_V + g_2)\,.$$

The parameters δ_{SV} and δ_{VD} give the adjusted correlation between the return of the option's underlying and the counterparty's assets and the adjusted correlation between the return of the counterparty's assets and liabilities, respectively:

$$\delta_{SV} = \frac{\bar{\rho}_{SV} - m_1}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}},$$

$$\delta_{VD} = \frac{\bar{\rho}_{VD} - m_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}}.$$

The parameters b, m_1, m_2, g_1 and g_2 depend on the type of the considered option. For European calls and puts, respectively, they are given as follows:

$$b^{\text{Call}} = \frac{\ln \frac{V_t}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K} - \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V},$$

$$m_1^{\text{Call}} = \frac{\bar{\sigma}_S}{\bar{\sigma}_V}\,\frac{S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K},$$

$$m_2^{\text{Call}} = \frac{\bar{\sigma}_D}{\bar{\sigma}_V}\,\frac{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K},$$

$$g_1^{\text{Call}} = \frac{-\bar{\sigma}_S\, S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K},$$

$$g_2^{\text{Call}} = \frac{-\bar{\sigma}_D\, D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}\, K},$$

$$b^{\text{Put}} = \frac{\ln \frac{V_t}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}} - \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V},$$

$$m_1^{\text{Put}} = -\frac{\bar{\sigma}_S}{\bar{\sigma}_V}\,\frac{S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}},$$

$$m_2^{\text{Put}} = \frac{\bar{\sigma}_D}{\bar{\sigma}_V}\,\frac{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}},$$

$$g_1^{\text{Put}} = \frac{\bar{\sigma}_S\, S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}},$$

$$g_2^{\text{Put}} = \frac{-\bar{\sigma}_D\, D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + B_{t,T}\, K - S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}.$$

Since a first order Taylor series expansion is used in the derivation to linearize and approximate both the default boundary and the denominator in the expected value's second summand, the valuation formulas given by Equations (4.59) and (4.60) are analytical approximations and depend on the points of expansion p_1 and p_2. In principle, the values for p_1 and p_2 can be chosen freely, however, this choice might have a decisive impact on the accuracy of the obtained option values.

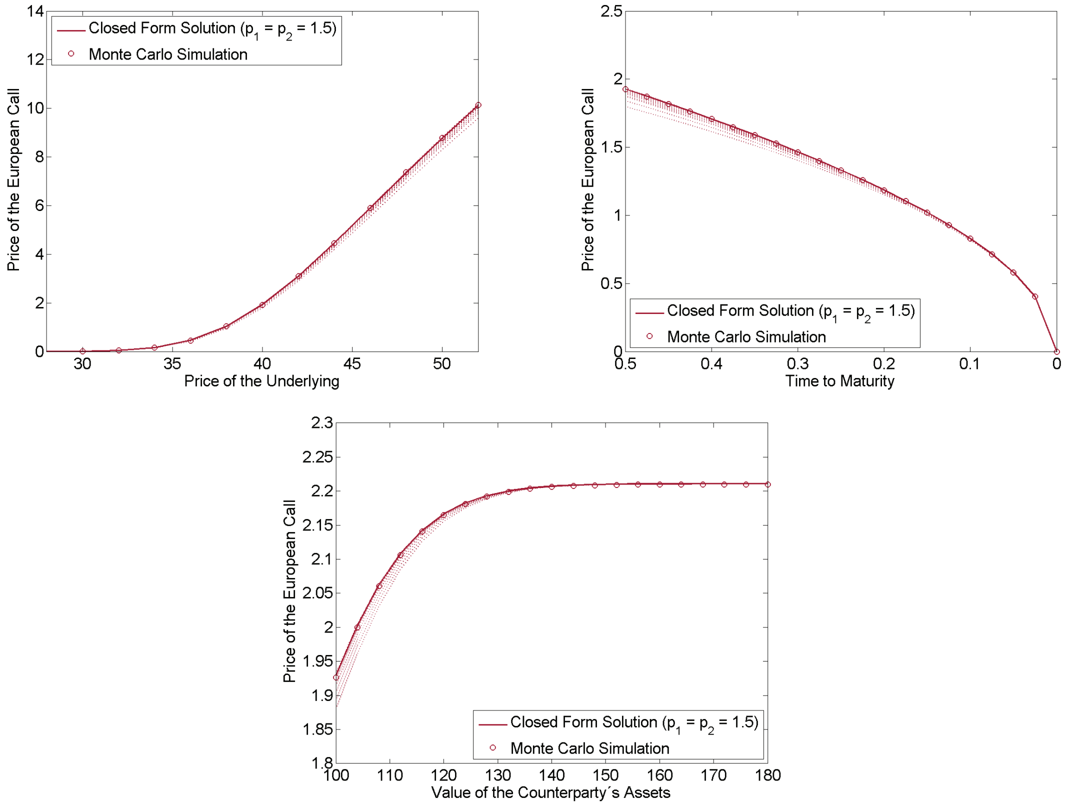

Figure 4.3: European Calls in the General Model

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values (red line) are generated using the approximate closed form solution given by Equation (4.59) based on $p_1 = p_2 = 1.5$. The numerical solution of the general model (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the points of expansion p_1 and p_2 ranging from 0 to 4.

Figures 4.3 and 4.4 provide insights to the impact of choosing a particular value for the points of expansion p_1 and p_2. In these two figures, the option values are depicted as functions of the price of the option's underlying, the time to maturity and the value of the counterparty's assets. These option values are obtained from our approximate closed form solutions given by Equations (4.59) and (4.60) using different values for the points of expansion. The approximate analytical solution and the numerical solution are almost identical for $p_1 = p_2 = 1.5$ and $p_1 = p_2 = -1.5$ in case of vulnerable European calls and puts, respectively. The same finding is also obtained based on several other numerical examples. Hence, the approximate closed form valuation formulas of the general model work quite well for a wide range of parameters if the values for the points of expansion are chosen appropriately.

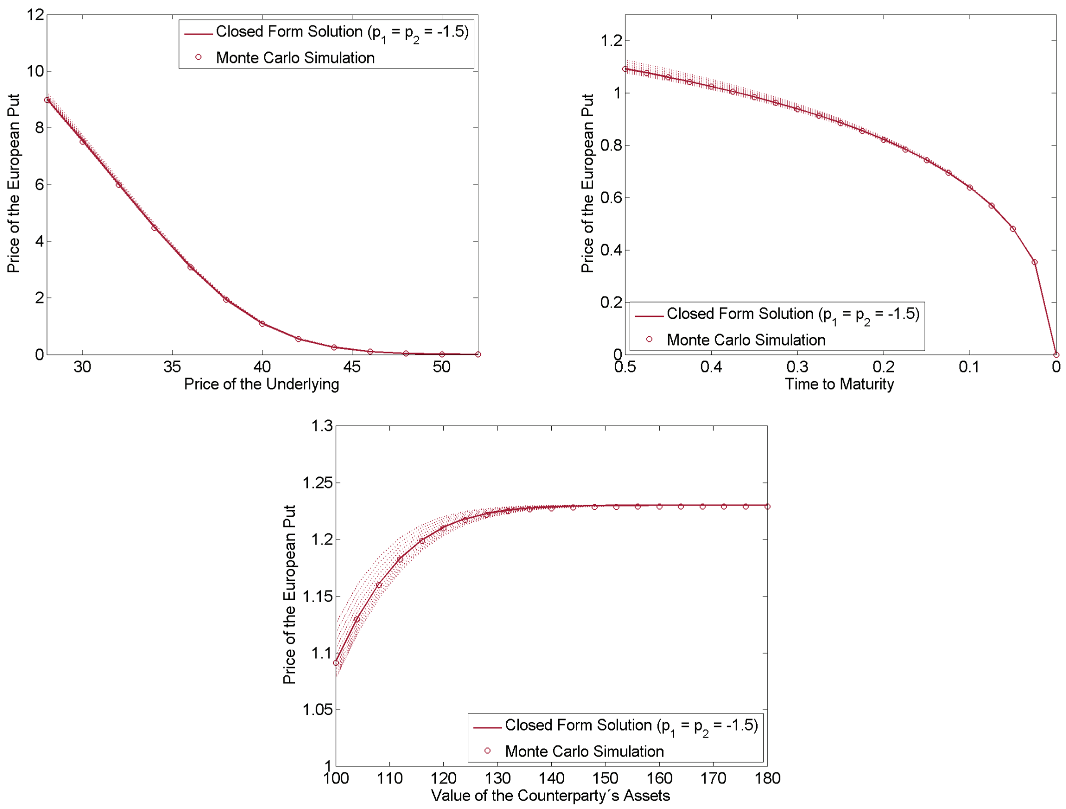

Figure 4.4: European Puts in the General Model

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values (red line) are generated using the approximate closed form solution given by Equation (4.60) based on $p_1 = p_2 = -1.5$. The numerical solution of the general model (circles) is calculated by Monte Carlo simulation ($N = 1\,000\,000$). The shaded area of the figure represents several possible approximate analytical solutions using different values for the points of expansion p_1 and p_2 ranging from -4 to 0.

	European Call			European Put		
	Approx. CF	Num. Sol.	Approx. Error	Approx. CF	Num. Sol.	Approx. Error
Base Case	1.9339	1.9303	+0.19%	1.0924	1.0914	+0.09%
$S = 45$	5.1786	5.1791	−0.01%	0.1658	0.1672	−0.86%
$S = 35$	0.2827	0.2812	+0.56%	3.7676	3.7544	+0.35%
$V = 105$	2.0246	2.0210	+0.18%	1.1391	1.1380	+0.09%
$V = 95$	1.8226	1.8190	+0.20%	1.0332	1.0323	+0.09%
$\rho_{SD} = 0.5$	1.9339	1.8058	+7.09%	1.0924	1.1603	−5.85%
$\rho_{SD} = -0.5$	1.9339	2.0516	−5.74%	1.0924	1.0136	+7.77%
$T - t = 1$	2.8673	2.8512	+0.56%	1.2911	1.2927	−0.13%
$T - t = 0.25$	1.3316	1.3303	+0.10%	0.8860	0.8851	+0.11%
$q = 0.02$	1.7316	1.7279	+0.21%	1.2343	1.2328	+0.13%
$r_0 = 0.08$	2.1957	2.1924	+0.15%	0.9094	0.9092	+0.03%
$r_0 = 0.02$	1.6892	1.6853	+0.23%	1.2995	1.2976	+0.15%
$\kappa = 0.8$	1.9333	1.9299	+0.18%	1.0919	1.0908	+0.10%
$\kappa = 0.2$	1.9346	1.9308	+0.20%	1.0930	1.0921	+0.08%
$\theta = 0.08$	1.9670	1.9635	+0.18%	1.0672	1.0664	+0.08%
$\theta = 0.02$	1.9010	1.8974	+0.19%	1.1180	1.1169	+0.09%
$\sigma_r = 0.08$	1.9434	1.9369	+0.34%	1.0998	1.1006	−0.07%
$\sigma_r = 0.02$	1.9287	1.9267	+0.10%	1.0884	1.0865	+0.18%
$\rho_{Sr} = 0.5$	1.9923	1.9791	+0.67%	1.1386	1.1433	−0.41%
$\rho_{Sr} = -0.5$	1.8734	1.8793	−0.31%	1.0439	1.0375	+0.62%
$\rho_{Vr} = 0.5$	1.9430	1.9394	+0.19%	1.0858	1.0848	+0.09%
$\rho_{Vr} = -0.5$	1.9247	1.9211	+0.19%	1.0990	1.0979	+0.10%
$\rho_{Dr} = 0.5$	1.9348	1.9217	+0.68%	1.0930	1.0976	−0.41%
$\rho_{Dr} = -0.5$	1.9329	1.9387	−0.30%	1.0918	1.0852	+0.60%

Table 4.2: Approx. Error in the General Model

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The approximate closed form solutions that are used to compute the option values are given by Equations (4.59) and (4.60), respectively. The points of expansion are chosen to be $p_1 = p_2 = 1.5$ in case of a European call and $p_1 = p_2 = -1.5$ in case of a European put. The numerical solution is calculated by Monte Carlo simulation ($N = 1\,000\,000$).

In Table 4.2, the option values for vulernable European calls and puts based on our general model are presented. The first two columns give the values of a vulnerable European call computed by the approximate valuation formula and Monte Carlo simulation (= numerical solution), respectively. The third column reports the approximation error which is measured as the percentage deviation of the approximate from the numerical solution. Most errors are smaller than $\pm 0.2\%$ with the highest errors being equal to -5.74% and $+7.09\%$. These errors are observed if the correlation between the return of the option's underlying and the counterparty's other liabilities is -0.5 and $+0.5$, respectively. This result is obvious, since the analytical approximation is based on the assumption of independence between these returns. Compared to the base case scenario, the magnitude of the approximation error considerably increases for out-of-the-money options ($S \downarrow$), a longer time to maturity ($T \uparrow$), an increased volatility of the risk-free interest rate ($\sigma_r \uparrow$) as well as for stronger correlations betwenn the risk-free interest rate and the return of both the option's underlying and the counterparty's other liabilities ($\rho_{Sr} \neq 0$ and $\rho_{Dr} \neq 0$). The remaining parameters characterizing the stochastic process of the risk-free interest rate do not influence the quality of the analytical approximation substantially. In the fourth and fifth columns, the values of a vulnerable European put computed by the approximate valuation formula and Monte Carlo simulation (= numerical solution), respectively, are presented. In the sixth column, the approximation error is given. Again, it is defined as the percentage deviation of the approximate solution from the numerical solution. Most errors are smaller than $\pm 0.2\%$ with the highest errors being equal to -5.85% and $+7.77\%$. These errors are observed if the correlation between the return of the option's underlying and the counterparty's other liabilities is 0.5 and -0.5, repsectively. This result is obvious, since the analytical approximation is based on the assumption of independence between these returns. Compared to the base case scenario, the magnitude of the approximation error considerably increases for in-the-money and out-of-the-money options ($S \downarrow$ and $S \uparrow$), a longer time to maturity ($T \uparrow$) as well as for for stronger correlations betwenn the risk-free interest rate and the return of both the option's underlying and the counterparty's other liabilities ($\rho_{Sr} \neq 0$ and $\rho_{Dr} \neq 0$). Like in the case of vulnerable European calls, the impact of the remaining parameters characterizing the stochastic process of the risk-free interest rate on the quality of the analytical approximation is negligible.

To conclude, the size of the approximation errors is relatively low for both vulnerable European calls and puts indicating that our general model's approximate valuation formulas work quite well for the given parameters. The size of the approximation errors is similarly high as in the extended model of Klein and Inglis (2001).

4.5 Numerical Examples

In this section, various numerical examples are presented to compare the results of the different valuation models for European options subject to counterparty risk. Since the full payoff on the option cannot be made if the option writer defaults, it should be expected that vulnerable options will have lower values than otherwise identical non-vulnerable options. Hence, the upper price limit is given by the default-free option value which is obtained from the model of Rabinovitch (1989) in the considered famework. Consequently, the value of a vulnerable European option can never be higher than this value irrespective of the considered valuation model.

The starting point of the following comparative analysis is a typical market situation for a European option. At time $t = 0$, the option is at the money ($S_0 = 40$, $K = 40$) and expires in six months ($T = 0.5$). The return volatility of the option's underlying equals 15% ($\sigma_S = 0.15$) and its dividend yield is zero ($q = 0$). The option writer is assumed to be highly levered ($V_0 = 100$, $D_0 = 90$). The return volatility of the counterparty's assets and liabilities is assumed to be 15% ($\sigma_V = 0.15$, $\sigma_D = 0.15$). The correlations between the returns of the option's underlying, the counterparty's assets and liabilities are zero ($\rho_{SV} = \rho_{VD} = \rho_{SD} = 0$). If the counterparty defaults, deadweight costs of 25% are applied ($\alpha = 0.25$). The risk-free interest rate is assumed to follow an mean-reverting Ornstein-Uhlenbeck process. The current risk-free interest rate equals 5% ($r_0 = 0.05$). The long-term mean is also equal to 5% ($\theta = 0.05$), while the reversion speed is 0.5 ($\kappa = 0.5$). The volatility of the risk-free interest rate is assumed to be 5% ($\sigma_r = 0.05$). The correlation between the risk-free interest rate and the returns of the option's underlying, the couterparty's assets and liabilities is assumed to be zero ($\rho_{Sr} = \rho_{Vr} = \rho_{Dr} = 0$).

Figures 4.5 and 4.6 depict the values of European calls and puts, respectively, as functions of the price of the option's underlying, the option's time to maturity and the value of the counterparty's assets for the valuation models presented in previous section. As expected, the option values obtained from the model of Klein and Inglis (1999), the extended models of Klein and Inglis (2001) and Liu and Liu (2011) as well as from the general model are always lower than the default-free option value given by the model of Rabinovitch (1989). In particular, the highest price reduction due to counterparty risk can be observed for our general model followed by the extended models of Klein and Inglis (2001) and Liu and Liu (2011). The smallest price reduction is found for the model of Klein and Inglis (1999).

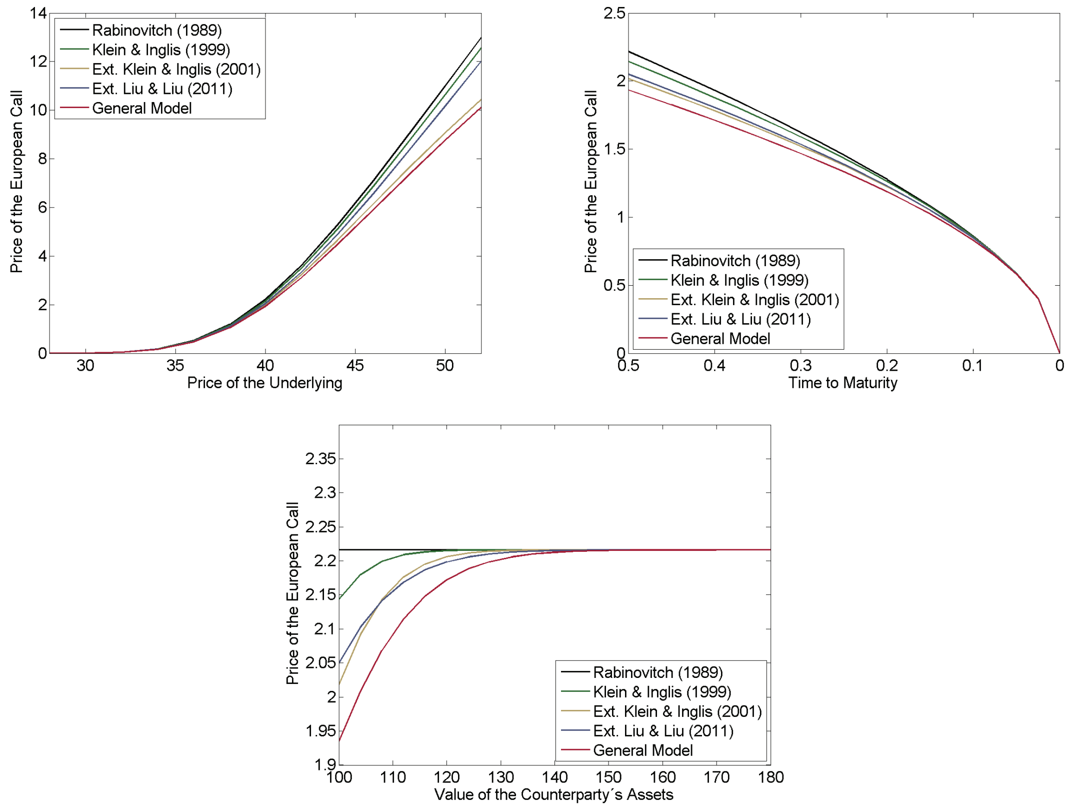

Figure 4.5: European Calls subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 4.4. The analytical approximations of the extended model of Klein and Inglis (2001) and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively.

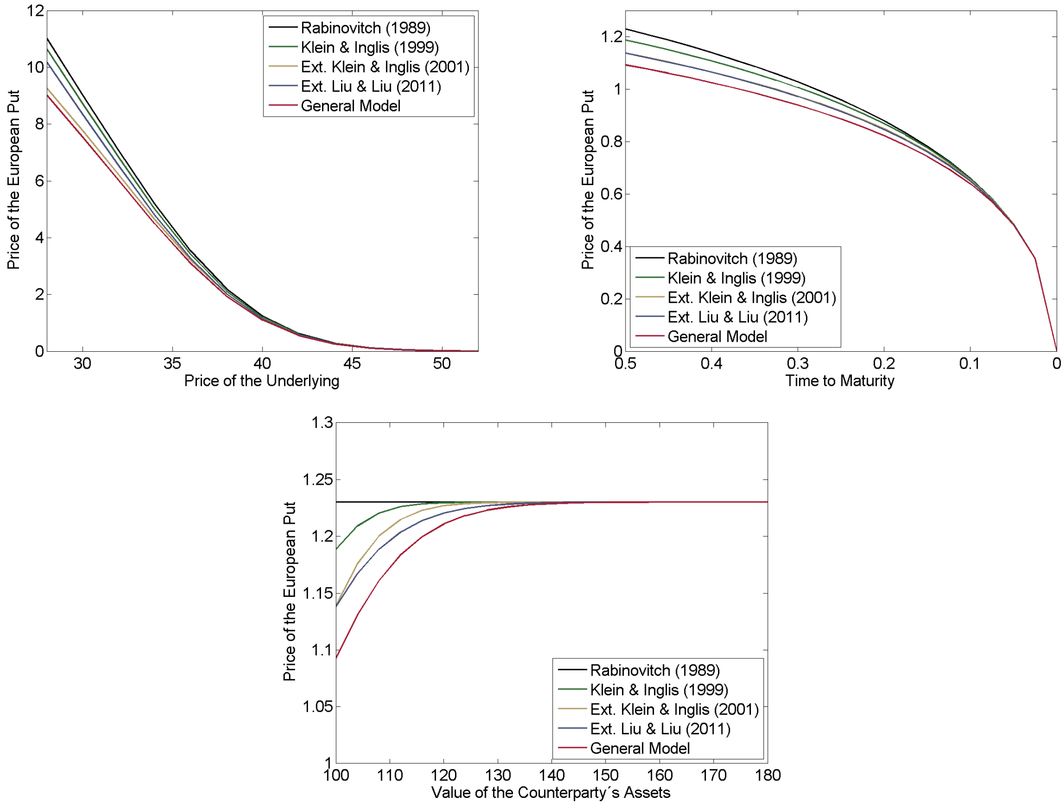

Figure 4.6: European Puts subject to Counterparty and Interest Rate Risk
Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 4.4. The analytical approximations of the extended model of Klein and Inglis (2001) and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively.

In the upper left diagram in Figures 4.5 and 4.6, the values of vulnerable European calls and puts, respectively, is plotted against the price of the option's underlying. It is obvious that the price difference between default-free and vulnerable European options increases if the option is deeper in the money. This behavior is applicable for all valuation models, but it is most prominent for the extended model of Klein and Inglis (2001) and the general model. We also observe that the price difference between these two models and the other models increases substantially if the considered European option is further in the money. This observation is attributed to the fact that the extended model of Klein and Inglis (2001) as well as our general model include the option itself directly in the default boundary which additionally increases the counterparty's default risk for in-the-money options. Refering to the upper right diagram in Figures 4.5 and 4.6, the effect of the time to maturity on the value of vulnerable European options can be analyzed. If the time to maturity decreases, the difference between the default-free and the vulnerable European call values is also reduced. This result is not surprising, since the counterparty is less likely to default if the option's maturity date gets closer. The lower diagram in Figures 4.5 and 4.6 shows the prices of a vulnerable European option converge to the default-free option price with increasing values for the counterparty's assets, since the probability that the value of the counterparty's assets hits the default barrier decreases. Our general model has the lowest convergence speed which is most likely explained by the fact that this model is the only one that incorporates three sources of default risk: a decrease in the value of the counterparty's assets, an increase in the counterparty's other liabilities and an increase in the option value.

Tables 4.3 and 4.4 present the option values for vulnerable European calls and puts, respectively, which are obtained from valuation models presented in Section 4.4. Once again it can be observed that the option values based on the model of Klein and Inglis (1999), the extended models of Klein and Inglis (2001) and Liu and Liu (2011) as well as based on the general model are always lower than the Rabinovitch option values. Furthermore, the option values obtained from our general model differ substantially from those of the other valuation models in most situations. This finding is explained by the construction of the general model's default boundary. The general model is the only one which incorporates three sources of risk simultaneously. First, a decrease in the value of the counterparty's assets might lead to the default of the option writer like in all the other valuation models. Second, the general model accounts for the potential increase in the counterparty risk induced by the option itself (unlike the model of Klein and Inglis (1999) and the extended model of Liu and Liu (2011)). Third, it is assumed that the counterparty's other liabilities are stochastic which creates an additional default risk (unlike the model of Klein and Inglis (1999) and the extended model of Klein and Inglis (2001)). Consequently, the option values based on our general model are the lowest, since it accounts for all possible sources of the counterparty's default risk.

	General Model	Ext. LL2011	Ext. KI2011	KI1999	R1989
Base Case	1.9339	2.0495	2.0168	2.1432	2.2161
$S = 45$	5.1786	5.7078	5.3895	5.9675	6.1719
$S = 35$	0.2827	0.2917	0.2945	0.3051	0.3154
$V = 105$	2.0246	2.1134	2.1067	2.1859	2.2161
$V = 95$	1.8226	1.9609	1.8908	2.0622	2.2161
$T - t = 1$	2.8673	3.0924	3.0262	3.2834	3.4584
$T - t = 0.25$	1.3316	1.3875	1.3781	1.4315	1.4551
$\alpha = 0.5$	1.7362	1.9268	1.8622	2.0834	2.2161
$\alpha = 0$	2.1315	2.1721	2.1714	2.2029	2.2161
$q = 0.02$	1.7316	1.8304	1.8059	1.9142	1.9793
$r_0 = 0.08$	2.1957	2.3330	2.3197	2.4562	2.5227
$r_0 = 0.02$	1.6892	1.7857	1.7374	1.8525	1.9309
$\kappa = 0.8$	1.9333	2.0490	2.0162	2.1426	2.2156
$\kappa = 0.2$	1.9346	2.0500	2.0175	2.1438	2.2167
$\theta = 0.08$	1.9670	2.0853	2.0549	2.1827	2.2549
$\theta = 0.02$	1.9010	2.0140	1.9791	2.1040	2.1778
$\sigma_r = 0.08$	1.9434	2.0571	2.0258	2.1518	2.2243
$\sigma_r = 0.02$	1.9287	2.0454	2.0119	2.1385	2.2117
$\rho_{Sr} = 0.5$	1.9923	2.1059	2.0780	2.2078	2.2772
$\rho_{Sr} = -0.5$	1.8734	1.9910	1.9535	2.0762	2.1528
$\rho_{Vr} = 0.5$	1.9430	2.0570	2.0219	2.1426	2.2161
$\rho_{Vr} = -0.5$	1.9247	2.0419	2.0116	2.1441	2.2161
$\rho_{Dr} = 0.5$	1.9348	2.0419	2.0168	2.1432	2.2161
$\rho_{Dr} = -0.5$	1.9329	2.0570	2.0168	2.1432	2.2161

Table 4.3: European Calls subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 4.4. The analytical approximations of the extended model of Klein and Inglis (2001) and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively. The abbreviations Ext. KI2001 and Ext. LL2011 stand for the extended models of Klein and Inglis (2001) as well as Liu and Liu (2011), whereas R1989 and KI1999 stand for Rabinovitch (1989) and Klein and Inglis (1999), respectively.

	General Model	Ext. LL2011	Ext. KI2011	KI1999	R1989
Base Case	1.0766	1.1377	1.1028	1.1883	1.2302
$S = 45$	0.1619	0.1720	0.1655	0.1796	0.1860
$S = 35$	3.7351	4.0039	3.8238	4.1833	4.3295
$V = 105$	1.1236	1.1732	1.1560	1.2128	1.2302
$V = 95$	1.0189	1.0885	1.0327	1.1425	1.2302
$T - t = 1$	1.2598	1.3579	1.2890	1.4338	1.5186
$T - t = 0.25$	0.8783	0.9139	0.9016	0.9427	0.9585
$\alpha = 0.5$	0.9710	1.0696	0.9956	1.1541	1.2302
$\alpha = 0$	1.1822	1.2058	1.2100	1.2226	1.2302
$q = 0.02$	1.2172	1.2867	1.2470	1.3440	1.3914
$r_0 = 0.08$	0.8959	0.9456	0.9300	0.9945	1.0224
$r_0 = 0.02$	1.2813	1.3560	1.2935	1.4048	1.4662
$\kappa = 0.8$	1.0761	1.1371	1.1024	1.1878	1.2295
$\kappa = 0.2$	1.0771	1.1384	1.1033	1.1890	1.2310
$\theta = 0.08$	1.0517	1.1112	1.0793	1.1618	1.2016
$\theta = 0.02$	1.1019	1.1646	1.1267	1.2153	1.2593
$\sigma_r = 0.08$	1.0838	1.1477	1.1089	1.1970	1.2410
$\sigma_r = 0.02$	1.0727	1.1323	1.0995	1.1836	1.2244
$\rho_{Sr} = 0.5$	1.1202	1.1942	1.1448	1.2436	1.2913
$\rho_{Sr} = -0.5$	1.0304	1.0792	1.0581	1.1307	1.1669
$\rho_{Vr} = 0.5$	1.0703	1.1324	1.0928	1.1810	1.2302
$\rho_{Vr} = -0.5$	1.0829	1.1429	1.1133	1.1955	1.2302
$\rho_{Dr} = 0.5$	1.0778	1.1429	1.1028	1.1883	1.2302
$\rho_{Dr} = -0.5$	1.0753	1.1324	1.1028	1.1883	1.2302

Table 4.4: European Puts subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 40$, $K = 40$, $V_0 = 100$, $D_0 = 90$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.15$, $\sigma_V = 0.15$, $\sigma_D = 0.15$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values are generated using the (approximate) closed form solutions presented in Section 4.4. The analytical approximations of the extended model of Klein and Inglis (2001) and the general model are based on $p = 1.5$ and $p_1 = p_2 = 1.5$, respectively. The abbreviations Ext. KI2001 and Ext. LL2011 stand for the extended models of Klein and Inglis (2001) as well as Liu and Liu (2011), whereas R1989 and KI1999 stand for Rabinovitch (1989) and Klein and Inglis (1999), respectively.

4.6 Summary

In this chapter, different valuation models for vulnerable European options under stochastic interest rates are presented and derived. First, the model of Klein and Inglis (1999) which is an extension of the model of Klein (1996) was discussed in greater deatils. Second, we extended the valuation models of Klein and Inglis (2001) and Liu and Liu (2011) in the same way as Klein and Inglis (1999) extended the model of Klein (1996). Third, we combined the features of these models in a general valuation model. Therefore, the general model is the only model which incorporates three sources of financial distress simultaneously: a decline in the value of the counterparty's assets, an increase in the value of the counterparty's other liabilities or an increase in the value of the option itself.

Despite the complexity of the default condition of our general model, we derived an approximate closed form solution for vulnerable European calls and puts. In particular, we approximated the default condition by employing a first order Taylor series expansion and assumed that the returns of the option's underlying and the counterparty's other liabilities are assumed to be uncorrelated. The obtained approximate valuation formula depends on the two points around which the Taylor series is expanded in the derivation. Choosing the points of expansion to be equal to $p_1 = p_2 = 1.5$ in case of a European call and to be equal to $p_1 = p_2 = -1.5$ in case of a European put, respectively, the approximate analytical solution is quite close to the numerical solution for a wide range of parameters.

Based on various numerical examples and graphical illustrations, we compared the option values obtained from our general model with those of the alternative models for vulnerable European options under stochastic interest rates. All the considered valuation models have in common that the reduction in the value of a vulnerable European option (compared to a default-free European option) increases if the option is deeper in the money, the time to maturity is longer and if the value of the counterparty's assets is low. The option values obtained from our general model are typically the lowest, since it is the only model which accounts for all possible sources of the counterparty's default.

5 American Options subject to Counterparty Risk

In this chapter, different valuation models for American options subject to counterparty risk are presented and discussed. Due to the early exercise features of American options, the counterparty's default is modeled using the structural approach of Black and Cox (1976) which allows for a default prior to the option's maturity. In particular, the counterparty's default is triggered by the value of its assets being below the value of its total liabilities for the first time.

Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) develop valuation models for vulnerable European options assuming a deterministic risk-free interest rate.[12] In the following, we extend these models to analyze the properties of American options subject to counterparty risk. In particular, we maintain their key characteristics, especially with respect to the default condition, but adjust them to be applicable in the context of vulnerable American options. Furthermore, we set up a general valuation model which incorporates all the features and particularities of the other models.

Due to the complexity of the models, closed form solutions cannot be derived. Therefore, numerical methods have to be applied to compute the value of a vulnerable American option. In particular, we use the least squares Monte Carlo simulation approach suggested by Longstaff and Schwartz (2001) and adapt it appropriately to be applicable to value vulnerable American options.

Section 5.1 outlines and discusses the assumptions of the considered theoretical framework. In Section 5.2, we derive the partial differential equation characterizing the price of an American option subject to counterparty risk. Section 5.3 explains how the Longstaff-Schwartz approach can be used to solve this partial differential equation in general. In Section 5.4, we extend the models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) to be applicable for American options subject to counterparty risk. Furthermore, we set up our general model. Section 5.5 provides a comparative analysis of the different valuation models based on numerical examples. Section 5.6 gives a summary of the main findings.

[12] In Chapter 3, the valuation models of Klein (1996), Klein and Inglis (2001) as well as of Liu and Liu (2011) are presented and discussed in greater details.

5.1 Assumptions

The assumptions that characterize the theoretical framework for the valuation of European options subject to counterparty risk are based on Black and Scholes (1973), Merton (1974), Black and Cox (1976), Klein (1996), Klein and Inglis (2001), Chang and Hung (2006), Klein and Yang (2010, 2013) as well as on Liu and Liu (2011).

1. The price of the option's underlying S_t follows a continuous-time geometric Brownian motion. Assuming that the option's underlying is a dividend-paying stock, its dynamics are given by

$$dS_t = (\mu_S - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S\,, \tag{5.1}$$

   where μ_S indicates the expected instantaneous return of the option's underlying, q denotes the continuous dividend yield, σ_S is the instantaneous return volatility and dW_S represents the standard Wiener process.

2. Likewise, the market value of the counterparty's assets V_t follows a continuous-time geometric Brownian motion. Its dynamics are given by

$$dV_t = \mu_V\, V_t\, dt + \sigma_V\, V_t\, dW_V\,, \tag{5.2}$$

   where μ_V is the expected instantaneous return of the counterparty's assets, σ_V gives the instantaneous return volatility and dW_V is a standard Wiener process. The instantaneous correlation between dW_S and dW_V equals ρ_{SV}.

3. The total liabilities D_t comprise all the obligations of the counterparty's, i.e. debt, short positions in financial securities and accruals. The dynamics follow a continuous-time geometric Brownian motion which is given by

$$dD_t = \mu_D\, D_t\, dt + \sigma_D\, D_t\, dW_D\,, \tag{5.3}$$

   where μ_D is the expected instantaneous return of the counterparty's liabilities, σ_D indicates the instantaneous return volatility and dW_D represents the standard Wiener process. The instantaneous correlation between dW_S and dW_D equals ρ_{SD} and ρ_{VD} between dW_V and dW_D, respectively.

4. The market is perfect and frictionless, i.e. it is free of transaction costs or taxes and the available securities are traded in continuous time.

5. The instantaneous risk-free interest rate r is assumed to be deterministic and constant over time.

6. The expected instantaneous return of the option's underlying as well as of the counterparty's assets and liabilities (μ_S, μ_V and μ_D) are deterministic and constant over time. The same applies for the dividend yield q of the option's underlying.

7. The instantaneous return volatilities of the option's underlying as well as of the counterparty's assets and liabilities (σ_S, σ_V and σ_D) are deterministic and constant over time. The instantaneous correlations ρ_{SV}, ρ_{SD} and ρ_{VD} are also constant and independent of time.

8. All the liabilities of the counterparty (i.e. debt, short positions in options, etc.) are assumed to be of equal rank. Consequently, all creditors receive the same proportion of their claim if the counterparty defaults.

9. Before the option's maturity (i.e. $t < T$), default occurs if the counterparty's assets V_t are less than the threshold level L:

$$V_t < \bar{L} \quad \text{or} \quad V_t < L(S_t,\, D_t). \tag{5.4}$$

   Depending on the considered valuation model, the threshold level L is characterized in different ways and is either a constant or a function of the stochastic variables S_t and D_t.

10. At the option's maturity (i.e. $t = T$), default occurs if the market value of the counterparty's assets V_T are less than the threshold level L:

$$V_T < \bar{L} \quad \text{or} \quad V_T < L(S_T,\, D_T). \tag{5.5}$$

   Depending on the considered valuation model, the threshold level L is characterized in different ways and is either a constant or a function of the stochastic variables S_T and D_T.

11. If the counterparty is in default, the option holder receives the fraction $1-\omega_t$ of the nominal claim, where ω_t represents the precentage write-down on the nominal claim at time t. The percentage write-down ω_t can be endogenized. Assuming that all the liabilities of the counterparty are ranked equally, the proportion of the option holder's claim which can be paid back is given by

$$(1-\omega_t) = \frac{(1-\alpha)\,V_t}{L(S_t,\,D_t)}, \tag{5.6}$$

where α represents the cost of default (e.g. bankruptcy or reorganization cost) as a percentage of the counterparty's assets.

5.2 Derivation of the Partial Differential Equation

Following the argument of Hull (2012: 309–312), we derive the partial differential equation governing the price evolution of a vulnerable European option. In the considered theoretical framework (see Section 5.1), the price of a vulnerable American option F_t must be a function of the underlying S_t, the counterparty's assets V_t, the counterparty's liabilities D_t and time t. According to Itô's lemma, the price evolution of a vulnerable American option is given by the following stochastic differential equation:

$$\begin{aligned} dF_t = {} & \frac{\partial F_t}{\partial t}dt + (\mu_S - q)S_t\frac{\partial F_t}{\partial S_t}dt + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2}dt + \sigma_S S_t\frac{\partial F_t}{\partial S_t}dW_S \\ & + \mu_V V_t\frac{\partial F_t}{\partial V_t}dt + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2}dt + \sigma_V V_t\frac{\partial F_t}{\partial V_t}dW_V + \mu_D D_t\frac{\partial F_t}{\partial D_t}dt \\ & + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2}dt + \sigma_D D_t\frac{\partial F_t}{\partial D_t}dW_D + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t\partial V_t}dt \\ & + \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t\partial D_t}dt + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t\partial D_t}dt. \end{aligned} \tag{5.7}$$

To eliminate the Wiener processes dW_S, dW_V and dW_D, a portfolio Π_t consisting of the American option F_t, the underlying S_t, the counterparty's assets V_t and the counterparty's liabilities D_t must be set up.[13] In particular, this portfolio consists

[13] To construct such a portfolio, it is necessary to assume that option's underlying as well as the counterparty's assets and liabilities are traded securities. This assumption is not questionable for the option's underlying, but it is for both the counterparty's assets and liabilities. As argued by Klein (1996), it is likely that the counterparty's assets and liabilities are not traded directly in the market, but that their market values behave similarly as if they were traded securities.

of a short position in the American option and long positions in the underlying, the counterparty's assets and liabilities. The amount of shares in the long positions are equal to $\partial F_t/\partial S_t$, $\partial F_t/\partial V_t$ and $\partial F_t/\partial D_t$, respectively. Hence, the value of the portfolio at time t is given by

$$\Pi_t = -F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t. \tag{5.8}$$

The change in the value of the portfolio over the time interval dt is characterized by the total differential which is equal to

$$d\Pi_t = -dF_t + \frac{\partial F_t}{\partial S_t} dS_t + \frac{\partial F_t}{\partial V_t} dV_t + \frac{\partial F_t}{\partial D_t} dD_t. \tag{5.9}$$

Substituting Equations (5.1) to (5.3) and (5.7) into Equation (5.9) yields

$$\begin{aligned} d\Pi_t = &-\frac{\partial F_t}{\partial t} dt + qS_t \frac{\partial F_t}{\partial S_t} - \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt - \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt \\ &- \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt - \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt \\ &- \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt - \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt. \end{aligned} \tag{5.10}$$

Since the portfolio dynamics are independent of the Wiener processes dW_S, dW_V and dW_D, the portfolio is riskless during the infinitesimal time interval dt. To avoid arbitrage opportunities, the portfolio must earn the same return as other short-term risk-free investments – namely the risk-free interest rate r:

$$r\Pi dt = d\Pi_t. \tag{5.11}$$

We substitute Equations (5.8) and (5.10) into Equation (5.11) which yields

$$\begin{aligned} &r\left(-F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t\right) dt \\ &= \frac{\partial F_t}{\partial t} dt - qS_t \frac{\partial F_t}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt + \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt \\ &\quad + \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt + \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt \\ &\quad + \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt. \end{aligned} \tag{5.12}$$

Rewriting Equation (5.12), the partial differential equation that characterizes the price of an American option whose payoff at time T is contigent upon the price of the option's underlying as well as upon the value of both the counterparty's assets and liabilities is obtained. It is given by

$$\begin{aligned}0 = &\frac{\partial F_t}{\partial t} - rF_t + (r-q)S_t\frac{\partial F_t}{\partial S_t} + rV_t\frac{\partial F_t}{\partial V_t} + rD_t\frac{\partial F_t}{\partial D_t} \\ &+ \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2} + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2} + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2} + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t \partial V_t} \\ &+ \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t \partial D_t} + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t \partial D_t}.\end{aligned} \tag{5.13}$$

To obtain a unique solution to the partial differential equation, we must set up the boundary conditions which specify the value of the American option based on Assumptions 9 to 11 (see Section 5.1). For the American call, the boundary conditions can be expressed as follows:

1. At the option's maturity (i.e. $t = T$), three different scenarios may occur. If the option expires in the money and the counterparty does not default, $S_T - K$ are paid out to the holder of an American call. If the option expires in the money and the counterparty is in default, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American call receives $((1-\alpha)\, V_T\, (S_T - K))\, / L(S_T,\, D_T)$. If the option is out of the money at maturity, the option holder receives nothing.

$$F_T = C_T = \begin{cases} S_T - K & \text{if } S_T \geq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(S_T - K) & \text{if } S_T \geq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{5.14}$$

2. If the counterparty defaults prior to maturity (i.e. $t < T$), the American option is immediately exercised. If the option is in the money at that point in time, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all

creditors receive the same proportion of their claims. Hence, the holder of an American call receives $\left(\left(1-\alpha\right)V_t\left(S_t-K\right)\right)/L(S_t,\,D_t)$. If the option is out of the money at that point in time, the option holder receives nothing.

$$F_t = C_t = \begin{cases} \dfrac{(1-\alpha)V_t}{L(S_t,\,D_t)}\,(S_t-K) & \text{if } S_t \geq K,\, V_t < L(S_t,\,D_t) \\ 0 & \text{otherwise} \end{cases} \tag{5.15}$$

3. It may be optimal to exercise an American call prior to maturity (i.e. $t < T$) even though the counterparty is not in default. Early exercise is optimal if the early exercise payoff $C_t^{\text{Ex}} = \max(S_t - K, 0)$ is larger than the conditional expected continuation value C_t^{Cont}, i.e. the expected future option payoff.

$$F_t = C_t = \begin{cases} S_t - K & \text{if } C_t^{\text{Ex}} > C_t^{\text{Cont}},\, V_t \geq L(S_t,\,D_t) \\ \text{No early exercise} & \text{otherwise} \end{cases} \tag{5.16}$$

The boundary conditions for the American put are given in analogy:

1. At the option's maturity (i.e. $t = T$), three different scenarios may occur. If the option expires in the money and the counterparty does not default, $K - S_T$ are paid out to the holder of an American put. If the option expires in the money and the counterparty is in default, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American put receives $\left(\left(1-\alpha\right)V_T\left(K-S_T\right)\right)/L(S_T,\,D_T)$. If the option is out of the money at maturity, the option holder receives nothing.

$$F_T = P_T = \begin{cases} K - S_T & \text{if } S_T \leq K,\, V_T \geq L(S_T,\,D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\,D_T)}\,(K-S_T) & \text{if } S_T \leq K,\, V_T < L(S_T,\,D_T) \\ 0 & \text{otherwise} \end{cases} \tag{5.17}$$

2. If the counterparty defaults prior to maturity (i.e. $t < T$), the American put is immediately exercised. If the option is in the money at that point in time,

the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American put receives $((1-\alpha)\,V_t\,(K-S_t))\,/L(S_t,\,D_t)$. If the option is out of the money at that point in time, the option holder receives nothing.

$$F_t = P_t = \begin{cases} \dfrac{(1-\alpha)V_t}{L(S_t,\,D_t)}\,(K-S_t) & \text{if } S_t \leq K,\, V_t < L(S_t,\,D_t) \\ 0 & \text{otherwise} \end{cases} \tag{5.18}$$

3. It may be optimal to exercise an American put prior to maturity (i.e. $t < T$) even though the counterparty is not in default. Early exercise is optimal if the early exercise payoff $P_t^{\text{Ex}} = \max(K - S_t, 0)$ is larger than the conditional expected continuation value P_t^{Cont}, i.e. the expected future option payoff.

$$F_t = P_t = \begin{cases} K - S_t & \text{if } P_t^{\text{Ex}} > P_t^{\text{Cont}},\, V_t \geq L(S_t,\,D_t) \\ \text{No early exercise} & \text{otherwise} \end{cases} \tag{5.19}$$

The actual characterization of the boundary conditions depends on the choice of a specific valuation model (see Section 5.4). In particular, the threshold level $L(S_t,\,D_t)$ must be defined according to the chosen model.

Referring to Equations (5.15) and (5.18), we assume that an American option is immediately exercised if the counterparty defaults at a given time t prior to the option's maturity. Chang and Hung (2006) as well as Klein and Yang (2010) also deal with the valuation of vulnerable American options. However, their assumptions with respect to the option payoff if the counterparty defaults prior to maturity differ from our assumption. In particular, Chang and Hung (2006) assume that the American option is not necessarily exercised in the case of the counterparty's default, i.e. the option holder has the opportunity to keep the American option unexercised until maturity, although the counterparty is insolvent. Klein and Yang (2010), in turn, suppose that only in-the-money American options are immediately exercised if the counterparty is in default prior to maturity. If the counterparty is in default and the American option is out of the money, it is not exercised.

5.3 Solution to the Partial Differential Equation

The partial differential equation given by Equation (5.13) depends on the price of the option's underlying, the counterparty's assets and liabilities, the risk-free interest rate, the dividend yield of the option's underlying as well as on the return volatilities. All these variables and parameters are independent of the risk preferences of the investors.[14] Since the risk preferences of the investors do not enter the partial differential equation, they cannot affect its solution. Consequently, any type of risk preferences can be used when solving the partial differential equation.

The partial differential equation given by Equation (5.13) subject to the boundary conditions specified by Equations (5.14) to (5.16) and (5.17) to (5.19), respectively, can be solved using the regression-based Monte Carlo simulation approach suggested by Longstaff and Schwartz (2001). Even though this approach has originally been derived to value plain vanilla American options, it can also be applied in more complex theoretical frameworks in which the price of the considered option depends on more than one stochastic variable (see Longstaff & Schwartz, 2001; Moreno & Navas, 2003). It is optimal to exercise an American option prior to its maturity if the option payoff based on the immediate exercise is greater than the option's conditional expected continuation value. Longstaff and Schwartz (2001) suggest to estimate the conditional expectation by a least-squares regression based on the cross-sectional information provided by Monte Carlo simulation. Consequently, sample paths need to be generated for the price of the option's underlying as well as for the market value of the counterparty's assets and liabilities.

Using the approach of Cox and Ross (1976) and Harrison and Pliska (1981), the risk-neutral stochastic processes for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities are equal to

$$dS_t = (r - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S, \tag{5.20}$$

[14] Following the argument of Hull (2012: 311–312), the partial differential equation given by Equation (5.13) would not be independent of risk preferences if it included the expected returns of the option's underlying, the counterparty's assets and the counterparty's liabilities. These parameters depend on risk preferences, since their magnitude represents the level of risk aversion of the investor: the higher the level of the investor's risk aversion, the higher the required expected return.

$$dV_t = r\, V_t\, dt + \sigma_V\, V_t\, dW_V \tag{5.21}$$

and

$$dD_t = r\, D_t\, dt + \sigma_D\, D_t\, dW_D, \tag{5.22}$$

where r denotes the risk-free interest rate and all other variables are defined as before.

Applying Itô's lemma to Equations (5.20) to (5.22), the stochastic processes for $\ln S_t$, $\ln V_t$ and $\ln D_t$ are obtained. They are given by

$$d\ln S_t = \left(r - q - \frac{1}{2}\sigma_S^2\right) dt + \sigma_S dW_S, \tag{5.23}$$

$$d\ln V_t = \left(r - \frac{1}{2}\sigma_V^2\right) dt + \sigma_V dW_V \tag{5.24}$$

and

$$d\ln D_t = \left(r - \frac{1}{2}\sigma_D^2\right) dt + \sigma_D dW_D. \tag{5.25}$$

Rewriting Equations (5.23) to (5.25), expressions for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities at every point in time can be derived. Using Δt as the time step, the evolution of the stochastic variables over time is given by

$$S_{t+\Delta t} = S_t\, e^{(r-q-\frac{1}{2}\sigma_S^2)\Delta t + \sigma_S \sqrt{\Delta t}\, x_S}, \tag{5.26}$$

$$V_{t+\Delta t} = V_t\, e^{(r-\frac{1}{2}\sigma_V^2)\Delta t + \sigma_V \sqrt{\Delta t}\, x_V} \tag{5.27}$$

and

$$D_{t+\Delta t} = D_t\, e^{(r-\frac{1}{2}\sigma_D^2)\Delta t + \sigma_D \sqrt{\Delta t}\, x_D}, \tag{5.28}$$

where the three random variables x_S, x_V and x_D are jointly standard normally distributed and their respective correlations are given by the coefficients ρ_{SV}, ρ_{SD} and ρ_{VD}.

Equations (5.26) to (5.28) can be used in the Monte Carlo simulation to generate sample paths for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities $(S_0,\, S_{\Delta t}, ..., \, S_t, ..., \, S_T)$, $(V_0,\, V_{\Delta t}, ..., \, V_t, ..., \, V_T)$ and $(D_0,\, D_{\Delta t}, ..., \, D_t, ..., \, D_T)$, where t denotes the time index and Δt is the discrete time step. At any time step t, the dynamic programming recursion functions for American calls and puts, respectively, are given by

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t}\,|\, S_t, V_t, D_t\right]\right) & \text{if } V_t \geq L(S_t,\, D_t) \\ \dfrac{(1-\alpha)\, V_t}{L(S_t,\, D_t)}\, \max\left(S_t - K, 0\right) & \text{if } V_t < L(S_t,\, D_t) \end{cases} \tag{5.29}$$

and

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, P_{t+\Delta t}\,|\, S_t, V_t, D_t\right]\right) & \text{if } V_t \geq L(S_t,\, D_t) \\ \dfrac{(1-\alpha)\, V_t}{L(S_t,\, D_t)}\, \max\left(K - S_t, 0\right) & \text{if } V_t < L(S_t,\, D_t) \end{cases} \tag{5.30}$$

If the counterparty defaults, the option is immediately exercised irrespective of whether the option is in the money or not. If the counterparty is not in default, however, the option holder must decide whether he wants to exercise the option prior to maturity. In particular, the option is exercised immediately if the option payoff is greater than the conditional expectation of continuation under the risk-neutral measure. Being at a given time step of the sample path, this decision, however, cannot be taken along an individual sample path, since the option holder cannot exploit knowledge of the future prices along that path. To avoid anticipativity, the total set of sample paths is used to approximate the conditional expected continuation value by regressing the conditional expectation against M basis functions $\psi_m(\,\cdot\,)$. At each time step, the same set of basis functions is used, but the coefficients $\beta_{m,t}$ are time-dependent. Consequently, the relationship between the expected option value one time step ahead and the basis functions are given by the following expressions:

$$\begin{aligned} \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t}\,|\, S_t, V_t, D_t\right] \approx \beta_{0,t} &+ \beta_{1,t}\, \psi_1(S_t, V_t, D_t) \\ &+ \cdots + \beta_{M,t}\, \psi_M(S_t, V_t, D_t), \end{aligned} \tag{5.31}$$

$$\begin{aligned} \mathbb{E}_t\left[e^{-r\Delta t}\, P_{t+\Delta t}\,|\, S_t, V_t, D_t\right] \approx \beta_{0,t} &+ \beta_{1,t}\, \psi_1(S_t, V_t, D_t) \\ &+ \cdots + \beta_{M,t}\, \psi_M(S_t, V_t, D_t). \end{aligned} \tag{5.32}$$

Since the coefficients $\beta_{m,t}$ are not related to a particular sample path, the decisions based on the approximated conditional expected continuation value of the considered American option are non-anticipative. The coefficients $\beta_{m,t}$ can be estimated by a simple least-squares regression minimizing the sum of the squared residuals. The sample paths for the option's underlying as well as for the counterparty's assets and liabilities are generated using Monte Carlo simulation, where S_t^i, V_t^i and D_t^i give the value of the respective stochastic variable at time t along sample path $i = 1, ..., N$. Based on these considerations, the regression model is equal to

$$\begin{aligned} e^{-r\Delta t}\, C^i_{t+\Delta t} = \beta_{0,\,t} + \beta_{1,\,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\ + \cdots + \beta_{M,\,t}\, \psi_M(S_t^i, V_t^i, D_t^i) + \varepsilon_i \end{aligned} \tag{5.33}$$

and

$$\begin{aligned} e^{-r\Delta t}\, P^i_{t+\Delta t} = \beta_{0,\,t} + \beta_{1,\,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\ + \cdots + \beta_{M,\,t}\, \psi_M(S_t^i, V_t^i, D_t^i) + \varepsilon_i, \end{aligned} \tag{5.34}$$

where ε_i is the residual for each sample path. The obtained estimators $\hat{\beta}_{k,\,t}$ can be used to approximate the conditional expected continuation value of the American option for each sample path i. For vulnerable American calls and puts, respectively, the approximation is given by

$$\begin{aligned} e^{-r\Delta t}\, C^i_{t+\Delta t} = \hat{\beta}_{0,\,t} + \hat{\beta}_{1,\,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\ + \cdots + \hat{\beta}_{M,\,t}\, \psi_M(S_t^i, V_t^i, D_t^i) \end{aligned} \tag{5.35}$$

and

$$\begin{aligned} e^{-r\Delta t}\, P^i_{t+\Delta t} = \hat{\beta}_{0,\,t} + \hat{\beta}_{1,\,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\ + \cdots + \hat{\beta}_{M,\,t}\, \psi_M(S_t^i, V_t^i, D_t^i). \end{aligned} \tag{5.36}$$

Since the regression-based approach of Longstaff and Schwartz (2001) is a dynamic programing method, the valuation problem must be solved recursively, i.e. the procedure starts at the option's maturity and goes backwards in time. Using the generated sample paths for the option's underlying as well as for the counterparty's assets and liabilities, the dynamic programming recursion functions at the option's maturity can be determined for each sample path i. At the option's expiration, these functions are simply given by the payoff of the vulnerable American option. For vulnerable American calls and puts, respectively, they are equal to

$$C_T^i = \begin{cases} \max\left(S_T^i - K, 0\right) & \text{if } V_T^i \geq L(S_T^i,\, D_T^i) \\ \dfrac{(1-\alpha)\, V_T^i}{L(S_T^i,\, D_T^i)} \max\left(S_T^i - K, 0\right) & \text{if } V_T^i < L(S_T^i,\, D_t^i) \end{cases} \tag{5.37}$$

and

$$P_T^i = \begin{cases} \max\left(K - S_T^i, 0\right) & \text{if } V_T^i \geq L(S_T^i,\, D_T^i) \\ \dfrac{(1-\alpha)\, V_T^i}{L(S_T^i,\, D_T^i)} \max\left(K - S_T^i, 0\right) & \text{if } V_T^i < L(S_T^i,\, D_T^i) \end{cases} \tag{5.38}$$

Longstaff and Schwartz (2001) argue that it is more efficient to consider only the subset of sample paths for which a decision must be taken at a given time step t when regressing the conditional expectation against the basis functions. Consequently, this subset must contain all the sample paths in which the option is in the money at the given time step t. This subset is denoted by $\mathcal{I}_t$. At the time step $T - \Delta t$, the regression model for American calls and puts, respectively, is thus given by

$$\begin{aligned} e^{-r\Delta t}\, C_t^i = {} & \beta_{0,\,T-\Delta t} + \beta_{1,\,T-\Delta t}\, \psi_1(S_{T-\Delta t}^i, V_{T-\Delta t}^i, D_{T-\Delta t}^i) \\ & + \cdots + \beta_{M,\,T-\Delta t}\, \psi_M(S_{T-\Delta t}^i, V_{T-\Delta t}^i, D_{T-\Delta t}^i) + \varepsilon_i \qquad i \,\epsilon\, \mathcal{I}_{T-\Delta t} \end{aligned} \tag{5.39}$$

and

$$\begin{aligned} e^{-r\Delta t}\, P_t^i = {} & \beta_{0,\,T-\Delta t} + \beta_{1,\,T-\Delta t}\, \psi_1(S_{T-\Delta t}^i, V_{T-\Delta t}^i, D_{T-\Delta t}^i) \\ & + \cdots + \beta_{M,\,T-\Delta t}\, \psi_M(S_{T-\Delta t}^i, V_{T-\Delta t}^i, D_{T-\Delta t}^i) + \varepsilon_i \qquad i \,\epsilon\, \mathcal{I}_{T-\Delta t} \end{aligned} \tag{5.40}$$

The estimated parameters $\hat{\beta}_{m,\,T-\Delta t}$ obtained from the least squares regression are used to compute the approximate continuation value of the option. Comparing this value with the payoff of immediate exercise, it can be decided whether the option should be exercised early.

The above procedure is repeated going backwards in time. On each sample path i, the cash flows resulting from early exercise decisions must be considered. At the time step t on sample path i, there may be a time step $t^* \geq t$ at which the American option has been exercised early. Taking this issue into account, the regression model for American calls and puts, respectively, can be rewritten as

$$\begin{aligned} e^{-r(t^*-t)}\, C^i_{t^*} &= \beta_{0,t} + \beta_{1,t}\, \psi_1(S^i_t, V^i_t, D^i_t) \\ &\quad + \cdots + \beta_{M,t}\, \psi_M(S^i_t, V^i_t, D^i_t) + \varepsilon_i \qquad i \in \mathcal{I}_t \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} e^{-r(t^*-t)}\, P^i_{t^*} &= \beta_{0,t} + \beta_{1,t}\, \psi_1(S^i_t, V^i_t, D^i_t) \\ &\quad + \cdots + \beta_{M,t}\, \psi_M(S^i_t, V^i_t, D^i_t) + \varepsilon_i \qquad i \in \mathcal{I}_t. \end{aligned} \tag{5.42}$$

Since there is at most one exercise time for each path, it may be the case that after comparing the payoff of immediate exercise with the approximate continuation value on a particular path, the exercise time t^* needs to be reset to a another period.

To apply the above approach to a valuation model for vulnerable American options, the threshold level $L(S_t, D_t)$ must be specified in accordance. Furthermore, the basis functions used in the linear regression must be chosen appropriately.

5.4 Valuation Models

Various valuation models for vulnerable European options have been developed over the last three decades based on the structural approach of Merton (1974). In this framework, the predominant valuation models are those of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011). In the following, we use the main ideas of these models to set up equivalent models for vulnerable American options. Due to the early exercise feature of American options, the counterparty's default may occur prior to maturity. Hence, the structural approach of Black and Cox (1976) need to be considered. Furthermore, we set up a general valuation model incorporating the features of the other models. To value the vulnerable American options, the least squares Monte Carlo simulation by Longstaff and Schwartz (2001) is applied.

In Section 5.3, we generally showed how the Longstaff-Schwartz approach is adjusted to value vulnerable American options. To apply this method to a particular valuation model, the dynamic programming recursion functions in Equations (5.29) and (5.30) as well as the basis functions $\psi_m(S_t, V_t, D_t)$ must be specified accordingly.

5.4.1 Absence of Default Risk

Longstaff and Schwartz (2001) originally derived the regression-based approach to value American options in the absence of counterparty risk. The dynamic programming recursion functions for default-free American calls and puts, respectively, are given by

$$C_t = \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t}\,|\, S_t\right]\right) \tag{5.43}$$

and

$$P_t = \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, P_{t+\Delta t}\,|\, S_t\right]\right). \tag{5.44}$$

An American option is exercised prior to maturity only if the payoff of an immediate exercise is larger than the option's continuation value. Consequently, the crucial point in the Longstaff-Schwartz approach is the estimation of the conditional expected continuation value. As shown in Equations (5.31) and (5.32), an approximation for the conditional expected continutaion value can be obtained by regressing the discounted expected future cash flows against a set of basis functions. Longstaff and Schwartz (2001) use Laguerre polynomials for these functions and argue that using more than three basis functions does not yield more accurate results. In particular, the basis functions are given as follows:

$$\begin{aligned} \psi_1 &= 1 - S_t, \\ \psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right), \\ \psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right). \end{aligned} \tag{5.45}$$

5.4.2 Deterministic Liabilities

Originally, Klein (1996) deals with the valuation of vulnerable European options and assumes that the counterparty defaults if its assets are lower than the total liabilities. The total liabilities of the counterparty are constant over time and must include the short position in the option by construction, since it obliges the option writer to deliver or purchase the option's underlying if the option is exercised. In the context of American options, we must account for the counterparty's default occuring prior to maturity and need adjust the default condition of Klein (1996) accordingly. Hence, the default boundary $L(S_t,\, D_t)$ must be given by

$$L(S_t,\, D_t) = \bar{L} = \bar{D} = D_0. \tag{5.46}$$

Inserting this expression into Equations (5.29) and (5.30), the dynamic programming recursion functions for vulnerable American calls and puts, respectively, based on the ideas of Klein (1996) are given by

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} \\ \dfrac{(1-\alpha)\, V_t}{\bar{D}} \max\left(S_t - K, 0\right) & \text{if } V_t < \bar{D} \end{cases} \tag{5.47}$$

and

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, P_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} \\ \dfrac{(1-\alpha)\, V_t}{\bar{D}} \max\left(K - S_t, 0\right) & \text{if } V_t < \bar{D} \end{cases} \tag{5.48}$$

Referring to the first line in Equations (5.47) and (5.48), the holder of an American option must decide whether the option should be exercised early at the given time step t if the counterparty is not in default. Early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. If the counterparty, however, defaults at the given time step t, the American option is immediately exercised irrespective of whether the option is in the money or not according to the second line in Equations (5.47) and (5.48). In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $\left((1-\alpha)\, V_t\right)/\bar{D}$. Consequently, the holder of a vulnerable American call receives $\left((1-\alpha)\, V_t \max(S_t - K, 0)\right)/\bar{D}$, whereas $\left((1-\alpha)\, V_t \max(K - S_t, 0)\right)/\bar{D}$ is paid out to the holder of a vulnerable American put.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions of the state variables as illustrated in Equations (5.31) and (5.32). Referring to Moreno and Navas (2003) as well as to Chang and Hung (2006), it is sufficient to use a total of nine basis functions in case of two state variables. In particular, the following Laguerre polynomials are used:

$$
\begin{aligned}
\psi_1 &= 1 - S_t, \qquad (5.49)\\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right),\\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right),\\
\psi_4 &= 1 - V_t,\\
\psi_5 &= \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right),\\
\psi_6 &= \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right),\\
\psi_7 &= 1 - S_t V_t,\\
\psi_8 &= \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right),\\
\psi_9 &= \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right).
\end{aligned}
$$

5.4.3 Deterministic Liabilities and Option induced Default Risk

Like Klein (1996), Klein and Inglis (2001) originally set up a valuation model for vulnerable European options in which the counterparty can only default at the option's maturity. They recognize that the short position in the option itself may cause additional financial distress. To account for this potential source of default risk, they split the counterparty's total liabilities into two components. In particular, the total liabilities consist of the short position in the option on the one hand and all the other liabilities on the other which are assumed to be constant over time. When dealing with the valuation of American options, it is reasonable to consider that the counterparty may default prior to maturity. If we account for this issue and maintain the key features of Klein and Inglis (2001), the time-dependent default boundary $L(S_t,\, D_t)$ for American calls and puts, respectively, is given as follows:

$$L(S_t,\, D_t) = L(S_t) = \bar{D} + S_t - K = D_0 + S_t - K, \qquad (5.50)$$

$$L(S_t,\, D_t) = L(S_t) = \bar{D} + K - S_t = D_0 + K - S_t. \qquad (5.51)$$

Inserting the above expressions into Equations (5.29) and (5.30), the dynamic programming recursion functions for the Longstaff-Schwartz approach based on the ideas of Klein and Inglis (2001) are obtained. For vulnerable American calls and puts, respectively, they are equal to

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} + S_t - K \\ \dfrac{(1-\alpha)\, V_t}{\bar{D} + S_t - K} \max\left(S_t - K, 0\right) & \text{if } V_t < \bar{D} + S_t - K \end{cases} \tag{5.52}$$

and

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, P_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} + K - S_t \\ \dfrac{(1-\alpha)\, V_t}{\bar{D} + K - S_t} \max\left(K - S_t, 0\right) & \text{if } V_t < \bar{D} + K - S_t \end{cases} \tag{5.53}$$

Like in the extended Klein model, the holder of the American option must decide whether the option should be immediately exercised if the counterparty is not in default at time t according to the first line in Equations (5.52) and (5.53). Early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. The second line in Equations (5.52) and (5.53) refers to the scenario in which the counterparty is in default at time t. In this case, the American option is immediately exercised irrespective of whether the option is in the money or not. The entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Due to the construction of the default boundary, this proportion depends on the type of the considered option. It is given by $\left((1-\alpha)\, V_t\right) / \left(\bar{D} + S_t - K\right)$ for a vulnerable American call, whereas it is equal to $\left((1-\alpha)\, V_t\right) / \left(\bar{D} + K - S_t\right)$ for a vulnerable American put. Consequently, the holder of a vulnerable American call receives $\left((1-\alpha)\, V_t \max(S_t - K, 0)\right) / \left(\bar{D} + S_t - K\right)$, whereas the holder of a vulnerable American put receives $\left((1-\alpha)\, V_t \max(K - S_t)\right) / \left(\bar{D} + K - S_t\right)$.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions of the state variables as illustrated in Equations (5.31) and (5.32). Like in the extended Klein model, the value of the vulnerable American option based on the extended Klein-Inglis model is driven by two state variables. Consequently, the same Laguerre polynomials as before can be used as basis functions:

$$
\begin{aligned}
\psi_1 &= 1 - S_t, \\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right), \\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right), \\
\psi_4 &= 1 - V_t, \\
\psi_5 &= \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right), \\
\psi_6 &= \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right), \\
\psi_7 &= 1 - S_t V_t, \\
\psi_8 &= \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right), \\
\psi_9 &= \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right).
\end{aligned} \tag{5.54}
$$

5.4.4 Stochastic Liabilities

Liu and Liu (2011) also suggest a valuation model for vulnerable European options. Like in the models of Klein (1996) and Klein and Inglis (2001), the counterparty's default can only occur at the option's maturity and is triggered by the counterparty's assets being lower than the total liabilities. In contrast to the previous models, Liu and Liu (2011) assume that the market value of the counterparty's total liabilities is stochastic and follows a geometric Brownian motion as given by Equation (5.3). It is important to note that the short position in the option is implicitly included in the counterparty's total liabilities, but its impact on the value of the couterparty's total liabilities is not explicitly modeled (unlike in the Klein-Inglis model). In the valuation of American options, it is important to consider that the counterparty may also default prior to maturity. If we consider this issue and follow the key aspects of Liu and Liu (2011), especially with respect to the default condition, the time-dependent default boundary $L(S_t,\, D_t)$ must be given by

$$
L(S_t,\, D_t) = L(D_t) = D_t. \tag{5.55}
$$

Inserting this expression into Equations (5.29) and (5.30), the dynamic programming recursion functions for vulnerable American calls and puts, respectively, based on the extended model of Liu and Liu (2011) are given by

$$C_t = \begin{cases} \max\Big(S_t - K,\, \mathbb{E}_t\Big[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t, D_t\Big]\Big) & \text{if } V_t \geq D_t \\ \dfrac{(1-\alpha)\, V_t}{D_t} \max\left(S_t - K, 0\right) & \text{if } V_t < D_t \end{cases} \tag{5.56}$$

and

$$P_t = \begin{cases} \max\Big(K - S_t,\, \mathbb{E}_t\Big[e^{-r\Delta t}\, P_{t+\Delta t} \,|\, S_t, V_t, D_t\Big]\Big) & \text{if } V_t \geq D_t \\ \dfrac{(1-\alpha)\, V_t}{D_t} \max\left(K - S_t, 0\right) & \text{if } V_t < D_t \end{cases} \tag{5.57}$$

Referring to the first line in Equations (5.56) and (5.57), the holder of an American option must decide whether the option should be exercised early at the given time step t if the counterparty is not in default. Early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. If the counterparty, however, defaults at the given time step t, the American option is immediately exercised irrespective of whether the option is in the money or not according to the second line in Equations (5.56) and (5.57). In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $\big((1-\alpha)\, V_t\big)/D_t$. Consequently, the holder of a vulnerable American call receives $\big((1-\alpha)\, V_t \max(S_t - K, 0)\big)/D_t$, whereas $\big((1-\alpha)\, V_t \max(K - S_t, 0)\big)/D_t$ is paid out to the holder of a vulnerable American put.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions of the state variables as illustrated in Equations (5.31) and (5.32). Unlike in the previously presented models, the value of the vulnerable American option is driven by three state variables in the extended Liu-Liu model, since the price of the option's underlying as well as the counterparty's assets and liabilities are stochastic. Consequently, more basis functions need to be used in the estimation of the option's conditional expected continuation value. Setting up the Laguerre polynomials in the trivariate case in analogy to the bivariate case of the extended Klein and Klein-Inglis model results in a total of 18 basis functions.[15] In particular, they are given as follows:

$$
\begin{aligned}
\psi_1 &= 1 - S_t \\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right) \\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right) \\
\psi_4 &= 1 - V_t \\
\psi_5 &= \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right) \\
\psi_6 &= \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right) \\
\psi_7 &= 1 - D_t \\
\psi_8 &= \frac{1}{2}\left(2 - 4\,D_t + D_t^2\right) \\
\psi_9 &= \frac{1}{6}\left(6 - 18\,D_t + 9\,D_t^2 - D_t^3\right) \\
\psi_{10} &= 1 - S_t V_t \\
\psi_{11} &= \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right) \\
\psi_{12} &= \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right) \\
\psi_{13} &= 1 - S_t D_t \\
\psi_{14} &= \frac{1}{2}\left(2 - 4\,S_t^2 D_t + (S_t^2 D_t)^2\right) \\
\psi_{15} &= \frac{1}{6}\left(6 - 18\,S_t D_t^2 + 9\,(S_t D_t^2)^2 - (S_t D_t^2)^3\right) \\
\psi_{16} &= 1 - V_t D_t \\
\psi_{17} &= \frac{1}{2}\left(2 - 4\,V_t^2 D_t + (V_t^2 D_t)^2\right) \\
\psi_{18} &= \frac{1}{6}\left(6 - 18\,V_t D_t^2 + 9\,(V_t D_t^2)^2 - (V_t D_t^2)^3\right)
\end{aligned}
\tag{5.58}
$$

[15] In the course of this dissertation, we also tested a higher number of Laguerre polynomials as well as different basis functions especially with respect to the combinations of the state variables' cross products. However, the effect on the accuracy of the results was only marginal. This result is consistent with Longstaff and Schwartz (2001), Moreno and Navas (2003) as well as Chang and Hung (2006).

5.4.5 General Model

In our general model, we pick up on the ideas of both Klein and Inglis (2001) and Liu and Liu (2011). In particular, we assume that the short position in the option may increase the counterparty's default risk and the market value of the counterparty's other liabilities follows a geometric Brownian motion as given by Equation (5.3). At time t, the counterparty's total liabilities are given by $D_t + S_t - K$ in the case of an American call and $D_t + K - S_t$ in the case of an American put, respectively. Consequently, the default boundary $L(S_t,\, D_t)$ indicating the default boundary depends on the type of the considered option. For vulnerable American calls and puts, respectively, it is given by the following expressions:

$$L(S_t,\, D_t) = D_t + S_t - K \tag{5.59}$$

$$L(S_t,\, D_t) = D_t + K - S_t. \tag{5.60}$$

Plugging these expressions into Equations (5.29) and (5.30), the dynamic programming recursion functions for vulnerable American calls and puts, respectively, based on the ideas of the general model are given as follows:

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t, D_t\right]\right) & \text{if } V_t \geq D_t + S_t - K \\ \dfrac{(1-\alpha)\, V_t}{D_t + S_t - K} \max\left(S_t - K, 0\right) & \text{if } V_t < D_t + S_t - K \end{cases} \tag{5.61}$$

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, P_{t+\Delta t} \,|\, S_t, V_t, D_t\right]\right) & \text{if } V_t \geq D_t + K - S_t \\ \dfrac{(1-\alpha)\, V_t}{D_t + K - S_t} \max\left(K - S_t, 0\right) & \text{if } V_t < D_t + K - S_t \end{cases} \tag{5.62}$$

In analogy to the previously presented valuation models, the holder of the American option must decide whether the option should be immediately exercised if the counterparty is not in default at time t. According to the first line in Equations (5.61) and (5.62), early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. The second line in Equations (5.61) and (5.62) refers to the scenario in which the counterparty is in default at time t. In this case, the American option is immediately exercised irrespective of whether the option is in the money or not. The counterparty's entire assets (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Due to the construction of the default boundary, this proportion depends on the type of the considered option. It is given by $\Big((1-\alpha)\,V_t\Big)\,/\,\Big(D_t + S_t - K\Big)$ for a vulnerable American call, whereas it is equal to $\Big((1-\alpha)\,V_t\Big)\,/\,\Big(D_t + K - S_t\Big)$ for a vulnerable American put. Consequently, the holder of a vulnerable American call receives $\Big((1-\alpha)\,V_t\,\max(S_t - K, 0)\Big)\,/\,\Big(D_t + S_t - K\Big)$, whereas the holder of a vulnerable American put receives $((1-\alpha)\,V_t\,\max(K - S_t))\,/\,\Big(D_t + K - S_t\Big)$.

Looking at Equations (5.61) and (5.62), it becomes clearly evident that our general valuation model incorporates the key features of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011). The communalities and differences between these models are summarized as follows:

1. If the counterparty's other liabilities are assumed to be constant over time, the general model is reduced to the extended model of Klein and Inglis (2001) represented by Equations (5.52) and (5.53), since then the default condition is given by $V_t < \bar{D} + S_t - K$ and $V_t < \bar{D} + K - S_t$, respectively.

2. If the option holder's claim $S_T - K$ and $K - S_T$, respectively, is removed from the default condition and the counterparty's other liabilities still follow a geometric Brownian motion, our general model collapses to the extended model of Liu and Liu (2011) specified by Equations (5.56) and (5.57), since the default condition is equal to $V_t < D_t$ in this case.

3. If the option holder's claim $S_t - K$ and $K - S_t$, respectively, is removed from the default condition and the market value of the counterparty's other liabilities is assumed to be constant over time, our general model is reduced to the extended model of Klein (1996) specified by Equations (5.47) and (5.48), since the default condition is equal to $V_t < \bar{D}$ in this case.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions of the state variables as illustrated in Equations (5.31) and (5.32). In the general model, the value of a vulnerable American option is driven by the same three state variables as in the Liu-Liu model. Consequently, the same 18 basis functions as in the model of Liu and Liu (2011) should be applied in the estimation of the option's conditional expected continuation value. These basis functions are given as follows:

$$
\begin{aligned}
\psi_1 &= 1 - S_t \\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right) \\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right) \\
\psi_4 &= 1 - V_t \\
\psi_5 &= \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right) \\
\psi_6 &= \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right) \\
\psi_7 &= 1 - D_t \\
\psi_8 &= \frac{1}{2}\left(2 - 4\,D_t + D_t^2\right) \\
\psi_9 &= \frac{1}{6}\left(6 - 18\,D_t + 9\,D_t^2 - D_t^3\right) \\
\psi_{10} &= 1 - S_tV_t \\
\psi_{11} &= \frac{1}{2}\left(2 - 4\,S_t^2V_t + (S_t^2V_t)^2\right) \\
\psi_{12} &= \frac{1}{6}\left(6 - 18\,S_tV_t^2 + 9\,(S_tV_t^2)^2 - (S_tV_t^2)^3\right) \\
\psi_{13} &= 1 - S_tD_t \\
\psi_{14} &= \frac{1}{2}\left(2 - 4\,S_t^2D_t + (S_t^2D_t)^2\right) \\
\psi_{15} &= \frac{1}{6}\left(6 - 18\,S_tD_t^2 + 9\,(S_tD_t^2)^2 - (S_tD_t^2)^3\right) \\
\psi_{16} &= 1 - V_tD_t \\
\psi_{17} &= \frac{1}{2}\left(2 - 4\,V_t^2D_t + (V_t^2D_t)^2\right) \\
\psi_{18} &= \frac{1}{6}\left(6 - 18\,V_tD_t^2 + 9\,(V_tD_t^2)^2 - (V_tD_t^2)^3\right)
\end{aligned}
\tag{5.63}
$$

5.5 Numerical Examples

In this section, we present various numerical examples to compare the results of the different valuation models for American options subject to counterparty risk. Since the entire payoff of the option cannot be made if the option writer defaults, it should be expected that vulnerable options will have lower values than otherwise identical non-vulnerable American options. Consequently, the upper limit for the value of a vulnerable American option is given by the default-free option price obtained from the standard Longstaff-Schwartz approach. Furthermore, a vulnerable American option must have a higher value than an otherwise identical vulnerable European option due to the early exercise features.

The starting point of the following comparative analysis is a typical market situation for an American option. At today's point in time ($t = 0$), the option is at the money ($S_0 = 200$, $K = 200$) and expires in six months ($T = 0.5$). The return volatility of the option's underlying equals 25% ($\sigma_S = 0.25$) and its dividend yield is zero ($q = 0$). The risk-free interest rate is assumed to be 5% ($r = 0.05$). The option writer is assumed to be highly levered ($V_0 = 1\,000$, $D_0 = 900$). The return volatility of both the counterparty's assets and liabilities is assumed to be 25% ($\sigma_V = 0.25$, $\sigma_D = 0.25$). The correlations between the returns of the option's underlying, the counterparty's assets and liabilities are assumed to be zero ($\rho_{SV} = \rho_{VD} = \rho_{SD} = 0$). If the counterparty defaults, deadweight costs of 25% are applied ($\alpha = 0.25$).

The price of the vulnerable American option is computed based on the different valuation models presented in Section 5.4 using the least squares Monte Carlo simulation. We use 10 000 sample paths with 50 time steps ($N_{\text{Sim}} = 10\,000$, $N_{\text{T}} = 50$) and obtain the value of the American option by computing the mean over 100 re-runs of the algorithm ($n = 100$).

In a first step, we analyze whether the parameters for the least squares Monte Carlo simulation are appropriately chosen and whether the obtained results are reasonably accurate. The confidence interval, for instance, can be used to examine the accuracy of the estimated option value. Assuming that the option values obtained from the least squares Monte Carlo simulation are normally distributed, the two-sided 95% confidence interval for the option value is given by

$$CI = \frac{1}{n} \sum_{j=1}^{n} AO_j \pm 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \tag{5.64}$$

where AO_j gives the value of the American option based on run $j = 1, ..., n$ and σ is the standard deviation of the obtained option values.

Table 5.1 gives the option value as well as the corresponding 95% confidence interval for the different valuation models using the previously mentioned numerical example. The confidence intervals of all the considered valuation models are relatively tight indicating that the computed option values are quite accurate. Hence, the parameters for the least squares Monte Carlo simulation ($N_{\text{Sim}} = 10\,000$, $N_{\text{T}} = 50$, $n = 100$) seem to be reasonably chosen.

	American Call	
	Option Value	95% Confidence Interval
Longstaff & Schwartz (2001)	16.4951	[16.4867; 16.5435]
Ext. Klein (1996)	12.6375	[12.6027; 12.6723]
Ext. Klein & Inglis (2001)	12.0900	[12.0579; 12.1221]
Ext. Liu & Liu (2011)	10.8535	[10.8240; 10.8830]
General Model	10.4402	[10.4131; 10.4673]

	American Put	
	Option Value	95% Confidence Interval
Longstaff & Schwartz (2001)	12.0813	[12.0521; 12.1105]
Ext. Klein (1996)	9.7293	[9.7037; 9.7549]
Ext. Klein & Inglis (2001)	9.5210	[9.4955; 9.5465]
Ext. Liu & Liu (2011)	8.5211	[8.4972; 8.5450]
General Model	8.3441	[8.3221; 8.3661]

Table 5.1: Confidence Intervals for the Monte Carlo Simulation

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times.

Figures 5.1 and 5.2 depict the values of American calls and puts, respectively, as functions of the price of the option's underlying, the option's time to maturity and the value of the counterparty's assets for the valuation models presented in the previous section. As expected, the option values obtained from the extended Klein, the extended Klein-Inglis, the extended Liu-Liu and our general model are always lower than the default-free option value given by the model of Longstaff and Schwartz (2001).

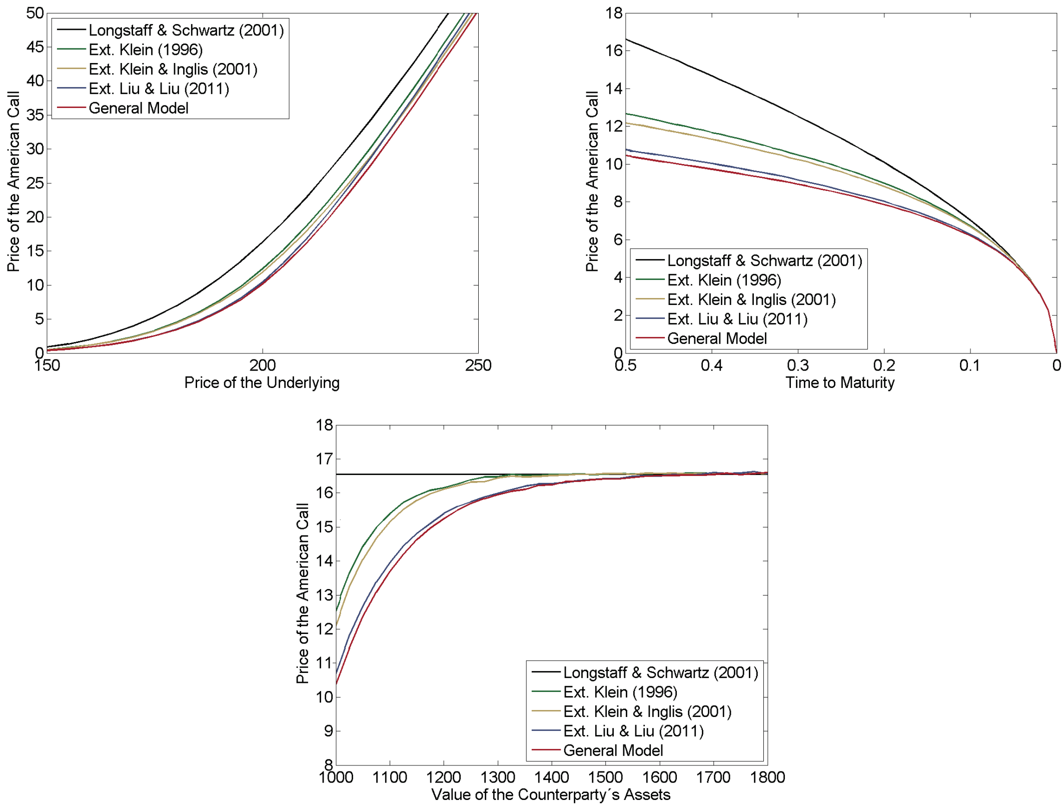

Figure 5.1: American Calls subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on $10\,000$ sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times.

In the upper left diagram of Figure 5.1, the value of the vulnerable American call is plotted against the price of the option's underlying. It can be seen that the price difference between default-free and vulnerable American calls is largest for at-the-money options. This price difference decreases if the American call is either further out of the money or further in the money. Additionally, it can be observed that option values obtained from the extended Klein-Inglis and our general model converge if the price of the option's underlying increases, i.e. if the American call is further in the money. This observation is attributed to the fact the option itself is included in the default boundary of both models. For deep in-the-money options, the counterparty's default risk is predominantly driven by the short position in the American option, since it takes an increasing share of the counterparty's total liabilities.

Referring to the upper left diagram of Figure 5.1, the effect of the time to maturity on the value of vulnerable American calls is analyzed. If the time to maturity decreases, the difference between the default-free and the vulnerable American call values is also reduced. This result is not surprising, since the counterparty is less likely to default if the option's maturity date gets closer.

The lower diagram of Figure 5.1 shows that the price of a vulnerable American call converges to the default-free option price if the value of the counterparty's assets increases, since the probability of hitting the default boundary is decreased in this case. Our general model has the lowest convergence speed which is most likely explained by the fact that this model is the only one that incorporates three sources of default risk simultaneously: a decrease in the value of the counterparty's assets, an increase in the counterparty's other liabilities as well as an increase in the option value itself.

A similar analysis can also be done for vulnerable American puts. In the upper left diagram of Figure 5.2, the value of the vulnerable American put is plotted against the price of the option's underlying. It can be seen that the price difference between default-free and vulnerable American puts is largest for at-the-money options. This price difference decreases if the American call is either further out of the money or further in the money. Moreover, it can be observed that option values obtained from the different valuation models converge if the price of the option's underlying decreases, i.e. if the American put is in the money. This observation is attributed to the fact it is optimal to immediately exercise the American put if it is sufficiently deep in the money.

Referring to the upper left diagram of Figure 5.2, the effect of the time to maturity on the value of vulnerable American puts is analyzed. If the time to maturity decreases, the difference between the default-free and the vulnerable American put values is also reduced. This result is not surprising, since the counterparty is less likely to default if the option's maturity date gets closer.

The lower diagram of Figure 5.2 shows that the price of a vulnerable American put converges to the default-free option price if the value of the counterparty's assets increases, since the probability of hitting the default boundary is decreased in this case. Our general model has the lowest convergence speed which is most likely explained by the fact that this model is the only one that incorporates three sources of default risk simultaneously: a decrease in the value of the counterparty's assets, an increase in the counterparty's other liabilities as well as an increase in the option value itself.

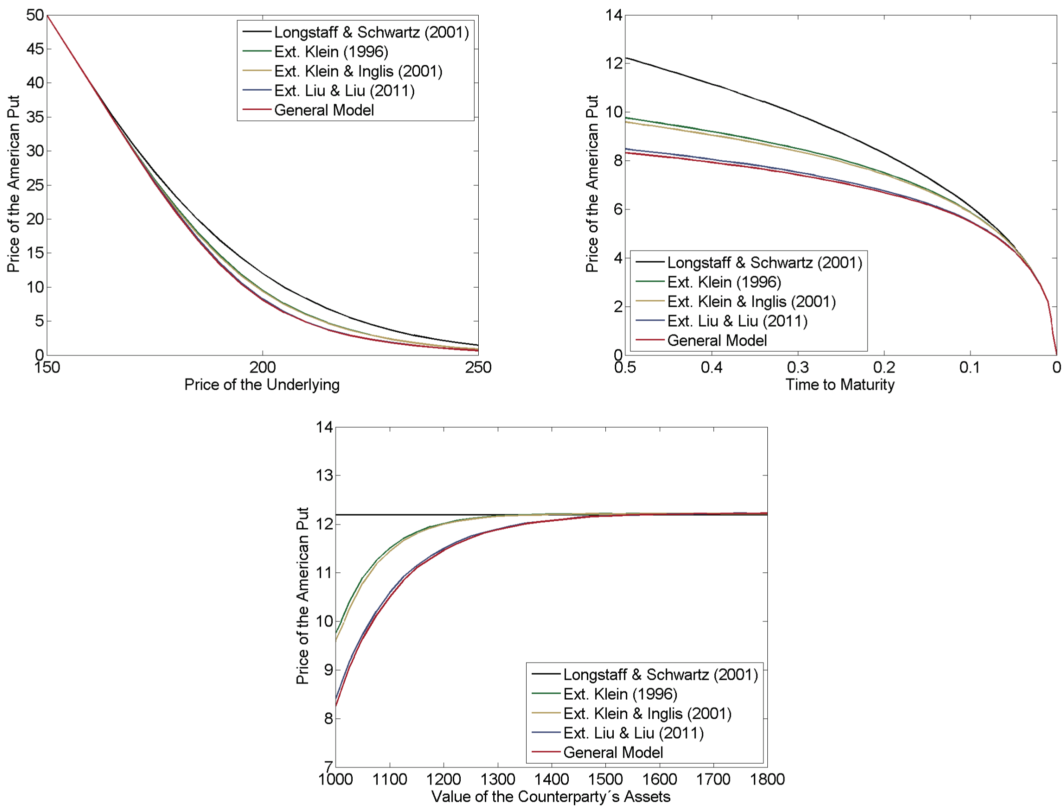

Figure 5.2: American Puts subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times.

	General Model	Ext. LL2011	Ext. KI2001	Ext. K1996	LS2001
Base Case	10.4402	10.8535	12.0900	12.6375	16.4951
$S_0 = 220$	23.4981	24.5592	25.0038	26.2932	30.3642
$S_0 = 180$	3.6585	3.7473	4.5756	4.7222	7.0391
$V_0 = 1050$	12.3895	12.7508	14.0343	14.4394	16.4951
$V_0 = 950$	7.5379	7.9800	8.8310	9.4802	16.4951
$\sigma_S = 0.3$	12.1776	12.7606	14.0768	14.8312	19.2536
$\sigma_S = 0.2$	8.6648	8.9539	10.1145	10.4801	13.7848
$\sigma_V = 0.3$	9.8798	10.2648	11.1291	11.6032	16.4951
$\sigma_V = 0.2$	10.9263	11.3786	13.2189	13.8150	16.4951
$\sigma_D = 0.3$	10.0513	10.4272	12.0900	12.6375	16.4951
$\sigma_D = 0.2$	10.8137	11.2653	12.0900	12.6375	16.4951
$\rho_{SV} = 0.5$	11.1566	11.4190	13.2129	13.4413	16.4951
$\rho_{SV} = -0.5$	9.7761	10.3426	11.2376	12.1270	16.4951
$\rho_{VD} = 0.5$	12.0035	12.5222	12.0900	12.6375	16.4951
$\rho_{VD} = -0.5$	9.5581	9.9021	12.0900	12.6375	16.4951
$\rho_{SD} = 0.5$	9.7782	10.3323	12.0900	12.6375	16.4951
$\rho_{SD} = -0.5$	11.0972	11.3691	12.0900	12.6375	16.4951
$T - t = 1$	12.9193	13.6092	15.2919	16.3010	24.7401
$T - t = 0.25$	8.3443	8.5748	9.4283	9.6747	11.1693
$\alpha = 0.5$	9.9718	10.4567	11.6996	12.3361	16.4951
$\alpha = 0$	11.2220	11.5015	12.8458	13.2100	16.4951
$r = 0.08$	11.0936	11.6053	13.2037	13.8931	18.1183
$r = 0.02$	9.8001	10.1396	11.1302	11.5642	15.0602
$q = 0.05$	9.2600	9.5494	10.6439	10.9841	13.8602

Table 5.2: American Calls subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The abbreviations Ext. K1996, Ext. KI2001 and Ext. LL2011 stand for the extended models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011), whereas LS2001 indicates the model of Longstaff and Schwartz (2001).

	General Model	Ext. LL2011	Ext. KI2001	Ext. K1996	LS2001
Base Case	8.3441	8.5211	9.5210	9.7293	12.0813
$S_0 = 220$	3.0955	3.1413	3.8209	3.8835	5.5936
$S_0 = 180$	20.9189	21.2153	21.5239	21.8079	23.2997
$V_0 = 1050$	9.6470	9.7743	10.7077	10.8332	12.0813
$V_0 = 950$	6.2159	6.4380	7.2067	7.5106	12.0813
$\sigma_S = 0.3$	10.1236	10.3936	11.5733	11.8974	14.8312
$\sigma_S = 0.2$	6.5227	6.6240	7.4299	7.5485	9.3347
$\sigma_V = 0.3$	7.9640	8.1282	8.8576	9.0508	12.0813
$\sigma_V = 0.2$	8.7177	8.8978	10.2903	10.4883	12.0813
$\sigma_D = 0.3$	8.0569	8.2222	9.5210	9.7293	12.0813
$\sigma_D = 0.2$	8.6093	8.8023	9.5210	9.7293	12.0813
$\rho_{SV} = 0.5$	8.0593	8.3453	9.2075	9.6262	12.0813
$\rho_{SV} = -0.5$	8.6409	8.7297	9.9799	10.0485	12.0813
$\rho_{VD} = 0.5$	9.4759	9.6841	9.5210	9.7293	12.0813
$\rho_{VD} = -0.5$	7.6867	7.8382	9.5210	9.7293	12.0813
$\rho_{SD} = 0.5$	8.6390	8.7327	9.5210	9.7293	12.0813
$\rho_{SD} = -0.5$	8.0720	8.3485	9.5210	9.7293	12.0813
$T - t = 1$	9.6148	9.8661	11.1577	11.4836	15.9446
$T - t = 0.25$	7.0369	7.1527	7.8350	7.9455	8.9611
$\alpha = 0.5$	8.0467	8.2771	9.2809	9.5352	12.0813
$\alpha = 0$	8.7679	8.8656	9.8824	9.9804	12.0813
$r = 0.08$	7.7776	7.9200	8.9500	9.1113	11.0517
$r = 0.02$	8.9126	9.1362	10.0826	10.3476	13.2374
$q = 0.05$	9.2480	9.4960	10.6528	10.9485	13.8716

Table 5.3: American Puts subject to Counterparty Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The abbreviations Ext. K1996, Ext. KI2001 and Ext. LL2011 stand for the extended models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011), whereas LS2001 indicates the model of Longstaff and Schwartz (2001).

Tables 5.2 and 5.3 present the option values for vulnerable American calls and puts, respectively, which are obtained from the least squares Monte Carlo simulation based on the valuation models presented in Section 5.4. Once again it can be observed that the option values based on the extended Klein, the extended Klein-Inglis, the extended Liu-Liu and our general valuation model are always lower than the default-free option value of the Longstaff-Schwartz model. Furthermore, the option values obtained from the extended general model are substantially lower than those of the other valuation models in most situations. This finding is explained by the construction of the general model's default boundary. Our general model is the only one which incorporates three sources of risk simultaneously. First, a decrease in the value of the counterparty's assets might lead to a default of the option writer like in all the other valuation models. Second, our general model accounts for the potential increase in the default risk induced by the option itself (unlike the extended Klein and the extended Liu-Liu model). Third, it is assumed that the counterparty's other liabilities are stochastic which creates an additional default risk (unlike the extended Klein and the extended Klein-Inglis model). Consequently, the option values based on our general model are the lowest, since it accounts for all possible sources of the counterparty's default risk.

Table 5.4 provides the values of default-free and vulnerable American puts for different prices of the option's underlying. Figure 5.2 already showed that the price of American puts obtained from the different valuation models converge if the price of the option's underlying decreases. This observation is attributed to the fact it is optimal to immediately exercise the American put if it is sufficiently deep in the money. Having a closer look at Table 5.4, it can easily be seen that all valuation models suggest an immediate exercise of the American put if the current price of the option's underlying is lower than 160. Furthermore, it can be observed that the critical stock price for which the American put is immediately exercised is highest for our general model ($S_0 = 170$). This aspect is explained by the fact that our model is the only one that incorporates three sources of default risk simultaneously. A similar analysis could also be performed for American calls. However, the option will only be exercised immediately if both the current price and the dividend yield of the option's underlying are sufficiently large (i.e. $S_0 \gg K$ and $q \gg 0$).

	General Model	Ext. LL2011	Ext. KI2001	Ext. K1996	LS2001
$S_0 = 157$	43*	43*	43*	43*	43*
$S_0 = 158$	42*	42*	42*	42*	42*
$S_0 = 159$	41*	41*	41*	41*	41*
$S_0 = 160$	40*	40*	40*	40*	40.0475
$S_0 = 161$	39*	39*	39*	39*	39.0759
$S_0 = 162$	38*	38*	38*	38.0176	38.1193
$S_0 = 163$	37*	37.0124	37*	37.0203	37.1874
$S_0 = 164$	36*	36.0199	36*	36.0341	36.2632
$S_0 = 165$	35*	35.0259	35*	35.0538	35.3347
$S_0 = 166$	34*	34.0441	34.0231	34.1017	34.4545
$S_0 = 167$	33*	33.0547	33.0332	33.1170	33.5448
$S_0 = 168$	32*	32.0884	32.0765	32.1853	32.7039
$S_0 = 169$	31*	31.1301	31.1182	31.2458	31.8282
$S_0 = 170$	30*	30.1560	30.1765	30.3220	30.9697
$S_0 = 171$	29.0247	29.2067	29.2487	29.4142	30.1459
$S_0 = 172$	28.0645	28.2910	28.3474	28.5101	29.3389
$S_0 = 173$	27.1231	27.3571	27.4326	27.6434	28.5377
$S_0 = 174$	26.2069	26.4562	26.5700	26.7804	27.7414
$S_0 = 175$	25.2875	25.5495	25.6913	25.9156	26.9836
$S_0 = 176$	24.3775	24.6372	24.8181	25.0606	26.2034
$S_0 = 177$	23.5018	23.7640	23.9611	24.2091	25.4654
$S_0 = 178$	22.6254	22.9058	23.1535	23.4232	24.7358
$S_0 = 179$	21.7483	22.0340	22.3165	22.5879	23.9899
$S_0 = 180$	20.9006	21.1860	21.5213	21.8107	23.2900

Table 5.4: Analysis of In-the-Money American Puts

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The immediate exercise of the American put is indicated by an asterisk. The abbreviations Ext. K1996, Ext. KI2001 and Ext. LL2011 stand for the extended models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011), whereas LS2001 indicates the model of Longstaff and Schwartz (2001).

	Call Options					Put Options				
	Base Case	$S_0 = 220$	$V_0 = 950$	$T-t = 1$	$q = 0.05$	Base Case	$S_0 = 180$	$V_0 = 950$	$T-t = 1$	$q = 0.05$
LS2001	16.4951 (16.4922)	30.3642 (30.3624)	16.4951 (16.4922)	24.7401 (24.7379)	13.8602 (13.7507)	12.0813 (11.6116)	23.2997 (22.0727)	12.0813 (11.5668)	15.9446 (14.8923)	13.8716 (13.7514)
Ext. K1996	12.6375 (11.9778)	26.2932 (24.3651)	9.4802 (8.8863)	16.3010 (15.4352)	10.9841 (10.1393)	9.7293 (8.7150)	21.8079 (18.6892)	7.5106 (6.6621)	11.4836 (9.9657)	10.9485 (10.1523)
Ext. KI2001	12.0900 (11.3091)	25.0038 (22.4065)	8.8310 (8.3206)	15.2919 (14.3635)	10.6439 (9.7092)	9.5210 (8.5151)	21.5239 (18.0219)	7.2067 (6.5073)	11.1577 (9.7997)	10.6528 (9.8200)
Ext. LL2011	10.8535 (10.1204)	24.5592 (22.1912)	7.9800 (7.4071)	13.6092 (12.7158)	9.5494 (8.6467)	8.5211 (7.4951)	21.2153 (17.6123)	6.4380 (5.6621)	9.8661 (8.4555)	9.4960 (8.6435)
GM	10.4402 (9.7071)	23.4981 (20.8052)	7.5379 (7.0769)	12.9193 (12.0816)	9.2600 (8.3830)	8.3441 (7.3725)	20.9189 (17.1487)	6.2159 (5.5806)	9.6148 (8.3639)	9.2480 (8.4479)

Table 5.5: American Options vs. European Options

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r = 0.05$, $q = 0$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 5.3 and 5.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The values in parantheses refer to the corresponding European options. The abbreviations Ext. K1996, Ext. KI2001 and Ext. LL2011 stand for the extended models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011), whereas LS2001 indicates the model of Longstaff and Schwartz (2001). The abbreviation GM stands for the general model.

In Table 5.5, we compare the values of American options with those of the corresponding European options for a selected set of parameters. For non-vulnerable American calls, we find the well-known result that the early exercise is not optimal as long as the dividend yield of the option's underlying is zero. In this case, the values of American and European calls are identical. If the dividend yield is positive, the value of an American call is greater than the value of a European call, i.e. the early exercise of the American call is optimal. The early exercise of non-vulnerable American puts is always optimal, since their values are higher than those of the corresponding European puts.

In contrast to that, we observe that the values of vulnerable American options are always greater than the values of the corresponding European options for all the considered valuation models. Hence, the early exercise may be optimal for both American calls and puts subject to counterparty risk irrespective of the used parameters.

Furthermore, the price difference between the vulnerable American options and the corresponding vulnerable European options is greater than the price difference between non-vulnerable American options and the correpsonding non-vulnerable European options. Consequently, we may conclude that the early exercise feature receives a greater recognition in case of vulnerable American options, since the option holder has the opportunity to avoid a potential write-down on his claim.

5.6 Summary

In this chapter, we picked up on the fundamental ideas of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) and set up equivalent models for vulnerable American options. Furthermore, we combine the features of these models in a general valuation model for vulnerable American options. It is the only model which incorporates three sources of financial distress simultaneously: a decline in the value of the counterparty's assets, an increase in the value of the counterparty's other liabilities or an increase in the value of the option itself.

Due to the early exercise feature of American options, the counterparty's default may occur prior to maturity. Consequently, the structural approach of Black and

Cox (1976) need to be considered. To value vulnerable American options in this framework using the different valuation models, we adjusted the least squares Monte Carlos simulation suggested by Longstaff and Schwartz (2001) to be applicable to the considered valuation problem.

Based on various numerical examples and graphical illustrations, we compared the results of our general model with those of the alternative models for vulnerable American options. All the considered valuation models have in common that the reduction in the value of a vulnerable American option (compared to a default-free American option) increases if the time to maturity is longer and if the value of the counterparty's assets is low. The deepest price reduction is oberserved for at-the-money options. The values for vulnerable American options obtained from our general model are typically the lowest, since it is the only model which accounts for all possible sources of the counterparty's default.

6 American Options subject to Counterparty and Interest Rate Risk

In this chapter, valuation models for American options subject to counterparty and interest rate risk are set up. Due to the early exercise features of American options, the counterparty's default is modeled using the structural approach of Black and Cox (1976) allowing for default prior to the option's maturity. In particular, the counterparty's default is triggered by the value of its assets being below the value of its total liabilities for the first time.

Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) develop valuation models for vulnerable European options under a deterministic interest rate.[16] In the following, we extend these models to analyze the properties of American options subject to counterparty risk. In particular, we maintain their key characteristics, especially with respect to the default condition, but adjust them to be applicable in the context of vulnerable American options. Additionally, we account for stochastic interest rates based on the model of Vasicek (1977). Finally, we develop a general model which incorporates the features of the other models.

Due to the complexity of the models, closed form solutions cannot be derived. Thus, numerical methods have to be applied to compute the value of a vulnerable American option. In particular, we use the least squares Monte Carlo simulation approach suggested by Longstaff and Schwartz (2001) and adapt it appropriately to be applicable to value vulnerable American options under stochastic interest rates.

Section 6.1 presents the considered theoretical framework. In Section 6.2, we derive the partial differential equation characterizing the price of a vulnerable American option under interest rate risk. Section 6.3 explains how this partial differential equation can be solved by the Longstaff-Schwartz approach. In Section 6.4, we extend the models of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) to be applicable for American options subject to counterparty and interest rate risk. Moreover, we set up our general model. Section 6.5 provides a comparative analysis of the different valuation models. Section 6.6 summarizes the main findings.

[16] In Chapter 3, the valuation models of Klein (1996), Klein and Inglis (2001) as well as of Liu and Liu (2011) are presented and discussed in greater details.

6.1 Assumptions

The assumptions for the valuation of American options subject to counterparty and interest rate risk are based on Merton (1973), Black and Cox (1976), Vasicek (1977), Rabinovitch (1989), Klein (1996), Klein and Inglis (1999, 2001), Chang and Hung (2006), Klein and Yang (2010, 2013) as well as on Liu and Liu (2011).

1. The price of the option's underlying S_t follows a continuous-time geometric Brownian motion. Assuming that the option's underlying is a dividend-paying stock, its dynamics are given by

$$dS_t = (\mu_S - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S\,, \tag{6.1}$$

   where μ_S indicates the expected instantaneous return of the option's underlying, q denotes the continuous dividend yield, σ_S is the instantaneous return volatility and dW_S represents the standard Wiener process.

2. Likewise, the market value of the counterparty's assets V_t follows a continuous-time geometric Brownian motion. Its dynamics are given by

$$dV_t = \mu_V\, V_t\, dt + \sigma_V\, V_t\, dW_V\,, \tag{6.2}$$

   where μ_V is the expected instantaneous return of the counterparty's assets, σ_V gives the instantaneous return volatility and dW_V is a standard Wiener process. The instantaneous correlation between dW_S and dW_V equals ρ_{SV}.

3. The total liabilities D_t comprise all the obligations of the counterparty's, i.e. debt, short positions in financial securities and accruals. The dynamics follow a continuous-time geometric Brownian motion which is given by

$$dD_t = \mu_D\, D_t\, dt + \sigma_D\, D_t\, dW_D\,, \tag{6.3}$$

   where μ_D is the expected instantaneous return of the counterparty's liabilities, σ_D indicates the instantaneous return volatility and dW_D represents the standard Wiener process. The instantaneous correlation between dW_S and dW_D equals ρ_{SD} and ρ_{VD} between dW_V and dW_D, respectively.

If the counterparty's total liabilities, however, are given by a zero bond only and the risk-free interest rate follows the Ornstein-Uhlenbeck, the expected instantaneous return μ_D as well as the instantaneous return volatility σ_D cannot be chosen arbitrarily anymore. In particular, μ_D and σ_D become time-dependent parameters which are given by the expressions specified in Equation (6.6).[17]

4. The market is perfect and frictionless, i.e. it is free of transaction costs or taxes and the available securities are traded in continuous time.

5. The instantaneous risk-free interest rate r_t follows the Ornstein-Uhlenbeck process of Vasicek (1977). The mean-reverting dynamics are given by

$$dr_t = \kappa \left(\theta - r_t\right) \, dt + \sigma_r \, dW_r \,, \tag{6.4}$$

where κ is the speed of reversion, θ represents the long-term mean of the risk-free interest rate, σ_r is the instantaneous volatility of the risk-free interest rate and dW_r represents the standard Wiener process. The instantaneous correlations between dW_r and dW_S, between dW_r and dW_V as well as between dW_r and dW_D are equal to ρ_{Sr}, ρ_{Vr} and ρ_{Dr}, respectively.

In the considered stochastic interest rate framework, a closed form solution for the price of a risk-free zero bond paying one dollar at maturity T can be derived (Vasicek, 1977; Mamon, 2004). Denoting the price at time t of a zero bond by $B_{t,T}$, the analytical bond price formula is given by

$$B_{t,T} = e^{A_{t,T} \, r_t + C_{t,T}} \tag{6.5}$$

where

$$A_{t,T} = \frac{1}{\kappa} \left(1 - e^{-\kappa \, (T-t)}\right)$$

$$C_{t,T} = \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right) \left(A_{t,T} - (T-t)\right) - \frac{\sigma_r^2 A_{t,T}^2}{4\kappa}$$

[17] This issue only affects the extended model of Liu and Liu (2011) as well as the general model, since it is assumed that the counterparty's liabilities are stochastic in these two models exclusively (see Sections 5.4.4 and 5.4.5).

The instantaneous expected return and the return volatility of the risk-free zero bond are time-dependent. In particular, they are given as follows:

$$\mu_B(t) = r_t + \frac{1 - e^{-\kappa(T-t)}}{\kappa}\sigma_r, \qquad \sigma_B(t) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}\sigma_r. \tag{6.6}$$

6. The expected instantaneous return of the option's underlying as well as of the counterparty's assets and liabilities (μ_S, μ_V and μ_D) are constant over time. The same applies for the dividend yield of the option's underlying.

7. The instantaneous return volatilities of the option's underlying as well as of the counterparty's assets and liabilities (σ_S, σ_V and σ_D) are constant over time. The same applies for the risk-free interest rate's instantanenous volatility σ_r as well as for the instantaneous correlations ρ_{SV}, ρ_{SD}, ρ_{VD}, ρ_{Sr}, ρ_{Vr} and ρ_{Dr}.

8. All the liabilities of the counterparty (i.e. debt, short positions in financial securties, etc.) are assumed to be of equal rank.

9. Before the option's maturity (i.e. $t < T$), default occurs if the counterparty's assets V_t are less than the threshold level L:

$$V_t < \bar{L} \quad \text{or} \quad V_t < L(S_t,\, D_t). \tag{6.7}$$

   Depending on the considered valuation model, the threshold level L is characterized in different ways and is either a constant or a function of the stochastic variables S_t and D_t.

10. At the option's maturity (i.e. $t = T$), default occurs if the market value of the counterparty's assets V_T are less than the threshold level L:

$$V_T < \bar{L} \quad \text{or} \quad V_T < L(S_T,\, D_T). \tag{6.8}$$

   Depending on the considered valuation model, the threshold level L is characterized in different ways and is either a constant or a function of the stochastic variables S_T and D_T.

11. If the counterparty is in default, the option holder receives the fraction $1 - \omega_t$ of the nominal claim, where ω_t represents the precentage write-down on the nominal claim at time t. The percentage write-down ω can be endogenized.

Assuming that all the liabilities of the counterparty are ranked equally, the amount payable to the holder of an American option is given by

$$(1-\omega_t) = \frac{(1-\alpha)\,V_t}{L(S_t,\, D_t)}, \tag{6.9}$$

where α represents the cost of default (e.g. bankruptcy or reorganization cost) as a percentage of the counterparty's assets.

6.2 Derivation of the Partial Differential Equation

Following the argument of Fang (2012), we derive the partial differential equation governing the price evolution of a vulnerable American option under stochastic interest rates. The the price of a vulnerable American option F_t must be a function of the underlying S_t, the counterparty's assets V_t, the counterparty's liabilities D_t, the risk-free interest rate r_t and time t. According to Itô's lemma, the corresponding stochastic differential equation for an American option is given as follows:

$$\begin{aligned} dF_t = {} & \frac{\partial F_t}{\partial t}dt + (\mu_S - q)S_t\frac{\partial F_t}{\partial S_t}dt + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2}dt + \sigma_S S_t\frac{\partial F_t}{\partial S_t}dW_S \\ & + \mu_V V_t\frac{\partial F_t}{\partial V_t}dt + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2}dt + \sigma_V V_t\frac{\partial F_t}{\partial V_t}dW_V + \mu_D D_t\frac{\partial F_t}{\partial D_t}dt \\ & + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2}dt + \sigma_D D_t\frac{\partial F_t}{\partial D_t}dW_D + \kappa(\theta - r_t)\frac{\partial F_t}{\partial r_t}dt + \frac{1}{2}\sigma_r^2\frac{\partial^2 F_t}{\partial r_t^2}dt \\ & + \sigma_r\frac{\partial F_t}{\partial r_t}dW_r + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t \partial V_t}dt + \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t \partial D_t}dt \\ & + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t \partial D_t}dt + \rho_{Sr}\sigma_S\sigma_r S_t\frac{\partial^2 F_t}{\partial S_t \partial r_t}dt \\ & + \rho_{Vr}\sigma_V\sigma_r V_t\frac{\partial^2 F_t}{\partial V_t \partial r_t}dt + \rho_{Dr}\sigma_D\sigma_r D_t\frac{\partial^2 F_t}{\partial D_t \partial r_t}dt. \end{aligned} \tag{6.10}$$

To eliminate the four Wiener processes dW_S, dW_V, dW_D and dW_B, a portfolio Π_t which consists of the American option F_t, the underlying S_t, the counterparty's assets V_t, the counterparty's liabilities D_t and the risk-free zero bond $B_{t,T}$ must be constructed.[18] In particular, this portfolio consists of a short position in the

[18] To set up such a portfolio, it is necessary to assume that option's underlying, the counterparty's assets and liabilities as well as the risk-free zero bond are traded securities. This assumption is not questionable for the option's underlying and the risk-free zero bond, but it is for both the counterparty's assets and liabilities. As argued by Klein (1996), it is likely that the counterparty's assets and liabilities are not traded directly in the market, but that their market values behave similarly as if they were traded securities.

American option and long positions in the underlying, the counterparty's assets and liabilities as well as in the risk-free zero bond. The amount of shares in the long positions are equal to $\partial F_t/\partial S_t$, $\partial F_t/\partial V_t$, $\partial F_t/\partial D_t$ and $\partial F_t/\partial r_t\,\partial r_t/\partial B_{t,T}$, respectively. Hence, the value of the portfolio at time t is given by

$$\Pi_t = -F_t + \frac{\partial F_t}{\partial S_t} S_t + \frac{\partial F_t}{\partial V_t} V_t + \frac{\partial F_t}{\partial D_t} D_t + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}} B_{t,T}. \tag{6.11}$$

The change in the value of the portfolio over the time interval dt is characterized by the total differential which is equal to

$$d\Pi_t = -dF_t + \frac{\partial F_t}{\partial S_t} dS_t + \frac{\partial F_t}{\partial V_t} dV_t + \frac{\partial F_t}{\partial D_t} dD_t + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}} dB_{t,T}. \tag{6.12}$$

Using Itô's lemma, the dynamics of the risk-free zero bond can be set up. The dynamics $dB_{t,T}$ are given by

$$dB_{t,T} = \frac{\partial B_{t,T}}{\partial t} dt + \kappa(\theta - r_t)\frac{\partial B_{t,T}}{\partial r_t} dt + \sigma_r \frac{\partial B_{t,T}}{\partial r_t} dW_r + \frac{1}{2}\sigma_r^2 \frac{\partial^2 B_{t,T}}{\partial r_t^2} dt. \tag{6.13}$$

Under the martingale measure, the dynamics of the risk-free zero bond given by Equation (6.13) can be rewritten as follows (see Fang, 2012):

$$dB_{t,T} = r_t B_{t,T} dt + \sigma_r \frac{\partial B_{t,T}}{\partial r_t} dW_r. \tag{6.14}$$

Substituting Equations (6.1) to (6.3), (6.10) and (6.14) into Equation (6.12) yields the following expression:

$$\begin{aligned} d\Pi_t = &-\frac{\partial F_t}{\partial t} dt + qS_t \frac{\partial F_t}{\partial S_t} dt - \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 F_t}{\partial S_t^2} dt - \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 F_t}{\partial V_t^2} dt \\ &- \frac{1}{2}\sigma_D^2 D_t^2 \frac{\partial^2 F_t}{\partial D_t^2} dt - \kappa(\theta - r_t)\frac{\partial F_t}{\partial r_t} dt - \frac{1}{2}\sigma_r^2 \frac{\partial^2 F_t}{\partial r_t^2} dt \\ &- \rho_{SV}\sigma_S\sigma_V S_t V_t \frac{\partial^2 F_t}{\partial S_t \partial V_t} dt - \rho_{SD}\sigma_S\sigma_D S_t D_t \frac{\partial^2 F_t}{\partial S_t \partial D_t} dt \\ &- \rho_{VD}\sigma_V\sigma_D V_t D_t \frac{\partial^2 F_t}{\partial V_t \partial D_t} dt - \rho_{Sr}\sigma_S\sigma_r S_t \frac{\partial^2 F_t}{\partial S_t \partial r_t} dt \\ &- \rho_{Vr}\sigma_V\sigma_r V_t \frac{\partial^2 F_t}{\partial V_t \partial r_t} dt - \rho_{Dr}\sigma_D\sigma_r D_t \frac{\partial^2 F_t}{\partial D_t \partial r_t} dt + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}} r_t B_{t,T} dt. \end{aligned} \tag{6.15}$$

Since the dynamics of portfolio Π_t are independent of the four Wiener processes dW_S, dW_V,dW_D and dW_B, the portfolio must be riskless during the infinitesimal time interval dt. Consequently, the portfolio must earn the same return as other short-term risk-free investments, namely the risk-free interest rate r_t, to avoid arbitrage opportunities:

$$r_t \Pi dt = d\Pi_t. \tag{6.16}$$

We substitute Equations (6.11) and (6.15) into Equation (6.16) which yields the following expression:

$$\begin{aligned}
&r_t\left(-F_t + \frac{\partial F_t}{\partial S_t}S_t + \frac{\partial F_t}{\partial V_t}V_t + \frac{\partial F_t}{\partial D_t}D_t + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}}B_{t,T}\right)dt \qquad (6.17)\\
&= -\frac{\partial F_t}{\partial t}dt + qS_t\frac{\partial F_t}{\partial S_t}dt - \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2}dt - \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2}dt - \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2}dt\\
&\quad - \kappa(\theta - r_t)\frac{\partial F_t}{\partial r_t}dt - \frac{1}{2}\sigma_r^2\frac{\partial^2 F_t}{\partial r_t^2}dt - \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t\partial V_t}dt\\
&\quad - \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t\partial D_t}dt - \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t\partial D_t}dt - \rho_{Sr}\sigma_S\sigma_r S_t\frac{\partial^2 F_t}{\partial S_t\partial r_t}dt\\
&\quad - \rho_{Vr}\sigma_V\sigma_r V_t\frac{\partial^2 F_t}{\partial V_t\partial r_t}dt - \rho_{Dr}\sigma_D\sigma_r D_t\frac{\partial^2 F_t}{\partial D_t\partial r_t}dt + \frac{\partial F_t}{\partial r_t}\frac{\partial r_t}{\partial B_{t,T}}r_t B_{t,T}dt.
\end{aligned}$$

Rewriting Equation (6.17), the partial differential equation that characterizes the price of an American option whose payoff is contigent upon the price of the option's underlying as well as upon the value of both the counterparty's assets and liabilities is obtained. It is given by

$$\begin{aligned}
0 &= \frac{\partial F_t}{\partial t} + (r_t - q)S_t\frac{\partial F_t}{\partial S_t} + \frac{1}{2}\sigma_S^2 S_t^2\frac{\partial^2 F_t}{\partial S_t^2} + r_t V_t\frac{\partial F_t}{\partial V_t} + \frac{1}{2}\sigma_V^2 V_t^2\frac{\partial^2 F_t}{\partial V_t^2} \qquad (6.18)\\
&\quad + r_t D_t\frac{\partial F_t}{\partial D_t} + \frac{1}{2}\sigma_D^2 D_t^2\frac{\partial^2 F_t}{\partial D_t^2} + \kappa(\theta - r_t)\frac{\partial F_t}{\partial r_t} + \frac{1}{2}\sigma_r^2\frac{\partial^2 F_t}{\partial r_t^2}\\
&\quad + \rho_{SV}\sigma_S\sigma_V S_t V_t\frac{\partial^2 F_t}{\partial S_t\partial V_t} + \rho_{SD}\sigma_S\sigma_D S_t D_t\frac{\partial^2 F_t}{\partial S_t\partial D_t}\\
&\quad + \rho_{VD}\sigma_V\sigma_D V_t D_t\frac{\partial^2 F_t}{\partial V_t\partial D_t} + \rho_{Sr}\sigma_S\sigma_r S_t\frac{\partial^2 F_t}{\partial S_t\partial r_t}\\
&\quad + \rho_{Vr}\sigma_V\sigma_r V_t\frac{\partial^2 F_t}{\partial V_t\partial r_t} + \rho_{Dr}\sigma_D\sigma_r D_t\frac{\partial^2 F_t}{\partial D_t\partial r_t} - r_t F_t.
\end{aligned}$$

To obtain a unique solution to the partial differential equation, we must set up the boundary conditions which specify the value of the American option based on Assumptions 9 to 11 (see Section 6.1). For the American call, the boundary conditions can be expressed as follows:

1. At the option's maturity (i.e. $t = T$), three different scenarios may occur. If the option expires in the money and the counterparty does not default, $S_T - K$ are paid out to the holder of an American call. If the option expires in the money and the counterparty is in default, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American call receives $((1-\alpha)\, V_T\, (S_T - K))\, / L(S_T,\, D_T)$. If the option is out of the money at maturity, the option holder receives nothing.

$$F_T = C_T = \begin{cases} S_T - K & \text{if } S_T \geq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(S_T - K) & \text{if } S_T \geq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{6.19}$$

2. If the counterparty defaults prior to maturity (i.e. $t < T$), the American option is immediately exercised. If the option is in the money at that point in time, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American call receives $((1-\alpha)\, V_t\, (S_t - K))\, / L(S_t,\, D_t)$. If the option is out of the money at that point in time, the option holder receives nothing.

$$F_t = C_t = \begin{cases} \dfrac{(1-\alpha)V_t}{L(S_t,\, D_t)}\,(S_t - K) & \text{if } S_t \geq K,\, V_t < L(S_t,\, D_t) \\ 0 & \text{otherwise} \end{cases} \tag{6.20}$$

3. It may be optimal to exercise an American call prior to maturity (i.e. $t < T$) even though the counterparty is not in default. Early exercise is optimal if

the early exercise payoff $C_t^{\text{Ex}} = \max(S_t - K, 0)$ is larger than the conditional expected continuation value C_t^{Cont}, i.e. the expected future option payoff.

$$F_t = C_t = \begin{cases} S_t - K & \text{if } C_t^{\text{Ex}} > C_t^{\text{Cont}},\, V_t \geq L(S_t,\, D_t) \\ \text{No early exercise} & \text{otherwise} \end{cases} \tag{6.21}$$

The boundary conditions for the American put are given in analogy:

1. At the option's maturity (i.e. $t = T$), three different scenarios may occur. If the option expires in the money and the counterparty does not default, $K - S_T$ are paid out to the holder of an American put. If the option expires in the money and the counterparty is in default, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American put receives $\left((1-\alpha)\, V_T\, (K - S_T)\right) / L(S_T,\, D_T)$. If the option is out of the money at maturity, the option holder receives nothing.

$$F_T = P_T = \begin{cases} K - S_T & \text{if } S_T \leq K,\, V_T \geq L(S_T,\, D_T) \\ \dfrac{(1-\alpha)V_T}{L(S_T,\, D_T)}\,(K - S_T) & \text{if } S_T \leq K,\, V_T < L(S_T,\, D_T) \\ 0 & \text{otherwise} \end{cases} \tag{6.22}$$

2. If the counterparty defaults prior to maturity (i.e. $t < T$), the American put is immediately exercised. If the option is in the money at that point in time, the entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Hence, the holder of an American put receives $\left((1-\alpha)\, V_t\, (K - S_t)\right) / L(S_t,\, D_t)$. If the option is out of the money at that point in time, the option holder receives nothing.

$$F_t = P_t = \begin{cases} \dfrac{(1-\alpha)V_t}{L(S_t,\, D_t)}\,(K - S_t) & \text{if } S_t \leq K,\, V_t < L(S_t,\, D_t) \\ 0 & \text{otherwise} \end{cases} \tag{6.23}$$

3. It may be optimal to exercise an American put prior to maturity (i.e. $t < T$) even though the counterparty is not in default. Early exercise is optimal if the early exercise payoff $P_t^{\text{Ex}} = \max(K - S_t, 0)$ is larger than the conditional expected continuation value P_t^{Cont} which is given by the expected future option payoff.

$$F_t = P_t = \begin{cases} K - S_t & \text{if } P_t^{\text{Ex}} > P_t^{\text{Cont}},\, V_t \geq L(S_t,\, D_t) \\ \text{No early exercise} & \text{otherwise} \end{cases} \tag{6.24}$$

The actual characterization of the boundary conditions depends on the choice of a specific valuation model (see Section 6.4). In particular, the variable $L(S_T,\, D_T)$ must be defined according to the chosen model in order to value vulnerable American options using the least squares Monte Carlo simulation.

Referring to Equations (6.20) and (6.23), we assume that an American option is immediately exercised if the counterparty defaults at a given time t prior to the option's maturity. Chang and Hung (2006) as well as Klein and Yang (2010) also deal with the valuation of vulnerable American options. However, their assumptions with respect to the option payoff if the counterparty defaults prior to maturity differ from our assumption. In particular, Chang and Hung (2006) assume that the American option is not necessarily exercised in the case of the counterparty's default, i.e. they assume that the option holder has the opportunity to keep the American option unexercised until maturity, although the counterparty is insolvent. Klein and Yang (2010), in turn, suppose that only in-the-money American options are immediately exercised if the counterparty is in default prior to maturity. If the counterparty is in default and the American option is out of the money, the option is not exercised.

6.3 Solution to the Partial Differential Equation

The partial differential equation given by Equation (6.18) depends on the price of the option's underlying, the counterparty's assets, the counterparty's liabilities, the risk-free interest rate, the dividend yield of the option's underlying as well as on the return volatilities. All these variables and parameters are independent of the investors' risk preferences.[19] Since the risk preferences of the investors do not enter the partial differential equation, they cannot affect its solution. Consequently, any type of risk preferences can be assumed when solving the partial differential equation.

The partial differential equation given by Equation (6.18) subject to the boundary conditions specified by Equations (6.19) to (6.21) and (6.22) to (6.24), respectively, can be solved using the regression-based Monte Carlo simulation approach suggested by Longstaff and Schwartz (2001). Even though this approach has originally been derived to value plain vanilla American options, it can also be applied in more complex theoretical frameworks in which the price of the considered option depends on more than one stochastic variable (see Longstaff & Schwartz, 2001; Moreno & Navas, 2003).

It is optimal to exercise an American option prior to its maturity if the option payoff based on the immediate exercise is greater than the option's conditional expected continuation value. Longstaff and Schwartz (2001) suggest to estimate the conditional expectation by a least-squares regression based on the cross-sectional information provided by Monte Carlo simulation. Consequently, sample paths need to be generated for the price of the option's underlying as well as for the market value of the counterparty's assets and liabilities.

Using the approach of Cox and Ross (1976) and Harrison and Pliska (1981), the risk-neutral stochastic processes for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities can be obtained. They are equal to

$$dS_t = (r_t - q)\, S_t\, dt + \sigma_S\, S_t\, dW_S, \tag{6.25}$$

$$dV_t = r_t\, V_t\, dt + \sigma_V\, V_t\, dW_V \tag{6.26}$$

[19] Following the argument of Hull (2012: 311–312), the partial differential equation given by Equation (6.18) would not be independent of risk preferences if it included the expected returns of the option's underlying, the counterparty's assets and the counterparty's liabilities. These parameters depend on risk preferences, since their magnitude represents the level of risk aversion of the investor: the higher the level of the investor's risk aversion, the higher the required expected return.

and

$$dD_t = r_t\, D_t\, dt + \sigma_D\, D_t\, dW_D, \tag{6.27}$$

where r_t is the risk-free interest rate at time t and all other variables are defined as before.

Applying Itô's lemma to Equations (6.25) to (6.27), the stochastic processes for $\ln S_t$, $\ln V_t$ and $\ln D_t$ are obtained. They are given by

$$d\ln S_t = \left(r_t - q - \frac{1}{2}\sigma_S^2\right) dt + \sigma_S dW_S, \tag{6.28}$$

$$d\ln V_t = \left(r_t - \frac{1}{2}\sigma_V^2\right) dt + \sigma_V dW_V \tag{6.29}$$

and

$$d\ln D_t = \left(r_t - \frac{1}{2}\sigma_D^2\right) dt + \sigma_D dW_D. \tag{6.30}$$

Rewriting Equations (6.28) to (6.30), expressions for the price of the option's underlying as well as for the market values of the counterparty's assets and liabilities at every point in time can be derived. Using Δt as the time step, the evolution of the stochastic variables over time is given by

$$S_{t+\Delta t} = S_t\, e^{(r_t - q - \frac{1}{2}\sigma_S^2)\Delta t + \sigma_S \sqrt{\Delta t}\, x_S}, \tag{6.31}$$

$$V_{t+\Delta t} = V_t\, e^{(r_t - \frac{1}{2}\sigma_V^2)\Delta t + \sigma_V \sqrt{\Delta t}\, x_V} \tag{6.32}$$

and

$$D_{t+\Delta t} = D_t\, e^{(r_t - \frac{1}{2}\sigma_D^2)\Delta t + \sigma_D \sqrt{\Delta t}\, x_D}, \tag{6.33}$$

where the three random variables x_S, x_V and x_D are jointly standard normally distributed and their respective correlations are given by the coefficients ρ_{SV}, ρ_{SD} and ρ_{VD}.

To set up sample paths for the price of the option's underlying as well as for the market value of both the counterparty's assets and liabilities, the evolution of the risk-free interest rate r_t is needed as well. Since the risk-free interest rate follows

Vasicek's Ornstein-Uhlenbeck process (see Equation (6.4)), its dynamics in discrete time are given by

$$\Delta r_t = \kappa\,(\theta - r_t)\;\Delta t + \sigma_r\,\sqrt{\Delta t}\,x_r\,, \tag{6.34}$$

where the random x_r is standard normally distributed and its correlation to x_S, x_V and x_D are equal to ρ_{Sr}, ρ_{Vr} and ρ_{Dr}, respectively.

Integrating Equation (6.34), the evolution of the risk-free interest rate over time can be computed using the following expression:

$$r_{t+\Delta t} = r_t\,e^{-\kappa\,\Delta t} + \theta\,\left(1 - e^{-\kappa\,\Delta t}\right) + \sqrt{\frac{\sigma_r^2\,\left(1 - e^{-2\,\kappa\,\Delta t}\right)}{2\,\kappa}}\,x_r\,. \tag{6.35}$$

Equations (6.31) to (6.35) can be used in the Monte Carlo simulation to generate sample paths for the price of the option's underlying, the market values of the counterparty's assets and liabilities as well as for the risk-free interest rate$(S_0,\,S_{\Delta t},\dots,\,S_t,\dots,\,S_T)$, $(V_0,\,V_{\Delta t},\dots,\,V_t,\dots,\,V_T)$, $(D_0,\,D_{\Delta t},\dots,\,D_t,\dots,\,D_T)$ and $(r_0,\,r_{\Delta t},\dots,\,r_t,\dots,\,r_T)$, where t denotes the time index and Δt is the discrete time step. At any time step t, the dynamic programming recursion functions for American calls and puts, respectively, are given by

$$C_t = \begin{cases} \max\left(S_t - K,\,\mathbb{E}_t\left[e^{-r_t\Delta t}\,C_{t+\Delta t}\,|\,S_t, V_t, D_t\right]\right) & \text{if } V_t \geq L(S_t,\,D_t) \\ \dfrac{(1-\alpha)\,V_t}{L(S_t,\,D_t)}\,\max\left(S_t - K, 0\right) & \text{if } V_t < L(S_t,\,D_t) \end{cases} \tag{6.36}$$

and

$$P_t = \begin{cases} \max\left(K - S_t,\,\mathbb{E}_t\left[e^{-r_t\Delta t}\,B_{t+\Delta t}\,|\,S_t, V_t, D_t\right]\right) & \text{if } V_t \geq L(S_t,\,D_t) \\ \dfrac{(1-\alpha)\,V_t}{L(S_t,\,D_t)}\,\max\left(K - S_t, 0\right) & \text{if } V_t < L(S_t,\,D_t) \end{cases} \tag{6.37}$$

If the counterparty defaults, the option is immediately exercised irrespective of whether the option is in the money or not. If the counterparty is not in default, however, the option holder must decide whether he wants to exercise the option prior to maturity. In particular, the option is exercised immediately if the option payoff is greater than the conditional expectation of continuation under the risk-neutral measure. Being at a given time step of the sample path, this decision, however, cannot be taken along an individual sample path, since the option holder cannot exploit knowledge of the future prices along that path. To avoid anticipativity, the total set of sample paths is used to approximate the conditional expected continuation value by regressing the conditional expectation against M basis functions $\psi_m(\,\cdot\,)$. At each time step, the same set of basis functions is used, but the coefficients $\beta_{m,t}$ are time-dependent. Consequently, the relationship between the expected option value one time step ahead and the basis functions are given by the following expressions:

$$\begin{aligned}\mathbb{E}_t\left[e^{-r_t\Delta t}\, C_{t+\Delta t}\,|\, S_t, V_t, D_t\right] \approx \beta_{0,\,t} + \beta_{1,\,t}\,\psi_1(S_t, V_t, D_t) \\ + \cdots + \beta_{M,\,t}\,\psi_M(S_t, V_t, D_t),\end{aligned} \tag{6.38}$$

$$\begin{aligned}\mathbb{E}_t\left[e^{-r_t\Delta t}\, B_{t+\Delta t}\,|\, S_t, V_t, D_t\right] \approx \beta_{0,\,t} + \beta_{1,\,t}\,\psi_1(S_t, V_t, D_t) \\ + \cdots + \beta_{M,\,t}\,\psi_M(S_t, V_t, D_t).\end{aligned} \tag{6.39}$$

The stochastic risk-free interest rate directly enters the price of the option's underlying as well as the market value of the counterparty's assets and liabilities. Therefore, the basis need not expliclity include the risk-free interest rate at the given time step t.

Since the coefficients $\beta_{m,t}$ are not related to a particular sample path, the decisions based on the approximated conditional expected continuation value of the considered American option are non-anticipative. The coefficients $\beta_{m,t}$ can be estimated by a simple least squares regression minimizing the sum of the squared residuals. The sample paths for the option's underlying as well as for the counterparty's assets and liabilities are generated using Monte Carlo simulation, where S_t^i, V_t^i and D_t^i give the value of the respective stochastic variable at time t along a sample path $i = 1, ..., N$.

Based on these considerations, the least squares regression model for the conditional expected continuation value at time t is equal to

$$\begin{aligned}e^{-r_t^i\Delta t}\, C_{t+\Delta t}^i = \beta_{0,\,t} + \beta_{1,\,t}\,\psi_1(S_t^i, V_t^i, D_t^i) \\ + \cdots + \beta_{M,\,t}\,\psi_M(S_t^i, V_t^i, D_t^i) + \varepsilon_i\end{aligned} \tag{6.40}$$

and

$$
\begin{aligned}
e^{-r_t^i \Delta t} B_{t+\Delta t}^i = \beta_{0,t} + \beta_{1,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\
+ \cdots + \beta_{M,t}\, \psi_M(S_t^i, V_t^i, D_t^i) + \varepsilon_i,
\end{aligned}
\tag{6.41}
$$

where ε_i is the residual for each sample path. The obtained estimators $\hat{\beta}_{m,t}$ can be used to approximate the conditional expected continuation value of the American option for each sample path i. For vulnerable American calls and puts, respectively, the approximation is given by

$$
\begin{aligned}
e^{-r_t^i \Delta t} C_{t+\Delta t}^i = \hat{\beta}_{0,t} + \hat{\beta}_{1,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\
+ \cdots + \hat{\beta}_{M,t}\, \psi_M(S_t^i, V_t^i, D_t^i)
\end{aligned}
\tag{6.42}
$$

and

$$
\begin{aligned}
e^{-r_t^i \Delta t} B_{t+\Delta t}^i = \hat{\beta}_{0,t} + \hat{\beta}_{1,t}\, \psi_1(S_t^i, V_t^i, D_t^i) \\
+ \cdots + \hat{\beta}_{M,t}\, \psi_M(S_t^i, V_t^i, D_t^i).
\end{aligned}
\tag{6.43}
$$

Since the regression-based approach of Longstaff and Schwartz (2001) is a dynamic programing method, the valuation problem must be solved recursively, i.e. the procedure starts at the option's maturity and goes backwards in time. Using the generated sample paths for the option's underlying as well as for the counterparty's assets and liabilities, the dynamic programming recursion functions at the option's maturity can be determined for each sample path i. At the option's expiration, these functions are simply given by the payoff of the vulnerable American option. For vulnerable American calls and puts, respectively, they are given by

$$
C_T^i = \begin{cases} \max\left(S_T^i - K, 0\right) & \text{if } V_T^i \geq L(S_T^i,\, D_T^i) \\ \dfrac{(1-\alpha)\, V_T^i}{L(S_T^i,\, D_T^i)} \max\left(S_T^i - K, 0\right) & \text{if } V_T^i < L(S_T^i,\, D_T^i) \end{cases}
\tag{6.44}
$$

and

$$
B_T^i = \begin{cases} \max\left(K - S_T^i, 0\right) & \text{if } V_T^i \geq L(S_T^i,\, D_T^i) \\ \dfrac{(1-\alpha)\, V_T^i}{L(S_T^i,\, D_T^i)} \max\left(K - S_T^i, 0\right) & \text{if } V_T^i < L(S_T^i,\, D_T^i) \end{cases}
\tag{6.45}
$$

Longstaff and Schwartz (2001) argue that it is more efficient to consider only the subset of sample paths for which a decision must be taken at a given time t when regressing the conditional expectation against the basis functions. Consequently, this subset must contain all the sample paths in which the option is in the money at the given time step t. This subset is denoted by $\mathcal{I}_t$. For time step $T - \Delta t$, the regression model is thus given by

$$\begin{aligned} e^{-r^i_{T-\Delta t}\Delta t}\, C^i_T = {} & \beta_{0,T-\Delta t} + \beta_{1,T-\Delta t}\,\psi_1(S^i_{T-\Delta t}, V^i_{T-\Delta t}, D^i_{T-\Delta t}) \\ & + \cdots + \beta_{M,T-\Delta t}\,\psi_M(S^i_{T-\Delta t}, V^i_{T-\Delta t}, D^i_{T-\Delta t}) + \varepsilon_i \qquad i \,\epsilon\, \mathcal{I}_{T-\Delta t} \end{aligned} \tag{6.46}$$

and

$$\begin{aligned} e^{-r^i_{T-\Delta t}\Delta t}\, B^i_T = {} & \beta_{0,T-\Delta t} + \beta_{1,T-\Delta t}\,\psi_1(S^i_{T-\Delta t}, V^i_{T-\Delta t}, D^i_{T-\Delta t}) \\ & + \cdots + \beta_{M,T-\Delta t}\,\psi_M(S^i_{T-\Delta t}, V^i_{T-\Delta t}, D^i_{T-\Delta t}) + \varepsilon_i \qquad i \,\epsilon\, \mathcal{I}_{T-\Delta t} \end{aligned} \tag{6.47}$$

for American calls and puts, respectively. The estimated parameters $\hat{\beta}_{m,T-\Delta t}$ obtained from the least squares regression are used to compute the approximate continuation value of the option. Comparing this value with the payoff of immediate exercise, it can be decided whether the option should be exercised early.

The above procedure is repeated going backwards in time. On each path i, the cash flows resulting from early exercise decisions must be considered. At the time step t on sample path i, there may be a time step $t^* \geq t$ at which the American option has been exercised early. Taking this issue into account, the regression model can be rewritten as

$$\begin{aligned} B^i_{t,t^*}\, C^i_{t^*} = {} & \beta_{0,t} + \beta_{1,t}\,\psi_1(S^i_t, V^i_t, D^i_t) \\ & + \cdots + \beta_{M,t}\,\psi_M(S^i_t, V^i_t, D^i_t) + \varepsilon_i \qquad i \,\epsilon\, \mathcal{I}_t \end{aligned} \tag{6.48}$$

and

$$\begin{aligned} B^i_{t,t^*}\, B^i_{t^*} = {} & \beta_{0,t} + \beta_{1,t}\,\psi_1(S^i_t, V^i_t, D^i_t) \\ & + \cdots + \beta_{M,t}\,\psi_M(S^i_t, V^i_t, D^i_t) + \varepsilon_i \qquad i \,\epsilon\, \mathcal{I}_t. \end{aligned} \tag{6.49}$$

for American calls and puts, respectively. The discount factor B^i_{t,t^*} is different for each sample path i and is given by the value of a risk-free zero bond at time t paying one dollar at maturity t^* for the given interest rate r_t^i on sample path i. Referring to Equation (6.5)), the value of this zero bond is given by

$$B_{t,\,t^*} = e^{A_{t,\,t^*}\, r_t + C_{t,\,t^*}} \tag{6.50}$$

where

$$\begin{aligned} A_{t,\,t^*} &= \frac{1}{\kappa}\left(1 - e^{-\kappa\,(t^*-t)}\right) \\ C_{t,\,t^*} &= \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right)\left(A_{t,\,t^*} - (t^* - t)\right) - \frac{\sigma_r^2 A_{t,\,t^*}^2}{4\kappa} \end{aligned}$$

Since there is at most one exercise time t^* for each sample path i, it may be the case that after comparing the payoff of immediate exercise with the approximate continuation value on a particular path, the exercise time t^* needs to be reset to a another period.

To apply the above approach to a valuation model for vulnerable American options, the threshold level $L(S_t, D_t)$ must be specified in accordance. Furthermore, the basis functions used in the linear regression must be chosen appropriately.

6.4 Valuation Models

Various valuation models for vulnerable European options have been developed over the last three decades based on the structural approach of Merton (1974). The predominant valuation models are those of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011). However, these models do not account for stochastic interest rates. In the following, we use the main ideas of these models to set up equivalent models for vulnerable American options. Additionally, we assume that the risk-free interest rate follows the mean-reverting Ornstein-Uhlenbeck process of Vasicek (1977). Finally, we set up a general valuation model incorporating the features of the other models.

When dealing with vulnerable American options, it is important to consider that the counterparty's default may occur prior to the option's maturity. Hence, the structural approach of Black and Cox (1976) need to be considered. To value the vulnerable American options in such a framework, the least squares Monte Carlo simulation by Longstaff and Schwartz (2001) is applied.

In Section 6.3, we generally showed how the Longstaff-Schwartz approach is used to value American options subject to counterparty and interest rate risk. To apply this method to a particular valuation model, the dynamic programming recursion functions in Equations (6.36) and (6.37) as well as the basis functions $\psi_m(S_t, V_t, D_t)$ must be specified accordingly.

6.4.1 Absence of Default Risk

Longstaff and Schwartz (2001) originally derived the least squares Monte Carlo simulation to value American options in the absence of counterparty and interest rate risk. Nevertheless, the approach can also be applied in a stochatsic interest rate framework (see Section 6.3). In a first step, the dynamic programming recursion functions for default-free American calls and puts, respectively, need to be set up:

$$C_t = \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t\right]\right) \tag{6.51}$$

$$P_t = \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, B_{t+\Delta t} \,|\, S_t\right]\right). \tag{6.52}$$

An American option is exercised prior to maturity only if the payoff of an immediate exercise is larger than the option's continuation value. Otherwise, the option is kep unexercised. Consequently, the crucial point in the Longstaff-Schwartz approach is the estimation of the conditional expected continuation value. As shown in Equations (6.38) and (6.39), an approximation for the conditional expected continutaion value can be obtained by regressing the discounted expected future cash flows against a set of basis functions. Since the stochastic interest rates are implicitly included in the price of the option's underlying, they do not have to be explicitly considered in the construction of the basis functions. Hence, the same basis functions as in the deterministic interest rate framework can be used. Longstaff and Schwartz (2001) choose the first three Laguerre polynomials as basis functions and argue that more than three basis functions do not yield more accurate results:

$$\begin{aligned}
\psi_1 &= 1 - S_t, \\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right), \\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right).
\end{aligned} \tag{6.53}$$

6.4.2 Deterministic Liabilities

In his original paper, Klein (1996) deals with the valuation of vulnerable European options under deterministic interest rates and assumes that the counterparty defaults if its assets are lower than the total liabilities at the option's maturity. The counterparty's total liabilities are assumed to be constant and must include the short position in the option by construction, since it obliges the option writer to deliver or purchase the option's underlying if the option is exercised.

In the context of American options, it is necessary to account for the counterparty's default occuring prior to maturity. If we adopt the core idea of Klein (1996) to the American option framework, the default barrier $L(S_t,\, D_t)$ must be given by

$$L(S_t,\, D_t) = \bar{L} = \bar{D} = D_0. \tag{6.54}$$

Inserting this expression into Equations (6.36) and (6.37), the dynamic programming recursion functions for vulnerable American calls and puts, respectively, for the extended model of Klein (1996) are given by

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} \\ \dfrac{(1-\alpha)\, V_t}{\bar{D}} \max\left(S_t - K, 0\right) & \text{if } V_t < \bar{D} \end{cases} \tag{6.55}$$

and

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, B_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} \\ \dfrac{(1-\alpha)\, V_t}{\bar{D}} \max\left(K - S_t, 0\right) & \text{if } V_t < \bar{D} \end{cases} \tag{6.56}$$

Referring to the first line in Equations (6.55) and (6.56), the holder of an American option must decide whether the option should be exercised early at the given time step t if the counterparty is not in default. Early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. If the counterparty, however, defaults at the given time step t, the American option is immediately exercised irrespective of whether the option is in the money or not according to the second line in Equations (6.55) and (6.56). In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $\left((1-\alpha)\,V_t\right)/\bar{D}$. Consequently, the holder of a vulnerable American call receives $\left((1-\alpha)\,V_t\,\max(S_t-K,0)\right)/\bar{D}$, whereas $\left((1-\alpha)\,V_t\,\max(K-S_t,0)\right)/\bar{D}$ is paid out to the holder of a vulnerable American put.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions as illustrated in Equations (6.38) and (6.39). Since the stochastic interest rates are implicitly included in the price of the option's underlying and the value of the counterparty's assets, they do not have to be explicitly considered in the basis functions. Consequently, the basis functions for the extended model of Klein (1996) must contain the price of the option's underlying, the market value of the counterparty's assets as well as their cross product. In this case, a total of nine basis functions is obtained.[20] Like in the case of default-free American options, Laguerre polynomials are used as basis functions. In particular, the following nine basis functions are applied:

$$\begin{aligned}
\psi_1 &= 1 - S_t, \\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right), \\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right), \\
\psi_4 &= 1 - V_t, \\
\psi_5 &= \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right), \\
\psi_6 &= \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right), \\
\psi_7 &= 1 - S_t V_t, \\
\psi_8 &= \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right), \\
\psi_9 &= \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right).
\end{aligned} \tag{6.57}$$

[20] According to Moreno and Navas (2003) as well as Chang and Hung (2006), it is sufficient to use a total of nine basis functions if the option price is driven by two stochastic variables.

6.4.3 Deterministic Liabilities and Option induced Default Risk

Like Klein (1996), Klein and Inglis (2001) originally set up a valuation model for vulnerable European options in which the counterparty can only default at the option's maturity and the risk-free interest rate is deterministic. They recognize that the short position in the option itself may cause additional financial distress. To account for this potential source of default risk, they split the counterparty's total liabilities into two components. In particular, the total liabilities consist of the short position in the option on the one hand and all the other liabilities on the other which are assumed to be constant over time. When dealing with the valuation of American options, it is reasonable to consider that the counterparty may default prior to maturity. If we account for this issue and maintain the key features the key features of Klein and Inglis (2001), the time-dependent default barrier $L(S_t,\, D_t)$ for American calls and puts, respectively, is given as follows:

$$L(S_t,\, D_t) = L(S_t) = \bar{D} + S_t - K = D_0 + S_t - K, \tag{6.58}$$

$$L(S_t,\, D_t) = L(S_t) = \bar{D} + K - S_t = D_0 + K - S_t. \tag{6.59}$$

Inserting the above expressions into Equations (6.36) and (6.37), the dynamic programming recursion functions for the Longstaff-Schwartz approach based on the extended model of Klein and Inglis (2001) are obtained. For vulnerable American calls and puts, respectively, they are equal to the following expressions:

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} + S_t - K \\ \dfrac{(1-\alpha)\, V_t}{\bar{D} + S_t - K}\, \max\left(S_t - K, 0\right) & \text{if } V_t < \bar{D} + S_t - K \end{cases} \tag{6.60}$$

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, B_{t+\Delta t} \,|\, S_t, V_t\right]\right) & \text{if } V_t \geq \bar{D} + K - S_t \\ \dfrac{(1-\alpha)\, V_t}{\bar{D} + K - S_t}\, \max\left(K - S_t, 0\right) & \text{if } V_t < \bar{D} + K - S_t \end{cases} \tag{6.61}$$

The holder of the American option must decide whether the option should be immediately exercised if the counterparty is not in default at the given time step t according to the first line in Equations (6.60) and (6.61). Early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. The second line in Equations (6.60) and (6.61) refers to the scenario in which the counterparty is in default at the given time step t. In this case, the American option is immediately exercised irrespective of whether the option is in the money or not. The entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Due to the construction of the default boundary, this proportion depends on the type of the considered option. It is given by $\big((1-\alpha)\,V_t\big)\,/\,\big(\bar{D}+S_t-K\big)$ for a vulnerable American call, whereas it is equal to $\big((1-\alpha)\,V_t\big)\,/\,\big(\bar{D}+K-S_t\big)$ for a vulnerable American put. Consequently, the holder of a vulnerable American call receives $\big((1-\alpha)\,V_t\,\max(S_t-K,0)\big)\,/\,\big(\bar{D}+S_t-K\big)$, whereas the holder of a vulnerable American put receives $\big((1-\alpha)\,V_t\,\max(K-S_t)\big)\,/\,\big(\bar{D}+K-S_t\big)$.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions of the state variables as illustrated in Equations (6.38) and (6.39). The stochastic interest rates are implicitly included in the price of the option's underlying as well as in the market values of the counterparty's assets and liabilities and therefore need not be explicitly considered in the construction of the basis functions. Since the option price is governed by the same two stochastic variables as in the extended model of Klein (1996), the same Laguerre polynomials as before can be used as basis functions:

$$\begin{aligned}
\psi_1 &= 1 - S_t, \qquad (6.62)\\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right),\\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right),\\
\psi_4 &= 1 - V_t,\\
\psi_5 &= \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right),\\
\psi_6 &= \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right),\\
\psi_7 &= 1 - S_t V_t,
\end{aligned}$$

$$\psi_8 = \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right),$$
$$\psi_9 = \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right).$$

6.4.4 Stochastic Liabilities

Liu and Liu (2011) also suggest a valuation model for vulnerable European options. Like in the models of Klein (1996) and Klein and Inglis (2001), the counterparty's default can only occur at the option's maturity and is triggered by the counterparty's assets being lower than the total liabilities. In contrast to the previous models, Liu and Liu (2011) assume that the market value of the counterparty's total liabilities is stochastic and follows a geometric Brownian motion as given by Equation (6.3). It is important to note that the short position in the option is implicitly included in the counterparty's total liabilities, but its impact on the value of the couterparty's total liabilities is not explicitly modeled (unlike in the Klein-Inglis model).

When pricing American options, it is important to consider that the counterparty may also default prior to maturity. If we consider this issue and follow the key aspects of Liu and Liu (2011), especially with respect to the default condition, the time-dependent default barrier $L(S_t,\, D_t)$ must be given by

$$L(S_t,\, D_t) = L(D_t) = D_t. \tag{6.63}$$

Inserting this expression into Equations (6.36) and (6.37), the dynamic programming recursion functions for vulnerable American calls and puts, respectively, based on the extended model of Liu and Liu (2011) are given as follows:

$$C_t = \begin{cases} \max\left(S_t - K,\, \mathbb{E}_t\left[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t, D_t\right]\right) & \text{if } V_t \geq D_t \\ \dfrac{(1-\alpha)\,V_t}{D_t}\, \max\left(S_t - K, 0\right) & \text{if } V_t < D_t \end{cases} \tag{6.64}$$

$$P_t = \begin{cases} \max\left(K - S_t,\, \mathbb{E}_t\left[e^{-r\Delta t}\, B_{t+\Delta t} \,|\, S_t, V_t, D_t\right]\right) & \text{if } V_t \geq D_t \\ \dfrac{(1-\alpha)\,V_t}{D_t}\, \max\left(K - S_t, 0\right) & \text{if } V_t < D_t \end{cases} \tag{6.65}$$

Referring to the first line in Equations (6.64) and (6.65), the holder of an American option must decide whether the option should be exercised early at the given time step t if the counterparty is not in default. Early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. If the counterparty, however, defaults at the given time step t, the American option is immediately exercised irrespective of whether the option is in the money or not according to the second line in Equations (6.64) and (6.65). In this case, the entire assets of the counterparty (less the default costs α) are distributed to all the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. This proportion is given by the ratio $\big((1-\alpha)\,V_t\big)\,/D_t$. Consequently, the holder of a vulnerable American call receives $\big((1-\alpha)\,V_t\,\max(S_t-K,0)\big)\,/D_t$, whereas $\big((1-\alpha)\,V_t\,\max(K-S_t,0)\big)\,/D_t$ is paid out to the holder of a vulnerable American put.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions of the state variables as illustrated in Equations (6.38) and (6.39). Since the stochastic interest rates are implicitly included in the price of the option's underlying as well as in the value of the counterparty's assets and liabilities, they do not have to be explicitly considered in the construction of the basis functions. Based on the extended model of Liu and Liu (2011), the value of a vulnerable American option is driven by the price of the option's underlying as well as by the value of the counterparty's assets and liabilities. Consequently, the basis functions must contain these three stochastic variables as well as their cross products which results in a total of 18 basis functions.[21] In particular, they are given as follows:

$$\begin{aligned}
\psi_1 &= 1 - S_t \\
\psi_2 &= \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right) \\
\psi_3 &= \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right) \\
\psi_4 &= 1 - V_t
\end{aligned} \tag{6.66}$$

[21] In the course of this dissertation, we also tested a higher number of Laguerre polynomials as well as different basis functions especially with respect to the combinations of the state variables' cross products. However, the effect on the accuracy of the results was only marginal. This result is consistent with Longstaff and Schwartz (2001), Moreno and Navas (2003) as well as Chang and Hung (2006).

$$\psi_5 = \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right)$$
$$\psi_6 = \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right)$$
$$\psi_7 = 1 - D_t$$
$$\psi_8 = \frac{1}{2}\left(2 - 4\,D_t + D_t^2\right)$$
$$\psi_9 = \frac{1}{6}\left(6 - 18\,D_t + 9\,D_t^2 - D_t^3\right)$$
$$\psi_{10} = 1 - S_t V_t$$
$$\psi_{11} = \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right)$$
$$\psi_{12} = \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right)$$
$$\psi_{13} = 1 - S_t D_t$$
$$\psi_{14} = \frac{1}{2}\left(2 - 4\,S_t^2 D_t + (S_t^2 D_t)^2\right)$$
$$\psi_{15} = \frac{1}{6}\left(6 - 18\,S_t D_t^2 + 9\,(S_t D_t^2)^2 - (S_t D_t^2)^3\right)$$
$$\psi_{16} = 1 - V_t D_t$$
$$\psi_{17} = \frac{1}{2}\left(2 - 4\,V_t^2 D_t + (V_t^2 D_t)^2\right)$$
$$\psi_{18} = \frac{1}{6}\left(6 - 18\,V_t D_t^2 + 9\,(V_t D_t^2)^2 - (V_t D_t^2)^3\right)$$

6.4.5 General Model

In our general model, we pick up on the ideas of both Klein and Inglis (2001) and Liu and Liu (2011). In particular, it is assumed that the short position in the option may increase the counterparty's default risk and the market value of the counterparty's other liabilities follows a geometric Brownian motion as given by Equation (6.3).

At time t, the counterparty's total liabilities are given by $D_t + S_t - K$ in the case of an American call and $D_t + K - S_t$ in the case of an American put, respectively. Consequently, the default boundary $L(S_t,\, D_t)$ indicating the default boundary depends on the type of the considered option. For vulnerable American calls and puts, respectively, it is given as follows:

$$L(S_t,\, D_t) = D_t + S_t - K \tag{6.67}$$
$$L(S_t,\, D_t) = D_t + K - S_t. \tag{6.68}$$

Plugging the above expressions into Equations (6.36) and (6.37), the dynamic programming recursion functions for the least squares Monte Carlo simulation based on the general model are obtained. They are given by

$$C_t = \begin{cases} \max\Big(S_t - K,\, \mathbb{E}_t\Big[e^{-r\Delta t}\, C_{t+\Delta t} \,|\, S_t, V_t, D_t\Big]\Big) & \text{if } V_t \geq D_t + S_t - K \\ \dfrac{(1-\alpha)\, V_t}{D_t + S_t - K}\, \max\left(S_t - K, 0\right) & \text{if } V_t < D_t + S_t - K \end{cases} \tag{6.69}$$

and

$$P_t = \begin{cases} \max\Big(K - S_t,\, \mathbb{E}_t\Big[e^{-r\Delta t}\, B_{t+\Delta t} \,|\, S_t, V_t, D_t\Big]\Big) & \text{if } V_t \geq D_t + K - S_t \\ \dfrac{(1-\alpha)\, V_t}{D_t + K - S_t}\, \max\left(K - S_t, 0\right) & \text{if } V_t < D_t + K - S_t \end{cases} \tag{6.70}$$

for vulnerable American calls and puts, respectively.

In analogy to the previously presented valuation models, the holder of the American option must decide whether the option should be immediately exercised if the counterparty is not in default at the given time step t. According to the first line in Equations (6.69) and (6.70), early exercise is optimal only if the conditional expected continuation value is lower than the option payoff of an immediate exercise. The second line in Equations (6.69) and (6.70) refers to the scenario in which the counterparty is in default at time t. In this case, the American option is immediately exercised irrespective of whether the option is in the money or not. The entire assets of the counterparty (less the default costs α) are distributed to the creditors. Since all liabilities of the counterparty are ranked equally, all creditors receive the same proportion of their claims. Due to the construction of the default boundary, this proportion depends on the type of the considered option. It is given by $\big((1-\alpha)\, V_t\big) \,/\, \big(D_t + S_t - K\big)$ for a vulnerable American call, whereas it is equal to $\big((1-\alpha)\, V_t\big) \,/\, \big(D_t + K - S_t\big)$ for a vulnerable American put. Consequently, the holder of a vulnerable American call receives $\big((1-\alpha)\, V_t \max(S_t - K, 0)\big) \,/\, \big(D_t + S_t - K\big)$, whereas the holder of a vulnerable American put receives $((1-\alpha)\, V_t \max(K - S_t)) \,/\, \big(D_t + K - S_t\big)$.

Looking at Equations (6.69) and (6.70), it is obvious that our general valuation model incorporates the extended models of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) as special cases. The communalities and differences between these models are summarized as follows:

1. If the market value of the counterparty's other liabilities is assumed to be constant over time, our general model is reduced to the extended model of Klein and Inglis (2001) specified by Equations (6.60) and (6.61). In this case, the condition for the counterparty's default is given by either $V_t < \bar{D} + S_t - K$ or $V_t < \bar{D} + K - S_t$ for American calls and puts, respectively.
2. If the option holder's claim ($S_t - K$ or $K - S_t$) is not explicitly considered in the counterparty's total liabilities and if the counterparty's liabilities still follow a geometric Brownian motion, our general model collapses to the extended model of Liu and Liu (2011) represented by Equations (6.64) and (6.65). In this case, the condition for the counterparty's default is given by to $V_t < D_t$.
3. If the option holder's claim ($S_t - K$ or $K - S_t$) is not explicitly considered in the counterparty's total liabilities and if the market value of the counterparty's liabilities is constant over time, our general model is reduced to the extended model of Klein (1996) which is specified by Equations (6.55) and (6.56). Consequently, the condition for the counterparty's default is equal to $V_t < \bar{D}$ in this case.

To decide whether it is optimal to exercise the American option prior to maturity if the counterparty is not in default, the conditional expected continuation value must be determined by regressing the discounted future cash flows against a set of basis functions as illustrated in Equations (6.38) and (6.39). The stochastic interest rates are implicitly included in the price of the option's underlying as well as in the value of the counterparty's assets and liabilities. Therefore, they do not have to be explicitly considered in the construction of the basis functions. Since the price of the vulnerable American option is governed by the same three stochastic variables as in the extended model of Liu and Liu (2011), the same Laguerre polynomials as before can be used as basis functions:

$$\psi_1 = 1 - S_t \tag{6.71}$$

$$\psi_2 = \frac{1}{2}\left(2 - 4\,S_t + S_t^2\right)$$

$$\psi_3 = \frac{1}{6}\left(6 - 18\,S_t + 9\,S_t^2 - S_t^3\right)$$

$$\psi_4 = 1 - V_t$$

$$\psi_5 = \frac{1}{2}\left(2 - 4\,V_t + V_t^2\right)$$

$$\psi_6 = \frac{1}{6}\left(6 - 18\,V_t + 9\,V_t^2 - V_t^3\right)$$

$$\psi_7 = 1 - D_t$$

$$\psi_8 = \frac{1}{2}\left(2 - 4\,D_t + D_t^2\right)$$

$$\psi_9 = \frac{1}{6}\left(6 - 18\,D_t + 9\,D_t^2 - D_t^3\right)$$

$$\psi_{10} = 1 - S_t V_t$$

$$\psi_{11} = \frac{1}{2}\left(2 - 4\,S_t^2 V_t + (S_t^2 V_t)^2\right)$$

$$\psi_{12} = \frac{1}{6}\left(6 - 18\,S_t V_t^2 + 9\,(S_t V_t^2)^2 - (S_t V_t^2)^3\right)$$

$$\psi_{13} = 1 - S_t D_t$$

$$\psi_{14} = \frac{1}{2}\left(2 - 4\,S_t^2 D_t + (S_t^2 D_t)^2\right)$$

$$\psi_{15} = \frac{1}{6}\left(6 - 18\,S_t D_t^2 + 9\,(S_t D_t^2)^2 - (S_t D_t^2)^3\right)$$

$$\psi_{16} = 1 - V_t D_t$$

$$\psi_{17} = \frac{1}{2}\left(2 - 4\,V_t^2 D_t + (V_t^2 D_t)^2\right)$$

$$\psi_{18} = \frac{1}{6}\left(6 - 18\,V_t D_t^2 + 9\,(V_t D_t^2)^2 - (V_t D_t^2)^3\right)$$

6.5 Numerical Examples

In this section, we present various numerical examples to compare the results of the different valuation models for American options subject to counterparty and interest rate risk. Since the entire payoff on the option cannot be made if the option writer defaults, it should be expected that vulnerable options will have lower values than otherwise identical non-vulnerable options. Consequently, the upper limit for the value of a vulnerable American option is given by the default-free option price obtained from the Longstaff-Schwartz approach which is adjusted to the stochastic interest rate framework.

The following comparative analysis of the different valuation models is based on a typical market situation for an American option. At today's point in time ($t = 0$), the option is at the money ($S_0 = 200$, $K = 200$) and expires in six months ($T = 0.5$). The return volatility of the option's underlying equals 25% ($\sigma_S = 0.25$) and its dividend yield is zero ($q = 0$). The option writer is assumed to be highly levered ($V_0 = 1\,000$, $D_0 = 900$). The return volatility of both the counterparty's assets and liabilities is assumed to be 25% ($\sigma_V = 0.25$, $\sigma_D = 0.25$). The correlations between the returns of the option's underlying, the counterparty's assets and liabilities are assumed to be zero ($\rho_{SV} = \rho_{VD} = \rho_{SD} = 0$). If the counterparty defaults, deadweight costs of 25% are applied ($\alpha = 0.25$). The risk-free interest rate is assumed to follow an mean-reverting Ornstein-Uhlenbeck process. The current risk-free interest rate equals 5% ($r_0 = 0.05$). The long-term mean is also equal to 5% ($\theta = 0.05$), while the reversion speed is 0.5 ($\kappa = 0.5$). The volatility of the risk-free interest rate is assumed to be 5% ($\sigma_r = 0.05$). The correlation between the risk-free interest rate and the returns of the option's underlying, the couterparty's assets and liabilities is assumed to be zero ($\rho_{Sr} = \rho_{Vr} = \rho_{Dr} = 0$).

The price of the vulnerable American option is computed based on the different valuation models presented in Section 5.4 using the least squares Monte Carlo simulation. We use 10 000 sample paths with 50 time steps ($N_{\text{Sim}} = 10\,000$, $N_{\text{T}} = 50$) and obtain the value of the American option by computing the mean over 100 re-runs of the algorithm ($n = 100$).

In a first step, we analyze whether the parameters for the least squares Monte Carlo simulation are appropriately chosen and whether the obtained results are reasonably accurate. The confidence interval, for instance, can be used to examine the accuracy of the estimated option value. Assuming that the option values obtained from the least squares Monte Carlo simulation are normally distributed, the two-sided 95% confidence interval for the option value is given by

$$CI = \frac{1}{n} \sum_{j=1}^{n} AO_j \pm 1.96 \cdot \frac{\sigma}{\sqrt{n}}, \tag{6.72}$$

where AO_j gives the value of the American option based on run $j = 1, ..., n$ and σ is the standard deviation of the obtained option values.

Table 6.1 gives the option value as well as the corresponding 95% confidence interval for the different valuation models using the previously mentioned numerical example. The confidence intervals of all the considered valuation models are relatively tight indicating that the computed option values are quite accurate. Hence, the parameters for the least squares Monte Carlo simulation ($N_{\text{Sim}} = 10\,000$, $N_{\text{T}} = 50$, $n = 100$) seem to be reasonably chosen.

	American Call	
	Option Value	95% Confidence Interval
Ext. Longstaff & Schwartz (2001)	16.5424	[16.4860; 16.5988]
Ext. Klein (1996)	12.6369	[12.5943; 12.6795]
Ext. Klein & Inglis (2001)	12.0933	[12.0545; 12.1321]
Ext. Liu & Liu (2011)	10.8314	[10.7980; 10.8648]
General Model	10.4139	[10.3842; 10.4436]

	American Put	
	Option Value	95% Confidence Interval
Ext. Longstaff & Schwartz (2001)	12.0840	[12.0516; 12.1164]
Ext. (1996)	9.7314	[9.7034; 9.7594]
Ext. Klein & Inglis (2001)	9.5287	[9.5028; 9.5546]
Ext. Liu & Liu (2011)	8.5265	[8.5033; 8.5497]
General Model	8.3509	[8.3287; 8.3731]

Table 6.1: Confidence Intervals for the Monte Carlo Simulation

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times.

Figures 6.1 and 6.2 depict the values of American calls and puts, respectively, as functions of the price of the option's underlying, the option's time to maturity and the value of the counterparty's assets for the valuation models presented in the previous section. As expected, the option values obtained from the extended Klein, the extended Klein-Inglis, the extended Liu-Liu and our general model are always lower than the default-free option value given by the model of Longstaff and Schwartz (2001).

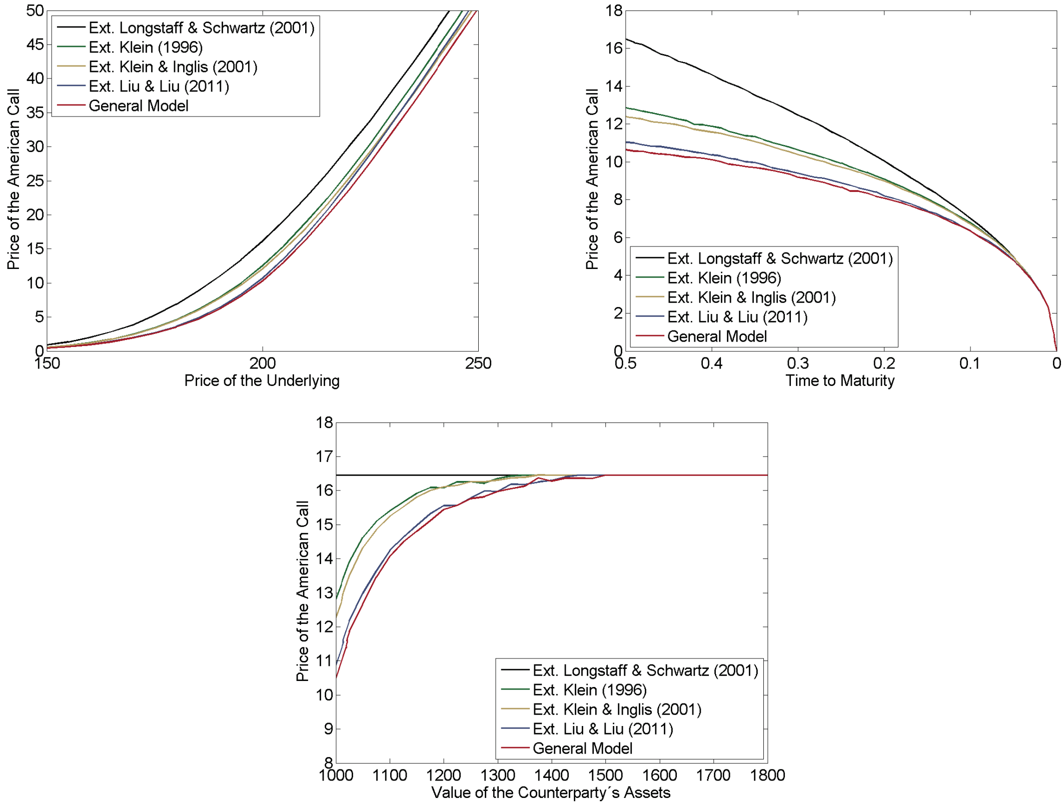

Figure 6.1: American Calls subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times.

In the upper left diagram of Figure 6.1, the value of the vulnerable American call is plotted against the price of the option's underlying. The price difference between default-free and vulnerable American calls is largest for at-the-money options. This price difference decreases if the American call is either further out of the money or further in the money. Moreover, it can be observed that option values obtained from the extended Klein-Inglis and our general model converge if the price of the option's underlying increases, i.e. if the American call is further in the money. This observation is attributed to the fact the option itself is included in the default boundary of both models. For deep in-the-money options, the counterparty's default risk is predominantly driven by the short position in the American option, since it takes an increasing share of the counterparty's total liabilities.

Refering to the upper left diagram of Figure 6.1, the effect of the time to maturity on the value of vulnerable American calls can be analyzed. If the time to maturity decreases, the difference between the default-free and the vulnerable American call values is also reduced. This result is not surprising, since the counterparty is less likely to default if the option's maturity date gets closer.

The lower diagram of Figure 6.1 shows that the price of a vulnerable American call converges to the default-free option price if the value of the counterparty's assets increases, since the probability of hitting the default boundary is decreased in this case. Our general model has the lowest convergence speed which is most likely explained by the fact that this model is the only one that incorporates three sources of default risk simultaneously: a decrease in the value of the counterparty's assets, an increase in the counterparty's other liabilities as well as an increase in the option value itself.

A similar analysis can also be done for vulnerable American puts. In the upper left diagram of Figure 6.2, the value of the vulnerable American put is plotted against the price of the option's underlying. It can be seen that the price difference between default-free and vulnerable American puts is largest for at-the-money options. This price difference decreases if the American call is either further out of the money or further in the money. Additionally, it can be observed that option values obtained from the different valuation models converge if the price of the option's underlying decreases, i.e. if the American put is in the money. This observation is attributed to the fact it is optimal to immediately exercise the American put if it is sufficiently deep in the money.

Refering to the upper left diagram of Figure 6.2, the effect of the time to maturity on the value of vulnerable American puts can be analyzed. If the time to maturity decreases, the difference between the default-free and the vulnerable American put values is also reduced. This result is not surprising, since the counterparty is less likely to default if the option's maturity date gets closer.

The lower diagram of Figure 6.2 shows that the price of a vulnerable American put converges to the default-free option price if the value of the counterparty's assets increases, since the probability of hitting the default boundary is decreased in this case. Our general model has the lowest convergence speed which is most likely explained by the fact that this model is the only one that incorporates three sources of default risk simultaneously: a decrease in the value of the counterparty's assets, an increase in the counterparty's other liabilities as well as an increase in the option value itself.

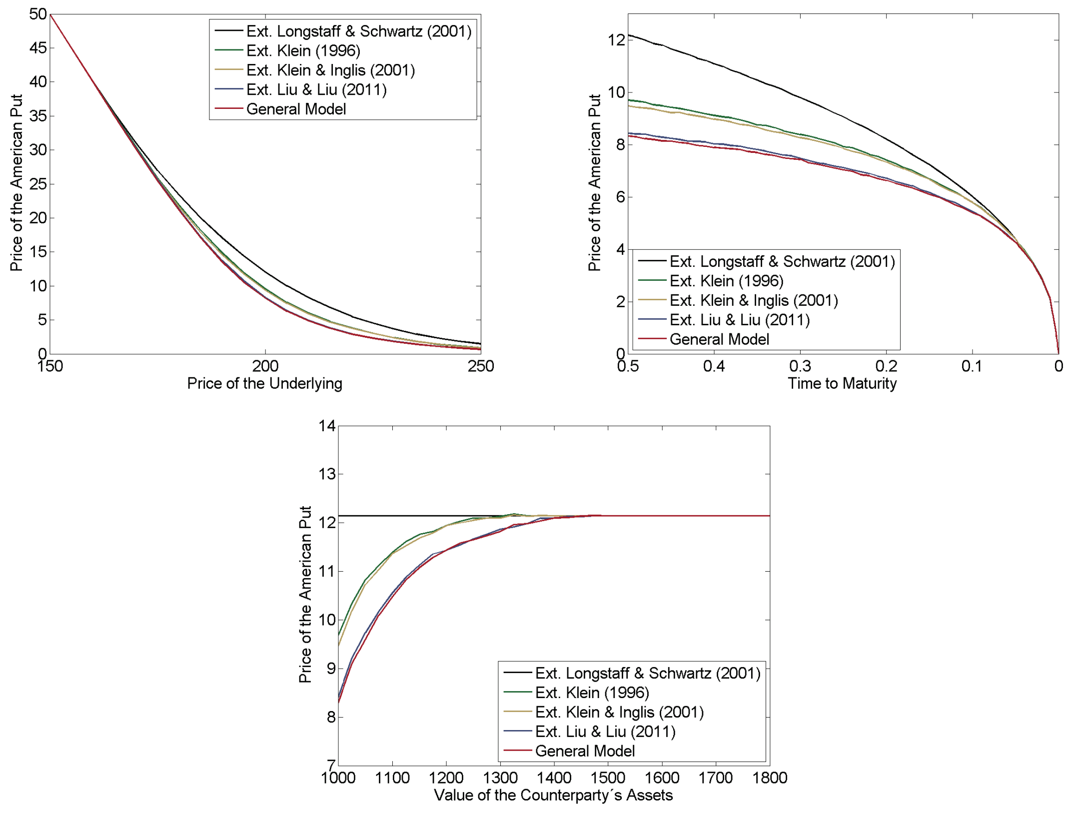

Figure 6.2: American Puts subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times.

	General Model	Ext. LL2011	Ext. KI2001	Ext. K1996	Ext. LS2001
Base Case	10.4139	10.8314	12.0933	12.6369	16.5424
$S_0 = 220$	23.4667	24.5009	24.9658	26.2454	30.3421
$S_0 = 180$	3.6591	3.7519	4.5811	4.7263	7.0328
$V_0 = 1050$	12.3674	12.7275	14.0076	14.4192	16.5424
$V_0 = 950$	7.5110	7.9489	8.8202	9.4617	16.5424
$T - t = 1$	12.9130	13.6198	15.3350	16.3624	24.8664
$T - t = 0.25$	8.3474	8.5738	9.4417	9.7043	11.1950
$\alpha = 0.5$	9.9773	10.4633	11.7324	12.3795	16.5424
$\alpha = 0$	11.2030	11.4920	12.8445	13.2046	16.5424
$q = 0.02$	9.2657	9.5479	10.6507	10.9925	13.9069
$r_0 = 0.08$	11.0743	11.5744	13.1359	13.8088	17.9090
$r_0 = 0.02$	9.8474	10.1916	11.2100	11.6533	15.2097
$\kappa = 0.8$	10.4439	10.8547	12.1293	12.6710	16.5655
$\kappa = 0.2$	10.4065	10.8165	12.0826	12.6065	16.5264
$\theta = 0.08$	10.5106	10.9345	12.2091	12.7515	16.7517
$\theta = 0.02$	10.3618	10.7726	12.0483	12.5758	16.3620
$\sigma_r = 0.08$	10.4073	10.8361	12.1218	12.6771	16.5426
$\sigma_r = 0.02$	10.4498	10.8557	12.1152	12.6586	16.5348
$\rho_{Sr} = 0.5$	10.5439	10.9825	12.2712	12.8859	16.9627
$\rho_{Sr} = -0.5$	10.3383	10.7270	11.9974	12.4938	16.1528
$\rho_{Vr} = 0.5$	10.5606	10.9837	12.3003	12.8550	16.5424
$\rho_{Vr} = -0.5$	10.3120	10.7189	11.9251	12.4639	16.5424
$\rho_{Dr} = 0.5$	10.3205	10.7226	12.0933	12.6369	16.5424
$\rho_{Dr} = -0.5$	10.5365	10.9622	12.0933	12.6369	16.5424

Table 6.2: American Calls subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The abbreviations Ext. LS2001, Ext. K1996, Ext. KI2001 and Ext. LL2011 stand for the extended models of Longstaff and Schwartz (2001), Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011).

	General Model	Ext. LL2011	Ext. KI2001	Ext. K1996	Ext. LS2001
Base Case	8.3509	8.5265	9.5287	9.7314	12.0840
$S_0 = 220$	3.0974	3.1451	3.8343	3.8960	5.6035
$S_0 = 180$	20.9163	21.2060	21.5388	21.8201	23.2999
$V_0 = 1050$	9.6657	9.7987	10.7228	10.8480	12.0840
$V_0 = 950$	6.2186	6.4390	7.1970	7.4973	12.0840
$T - t = 1$	9.6856	9.9380	11.1855	11.5016	15.9894
$T - t = 0.25$	7.0366	7.1483	7.8316	7.9397	8.9591
$\alpha = 0.5$	8.0424	8.2626	9.2727	9.5313	12.0840
$\alpha = 0$	8.7601	8.8605	9.8805	9.9746	12.0840
$q = 0.02$	9.2320	9.4891	10.6191	10.9132	13.8661
$r_0 = 0.08$	7.8277	7.9727	8.9986	9.1636	11.1570
$r_0 = 0.02$	8.8725	9.0836	10.0136	10.2705	13.1226
$\kappa = 0.8$	8.3817	8.5319	9.5471	9.7471	12.0908
$\kappa = 0.2$	8.3378	8.5021	9.5007	9.7035	12.0708
$\theta = 0.08$	8.2885	8.4573	9.4638	9.6644	11.9935
$\theta = 0.02$	8.3694	8.5523	9.5502	9.7628	12.1841
$\sigma_r = 0.08$	8.3628	8.5414	9.5266	9.7295	12.1077
$\sigma_r = 0.02$	8.3201	8.4972	9.4991	9.7014	12.0655
$\rho_{Sr} = 0.5$	8.3815	8.5622	9.5523	9.7669	12.1516
$\rho_{Sr} = -0.5$	8.2842	8.4571	9.4499	9.6463	11.9859
$\rho_{Vr} = 0.5$	8.2722	8.4485	9.4206	9.6219	12.0840
$\rho_{Vr} = -0.5$	8.4071	8.5831	9.6160	9.8199	12.0840
$\rho_{Dr} = 0.5$	8.3932	8.5747	9.5287	9.7314	12.0840
$\rho_{Dr} = -0.5$	8.2716	8.4529	9.5287	9.7314	12.0840

Table 6.3: American Puts subject to Counterparty and Interest Rate Risk

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The abbreviations Ext. LS2001, Ext. K1996, Ext. KI2001 and Ext. LL2011 stand for the extended models of Longstaff and Schwartz (2001), Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011).

Tables 6.2 and 6.3 present the option values for vulnerable American calls and puts, respectively, which are obtained from the least squares Monte Carlo simulation based on the valuation models presented in Section 6.4. Once again it can be observed that the option values based on the extended Klein, the extended Klein-Inglis, the extended Liu-Liu and our general valuation model are always lower than the default-free option value of the Longstaff-Schwartz model. Furthermore, the option values obtained from our general model are substantially lower than those of the other valuation models in most situations. This finding is explained by the construction of the general model's default boundary. Our general model is the only one which incorporates three sources of risk simultaneously. First, a decrease in the value of the counterparty's assets might lead to a default of the option writer like in all the other valuation models. Second, the general model accounts for the potential increase in the default risk induced by the option itself (unlike the extended Klein and the extended Liu-Liu model). Third, it is assumed that the counterparty's other liabilities are stochastic which creates an additional default risk (unlike the extended Klein and the extended Klein-Inglis model). Consequently, the option values based on the general model are the lowest, since it accounts for all possible sources of the counterparty's default risk.

Table 6.4 provides the values of default-free and vulnerable American puts for different prices of the option's underlying. Figure 6.2 already showed that the price of American puts obtained from the different valuation models converge if the price of the option's underlying decreases. This observation is attributed to the fact it is optimal to immediately exercise the American put if it is sufficiently deep in the money. Having a closer look at Table 6.4, it can easily be seen that all valuation models suggest an immediate exercise of the American put if the current price of the option's underlying is lower than 160. Furthermore, it can be observed that the critical stock price for which the American put is immediately exercised is highest for our general model ($S_0 = 168$). This aspect is explained by the fact that this model is the only one that incorporates three sources of default risk simultaneously. A similar analysis could also be performed for American calls. However, the option will only be exercised immediately if both the current price and the dividend yield of the option's underlying are sufficiently large (i.e. $S_0 \gg K$ and $q \gg 0$).

	General Model	Ext. LL2011	Ext. KI2001	Ext. K1996	Ext. LS2001
$S_0 = 158$	42*	42*	42*	42*	42*
$S_0 = 159$	41*	41*	41*	41*	41.0232
$S_0 = 160$	40*	40*	40*	40*	40.0447
$S_0 = 161$	39*	39*	39*	39*	39.0755
$S_0 = 162$	38*	38*	38*	38.0148	38.1106
$S_0 = 163$	37*	37.0175	37*	37.0317	37.2035
$S_0 = 164$	36*	36.0208	36*	36.0445	36.2413
$S_0 = 165$	35*	35.0363	35.0127	35.0690	35.3613
$S_0 = 166$	34*	34.0455	34.0240	34.0917	34.4421
$S_0 = 167$	33*	33.0611	33.0404	33.1297	33.5627
$S_0 = 168$	32*	32.0905	32.0730	32.1837	32.6957
$S_0 = 169$	31.0187	31.1306	31.1255	31.2472	31.8257
$S_0 = 170$	30.0227	30.1788	30.1815	30.3354	30.9917
$S_0 = 171$	29.0302	29.2284	29.2554	29.4213	30.1529
$S_0 = 172$	28.0526	28.2838	28.3382	28.5176	29.3305
$S_0 = 173$	27.1107	27.3511	27.4361	27.6119	28.5237
$S_0 = 174$	26.2020	26.4331	26.5500	26.7539	27.7198
$S_0 = 175$	25.2824	25.5321	25.6814	25.9015	26.9667
$S_0 = 176$	24.3845	24.6552	24.8137	25.0542	26.1978
$S_0 = 177$	23.4853	23.7601	23.9714	24.2110	25.4605
$S_0 = 178$	22.6149	22.8983	23.1358	23.4027	24.7359
$S_0 = 179$	21.7631	22.0480	22.3311	22.6018	24.0018
$S_0 = 180$	20.9145	21.2074	21.5337	21.8225	23.3154

Table 6.4: Analysis of In-the-Money American Puts

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The immediate exercise of the American put is indicated by an asterisk. The abbreviations Ext. LS2001, Ext. K1996, Ext. KI2001 and LL2011 stand for the extended models of Longstaff and Schwartz (2001), Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011).

	Call Options					Put Options				
	Base Case	$S_0 = 220$	$V_0 = 950$	$T-t=1$	$q = 0.05$	Base Case	$S_0 = 180$	$V_0 = 950$	$T-t=1$	$q = 0.05$
Ext. LS2001	16.5424 (16.5396)	30.3421 (30.3384)	16.5424 (16.5396)	24.8664 (24.8628)	13.9069 (13.7682)	12.0840 (11.6205)	23.2999 (22.1376)	12.0840 (11.6132)	15.9894 (15.0519)	13.8661 (13.7665)
Ext. K1996	12.6369 (11.9793)	26.2454 (24.3248)	9.4617 (8.8780)	16.3624 (15.4942)	10.9925 (10.1839)	9.7314 (8.7234)	21.8201 (18.7215)	7.4973 (6.6580)	11.5016 (10.0247)	10.9132 (10.1414)
Ext. KI2001	12.0933 (11.3208)	24.9658 (22.3753)	8.8202 (8.3135)	15.3350 (14.4134)	10.6507 (9.7413)	9.5287 (8.5181)	21.5388 (18.0305)	7.1970 (6.5059)	11.1855 (9.8468)	10.6191 (9.8045)
Ext. LL2011	10.8314 (10.1023)	24.5009 (22.1576)	7.9489 (7.3905)	13.6198 (12.7498)	9.5479 (8.6644)	8.5265 (7.5118)	21.2060 (17.6291)	6.4390 (5.6701)	9.9380 (8.5529)	9.4891 (8.6542)
GM	10.4139 (9.6951)	23.4667 (20.7722)	7.5110 (7.0548)	12.9130 (12.0921)	9.2657 (8.3968)	8.3509 (7.3928)	20.9163 (17.1581)	6.2186 (5.5886)	9.6856 (8.4632)	9.2320 (8.4497)

Table 6.5: American Options vs. European Options

Unless otherwise noted, the calculations are based on the following parameters: $S_0 = 200$, $K = 200$, $V_0 = 1\,000$, $D_0 = 900$, $T - t = 0.5$, $r_0 = 0.05$, $q = 0$, $\kappa = 0.5$, $\theta = 0.05$, $\sigma_S = 0.25$, $\sigma_V = 0.25$, $\sigma_D = 0.25$, $\sigma_r = 0.05$, $\rho_{SV} = 0$, $\rho_{SD} = 0$, $\rho_{VD} = 0$, $\rho_{Sr} = 0$, $\rho_{Vr} = 0$, $\rho_{Dr} = 0$ and $\alpha = 0.25$. The option values for the different valuation models are computed by the least squares Monte Carlo simulation approach described in Sections 6.3 and 6.4. The simulation is based on 10 000 sample paths with 50 time steps. To improve the accuracy of the obtained option values the algorithm is re-run 100 times. The values in parantheses refer to the corresponding European options. The abbreviations Ext. LS2001, Ext. K1996, Ext. KI2001 and LL2011 stand for the extended models of Longstaff and Schwartz (2001), Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011). The abbreviation GM stands for the general model.

Table 6.5 provides a comparative analysis of American and European options based on the considered stochastic interest rate framework. Like in Chapter 5, we find that the values of American and European calls are identical if the dividend yield of the option's underlying is zero. In this case, it is not optimal to exercise the American call prior to maturity. In contrast to that, the early exercise of non-vulnerable American puts is optimal, since their values are always higher than those of the corresponding European puts.

For American options subject to counterparty and interest rate risk, we find different results. In particular, we observe that the values of vulnerable American options are always greater than the values of the corresponding European options for all the considered valuation models. This observation is consistent with Chapter 5.

Furthermore, the price difference between the American and the corresponding European option seems to be greater for vulnerable than for non-vulnerable options. Based on this finding, we may conclude that the early exercise feature receives a greater recognition in case of vulnerable American options. In particular, the holder of an American option subject to counterparty risk gets the opportunity to avoid a potential write-down on his claim by exercising the option prior to maturity.

6.6 Summary

In this chapter, we picked up on the fundamental ideas of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) to develop equivalent models for vulnerable American options. Furthermore, we accounted for stochastic interest rates which are modelled using the Ornstein-Uhlenbeck process suggested by Vasicek (1977). Finally, we set up a general valuation model for American options subject to counterparty and interest rate risk which combines the key characteristics of the other models. Our general model is the only model incorporating three sources of financial distress simultaneously: a decline in the counterparty's assets, an increase in the counterparty's other liabilities or an increase in the value of the option itself.

Due to the early exercise feature of American options, the counterparty's default may occur also prior to maturity. Consequently, the structural approach of Black and Cox (1976) need to be considered. To value vulnerable American options in this framework, the least squares Monte Carlos simulation suggested by Longstaff and Schwartz (2001) is extended to the stochastic interest rate framework and additionally adopted to the different valuation models for vulnerable American options.

Based on various numerical examples and graphical illustrations, we compared the results of our general model with those of the alternative models for American options subject to counterparty and interest rate risk. All the considered valuation models have in common that the reduction in the value of a vulnerable American option (compared to a default-free American option) increases if the time to maturity is longer and if the value of the counterparty's assets is low. The deepest price reduction is oberserved for at-the-money options. The values for vulnerable American options obtained from our general model are typically the lowest, since it is the only model which accounts for all possible sources of the counterparty's default.

7 Conclusion

In this dissertation, we addressed the valuation of European and American options subject to counterparty risk under deterministic and stochastic interest rate frameworks. Due to the lack of a central clearing house, the risk of the option writer's default must be taken into consideration when valuing OTC options. Based on the structural model of Merton (1974) and Black and Cox (1976), we presented and discussed several valuation models in the previous chapters.

First, we introduced the valuation models of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) for vulnerable European options. Combining the key characteristics of these models, we developed a general valuation model for European options subject to counterparty risk. Despite the complexity, we derived an approximate closed form solution for our general model. Numerical examples show that the price of vulnerable European options is substantially lower than the price of otherwise identical default-free European options. The option values obtained from our general model are the lowest, since it is the only model that accounts for three potential sources of the counterparty's default simultaneously. An overview of the different valuation models for European options subject to counterparty risk as well as of my personal contributions is given in Table 7.1.

Model	Default Condition	Remarks
Klein (1996)	$V_T < \bar{D}$	$\bar{D}$ is constant: $\bar{D} = D_0$. r is constant.
Klein & Inglis (2001)	$V_T < \bar{D} + S_T - K$ $V_T < \bar{D} + K - S_T$	$\bar{D}$ is constant: $\bar{D} = D_0$. r is constant.
Liu & Liu (2011)	$V_T < D_T$	D_T is driven by a GBM. r is constant.
General Model*	$V_T < D_T + S_T - K$ $V_T < D_T + K - S_T$	D_T is driven by a GBM. r is constant.

Table 7.1: Overview of the Models presented in Chapter 3

The considered valuation models are intensively discussed in Chapter 3. The risk-free interest rate is deterministic and constant over time. Personal contributions are indicated by an asterisk.

Second, we extended the valuation models of Klein (1996), Klein and Inglis (2001) and Liu and Liu (2011) to a stochastic interest rate framework. In particular, it was assumed that the interest rate is governed by the Ornstein-Uhlenbeck process of Vasicek (1977). Once again, we set up a general model incorporating the fundamental ideas of the other models and derived the corresponding approximate closed form solution. Using numerical examples, the impact of stochastic interest rates on the value of vulnerable European options was analyzed. Table 7.2 gives an overview of the different valuation models for European options subject to counterparty and interest rate risk as well as of my personal contributions.

Model	Default Condition	Remarks
Klein & Inglis (1999)	$V_T < \bar{D}$	$\bar{D}$ is constant: $\bar{D} = D_0$. r_t is driven by an OU.
Extended Version of Klein & Inglis (2001)*	$V_T < \bar{D} + S_T - K$ $V_T < \bar{D} + K - S_T$	$\bar{D}$ is constant: $\bar{D} = D_0$. r_t is driven by an OU.
Extended Version of Liu & Liu (2011)*	$V_T < D_T$	D_T is driven by a GBM. r_t is driven by an OU.
General Model*	$V_T < D_T + S_T - K$ $V_T < D_T + K - S_T$	D_T is driven by a GBM. r_t is driven by an OU.

Table 7.2: Overview of the Models presented in Chapter 4

The considered valuation models are intensively discussed in Chapter 4. The risk-free interest rate follows the Ornstein-Uhlenbeck process (OU) of Vasicek (1977): $dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_r$. Personal contributions are indicated by an asterisk.

Third, the valuation of vulnerable American options was addressed. In particular, we picked up on the key features of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011) to set up their equivalent models for American options. Furthermore, we developed a general valuation model. Due to the early exercise features of American options, closed form solutions could not be derived. Instead, the options are priced using the least squares Monte Carlo simulation suggested by Longstaff and Schwartz (2001). This approach was originally designed to value American options, but can also be applied to more complex problems such as the valuation of vulnerable American options. Based on numerical examples, we observed that the price for vulnerable American options is substantially lower than the price of otherwise identical default-free American options. The sharpest price reduction was found for our general model, since it is the only model considering three potential sources of the counterparty's default simultaneously. An overview of the different valuation models for American options subject to counterparty risk as well as of my personal contributions is given in Table 7.3.

Model	Default Condition	Remarks
Extended Version of Klein (1996)*	$V_t < \bar{D}$	$\bar{D}$ is constant: $\bar{D} = D_0$. r is constant.
Extended Version of Klein & Inglis (2001)*	$V_t < \bar{D} + S_t - K$ $V_t < \bar{D} + K - S_t$	$\bar{D}$ is constant: $\bar{D} = D_0$. r is constant.
Extended Version of Liu & Liu (2011)*	$V_t < D_t$	D_t is driven by a GBM. r is constant.
General Model*	$V_t < D_t + S_t - K$ $V_t < D_t + K - S_t$	D_t is driven by a GBM. r is constant.

Table 7.3: Overview of the Models presented in Chapter 5

The considered valuation models are intensively discussed in Chapter 5. The risk-free interest rate is deterministic and constant over time. Personal contributions are indicated by an asterisk.

Finally, we discussed the valuation of American options subject to counterparty and interest rate risk. Assuming that the risk-free interest rate follows an Ornstein-Uhlenbeck process, we set up models to price vulnerable American options built on the ideas of Klein (1996), Klein and Inglis (2001) as well as Liu and Liu (2011). Moreover, we developed a general model combining the features of the previously mentioned models. The least squares Monte Carlo simulation suggested by Longstaff and Schwartz (2001) was adapted to the considered framework and used to price the vulnerable American options. Several numerical examples showed the impact of stochastic interest rates on the option values. Table 7.4 gives an overview of the discussed models for American options subject to counterparty and interest rate risk as well as of my personal contributions.

Model	Default Condition	Remarks
Extended Version of Klein (1996)*	$V_t < \bar{D}$	$\bar{D}$ is constant: $\bar{D} = D_0$. r_t is driven by an OU.
Extended Version of Klein & Inglis (2001)*	$V_t < \bar{D} + S_t - K$ $V_t < \bar{D} + K - S_t$	$\bar{D}$ is constant: $\bar{D} = D_0$. r_t is driven by an OU.
Extended Version of Liu & Liu (2011)*	$V_t < D_t$	D_t is driven by a GBM. r_t is driven by an OU.
General Model*	$V_t < D_t + S_t - K$ $V_t < D_t + K - S_t$	D_t is driven by a GBM. r_t is driven by an OU.

Table 7.4: Overview of the Models presented in Chapter 6

The considered valuation models are intensively discussed in Chapter 6. The risk-free interest rate follows the Ornstein-Uhlenbeck (OU) process of Vasicek (1977): $dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_r$. Personal contributions are indicated by an asterisk.

As discussed in Chapter 2, valuation models on vulnerable American options are rather scarce. Consequently, this area offers broad research opportunities. In particular, the existing models for vulnerable American options can be extended to other price processes (e.g. jump diffusion processes), other stochastic interest rate or stochastic volatility models. Furthermore, an imperfect market framework can be considered to additionally account for liquidity risk. In the context of vulnerable European options, extensions to other stochastic interest rate models are possible. Furthermore, the valuation of exotic options (e.g. barrier or binary options) subject to counterparty risk can be addressed.

Appendix

Appendix 1

In the following, the approximate closed form valuation formula of the general model for vulnerable European options under deterministic interest rates is derived based on the general model. The derivation of the valuation formula is only given for vulnerable European calls, but the same procedure can also be used to get the valuation formula for vulnerable European puts. To obtain the valuation formula, it must be assumed that the returns of the option's underlying and the counterparty's other liabilities are uncorrelated (i.e. $\rho_{SD} = 0$).

The pricing equation for a vulnerable European call based on the general model can be written as follows:

$$\begin{aligned} C = e^{-r(T-t)} \Bigg(& E\left[S_T - K \mid S_T \geq K,\, V_T \geq D_T + S_T - K \right] \\ & + E\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{D_T + S_T - K} \mid S_T \geq K,\, V_T < D_T + S_T - K \right] \Bigg) . \end{aligned}$$

Using the risk-neutral pricing approach, the value of the vulnerable European call is given by

$$\begin{aligned} C = e^{-r(T-t)} \Bigg(& \int_K^\infty \int_{D_T+S_T-K}^\infty \int_0^\infty S_T\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \\ & - \int_K^\infty \int_{D_T+S_T-K}^\infty \int_0^\infty K\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \\ & + \int_K^\infty \int_0^{D_T+S_T-K} \int_0^\infty \frac{(1-\alpha)\, V_T\, S_T}{D_T + S_T - K}\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \\ & - \int_K^\infty \int_0^{D_T+S_T-K} \int_0^\infty \frac{(1-\alpha)\, V_T\, K}{D_T + S_T - K}\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \Bigg) , \end{aligned}$$

where $\Phi(\,\cdot\,)$ is the joint trivariate lognormal distribution function of the random variables S_T, V_T and D_T.

Applying the standard log transformation, standardizing the normal distribution and collecting terms yields

$$
\begin{aligned}
C = & \int_{-a}^{\infty} \int_{f(u,w)}^{\infty} \int_{-\infty}^{\infty} S_t \, e^{(-q-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S \sqrt{T-t}\, u} \, n_3(u,v,w) \, dw \, dv \, du \\
& - \int_{-a}^{\infty} \int_{f(u,w)}^{\infty} \int_{-\infty}^{\infty} e^{-r(T-t)} \, K \, n_3(u,v,w) \, dw \, dv \, du \\
& + \int_{-a}^{\infty} \int_{-\infty}^{f(u,w)} \int_{-\infty}^{\infty} \frac{(1-\alpha) \, S_t \, V_t \, e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)+\sigma_S \sqrt{T-t}\, u+\sigma_V \sqrt{T-t}\, v}}{D_t \, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D \sqrt{T-t}\, w} + S_t \, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S \sqrt{T-t}\, u} - K} \\
& \qquad \cdot \, n_3(u,v,w) \, dw \, dv \, du \\
& - \int_{-a}^{\infty} \int_{-\infty}^{f(u,w)} \int_{-\infty}^{\infty} \frac{(1-\alpha) \, K \, V_t \, e^{\left(r-\frac{1}{2}\sigma_V^2\right)(T-t)+\sigma_V \sqrt{T-t}\, v}}{D_t \, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D \sqrt{T-t}\, w} + S_t \, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S \sqrt{T-t}\, u} - K} \\
& \qquad \cdot \, n_3(u,v,w) \, dw \, dv \, du,
\end{aligned}
$$

where $n_3(\,\cdot\,)$ is the joint trivariate standard normal density function of the random variables u, v and w which is given by

$$
\begin{aligned}
n_3(u,v,w) &= n_3(u,v,w,0,0,0,1,1,1,\rho_{SV},\rho_{SD}=0,\rho_{VD}) \\
&= \frac{e^{-\frac{1}{2\left(1-\rho_{SV}^2-\rho_{VD}^2\right)}\left((1-\rho_{VD}^2)u^2+v^2+(1-\rho_{SV}^2)w^2-2\rho_{SV}\, uv+2\rho_{SV}\rho_{VD}uw-2\rho_{VD}vw\right)}}{\sqrt{8\pi^3}\sqrt{1-\rho_{SV}^2-\rho_{VD}^2}}
\end{aligned}
$$

The parameter a as well as the function $f(\,\cdot\,)$ are given as follows:

$$
a = \frac{\ln \frac{S_t}{K} + \left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)}{\sigma_S \sqrt{T-t}}
$$

$$
f(u,w) = \frac{\ln \frac{D_t \, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D \sqrt{T-t}\, w} + S_t \, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S \sqrt{T-t}\, u} - K}{V_t} - \left(r-\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \sqrt{T-t}}
$$

In the next step, the function $f(u,w)$ is linearized using Taylor series expansion.

$$\begin{aligned} f(u,w) &\approx f(p_1,p_2) + \frac{\partial f(p_1,p_2)}{\partial p_1}(u-p_1) + \frac{\partial f(p-1,p_2)}{\partial p_2}(w-p_2) \\ &= b + m_1(u-p_1) + m_2(w-p_2) \end{aligned}$$

where the parameters b, m_1 and m_2 are given as follows:

$$\begin{aligned} b &= \frac{\ln \frac{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}{V_t} - \left(r-\frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V\sqrt{T-t}}, \\ m_1 &= \frac{\sigma_S}{\sigma_V}\frac{S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}, \\ m_2 &= \frac{\sigma_D}{\sigma_V}\frac{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}. \end{aligned}$$

Furthermore, the denominator in the third and fourth integral needs to be modified as well using the first order Taylor series expansion.

$$\begin{aligned} F(u,w) &= \frac{1}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,w} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,u} - K} \\ G(u,w) &= \ln \frac{1}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,w} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,u} - K} \\ &\approx G(p_1,p_2) + \frac{\partial G(p_1,p_2)}{\partial p_1}(u-p_1) + \frac{\partial G(p_1,p_2)}{\partial p_2}(w-p_2) \\ &= h + g_1(u-p_1) + g_2(w-p_2) \end{aligned}$$

with

$$\begin{aligned} h &= \ln \frac{1}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \\ g_1 &= \frac{-\sigma_S\sqrt{T-t}\,S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \\ g_2 &= \frac{-\sigma_D\sqrt{T-t}\,D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \end{aligned}$$

Using the Taylor series approximations, the pricing equation for a vulnerable European call can be rewritten as follows:

$$C = S_t e^{(-q-\frac{1}{2}\sigma_S^2)(T-t)} \int_{-a}^{\infty} \int_{b+m_1(u-p_1)+m_2(w-p_2)}^{\infty} \int_{-\infty}^{\infty} e^{\sigma_S \sqrt{T-t}\, u}\, n_3(u,v,w)\, dw\, dv\, du$$

$$- K e^{-r(T-t)} \int_{-a}^{\infty} \int_{b+m_1(u-p_1)+m_2(w-p_2)}^{\infty} \int_{-\infty}^{\infty} n_3(u,v,w)\, dw\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{\left(r-q\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1 p_1-g_2 p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\, p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\, p_1} - K}$$

$$\int_{-a}^{\infty} \int_{-\infty}^{b+m_1(u-p_1)+m_2(w-p_2)} \int_{-\infty}^{\infty} e^{(g_1+\sigma_S\sqrt{T-t})\, u+\sigma_V\sqrt{T-t}\, v+g_2 w}\, n_3(u,v,w)\, dw\, dv\, du$$

$$- \frac{(1-\alpha)\, K V_t\, e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1 p_1-g_2 p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\, p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\, p_1} - K}$$

$$\int_{-a}^{\infty} \int_{-\infty}^{b+m_1(u-p_1)+m_2(w-p_2)} \int_{-\infty}^{\infty} e^{g_1 u+\sigma_V\sqrt{T-t}\, v+g_2 w}\, n_3(u,v,w)\, dw\, dv\, du$$

Using appropriate substitutions for u, v and w, the stochastic component in the integral boundaries can be eliminated. The random variables u, v and w are substituted by

$$u = \frac{x}{\sqrt{1+m_1^2}},$$

$$v = y + \frac{m_1 x}{\sqrt{1+m_1^2}} + \frac{m_2 z}{\sqrt{1+m_2^2}}$$

and

$$w = \frac{z}{\sqrt{1+m_2^2}},$$

where x, y and z are also jointly standard normally distributed.

Applying these substitutions to the pricing equation yields

$$C = S_t e^{(-q-\frac{1}{2}\sigma_S^2)(T-t)} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{b-m_1p_1-m_2p_2}^{\infty} \int\limits_{-\infty}^{\infty} e^{\frac{\sigma_S\sqrt{T-t}}{\sqrt{1+m_1^2}}x} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx$$

$$- K e^{-r(T-t)} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{b-m_1p-m_2p_2}^{\infty} \int\limits_{-\infty}^{\infty} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1p_1-g_2p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{-\infty}^{b-m_1p_1-m_2p_2} \int\limits_{-\infty}^{\infty} e^{\frac{g_1+(\sigma_S+m_1\sigma_V)\sqrt{T-t}}{\sqrt{1+m_1^2}}x+\sigma_V\sqrt{T-t}\,y+\frac{g_2+m_2\sigma_V\sqrt{T-t}}{\sqrt{1+m_2^2}}z} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx$$

$$- \frac{(1-\alpha)\, K V_t\, e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1p_1-g_2p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{-\infty}^{b-m_1p-m_2p_2} \int\limits_{-\infty}^{\infty} e^{\frac{g_1+m_1\sigma_V\sqrt{T-t}}{\sqrt{1+m_1^2}}x+\sigma_V\sqrt{T-t}\,y+\frac{g_2+m_2\sigma_V\sqrt{T-t}}{\sqrt{1+m_2^2}}z} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx,$$

where

$$\Omega(x,y,z) = e^{-\frac{1}{2\left(1-\rho_{SV}^2-\rho_{VD}^2\right)}} e^{(1-\rho_{VD}^2)\left(\frac{x}{\sqrt{1+m_1^2}}\right)^2+\left(y+\frac{m_1x}{\sqrt{1+m_1^2}}+\frac{m_2z}{\sqrt{1+m_2^2}}\right)^2+(1-\rho_{SV}^2)\left(\frac{z}{\sqrt{1+m_2^2}}\right)^2}$$
$$e^{-2\rho_{SV}\frac{x}{\sqrt{1+m_1^2}}\left(y+\frac{m_1x}{\sqrt{1+m_1^2}}+\frac{m_2z}{\sqrt{1+m_2^2}}\right)} e^{2\rho_{SV}\rho_{VD}\frac{x}{\sqrt{1+m_1^2}}\frac{z}{\sqrt{1+m_2^2}}}$$
$$e^{-2\rho_{VD}\frac{z}{\sqrt{1+m_2^2}}\left(y+\frac{m_1x}{\sqrt{1+m_1^2}}+\frac{m_2z}{\sqrt{1+m_2^2}}\right)}$$

and

$$\Theta = \sqrt{1+m_1^2}\,\sqrt{1+m_2^2}\,\sqrt{8\pi^3}\sqrt{1-\rho_{SV}^2-\rho_{VD}^2}.$$

The previous expression can be rewritten as follows:

$$C = S_t e^{(-q-\frac{1}{2}\sigma_S^2)(T-t)} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{b-m_1 p_1 - m_2 p_2}^{\infty} \int\limits_{-\infty}^{\infty} e^{\frac{\sigma_S\sqrt{T-t}}{\sqrt{1+m_1^2}}x} \, \frac{\Psi(x,y,z)}{\Theta} \, dz\, dy\, dx$$

$$- K e^{-r(T-t)} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{b-m_1 p - m_2 p_2}^{\infty} \int\limits_{-\infty}^{\infty} \frac{\Psi(x,y,z)}{\Theta} \, dz\, dy\, dx$$

$$+ \frac{(1-\alpha)\, S_t V_t \, e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1 p_1 - g_2 p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}$$

$$\int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{-\infty}^{b-m_1 p_1 - m_2 p_2} \int\limits_{-\infty}^{\infty} e^{\frac{g_1+(\sigma_S+m_1\sigma_V)\sqrt{T-t}}{\sqrt{1+m_1^2}}x + \sigma_V\sqrt{T-t}\,y + \frac{g_2+m_2\sigma_V\sqrt{T-t}}{\sqrt{1+m_2^2}}z}$$

$$\frac{\Psi(x,y,z)}{\Theta} \, dz\, dy\, dx$$

$$- \frac{(1-\alpha)\, K V_t \, e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1 p_1 - g_2 p_2}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}$$

$$\int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{-\infty}^{b-m_1 p - m_2 p_2} \int\limits_{-\infty}^{\infty} e^{\frac{g_1+m_1\sigma_V\sqrt{T-t}}{\sqrt{1+m_1^2}}x + \sigma_V\sqrt{T-t}\,y + \frac{g_2+m_2\sigma_V\sqrt{T-t}}{\sqrt{1+m_2^2}}z}$$

$$\frac{\Psi(x,y,z)}{\Theta} \, dz\, dy\, dx,$$

where

$$\Psi(x,y,z) = e^{-\frac{1}{2\left(1-\delta_{SV}^2-\delta_{VD}^2\right)}} \; e^{(1-\delta_{VD}^2)\left(\frac{x}{\sqrt{1+m_1^2}}\right)^2} \; e^{\left(\frac{y}{\sqrt{1-2\rho_{SV}m_1-2\rho_{VD}m_2+m_1^2+m_2^2}}\right)^2}$$

$$e^{(1-\delta_{SV}^2)\left(\frac{z}{\sqrt{1+m_2^2}}\right)^2} \; e^{-2\delta_{SV}\frac{x}{\sqrt{1+m_1^2}}\frac{y}{\sqrt{1-2\rho_{SV}m_1-2\rho_{VD}m_2+m_1^2+m_2^2}}}$$

$$e^{2\delta_{SV}\delta_{VD}\frac{x}{\sqrt{1+m_1^2}}\frac{z}{\sqrt{1+m_2^2}}} \; e^{-2\delta_{VD}\frac{z}{\sqrt{1+m_2^2}}\frac{y}{\sqrt{1-2\delta_{SV}m_1-2\delta_{VD}m_2+m_1^2+m_2^2}}}$$

and

$$\delta_{SV} = \frac{\rho_{SV}-m_1}{\sqrt{1-2\rho_{SV}m_1-2\rho_{VD}m_2+m_1^2+m_2^2}},$$

$$\delta_{VD} = \frac{\rho_{VD}-m_2}{\sqrt{1-2\rho_{SV}m_1-2\rho_{VD}m_2+m_1^2+m_2^2}}.$$

Using appropriate substitutions for x, y and z, the previous expression can be rewritten once again. In particular, the variables x, y and z are substituted by

$$x = \sqrt{1 + m_1^2}\, u,$$
$$y = \sqrt{1 - 2\,\rho_{SV}\, m_1 - 2\,\rho_{VD}\, m_2 + m_1^2 + m_2^2}\, v$$

and

$$z = \sqrt{1 + m_2^2}\, w,$$

where u, v and w are also jointly standard normally distributed.

Applying these substitutions to the pricing equation yields

$$C = S_t e^{(-q - \frac{1}{2}\sigma_S^2)(T-t)} \int_{-a}^{\infty}\int_{c}^{\infty}\int_{-\infty}^{\infty} e^{\sigma_S \sqrt{T-t} u}\, \Gamma(u,v,w)\, dw\, dv\, du$$

$$- K e^{-r(T-t)} \int_{-a}^{\infty}\int_{c}^{\infty}\int_{-\infty}^{\infty} \Gamma(u,v,w)\, dw\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{\left(r - q - \frac{1}{2}\sigma_S^2 - \frac{1}{2}\sigma_V^2\right)(T-t) - g_1 p_1 - g_2 p_2}}{D_t\, e^{\left(r - \frac{1}{2}\sigma_D^2\right)(T-t) + \sigma_D \sqrt{T-t}\, p_2} + S_t\, e^{\left(r - q - \frac{1}{2}\sigma_S^2\right)(T-t) + \sigma_S \sqrt{T-t}\, p_1} - K}$$
$$\int_{-a}^{\infty}\int_{-\infty}^{c}\int_{-\infty}^{\infty} e^{\left(g_1 + (\sigma_S + m_1 \sigma_V)\sqrt{T-t}\right)u}\; e^{\sigma_V \sqrt{T-t}\sqrt{1 - 2\,\rho_{SV}\, m_1 - 2\,\rho_{VD}\, m_2 + m_1^2 + m_2^2}\, v}$$
$$e^{\left(g_2 + m_2 \sigma_V \sqrt{T-t}\right)w}\, \Gamma(u,v,w)\, dw\, dv\, du$$

$$- \frac{(1-\alpha)\, K V_t\, e^{-\frac{1}{2}\sigma_V^2 (T-t) - g_1 p_1 - g_2 p_2}}{D_t\, e^{\left(r - \frac{1}{2}\sigma_D^2\right)(T-t) + \sigma_D \sqrt{T-t}\, p_2} + S_t\, e^{\left(r - q - \frac{1}{2}\sigma_S^2\right)(T-t) + \sigma_S \sqrt{T-t}\, p_1} - K}$$
$$\int_{-a}^{\infty}\int_{-\infty}^{c}\int_{-\infty}^{\infty} e^{\left(g_1 + m_1 \sigma_V \sqrt{T-t}\right)u}\; e^{\sigma_V \sqrt{T-t}\sqrt{1 - 2\,\rho_{SV}\, m_1 - 2\,\rho_{VD}\, m_2 + m_1^2 + m_2^2}\, v}$$
$$e^{\left(g_2 + m_2 \sigma_V \sqrt{T-t}\right)w}\, \Gamma(u,v,w)\, dw\, dv\, du$$

where

$$c = \frac{b - m_1 p_1 - m_2 p_2}{\sqrt{1 - 2\,\rho_{SV}\, m_1 - 2\,\rho_{VD}\, m_2 + m_1^2 + m_2^2}}$$

and

$$\Gamma(u,v,w) = n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},\delta_{SD}=0,\delta_{VD})$$

$$= \frac{e^{-\frac{1}{2\left(1-\delta_{SV}^2-\delta_{VD}^2\right)}\left((1-\delta_{VD}^2)u^2+v^2+(1-\delta_{SV}^2)w^2-2\delta_{SV}\,uv+2\delta_{SV}\delta_{VD}uw-2\delta_{VD}vw\right)}}{\sqrt{8\pi^3}\sqrt{1-\delta_{SV}^2-\delta_{VD}^2}}$$

Completing the square yields

$$C = S_t e^{-q(T-t)} \int_{-a}^{\infty}\int_{c}^{\infty}\int_{-\infty}^{\infty} n_3(u,v,w,\sigma_s\sqrt{T-t},\delta_{SV}\sigma_s\sqrt{T-t},0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$- Ke^{-r(T-t)} \int_{-a}^{\infty}\int_{c}^{\infty}\int_{-\infty}^{\infty} n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$+ \frac{(1-\alpha)\,S_tV_t\,e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1p_1-g_2p_2}\;e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\,e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2}+S_t\,e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1}-K}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{c}\int_{-\infty}^{\infty} n_3(u,v,w,\eta+\delta_{SV}\,\phi,\phi+\delta_{SV}\,\eta+\delta_{VD}\,\lambda,\lambda+\delta_{VD}\,\phi,\ldots$$

$$\ldots,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$- \frac{(1-\alpha)\,KV_t\,e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1p_1-g_2p_2}\;e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\,e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2}+S_t\,e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1}-K}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{c}\int_{-\infty}^{\infty} n_3(u,v,w,\xi+\delta_{SV}\,\phi,\phi+\delta_{SV}\,\xi+\delta_{VD}\,\lambda,\lambda+\delta_{VD}\,\phi,\ldots$$

$$\ldots,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du,$$

where

$$\eta = g_1+\sigma_S\sqrt{T-t}+m_1\,\sigma_V\sqrt{T-t},$$

$$\xi = \eta-\sigma_S\sqrt{T-t},$$

$$\phi = \sigma_V\sqrt{T-t}\sqrt{1-2\,\rho_{SV}\,m_1-2\,\rho_{VD}\,m_2+m_1^2+m_2^2},$$

$$\lambda = g_2+m_2\,\sigma_V\sqrt{T-t}.$$

Standardizing the normal distribution yields

$$C = S_t e^{-q(T-t)} \int\limits_{-a-\sigma_s\sqrt{T-t}}^{\infty} \int\limits_{c-\delta_{SV}\sigma_s\sqrt{T-t}}^{\infty} \int\limits_{-\infty}^{\infty} n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$- K e^{-r(T-t)} \int\limits_{-a}^{\infty}\int\limits_{c}^{\infty}\int\limits_{-\infty}^{\infty} n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$+ \frac{(1-\alpha)\,S_t V_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1p_1-g_2p_2}\; e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}$$

$$\int\limits_{-a-\eta-\delta_{SV}\,\phi}^{\infty} \int\limits_{-\infty}^{c-\phi-\delta_{SV}\,\eta-\delta_{VD}\,\lambda} \int\limits_{-\infty}^{\infty} n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$- \frac{(1-\alpha)\,K V_t\, e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1p_1-g_2p_2}\; e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}$$

$$\int\limits_{-a-\xi-\delta_{SV}\,\phi}^{\infty} \int\limits_{-\infty}^{c-\phi-\delta_{SV}\,\xi-\delta_{VD}\,\lambda} \int\limits_{-\infty}^{\infty} n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du.$$

Computing the triple integrals yields the approximate closed form solution for vulnerable European calls based on the general model. It is given by

$$C = S_t e^{-q(T-t)}\, N_3(a+\sigma_S\sqrt{T-t}, -c+\delta_{SV}\sigma_S\sqrt{T-t}, +\infty, \delta_{SV}, 0, \delta_{VD})$$

$$- K e^{-r(T-t)}\, N_3(a, -c, +\infty, \delta_{SV}, 0, \delta_{VD})$$

$$+ (1-\alpha)\, \frac{S_t V_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1p_1-g_2p_2}\; e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}$$

$$N_3(a+\eta+\delta_{SV}\phi, c-\phi-\delta_{SV}\eta-\delta_{VD}\lambda, +\infty, -\delta_{SV}, 0, -\delta_{VD})$$

$$- (1-\alpha)\, \frac{K V_t\, e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1p_1-g_2p_2}\; e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K}$$

$$N_3(a+\xi+\delta_{SV}\phi, c-\phi-\delta_{SV}\xi-\delta_{VD}\lambda, +\infty, -\delta_{SV}, 0, -\delta_{VD}),$$

where $N_3(\,\cdot\,)$ gives the trivariate cumulative normal distribution function.

Since the stochastic variable $\ln D_t$ can assume any value between $-\infty$ and $+\infty$, the trivariate cumulative normal ditribution becomes a bivariate cumulative normal distribution. Hence, the approximate closed form solution is given by

$$
\begin{aligned}
C = {} & S_t e^{-q(T-t)}\, N_2(a + \sigma_S\sqrt{T-t}, -c + \delta_{SV}\sigma_S\sqrt{T-t}, \delta_{SV}) \\
& - K e^{-r(T-t)}\, N_2(a, -c, \delta_{SV}) \\
& + (1-\alpha)\, \frac{S_t V_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2-\frac{1}{2}\sigma_V^2\right)(T-t)-g_1p_1-g_2p_2}\; e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \\
& \quad N_2(a+\eta+\delta_{SV}\phi, c-\phi-\delta_{SV}\eta-\delta_{VD}\lambda, -\delta_{SV}) \\
& - (1-\alpha)\, \frac{K V_t\, e^{-\frac{1}{2}\sigma_V^2(T-t)-g_1p_1-g_2p_2}\; e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{\left(r-\frac{1}{2}\sigma_D^2\right)(T-t)+\sigma_D\sqrt{T-t}\,p_2} + S_t\, e^{\left(r-q-\frac{1}{2}\sigma_S^2\right)(T-t)+\sigma_S\sqrt{T-t}\,p_1} - K} \\
& \quad N_2(a+\xi+\delta_{SV}\phi, c-\phi-\delta_{SV}\xi-\delta_{VD}\lambda, -\delta_{SV})
\end{aligned}
$$

where $N_2(\,\cdot\,)$ gives the bivariate cumulative normal distribution function.

Collecting and rearranging terms yields the approximate closed form valuation formula given by Equation (3.54).

Appendix 2

In the following, the approximate closed form valuation formula for vulnerable European options under the stochastic interest rate framework of Vasicek (1977) is derived based on the extended model of Klein and Inglis (2001). The derivation of the valuation formula is only given for vulnerable European calls, but the same procedure can also be used to get the valuation formula for vulnerable European puts.

The pricing equation for a vulnerable European call based on the general model is equal to

$$C = B_{t,T}\left(E\left[S_T - K \mid S_T \geq K,\, V_T \geq \bar{D} + S_T - K\right] + E\left[\frac{(1-\alpha)\,V_T\,(S_T - K)}{\bar{D} + S_T - K} \mid S_T \geq K,\, V_T < \bar{D} + S_T - K\right]\right).$$

where $\bar{D} = D_t$.

Using the risk-neutral pricing approach, the value of the vulnerable European call is given by

$$\begin{aligned} C = B_{t,T}\Bigg(& \int_K^{\infty}\int_{\bar{D}+S_T-K}^{\infty} S_T\,\Phi(S_T, V_T)\,dV_T\,dS_T \\ & - \int_K^{\infty}\int_{\bar{D}+S_T-K}^{\infty} K\,\Phi(S_T, V_T)\,dV_T\,dS_T \\ & + \int_K^{\infty}\int_0^{\bar{D}+S_T-K} \frac{(1-\alpha)\,V_T\,S_T}{\bar{D}+S_T-K}\,\Phi(S_T, V_T)\,dV_T\,dS_T \\ & - \int_K^{\infty}\int_0^{\bar{D}+S_T-K} \frac{(1-\alpha)\,V_T\,K}{\bar{D}+S_T-K}\,\Phi(S_T, V_T)\,dV_T\,dS_T \Bigg), \end{aligned}$$

where $\Phi(\,\cdot\,)$ is the joint bivariate lognormal distribution function of the random variables S_T and V_T.

Applying the standard log transformation, standardizing the normal distribution and collecting terms yields

$$C = \int_{-a}^{\infty}\int_{f(u)}^{\infty} S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u}\, n_2(u,v)\, dv\, du - \int_{-a}^{\infty}\int_{f(u)}^{\infty} B_{t,T}\, K\, n_2(u,v)\, dv\, du$$

$$+ \int_{-a}^{\infty}\int_{-\infty}^{f(u)} \frac{(1-\alpha)\, S_t\, V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2+\bar{\sigma}_S\, u+\bar{\sigma}_V\, v}}{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u} - B_{t,T}\, K}\, n_2(u,v)\, dv\, du$$

$$- \int_{-a}^{\infty}\int_{-\infty}^{f(u)} \frac{(1-\alpha)\, B_{t,T}\, K\, V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2+\bar{\sigma}_V\, v}}{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u} - B_{t,T}\, K}\, n_2(u,v)\, dv\, du,$$

where $n_2(\,\cdot\,)$ is the joint bivariate standard normal density function of the random variables u and v. It is given by

$$n_2(u,v) = n_2(u,v,0,0,1,1,\bar{\rho}_{SV}) = \frac{1}{\sqrt{4\pi^2}\sqrt{1-\bar{\rho}_{SV}^2}}\, e^{-\frac{1}{2\left(1-\bar{\rho}_{SV}^2\right)}\left(u^2+v^2-2\bar{\rho}_{SV}\, uv\right)}$$

The parameter a and the function $f(\,\cdot\,)$ are given as follows:

$$a = \frac{\ln \frac{S_t}{B_{t,T}K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S}$$

$$f(u) = \frac{\ln \frac{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u} - B_{t,T}K}{V_t} + \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V}$$

In the next step, the function $f(u,w)$ is linearized using Taylor series expansion.

$$f(u) \approx f(p) + \frac{\partial f(p)}{\partial p}(u-p) = b + m(u-p)$$

where the parameters b and m are given as follows:

$$b = \frac{\ln \frac{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}{V_t} + \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V},$$

$$m = \frac{\bar{\sigma}_S}{\bar{\sigma}_V}\frac{S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p}}{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K},$$

Furthermore, the denominator in the third and fourth integral needs to be modified as well using the first order Taylor series expansion.

$$F(u) = \frac{1}{B_{t,T}\,\bar{D} + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,u} - B_{t,T}K}$$

$$G(u) = \ln\frac{1}{B_{t,T}\,\bar{D} + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,u} - B_{t,T}K}$$

$$\approx G(p) + \frac{\partial G(p)}{\partial p}(u-p) = h + g(u-p)$$

with

$$h = \ln\frac{1}{B_{t,T}\,\bar{D} + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}K}$$

$$g = \frac{-\bar{\sigma}_S\,S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p}}{B_{t,T}\,\bar{D} + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}K}$$

Using the Taylor series approximations, the pricing equation for a vulnerable European call can be rewritten as follows:

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2}\int_{-a}^{\infty}\int_{b+m(u-p)}^{\infty} e^{\bar{\sigma}_S\,u}\,n_2(u,v)\,dv\,du$$

$$- B_{t,T}\,K\int_{-a}^{\infty}\int_{b+m(u-p)}^{\infty} n_2(u,v)\,dv\,du$$

$$+ \frac{(1-\alpha)\,S_t V_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\,\bar{D} + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}K}\int_{-a}^{\infty}\int_{-\infty}^{b+m(u-p)} e^{(g+\bar{\sigma}_S)\,u+\bar{\sigma}_V\,v}\,n_2(u,v)\,dv\,du$$

$$- \frac{(1-\alpha)\,B_{t,T}KV_t\,e^{-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\,\bar{D} + S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p} - B_{t,T}K}\int_{-a}^{\infty}\int_{-\infty}^{b+m(u-p)} e^{gu+\bar{\sigma}_V\,v}\,n_2(u,v)\,dv\,du$$

Using appropriate substitutions for u and v, the stochastic component in the integral boundaries can be eliminated. The random variables u, v and w are substituted by

$$u = \frac{x}{\sqrt{1+m^2}},$$

and

$$v = y + \frac{mx}{\sqrt{1+m^2}}$$

where x and y are also jointly standard normally distributed.

Applying these substitutions to the pricing equation yields

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{b-mp}^{\infty} e^{\frac{\bar{\sigma}_S}{\sqrt{1+m^2}}x} \frac{\Omega(x,y)}{\Theta}\, dy\, dx$$

$$- B_{t,T}\, K \int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{b-mp-m_2p_2}^{\infty} \frac{\Omega(x,y)}{\Theta}\, dy\, dx$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$\int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{-\infty}^{b-mp} e^{\frac{g+\bar{\sigma}_S+m\bar{\sigma}_V}{\sqrt{1+m^2}}x+\bar{\sigma}_V\, y} \frac{\Omega(x,y)}{\Theta}\, dy\, dx$$

$$- \frac{(1-\alpha)\, B_{t,T} K V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$\int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{-\infty}^{b-mp} e^{\frac{g+m\bar{\sigma}_V}{\sqrt{1+m^2}}x+\bar{\sigma}_V\, y} \frac{\Omega(x,y)}{\Theta}\, dy\, dx,$$

where

$$\Omega(x,y) = e^{-\frac{1}{2\left(1-\bar{\rho}_{SV}^2\right)}\left(\left(\frac{x}{\sqrt{1+m^2}}\right)^2+\left(y+\frac{mx}{\sqrt{1+m^2}}\right)^2-2\,\bar{\rho}_{SV}\frac{x}{\sqrt{1+m^2}}\left(y+\frac{mx}{\sqrt{1+m^2}}\right)\right)}$$

and

$$\Theta = \sqrt{1+m^2}\,\sqrt{4\pi^2}\sqrt{1-\bar{\rho}_{SV}^2}.$$

The previous expression can be rewritten as follows:

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{b-mp}^{\infty} e^{\frac{\bar{\sigma}_S \sqrt{T-t}}{\sqrt{1+m^2}} x} \, \frac{\Psi(x,y)}{\Theta} \, dy \, dx$$

$$- B_{t,T}\, K \int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{b-mp}^{\infty} \frac{\Psi(x,y)}{\Theta} \, dy \, dx$$

$$+ \frac{(1-\alpha)\, S_t V_t \, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\, \bar{D} + S_t \, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T} K}$$

$$\int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{-\infty}^{b-mp} e^{\frac{g+\bar{\sigma}_S+m\bar{\sigma}_V}{\sqrt{1+m^2}} x + \bar{\sigma}_V\, y} \, \frac{\Psi(x,y,z)}{\Theta} \, dy \, dx$$

$$- \frac{(1-\alpha)\, B_{t,T} K V_t \, e^{-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\, \bar{D} + S_t \, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T} K}$$

$$\int\limits_{-a\sqrt{1+m^2}}^{\infty} \int\limits_{-\infty}^{b-mp} e^{\frac{g+m\bar{\sigma}_V}{\sqrt{1+m^2}} x + \bar{\sigma}_V\, y} \, \frac{\Psi(x,y)}{\Theta} \, dy \, dx,$$

where

$$\Psi(x,y) = e^{-\frac{1}{2\left(1-\delta_{SV}^2\right)} + \left(\frac{x}{\sqrt{1+m^2}}\right)^2 + \left(\frac{y}{\sqrt{1-2\,\bar{\rho}_{SV}\, m+m^2}}\right)^2 - 2\,\delta_{SV}\, \frac{x}{\sqrt{1+m^2}} \frac{y}{\sqrt{1-2\,\bar{\rho}_{SV}\, m+m^2}}}$$

and

$$\delta_{SV} = \frac{\bar{\rho}_{SV} - m}{\sqrt{1-2\,\bar{\rho}_{SV} + m^2}},$$

Using appropriate substitutions for x, y and z, the previous expression can be rewritten once again. In particular, the variables x, y and z are substituted by

$$x = \sqrt{1 + m^2}\, u,$$

and

$$y = \sqrt{1 - 2\,\bar{\rho}_{SV}\, m - 2\,\bar{\rho}_{VD}\, m_2 + m^2 + m_2^2}\, v$$

where u and v are also jointly standard normally distributed.

Applying these substitutions to the pricing equation yields

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int_{-a}^{\infty}\int_{c}^{\infty} e^{\bar{\sigma}_S\, u}\, \Gamma(u,v)\, dv\, du$$

$$- B_{t,T}\, K \int_{-a}^{\infty}\int_{c}^{\infty} \Gamma(u,v)\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{c} e^{(g+\bar{\sigma}_S+m\bar{\sigma}_V)u}\; e^{\bar{\sigma}_V\,\sqrt{1-2\,\bar{\rho}_{SV}\, m+m^2}\, v}\, \Gamma(u,v)\, dv\, du$$

$$- \frac{(1-\alpha)\, B_{t,T} K V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-gp}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{c} e^{(g+m\bar{\sigma}_V)u}\; e^{\bar{\sigma}_V\,\sqrt{1-2\,\bar{\rho}_{SV}\, m+m^2}\, v}\, \Gamma(u,v)\, dv\, du$$

where

$$c = \frac{b - mp}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, mm^2}}$$

and

$$\Gamma(u,v) = n_2(u,v,0,0,1,1,\delta_{SV}) = \frac{1}{\sqrt{4\pi^2}\sqrt{1-\delta_{SV}^2}}\, e^{u^2+v^2-2\delta_{SV}\, uv}$$

Completing the square yields

$$C = S_t e^{-q(T-t)} \int\limits_{-a}^{\infty} \int\limits_{c}^{\infty} n_2(u, v, \bar{\sigma}_s, \delta_{SV} \bar{\sigma}_s, 1, 1, \delta_{SV})\, dv\, du$$

$$- B_{t,T}\, K \int\limits_{-a}^{\infty} \int\limits_{c}^{\infty} n_2(u, v, 0, 0, 1, 1, \delta_{SV})\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 - \frac{1}{2}\bar{\sigma}_V^2 - gp}\, e^{\frac{1}{2}\left(\eta^2 + \phi^2 + 2\delta_{SV}\, \eta\, \phi\right)}}{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p} - B_{t,T} K}$$

$$\int\limits_{-a}^{\infty} \int\limits_{-\infty}^{c} n_2(u, v, \eta + \delta_{SV}\, \phi, \phi + \delta_{SV}\, \eta, \delta_{SV})\, dv\, du$$

$$- \frac{(1-\alpha)\, B_{t,T} K V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2 - gp}\, e^{\frac{1}{2}\left(\xi^2 + \phi^2 + \lambda^2 + 2\delta_{SV}\, \xi\, \phi + 2\, \delta_{VD}\, \phi\, \lambda\right)}}{B_{t,T}\, \bar{D} + S_t\, e^{-q(T-t) - \frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p} - B_{t,T} K}$$

$$\int\limits_{-a}^{\infty} \int\limits_{-\infty}^{c} n_2(u, v, \xi + \delta_{SV}\, \phi, \phi + \delta_{SV}\, \xi, 1, 1, delta_{SV})\, dv\, du,$$

where

$$\eta = g + \bar{\sigma}_S + m\, \bar{\sigma}_V,$$

$$\xi = \eta - \bar{\sigma}_S,$$

$$\phi = \bar{\sigma}_V\, \sqrt{1 - 2\, \bar{\rho}_{SV}\, m + m^2}.$$

Standardizing the normal distribution yields

$$C = S_t e^{-q(T-t)} \int_{-a-\bar{\sigma}_s}^{\infty} \int_{c-\delta_{SV}\bar{\sigma}_s}^{\infty} n_2(u, v, 0, 0, 1, 1, \delta_{SV})\, dv\, du$$

$$- B_{t,T}\, K \int_{-a}^{\infty} \int_{c}^{\infty} n_2(u, v, 0, 0, 1, 1, \delta_{SV})\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-gp}\, e^{\frac{1}{2}\left(\eta^2+\phi^2+2\delta_{SV}\,\eta\,\phi\right)}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$\int_{-a-\eta-\delta_{SV}\,\phi}^{\infty} \int_{-\infty}^{c-\phi-\delta_{SV}\,\eta} n_2(u, v, 0, 0, 1, 1, \delta_{SV})\, dv\, du$$

$$- \frac{(1-\alpha)\, B_{t,T} K V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-gp}\, e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$\int_{-a-\xi-\delta_{SV}\,\phi}^{\infty} \int_{-\infty}^{c-\phi-\delta_{SV}\,\xi} n_2(u, v, 0, 0, 1, 1, \delta_{SV})\, dv\, du.$$

Computing the double integrals yields the approximate closed form solution for vulnerable European calls based on the extended model of Klein and Inglis (2001). It is given by

$$C = S_t e^{-q(T-t)}\, N_2(a + \bar{\sigma}_S, -c + \delta_{SV}\bar{\sigma}_S, \delta_{SV}) - B_{t,T}\, K\, N_2(a, -c, \delta_{SV})$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-gp}\, e^{\frac{1}{2}\left(\eta^2+\phi^2+2\delta_{SV}\,\eta\,\phi\right)}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$N_2(a + \eta + \delta_{SV}\phi, c - \phi - \delta_{SV}\eta, -\delta_{SV})$$

$$- \frac{(1-\alpha)\, B_{t,T} K V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-gp}\, e^{\frac{1}{2}\left(\xi^2+\phi^2+2\delta_{SV}\,\xi\,\phi\right)}}{B_{t,T}\,\bar{D} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p} - B_{t,T}K}$$

$$N_2(a + \xi + \delta_{SV}\phi, c - \phi - \delta_{SV}\xi, -\delta_{SV})$$

where $N_2(\,\cdot\,)$ gives the bivariate cumulative normal distribution function.

Collecting and rearranging terms yields the approximate closed form valuation formula given by Equation (4.48).

Appendix 3

In the following, the closed form valuation formula for vulnerable European options under the stochastic interest rate framework of Vasicek (1977) is derived based on the extended model of Liu and Liu (2011). The derivation of the valuation formula is only given for vulnerable European calls, but the same procedure can also be used to get the valuation formula for vulnerable European puts.

The pricing equation for a vulnerable European call based on the extended model of Liu and Liu (2011) can be written as follows:

$$C = B_{t,T}\left(E\left[S_T - K \mid S_T \geq K,\, V_T \geq D_T\right] \right.$$
$$\left. + E\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{D_T}) \mid S_T \geq K,\, V_T < D_T\right]\right).$$

Defining the debt ratio as $R_t = V_t/D_t$, the pricing equation can be rewritten as follows:

$$C = B_{t,T}\left(E\left[S_T - K \mid S_T \geq K,\, R_T \geq 1\right] \right.$$
$$\left. + E\left[(1-\alpha)\, R_T\, (S_T - K) \mid S_T \geq K,\, R_T < 1\right]\right).$$

Using the risk-neutral pricing approach, the value of the vulnerable European call is given by

$$C = B_{t,T}\left(\int_K^{\infty}\int_1^{\infty} S_T\, \Phi(S_T, R_T)\, dR_T\, dS_T \right.$$
$$- \int_K^{\infty}\int_1^{\infty} K\, \Phi(S_T, R_T)\, dR_T\, dS_T$$
$$+ \int_K^{\infty}\int_0^{1} (1-\alpha)\, R_T\, S_T\, \Phi(S_T, R_T)\, dR_T\, dS_T$$
$$\left. - \int_K^{\infty}\int_0^{1} (1-\alpha)\, R_T\, K\, \Phi(S_T, R_T)\, dR_T\, dS_T\right),$$

where $\Phi(\,\cdot\,)$ is the joint bivariate lognormal distribution function of S_T and R_T.

Applying the standard log transformation, standardizing the normal distribution and collecting terms yields

$$
\begin{aligned}
C = & \int_{-a}^{\infty}\int_{-b}^{\infty} S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u}\, n_2(u,v)\, dv\, du \\
& - \int_{-a}^{\infty}\int_{-b}^{\infty} B_{t,T}\, K\, n_2(u,v)\, dv\, du \\
& + \int_{-a}^{\infty}\int_{-\infty}^{-b} (1-\alpha)\, S_t\, R_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\left(\bar{\sigma}_V^2-\bar{\sigma}_D^2\right)+\bar{\sigma}_S\, u+\bar{\sigma}_R\, v}\, n_2(u,v)\, dv\, du \\
& - \int_{-a}^{\infty}\int_{-\infty}^{-b} (1-\alpha)\, B_{t,T}\, K\, R_t\, e^{-\frac{1}{2}\left(\bar{\sigma}_V^2-\bar{\sigma}_D^2\right)+\bar{\sigma}_R\, v}\, n_2(u,v)\, dv\, du,
\end{aligned}
$$

where $n_2(\,\cdot\,)$ is the joint bivariate standard normal density function of the random variables u and v which is given by

$$
\begin{aligned}
& n_2(u,v) = n_2(u,v,0,0,1,1,\delta_{SR}) = \frac{1}{\sqrt{4\pi^2}\sqrt{1-\delta_{SR}^2}}\, e^{-\frac{1}{2\left(1-\delta_{SR}^2\right)}\left(u^2+v^2-2\delta_{SR}\, uv\right)}, \\
& \bar{\sigma}_R = \sqrt{\bar{\sigma}_V^2+\bar{\sigma}_D^2-2\bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D}
\end{aligned}
$$

and

$$
\delta_{SR} = \frac{\bar{\rho}_{SV}\bar{\sigma}_V - \bar{\rho}_{SD}\bar{\sigma}_D}{\bar{\sigma}_R}.
$$

The parameters a and b are given as follows:

$$
\begin{aligned}
a &= \frac{\ln \frac{S_t}{-b_{t,T}K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S}, \\
b &= \frac{\ln R_t - \frac{1}{2}\left(\bar{\sigma}_V^2 - \sigma_D^2\right)}{\bar{\sigma}_R}
\end{aligned}
$$

Completing the square yields

$$C = S_t\, e^{-q(T-t)} \int_{-a}^{\infty}\int_{-b}^{\infty} n_2(u, v, \bar\sigma_S, \delta_{SR}\bar\sigma_S, 1, 1, \delta_{SR})\, dv\, du$$

$$- B_{t,T}\, K \int_{-a}^{\infty}\int_{-b}^{\infty} n_2(u, v, 0, 0, 1, 1, \delta_{SR})\, dv\, du$$

$$+ (1-\alpha)\, S_t\, R_t\, e^{-q(T-t)+\sigma_D^2+\bar\rho_{SV}\bar\sigma_S\bar\sigma_V-\bar\rho_{SD}\bar\sigma_S\bar\sigma_D-\bar\rho_{VD}\bar\sigma_V\bar\sigma_D}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{-b} n_2(u, v, \bar\sigma_S+\delta_{SR}\bar\sigma_R, \bar\sigma_R+\delta_{SR}\bar\sigma_S, 1, 1, \delta_{SR})\, dv\, du$$

$$- (1-\alpha)\, B_{t,T}\, K\, R_t\, e^{\bar\sigma_D^2-\bar\rho_{VD}\bar\sigma_V\bar\sigma_D}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{-b} n_2(u, v, \delta_{SR}\bar\sigma_R, \bar\sigma_R, 1, 1, \delta_{SR})\, dv\, du.$$

Standardizing the normal distribution gives

$$C = S_t\, e^{-q(T-t)} \int_{-a-\bar\sigma_S}^{\infty}\int_{-b-\delta_{SR}\bar\sigma_S}^{\infty} n_2(u, v, 0, 0, 1, 1, \delta_{SR})\, dv\, du$$

$$- B_{t,T}\, K \int_{-a}^{\infty}\int_{-b}^{\infty} n_2(u, v, 0, 0, 1, 1, \delta_{SR})\, dv\, du$$

$$+ (1-\alpha)\, S_t\, R_t\, e^{-q(T-t)+\sigma_D^2+\bar\rho_{SV}\bar\sigma_S\bar\sigma_V-\bar\rho_{SD}\bar\sigma_S\bar\sigma_D-\bar\rho_{VD}\bar\sigma_V\bar\sigma_D}$$

$$\int_{-a-\bar\sigma_S-\delta_{SR}\bar\sigma_R}^{\infty}\int_{-\infty}^{-b-\bar\sigma_R-\delta_{SR}\bar\sigma_S} n_2(u, v, 0, 0, 1, 1, \delta_{SR})\, dv\, du$$

$$- (1-\alpha)\, B_{t,T}\, K\, R_t\, e^{\bar\sigma_D^2-\bar\rho_{VD}\bar\sigma_V\bar\sigma_D}$$

$$\int_{-a-\delta_{SR}\bar\sigma_R}^{\infty}\int_{-\infty}^{-b-\bar\sigma_R} n_2(u, v, 0, 0, 1, 1, \delta_{SR})\, dv\, du.$$

Computing the double integrals yields

$$
\begin{aligned}
C = {} & S_t\, e^{-q(T-t)} N_2\left(a + \bar{\sigma}_S, b + \delta_{SR}\bar{\sigma}_S, \delta_{SR}\right) \\
& - B_{t,T}\, K N_2\left(a, b, \delta_{SR}\right) \\
& + (1-\alpha)\, S_t\, R_t\, e^{-q(T-t) + \sigma_D^2 + \bar{\rho}_{SV}\bar{\sigma}_S\bar{\sigma}_V - \bar{\rho}_{SD}\bar{\sigma}_S\bar{\sigma}_D - \bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D} \\
& \qquad N_2\left(a + \bar{\sigma}_S + \delta_{SR}\bar{\sigma}_R, -b - \bar{\sigma}_R - \delta_{SR}\bar{\sigma}_S, -\delta_{SR}\right) \\
& - (1-\alpha)\, B_{t,T}\, K\, R_t\, e^{\bar{\sigma}_D^2 - \bar{\rho}_{VD}\bar{\sigma}_V\bar{\sigma}_D} \\
& \qquad N_2\left(a + \delta_{SR}\bar{\sigma}_R, -b - \bar{\sigma}_R, -\delta_{SR}\right),
\end{aligned}
$$

where $N_2(\,\cdot\,)$ gives the bivariate cumulative normal distribution function.

Collecting and rearranging terms yields the closed form valuation formula given by Equation (4.53).

Appendix 4

In the following, the approximate closed form valuation formula for vulnerable European options under the stochastic interest rate framework of Vasicek (1977) is derived based on the general model. The derivation of the valuation formula is only given for vulnerable European calls, but the same procedure can also be used to get the valuation formula for vulnerable European puts. To obtain the valuation formula, it must be assumed that the returns of the option's underlying and the counterparty's other liabilities are uncorrelated (i.e. $\rho_{SD} = 0$).

The pricing equation for a vulnerable European call based on the general model can be written as follows:

$$C = B_{t,T} \Bigg(E\left[S_T - K \mid S_T \geq K,\, V_T \geq D_T + S_T - K \right] + E\left[\frac{(1-\alpha)\, V_T\, (S_T - K)}{D_T + S_T - K} \mid S_T \geq K,\, V_T < D_T + S_T - K \right] \Bigg).$$

Using the risk-neutral pricing approach, the value of the vulnerable European call is given by

$$\begin{aligned} C = B_{t,T} \Bigg(& \int_K^{\infty} \int_{D_T+S_T-K}^{\infty} \int_0^{\infty} S_T\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \\ & - \int_K^{\infty} \int_{D_T+S_T-K}^{\infty} \int_0^{\infty} K\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \\ & + \int_K^{\infty} \int_0^{D_T+S_T-K} \int_0^{\infty} \frac{(1-\alpha)\, V_T\, S_T}{D_T + S_T - K}\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \\ & - \int_K^{\infty} \int_0^{D_T+S_T-K} \int_0^{\infty} \frac{(1-\alpha)\, V_T\, K}{D_T + S_T - K}\, \Phi(S_T, V_T, D_T)\, dD_T\, dV_T\, dS_T \Bigg), \end{aligned}$$

where $\Phi(\,\cdot\,)$ represents the joint trivariate lognormal distribution function of S_T, V_T and D_T.

Applying the standard log transformation, standardizing the normal distribution and collecting terms yields

$$
\begin{aligned}
C = & \int_{-a}^{\infty} \int_{f(u,w)}^{\infty} \int_{-\infty}^{\infty} S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u}\, n_3(u,v,w)\, dw\, dv\, du \\
& - \int_{-a}^{\infty} \int_{f(u,w)}^{\infty} \int_{-\infty}^{\infty} B_{t,T}\, K\, n_3(u,v,w)\, dw\, dv\, du \\
& + \int_{-a}^{\infty} \int_{-\infty}^{f(u,w)} \int_{-\infty}^{\infty} \frac{(1-\alpha)\, S_t\, V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2+\bar{\sigma}_S\, u+\bar{\sigma}_V\, v}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, w}+S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u}-B_{t,T}\, K} \\
& \qquad \cdot\, n_3(u,v,w)\, dw\, dv\, du \\
& - \int_{-a}^{\infty} \int_{-\infty}^{f(u,w)} \int_{-\infty}^{\infty} \frac{(1-\alpha)\, B_{t,T}\, K\, V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2+\bar{\sigma}_V\, v}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, w}+S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u}-B_{t,T}\, K} \\
& \qquad \cdot\, n_3(u,v,w)\, dw\, dv\, du,
\end{aligned}
$$

where $n_3(\,\cdot\,)$ is the joint trivariate standard normal density function of the random variables u, v and w which is given by

$$
\begin{aligned}
n_3(u,v,w) &= n_3(u,v,w,0,0,0,1,1,1,\bar{\rho}_{SV},\bar{\rho}_{SD}=0,\bar{\rho}_{VD}) \\
&= \frac{e^{-\frac{1}{2\left(1-\bar{\rho}_{SV}^2-\bar{\rho}_{VD}^2\right)}\left((1-\bar{\rho}_{VD}^2)u^2+v^2+(1-\bar{\rho}_{SV}^2)w^2-2\bar{\rho}_{SV}\, uv+2\bar{\rho}_{SV}\bar{\rho}_{VD}uw-2\bar{\rho}_{VD}vw\right)}}{\sqrt{8\pi^3}\sqrt{1-\bar{\rho}_{SV}^2-\bar{\rho}_{VD}^2}}
\end{aligned}
$$

The parameter a as well as the function $f(\,\cdot\,)$ are given as follows:

$$
a = \frac{\ln \frac{S_t}{B_{t,T}K} - q(T-t) - \frac{1}{2}\bar{\sigma}_S^2}{\bar{\sigma}_S}
$$

$$
f(u,w) = \frac{\ln \frac{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, w}+S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, u}-B_{t,T}K}{V_t} + \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V}
$$

In the next step, the function $f(u,w)$ is linearized using Taylor series expansion.

$$
\begin{aligned}
f(u,w) &\approx f(p_1,p_2) + \frac{\partial f(p_1,p_2)}{\partial p_1}(u-p_1) + \frac{\partial f(p-1,p_2)}{\partial p_2}(w-p_2) \\
&= b + m_1(u-p_1) + m_2(w-p_2)
\end{aligned}
$$

where the parameters b, m_1 and m_2 are given as follows:

$$
b = \frac{\ln \frac{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}K}{V_t} + \frac{1}{2}\bar{\sigma}_V^2}{\bar{\sigma}_V},
$$

$$
m_1 = \frac{\bar{\sigma}_S}{\bar{\sigma}_V} \frac{S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}K},
$$

$$
m_2 = \frac{\bar{\sigma}_D}{\bar{\sigma}_V} \frac{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}K}.
$$

Furthermore, the denominator in the third and fourth integral needs to be modified as well using the first order Taylor series expansion.

$$
F(u,w) = \frac{1}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, w} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, u} - B_{t,T}K}
$$

$$
\begin{aligned}
G(u,w) &= \ln \frac{1}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, w} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, u} - B_{t,T}K} \\
&\approx G(p_1,p_2) + \frac{\partial G(p_1,p_2)}{\partial p_1}(u-p_1) + \frac{\partial G(p_1,p_2)}{\partial p_2}(w-p_2) \\
&= h + g_1(u-p_1) + g_2(w-p_2)
\end{aligned}
$$

with

$$
h = \ln \frac{1}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}K}
$$

$$
g_1 = \frac{-\bar{\sigma}_S\, S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}K}
$$

$$
g_2 = \frac{-\bar{\sigma}_D\, D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S\, p_1} - B_{t,T}K}
$$

Using the Taylor series approximations, the pricing equation for a vulnerable European call can be rewritten as follows:

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int_{-a}^{\infty} \int_{b+m_1(u-p_1)+m_2(w-p_2)}^{\infty} \int_{-\infty}^{\infty} e^{\bar{\sigma}_S\, u}\, n_3(u,v,w)\, dw\, dv\, du$$

$$- B_{t,T}\, K \int_{-a}^{\infty} \int_{b+m_1(u-p_1)+m_2(w-p_2)}^{\infty} \int_{-\infty}^{\infty} n_3(u,v,w)\, dw\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int_{-a}^{\infty} \int_{-\infty}^{b+m_1(u-p_1)+m_2(w-p_2)} \int_{-\infty}^{\infty} e^{(g_1+\bar{\sigma}_S)\, u+\bar{\sigma}_V\, v+g_2w}\, n_3(u,v,w)\, dw\, dv\, du$$

$$- \frac{(1-\alpha)\, B_{t,T}KV_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int_{-a}^{\infty} \int_{-\infty}^{b+m_1(u-p_1)+m_2(w-p_2)} \int_{-\infty}^{\infty} e^{g_1u+\bar{\sigma}_V\, v+g_2w}\, n_3(u,v,w)\, dw\, dv\, du$$

Using appropriate substitutions for u, v and w, the stochastic component in the integral boundaries can be eliminated. The random variables u, v and w are substituted by

$$u = \frac{x}{\sqrt{1+m_1^2}},$$

$$v = y + \frac{m_1 x}{\sqrt{1+m_1^2}} + \frac{m_2 z}{\sqrt{1+m_2^2}}$$

and

$$w = \frac{z}{\sqrt{1+m_2^2}},$$

where x, y and z are also jointly standard normally distributed.

Applying these substitutions to the pricing equation yields

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{b-m_1p_1-m_2p_2}^{\infty} \int\limits_{-\infty}^{\infty} e^{\frac{\bar{\sigma}_S}{\sqrt{1+m_1^2}}x} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx$$

$$- B_{t,T}\, K \int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{b-m_1p-m_2p_2}^{\infty} \int\limits_{-\infty}^{\infty} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{-\infty}^{b-m_1p_1-m_2p_2} \int\limits_{-\infty}^{\infty} e^{\frac{g_1+\bar{\sigma}_S+m_1\bar{\sigma}_V}{\sqrt{1+m_1^2}}x+\bar{\sigma}_V\, y+\frac{g_2+m_2\bar{\sigma}_V}{\sqrt{1+m_2^2}}z} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx$$

$$- \frac{(1-\alpha)\, B_{t,T}KV_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int\limits_{-a\sqrt{1+m_1^2}}^{\infty} \int\limits_{-\infty}^{b-m_1p-m_2p_2} \int\limits_{-\infty}^{\infty} e^{\frac{g_1+m_1\bar{\sigma}_V}{\sqrt{1+m_1^2}}x+\bar{\sigma}_V\, y+\frac{g_2+m_2\bar{\sigma}_V}{\sqrt{1+m_2^2}}z} \frac{\Omega(x,y,z)}{\Theta}\, dz\, dy\, dx,$$

where

$$\Omega(x,y,z) = e^{-\frac{1}{2\left(1-\bar{\rho}_{SV}^2-\bar{\rho}_{VD}^2\right)}} \; e^{(1-\bar{\rho}_{VD}^2)\left(\frac{x}{\sqrt{1+m_1^2}}\right)^2+\left(y+\frac{m_1x}{\sqrt{1+m_1^2}}+\frac{m_2z}{\sqrt{1+m_2^2}}\right)^2+(1-\bar{\rho}_{SV}^2)\left(\frac{z}{\sqrt{1+m_2^2}}\right)^2}$$

$$e^{-2\,\bar{\rho}_{SV}\frac{x}{\sqrt{1+m_1^2}}\left(y+\frac{m_1x}{\sqrt{1+m_1^2}}+\frac{m_2z}{\sqrt{1+m_2^2}}\right)} \; e^{2\,\bar{\rho}_{SV}\,\bar{\rho}_{VD}\frac{x}{\sqrt{1+m_1^2}}\frac{z}{\sqrt{1+m_2^2}}}$$

$$e^{-2\,\bar{\rho}_{VD}\frac{z}{\sqrt{1+m_2^2}}\left(y+\frac{m_1x}{\sqrt{1+m_1^2}}+\frac{m_2z}{\sqrt{1+m_2^2}}\right)}$$

and

$$\Theta = \sqrt{1+m_1^2}\,\sqrt{1+m_2^2}\,\sqrt{8\pi^3}\sqrt{1-\bar{\rho}_{SV}^2-\bar{\rho}_{VD}^2}.$$

The previous expression can be rewritten as follows:

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int_{-a\sqrt{1+m_1^2}}^{\infty} \int_{b-m_1 p_1 - m_2 p_2}^{\infty} \int_{-\infty}^{\infty} e^{\frac{\bar{\sigma}_S \sqrt{T-t}}{\sqrt{1+m_1^2}} x} \frac{\Psi(x,y,z)}{\Theta} \, dz \, dy \, dx$$

$$- B_{t,T} \, K \int_{-a\sqrt{1+m_1^2}}^{\infty} \int_{b-m_1 p - m_2 p_2}^{\infty} \int_{-\infty}^{\infty} \frac{\Psi(x,y,z)}{\Theta} \, dz \, dy \, dx$$

$$+ \frac{(1-\alpha) \, S_t V_t \, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 - \frac{1}{2}\bar{\sigma}_V^2 - g_1 p_1 - g_2 p_2}}{D_t \, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D \, p_2} + S_t \, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S \, p_1} - B_{t,T} K}$$

$$\int_{-a\sqrt{1+m_1^2}}^{\infty} \int_{-\infty}^{b-m_1 p_1 - m_2 p_2} \int_{-\infty}^{\infty} e^{\frac{g_1 + \bar{\sigma}_S + m_1 \bar{\sigma}_V}{\sqrt{1+m_1^2}} x + \bar{\sigma}_V \, y + \frac{g_2 + m_2 \bar{\sigma}_V}{\sqrt{1+m_2^2}} z} \frac{\Psi(x,y,z)}{\Theta} \, dz \, dy \, dx$$

$$- \frac{(1-\alpha) \, B_{t,T} K V_t \, e^{-\frac{1}{2}\bar{\sigma}_V^2 - g_1 p_1 - g_2 p_2}}{D_t \, e^{-\frac{1}{2}\bar{\sigma}_D^2 + \bar{\sigma}_D \, p_2} + S_t \, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2 + \bar{\sigma}_S \, p_1} - B_{t,T} K}$$

$$\int_{-a\sqrt{1+m_1^2}}^{\infty} \int_{-\infty}^{b-m_1 p - m_2 p_2} \int_{-\infty}^{\infty} e^{\frac{g_1 + m_1 \bar{\sigma}_V}{\sqrt{1+m_1^2}} x + \bar{\sigma}_V \, y + \frac{g_2 + m_2 \bar{\sigma}_V}{\sqrt{1+m_2^2}} z} \frac{\Psi(x,y,z)}{\Theta} \, dz \, dy \, dx,$$

where

$$\Psi(x,y,z) = e^{-\frac{1}{2\left(1-\delta_{SV}^2-\delta_{VD}^2\right)}} \, e^{(1-\delta_{VD}^2)\left(\frac{x}{\sqrt{1+m_1^2}}\right)^2} \, e^{\left(\frac{y}{\sqrt{1-2\,\bar{\rho}_{SV}\,m_1 - 2\,\bar{\rho}_{VD}\,m_2 + m_1^2 + m_2^2}}\right)^2}$$

$$e^{(1-\delta_{SV}^2)\left(\frac{z}{\sqrt{1+m_2^2}}\right)^2} \, e^{-2\,\delta_{SV} \frac{x}{\sqrt{1+m_1^2}} \frac{y}{\sqrt{1-2\,\bar{\rho}_{SV}\,m_1 - 2\,\bar{\rho}_{VD}\,m_2 + m_1^2 + m_2^2}}}$$

$$e^{2\,\delta_{SV}\,\delta_{VD} \frac{x}{\sqrt{1+m_1^2}} \frac{z}{\sqrt{1+m_2^2}}} \, e^{-2\,\delta_{VD} \frac{z}{\sqrt{1+m_2^2}} \frac{y}{\sqrt{1-2\,\delta_{SV}\,m_1 - 2\,\delta_{VD}\,m_2 + m_1^2 + m_2^2}}}$$

and

$$\delta_{SV} = \frac{\bar{\rho}_{SV} - m_1}{\sqrt{1 - 2\,\bar{\rho}_{SV}\,m_1 - 2\,\bar{\rho}_{VD}\,m_2 + m_1^2 + m_2^2}},$$

$$\delta_{VD} = \frac{\bar{\rho}_{VD} - m_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\,m_1 - 2\,\bar{\rho}_{VD}\,m_2 + m_1^2 + m_2^2}}.$$

Using appropriate substitutions for x, y and z, the previous expression can be rewritten once again. In particular, the variables x, y and z are substituted by

$$x = \sqrt{1+m_1^2}\, u,$$

$$y = \sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}\, v$$

and

$$z = \sqrt{1+m_2^2}\, w,$$

where u, v and w are also jointly standard normally distributed.

Applying these substitutions to the pricing equation yields

$$C = S_t e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2} \int\limits_{-a}^{\infty}\int\limits_{c}^{\infty}\int\limits_{-\infty}^{\infty} e^{\bar{\sigma}_S\, u}\, \Gamma(u,v,w)\, dw\, dv\, du$$

$$- B_{t,T}\, K \int\limits_{-a}^{\infty}\int\limits_{c}^{\infty}\int\limits_{-\infty}^{\infty} \Gamma(u,v,w)\, dw\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int\limits_{-a}^{\infty}\int\limits_{-\infty}^{c}\int\limits_{-\infty}^{\infty} e^{(g_1+\bar{\sigma}_S+m_1\bar{\sigma}_V)u}\; e^{\bar{\sigma}_V\,\sqrt{1-2\,\bar{\rho}_{SV}\, m_1-2\,\bar{\rho}_{VD}\, m_2+m_1^2+m_2^2}\, v}$$

$$e^{(g_2+m_2\bar{\sigma}_V)w}\, \Gamma(u,v,w)\, dw\, dv\, du$$

$$- \frac{(1-\alpha)\, B_{t,T}KV_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int\limits_{-a}^{\infty}\int\limits_{-\infty}^{c}\int\limits_{-\infty}^{\infty} e^{(g_1+m_1\bar{\sigma}_V)u}\; e^{\bar{\sigma}_V\,\sqrt{1-2\,\bar{\rho}_{SV}\, m_1-2\,\bar{\rho}_{VD}\, m_2+m_1^2+m_2^2}\, v}$$

$$e^{(g_2+m_2\bar{\sigma}_V)w}\, \Gamma(u,v,w)\, dw\, dv\, du$$

where

$$c = \frac{b - m_1p_1 - m_2p_2}{\sqrt{1 - 2\,\bar{\rho}_{SV}\, m_1 - 2\,\bar{\rho}_{VD}\, m_2 + m_1^2 + m_2^2}}$$

and

$$\Gamma(u,v,w) = n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},\delta_{SD}=0,\delta_{VD})$$

$$= \frac{e^{-\frac{1}{2\left(1-\delta_{SV}^2-\delta_{VD}^2\right)}\left((1-\delta_{VD}^2)u^2+v^2+(1-\delta_{SV}^2)w^2-2\delta_{SV}\,uv+2\delta_{SV}\delta_{VD}uw-2\delta_{VD}vw\right)}}{\sqrt{8\pi^3}\sqrt{1-\delta_{SV}^2-\delta_{VD}^2}}$$

Completing the square yields

$$C = S_t e^{-q(T-t)} \int_{-a}^{\infty}\int_{c}^{\infty}\int_{-\infty}^{\infty} n_3(u,v,w,\bar{\sigma}_s,\delta_{SV}\bar{\sigma}_s,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$- B_{t,T}\,K \int_{-a}^{\infty}\int_{c}^{\infty}\int_{-\infty}^{\infty} n_3(u,v,w,0,0,0,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$+ \frac{(1-\alpha)\,S_t V_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}\,e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\,e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\,p_2}+S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p_1}-B_{t,T}K}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{c}\int_{-\infty}^{\infty} n_3(u,v,w,\eta+\delta_{SV}\,\phi,\phi+\delta_{SV}\,\eta+\delta_{VD}\,\lambda,\lambda+\delta_{VD}\,\phi,...$$

$$...,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du$$

$$- \frac{(1-\alpha)\,B_{t,T}KV_t\,e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}\,e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\,e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\,p_2}+S_t\,e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\,p_1}-B_{t,T}K}$$

$$\int_{-a}^{\infty}\int_{-\infty}^{c}\int_{-\infty}^{\infty} n_3(u,v,w,\xi+\delta_{SV}\,\phi,\phi+\delta_{SV}\,\xi+\delta_{VD}\,\lambda,\lambda+\delta_{VD}\,\phi,...$$

$$...,1,1,1,\delta_{SV},0,\delta_{VD})\,dw\,dv\,du,$$

where

$$\eta = g_1+\bar{\sigma}_S+m_1\,\bar{\sigma}_V,$$

$$\xi = \eta-\bar{\sigma}_S,$$

$$\phi = \bar{\sigma}_V\,\sqrt{1-2\,\bar{\rho}_{SV}\,m_1-2\,\bar{\rho}_{VD}\,m_2+m_1^2+m_2^2},$$

$$\lambda = g_2+m_2\,\bar{\sigma}_V.$$

Standardizing the normal distribution yields

$$C = S_t e^{-q(T-t)} \int\limits_{-a-\bar{\sigma}_s}^{\infty} \int\limits_{c-\delta_{SV}\bar{\sigma}_s}^{\infty} \int\limits_{-\infty}^{\infty} n_3(u, v, w, 0, 0, 0, 1, 1, 1, \delta_{SV}, 0, \delta_{VD})\, dw\, dv\, du$$

$$- B_{t,T}\, K \int\limits_{-a}^{\infty} \int\limits_{c}^{\infty} \int\limits_{-\infty}^{\infty} n_3(u, v, w, 0, 0, 0, 1, 1, 1, \delta_{SV}, 0, \delta_{VD})\, dw\, dv\, du$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}\, e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int\limits_{-a-\eta-\delta_{SV}\,\phi}^{\infty} \int\limits_{-\infty}^{c-\phi-\delta_{SV}\,\eta-\delta_{VD}\,\lambda} \int\limits_{-\infty}^{\infty} n_3(u, v, w, 0, 0, 0, 1, 1, 1, \delta_{SV}, 0, \delta_{VD})\, dw\, dv\, du$$

$$- \frac{(1-\alpha)\, B_{t,T}KV_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}\, e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$\int\limits_{-a-\xi-\delta_{SV}\,\phi}^{\infty} \int\limits_{-\infty}^{c-\phi-\delta_{SV}\,\xi-\delta_{VD}\,\lambda} \int\limits_{-\infty}^{\infty} n_3(u, v, w, 0, 0, 0, 1, 1, 1, \delta_{SV}, 0, \delta_{VD})\, dw\, dv\, du.$$

Computing the triple integrals yields the approximate closed form solution for vulnerable European calls based on the general model. It is given by

$$C = S_t e^{-q(T-t)}\, N_3(a+\bar{\sigma}_S, -c+\delta_{SV}\bar{\sigma}_S+\infty, \delta_{SV}, 0, \delta_{VD})$$

$$- B_{t,T}\, K\, N_3(a, -c, +\infty, \delta_{SV}, 0, \delta_{VD})$$

$$+ \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}\, e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$N_3(a+\eta+\delta_{SV}\phi, c-\phi-\delta_{SV}\eta-\delta_{VD}\lambda, +\infty, -\delta_{SV}, 0, -\delta_{VD})$$

$$- \frac{(1-\alpha)\, B_{t,T}KV_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1p_1-g_2p_2}\, e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K}$$

$$N_3(a+\xi+\delta_{SV}\phi, c-\phi-\delta_{SV}\xi-\delta_{VD}\lambda, +\infty, -\delta_{SV}, 0, -\delta_{VD}),$$

where $N_3(\,\cdot\,)$ gives the trivariate cumulative normal distribution function.

Since the stochastic variable $\ln D_t$ can assume any value between $-\infty$ and $+\infty$, the trivariate cumulative normal ditribution becomes a bivariate cumulative normal distribution. Hence, the approximate closed form solution is given by

$$
\begin{aligned}
C = {} & S_t e^{-q(T-t)}\, N_2(a + \bar{\sigma}_S, -c + \delta_{SV}\bar{\sigma}_S, \delta_{SV}) \\
& - B_{t,T}\, K\, N_2(a, -c, \delta_{SV}) \\
& + \frac{(1-\alpha)\, S_t V_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2-\frac{1}{2}\bar{\sigma}_V^2-g_1 p_1-g_2 p_2}\; e^{\frac{1}{2}\left(\eta^2+\phi^2+\lambda^2+2\delta_{SV}\,\eta\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K} \\
& \qquad N_2(a + \eta + \delta_{SV}\phi, c - \phi - \delta_{SV}\eta - \delta_{VD}\lambda, -\delta_{SV}) \\
& - \frac{(1-\alpha)\, B_{t,T} K V_t\, e^{-\frac{1}{2}\bar{\sigma}_V^2-g_1 p_1-g_2 p_2}\; e^{\frac{1}{2}\left(\xi^2+\phi^2+\lambda^2+2\delta_{SV}\,\xi\,\phi+2\,\delta_{VD}\,\phi\,\lambda\right)}}{D_t\, e^{-\frac{1}{2}\bar{\sigma}_D^2+\bar{\sigma}_D\, p_2} + S_t\, e^{-q(T-t)-\frac{1}{2}\bar{\sigma}_S^2+\bar{\sigma}_S\, p_1} - B_{t,T}K} \\
& \qquad N_2(a + \xi + \delta_{SV}\phi, c - \phi - \delta_{SV}\xi - \delta_{VD}\lambda, -\delta_{SV})
\end{aligned}
$$

where $N_2(\,\cdot\,)$ gives the bivariate cumulative normal distribution function.

Collecting and rearranging terms yields the approximate closed form valuation formula given by Equation (4.59).

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